Quantum Mechanics as a Gauge Theory of Metaplectic Spinor Fields

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Abstract

A hidden gauge theory structure of quantum mechanics which is invisible in its conventional formulation is uncovered. Quantum mechanics is shown to be equivalent to a certain Yang–Mills theory with an infinite–dimensional gauge group and a nondynamical connection. It is defined over an arbitrary symplectic manifold which constitutes the phase–space of the system under consideration. The ”matter fields” are local generalizations of states and observables; they assume values in a family of local Hilbert spaces (and their tensor products) which are attached to the points of phase–space. Under local frame rotations they transform in the spinor representation of the metaplectic group $Mp(2N)$, the double covering of $Sp(2N)$. The rules of canonical quantization are replaced by two independent postulates with a simple group theoretical and differential geometrical interpretation. A novel background–quantum split symmetry plays a central rôle.

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1 Introduction

Both in general relativity and in Yang–Mills theory the principle of local gauge invariance plays a central rôle. At the classical level, these theories are based upon local structures such as the tangent spaces to the space–time manifold or local "color" spaces, as well as on connections (gauge fields) which provide a link between such spaces sitting at different points of space–time. However, at the quantum level, the importance of such local concepts is considerably reduced. One has to introduce nonlocal objects which are not related to any specific point of space–time, and which have no natural interpretation within the classical theory of fiber bundles. The most important object of this kind is the Hilbert space of physical states. According to standard canonical quantization, every quantum system is described by a single "global" Hilbert space whose elements (wave functions) give rise to a probability density over configuration–space. It is one and the same Hilbert space which governs the probabilities at all points of configuration space and it is not possible to associate this Hilbert space to a specific point of configuration space.

Obviously there is a remarkable conceptual clash between the classical geometry of the field theories employed in particle physics and gravity on one hand and the standard formulations of quantum mechanics on the other. In the former case one is dealing with fiber bundles over space–time. The matter fields are "living" in certain vector spaces ("fibers") which are erected over each point of space–time. Vector fields, say, assume values in the tangent spaces of the base manifold, or matter fields carrying nonabelian gauge charge live in local representation spaces of the gauge group. Gauge theories are covariant with respect to independent basis changes (frame rotations) in the fibers over different points of space–time. This covariance is achieved by introducing a connection, or a gauge field, which defines a parallel transport from one fiber to another.

The mathematical framework of quantum (field) theory on the other hand is rather different. Quantizing a free relativistic field theory, say, involves selecting a foliation of space–like hypersurfaces in space–time, and an expansion of the field operators in terms of the normal modes on these hypersurfaces. The normal modes are in one–to–one correspondence with the creation and annihilation operators which act on the Hilbert space of states. It is quite obvious what "nonlocality" of
the Hilbert space means in this case: in the Schrödinger picture, its elements are wave functionals whose arguments are functions defined over an entire space–like hypersurface.

Inspired by the geometric structure of classical gauge theories it is natural to ask if there exists a formulation of quantum mechanics, or a generalization thereof, in which there is not only one global Hilbert space, but rather a bundle of Hilbert spaces with one such space associated to each point of the base manifold. It is not clear a priori what this base manifold should be. In a field theory context, a plausible choice is to identify it with space–time. In particular if one tries to construct a consistent theory of quantum gravity it might prove helpful to recast the rules of quantum mechanics in a language similar to that of classical general relativity. There have been various suggestions along these lines in the literature [1, 2] but no complete theory has emerged so far. It seems clear that using only classical field theory and standard quantum mechanics as an input does not lead to a unique theory, and depending on which additional assumptions are made different models with different physical interpretations arise.

Another option is associating a Hilbert space to each point of configuration–space. A theory of this kind could be applied to all quantum systems, not only to field theories (in which case the configuration–space is infinite dimensional). Such families of Hilbert spaces have played a certain rôle in connection with Berry’s phase [3] where the configuration–space is the one pertaining to the slow degrees of freedom, but no general theory has been developed so far.

In the present paper we investigate an even more general setting where a local Hilbert space is ascribed to each point of phase–space. There are various motivations for this choice [4]. The most important one is that this setting allows for an intriguing reformulation of quantum mechanics which, while being strictly equivalent to the usual one, gives a remarkable new interpretation to the process of ”quantization”.

The theory which we are going to develop is, on the one hand, a Yang–Mills type gauge theory over phase–space with an emphasis on local geometric structures. On the other hand, it can be shown to be equivalent to standard quantum mechanics with a single global Hilbert space. The typical fiber at each point of phase–space is taken to be a copy of the ordinary quantum mechanical Hilbert space, henceforth denoted \( \mathcal{V} \). In each one of those infinite dimensional
spaces we can perform independent changes of their bases by means of a unitary transformation $U$. We shall denote local coordinates on phase–space by $\phi \equiv (\phi^a)$ and write $\mathcal{V}_\phi$ for the Hilbert space at $\phi$. Then the position–dependent unitary transformation $\phi \mapsto U(\phi)$ is precisely a local gauge transformation in the sense of Yang–Mills theory. The gauge group is the infinite dimensional group of all unitary transformations on $\mathcal{V}$, and the corresponding Lie algebra consists of hermitian operators satisfying the commutator relations of a (generalized) $W_\infty$–algebra [5, 6].

Hence a connection can be locally represented by a 1–form $\Gamma = \Gamma_a(\phi)d\phi^a$ where the ”gauge field” $\Gamma_a(\phi)$ is a hermitian operator on $\mathcal{V}_\phi$ (for $a$ and $\phi$ fixed).

The crucial question is which principle determines the connection $\Gamma$. Is it dynamical as in the gauge theories which we use in particle physics, or is it fixed to have a universal form? In this paper we shall demonstrate that, to a large extent, $\Gamma$ is fixed by a deep physical principle, invariance under the ”background–quantum split symmetry”. We shall see that the implementation of this symmetry partially replaces the usual process of quantization.

How can we reconcile then the standard single–Hilbert space description of quantum mechanics with the picture of the local Hilbert spaces drawn above? It is clear that if we had a parallel transport at our disposal by means of which a vector in $\mathcal{V}_\phi$ can be transported consistently from $\phi$ to the Hilbert space $\mathcal{V}_{\bar{\phi}}$ at an arbitrary point $\bar{\phi}$, then the infinitely many Hilbert spaces were redundant. In this case all vectors and operators of $\mathcal{V}_{\bar{\phi}}$ can be obtained from those of $\mathcal{V}_\phi$ by a known unitary transformation. This procedure is fully consistent only if the parallel transport is path–independent, i.e., if the pertinent connection $\Gamma$ has a vanishing curvature. However, in our case it will not be necessary to insist on a completely ”flat” connection. It is sufficient to require consistency up to a physically irrelevant phase which means that the curvature of $\Gamma$, $\Omega_{ab}(\Gamma)$, may be proportional to the unit operator. Connections with this property are called abelian since their curvature $\Omega_{ab}(\Gamma)$ commutes with any other operator. We shall see that the connection which is dictated by the background–quantum split symmetry is indeed an abelian one, and that the associated parallel transport can be used in order to prove the equivalence of the local theory with standard quantum mechanics.

In our approach the rules of canonical quantization are replaced by two new, independent postulates. They are not borrowed from standard quantum mechanics, but rather are a mathematically very natural option if one works within the
gauge theory framework. This will shed new light on what is means to "quantize" a hamiltonian system [7]. In a nutshell, our first postulate is that in order to go from classical to quantum mechanics one has to replace the vector representation of the group of linear canonical transformations, $Sp(2\mathcal{N})$, by its spinor representation. This leads us, by pure group theory, from classical mechanics to the semiclassical approximation of quantum mechanics. The second postulate is that the gauge theory should respect the background–quantum split symmetry. Imposing this symmetry, we recover full–fledged, exact quantum mechanics from the semiclassical theory resulting from the first postulate.

Let us be more explicit about the "matter fields" which will appear in our gauge theory. We choose them to be a generalization of the metaplectic spinor fields, which are a kind of phase–space analogue of the ordinary spinor fields over space–time. We consider hamiltonian systems with $N$ degrees of freedom and a $2\mathcal{N}$–dimensional phase–space $M_{2\mathcal{N}}$. Then tensor fields over $M_{2\mathcal{N}}$ transform under local frame rotations in the tangent spaces $T_{\phi}M_{2\mathcal{N}}$ according to tensor products of the vector representation of $Sp(2\mathcal{N})$. Here the group $Sp(2\mathcal{N})$ plays a rôle analogous to the Lorentz group $SO(1, n − 1)$, and the tensor fields on $M_{2\mathcal{N}}$ are the analogs of the integer-spin fields on space–time. Spinors on space–time, on the other hand, transform under local frame rotations according to the double covering of the Lorentz group, $Spin(1, n − 1)$. Metaplectic spinors on $M_{2\mathcal{N}}$ are defined in a very similar fashion: under local frame rotations they transform according to the covering group of $Sp(2\mathcal{N})$, i.e., the metaplectic group $Mp(2\mathcal{N})$. There exists a two–to–one homomorphism between $Mp(2\mathcal{N})$ and $Sp(2\mathcal{N})$. Unlike $Spin(1, n − 1)$ which has finite dimensional matrix representations, the representations of $Mp(2\mathcal{N})$ are all infinite dimensional. We shall be interested in unitary representations on the Hilbert space $\mathcal{V}$. The infinite dimensional space $\mathcal{V}$ will serve as a typical fiber for the Hilbert bundles we construct. At each point $\phi$ of $M_{2\mathcal{N}}$ there will be a local tangent space $T_{\phi}M_{2\mathcal{N}}$ and a local Hilbert space $\mathcal{V}_{\phi}$ whose elements respond to a frame rotation at $\phi$ by a $Sp(2\mathcal{N})$ and a $Mp(2\mathcal{N})$ transformation, respectively.

In order to find representations of $Mp(2\mathcal{N})$ we have to associate to all matrices $S \equiv (S^{a}_{b}) \in Sp(2\mathcal{N})$ a unitary operator $M(S)$ such that $M(S_{1})M(S_{2}) = \pm M(S_{1}S_{2})$ [8, 9, 10]. These operators can be found by starting from the Clifford
\[ \gamma^a \gamma^b - \gamma^b \gamma^a = 2i \omega^{ab} \] (1.1)

where in terms of \( N \times N \) blocks,

\[
(\omega^{ab}) = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\] (1.2)

Here \( \omega^{ab} \) is an antisymmetric analogue of the inverse metric tensor on Minkowski space. The metaplectic gamma-“matrices” \( \gamma^a \) are supposed to be hermitian operators on \( \mathcal{V} \) which transform as a vector of \( Sp(2N) \):

\[
M(S)^{-1} \gamma^a M(S) = S^a_b \gamma^b
\] (1.3)

To solve this equation, we assume that \( S \) is infinitesimally close to the identity, i.e., that \( S^a_b = \delta^a_b + \omega^{ac} \kappa_{cb} \) with symmetric coefficients \( \kappa_{ab} = \kappa_{ba} \). If we make the ansatz

\[
M(S) = 1 - \frac{i}{2} \kappa_{ab} \Sigma^{ab}
\] (1.4)

then (1.3) implies the following condition for the generators \( \Sigma^{ab} \):

\[
[\gamma^a, \Sigma^{bc}] = i (\omega^{ab} \gamma^c + \omega^{ac} \gamma^b)
\] (1.5)

It is easy check that this equation has the solution

\[
\Sigma^{ab} = \frac{1}{4} (\gamma^a \gamma^b + \gamma^b \gamma^a)
\] (1.6)

and that the generators satisfy the desired commutator relations:

\[
[\Sigma^{ab}, \Sigma^{cd}] = i \left( \omega^{ac} \Sigma^{bd} + \omega^{bc} \Sigma^{ad} + \omega^{ad} \Sigma^{bc} + \omega^{bd} \Sigma^{ac} \right)
\] (1.7)

Thus every representation of the metaplectic Clifford algebra in terms of hermitian \( \gamma^- \) “matrices” leads to hermitian generators \( \Sigma^{ab} = \Sigma^{ba} \) and to unitary operators \( \exp \left( -\frac{i}{2} \kappa_{ab} \Sigma^{ab} \right) \in Mp(2N) \) acting on \( \mathcal{V} \).

The crucial difference compared to spinors on Minkowski space is the minus sign on the LHS of the Clifford algebra (1.1). It has the consequence that there can be no finite dimensional matrix representations, and it also means that the metaplectic Clifford algebra is basically the same object as the Heisenberg algebra. In fact, consider a set of position operators \( \hat{x}^k \) and momentum operators \( \hat{p}^k \), \( k = 1, 2, \ldots, 2N \).
1, \cdots, N, \text{ which act on } \mathcal{V} \text{ and satisfy canonical commutation relations } [\hat{x}^j, \hat{\pi}^k] = i\hbar \delta^{j,k} \text{ with the other commutators vanishing. If we combine } \hat{x}^k \text{ and } \hat{\pi}^k \text{ into }

\hat{\phi}^a \equiv (\hat{\pi}^k, \hat{x}^k), \quad a = 1, \cdots, 2N \quad (1.8)

then the commutation relations read

[\hat{\phi}^a, \hat{\phi}^b] = i\hbar \omega^{ab} \quad (1.9)

Obviously we can realize the \gamma\text{” matrices” in terms of those position and momentum operators:}

\gamma^a = (2/\hbar)^{1/2} \hat{\phi}^a \quad (1.10)

If we choose an arbitrary basis \{ |x\rangle \} in \mathcal{V}, \gamma^a \text{ is represented by the matrix}

(\gamma^a)_x^y = (2/\hbar)^{1/2} \langle x | \hat{\phi}^a | y \rangle \quad (1.11)

We shall use both the matrix and the bra–ket notation, with bra (ket) vectors corresponding to upper (lower) indices. The components of a vector |\psi\rangle \in \mathcal{V} \text{ are written as}

\psi^x = \langle x | \psi \rangle \quad (1.12)

and those of the dual vector \langle \psi | \in \mathcal{V}^* \text{ read correspondingly}

\psi_x = \langle \psi | x \rangle = (\psi^x)^* \quad (1.13)

As the notation suggests already, we shall often use the representation in which the \hat{x}^k\text{’s are diagonal and the label } x \equiv (x^k) \in \mathbb{R}^N \text{ is the set of their eigenvalues.}

Then the dual pairing

\langle \chi | \psi \rangle \equiv \chi_x \psi^x \equiv (\chi^x)^* \psi^x \equiv \int d^N x \ (\chi^x)^* \psi^x \quad (1.14)

is the standard inner product on \(L^2(\mathbb{R}^N, d^N x)\). It is convenient to use a formal matrix notation where the integration over repeated indices \(x, y, \cdots\) is understood.

In this representation, \langle x | \hat{x}^k | y \rangle = x^k \delta(x-y) \text{ and } \langle x | \hat{\pi}^k | y \rangle = -i\hbar \partial_k \delta(x-y) \text{ so that the generators}

\Sigma^{ab} = \frac{1}{2\hbar} \left( \hat{\phi}^a \hat{\phi}^b + \hat{\phi}^b \hat{\phi}^a \right) \quad (1.15)

become Schrödinger Hamiltonians (second order differential operators) with a quadratic potential.
To summarize: The tangent bundle \( T\mathcal{M}_{2N} \) has the base manifold \( \mathcal{M}_{2N} \), and the fiber at the point \( \phi \in \mathcal{M}_{2N} \), the tangent space \( T_{\phi}\mathcal{M}_{2N} \), is a copy of \( \mathbb{R}^{2N} \). The structure group \( Sp(2N) \) acts on this space in its vector representation. The associated spin bundle has the same base manifold, but the fiber at \( \phi \) is the infinite dimensional space \( \mathcal{V}_{\phi} \), a copy of the Hilbert space \( \mathcal{V} \). The structure group acts on \( \mathcal{V}_{\phi} \) in its spinor (i.e., metaplectic) representation. The typical fiber \( \mathcal{V} \) can be realized as the space of square–integrable functions \( L^{2}(\mathbb{R}^{N},d^{N}x) \), for instance.

A section through the spin bundle is locally described by a function

\[
\psi : \mathcal{M}_{2N} \to \mathcal{V} , \quad \phi \mapsto |\psi\rangle_{\phi} \in \mathcal{V}_{\phi}
\]

(1.16)

The notation \( |\cdot\rangle_{\phi} \) indicates that the vector \( |\cdot\rangle_{\phi} \) is an element of the local Hilbert space at \( \phi \). In a concrete basis, this spinor field has the components

\[
\psi^{x}(\phi) = \langle x|\psi \rangle_{\phi}
\]

(1.17)

If we use the dual space \( \mathcal{V}^{*} \) instead of \( \mathcal{V} \) we arrive at the dual spin bundle. Locally its sections are functions

\[
\chi_{x}(\phi) = \phi \langle x|\chi \rangle \quad , \quad \phi \langle x| \in \mathcal{V}_{\phi}^{*}
\]

(1.18)

We shall also consider multispinor fields

\[
\phi \mapsto \chi^{x_{1} \cdots x_{r}}_{y_{1} \cdots y_{s}}(\phi)
\]

(1.19)

which describe sections through the tensor product bundle whose typical fiber is \( \mathcal{V} \otimes \mathcal{V} \otimes \cdots \mathcal{V}^{*} \otimes \mathcal{V}^{*} \otimes \cdots \) with \( r \) factors of \( \mathcal{V} \) and \( s \) factors of \( \mathcal{V}^{*} \). We refer to (1.19) as a \((r, s)\)–multispinor. If we consider, for instance, a family of operators which are labeled by the points of \( \mathcal{M}_{2N} \), \( A(\phi) : \mathcal{V}_{\phi} \to \mathcal{V}_{\phi} \), then its component representation

\[
A^{x}_{y}(\phi) = \langle x|A(\phi)|y \rangle
\]

(1.20)

is a \((1, 1)\)–multispinor field.

The operators \( \hat{\varphi}^{a} = (\hat{\pi}^{k}, \hat{x}^{k}) \) should not be confused with the momentum and position operators which result from the canonical quantization of a system with phase–space \( \mathcal{M}_{2N} \). The \( \hat{\varphi}^{a} \)’s operate on the infinitely many fibers \( \mathcal{V}_{\phi} \) whose relation to the quantum mechanical Hilbert space \( \mathcal{V}_{\text{QM}} \) is not known yet. In the simplest case of a flat phase–space \( \mathcal{M}_{2N} = \mathbb{R}^{2N} \) one can introduce canonical
coordinates $\phi^a \equiv (p^k, q^k)$, and then quantization amounts to introducing operators $\hat{\phi}^a \equiv (\hat{p}^k, \hat{q}^k)$ with $[\hat{q}^i, \hat{p}^k] = i\hbar \delta^{ik}$. The main issue of this paper will be to understand the interrelation between $\hat{\phi}^a$ and $\hat{\varphi}^a$ and between $V_QM$ and the family of spaces $\{V_\phi, \phi \in \mathcal{M}_{2N}\}$.

We shall see that conventional semiclassical quantization provides a natural motivation for studying (generalized) metaplectic spinors. It will turn out, however, that the transition to full quantum mechanics makes it necessary to replace the structure group $Mp(2N)$ by the group of all unitary transformations on $V$.

The remaining sections of this paper are organized as follows. In section 2 we show that switching from the vector to the spinor representation brings us from classical mechanics to the semiclassical quantum theory. Then, in the following sections, we describe the construction of full–fledged quantum mechanics by sewing together the local semiclassical approximations obtained in this way. To do this, we need a connection on the bundle of Hilbert spaces. It is introduced in section 3 where also some of its properties are described which will be needed later on. In section 4 we demonstrate that, within the present approach, exact quantum mechanics is recovered from the semiclassical approximations if one requires invariance under the background–quantum split symmetry. This condition forces the connection to be abelian. We show that by virtue of the corresponding parallel transport all local Hilbert spaces can be identified with a single reference Hilbert space, and that the resulting reduced theory coincides with conventional quantum mechanics. In section 5 we continue the discussion of the bundle approach for the special case of a flat phase–space, and section 6 contains the conclusions.

For a first reading, the reader who is mostly interested in the general results can proceed directly to the summary of section 6 before studying the details of their derivation.

2 Lie–Derivative and Semiclassical Approximation

In this section we construct a Lie–derivative for metaplectic spinor fields with respect to an arbitrary globally hamiltonian vector field. This Lie–derivative is the natural link between the Hilbert bundle approach and the semiclassical
approximation of standard quantum mechanics. From a slightly different point of view, it has been discussed in ref.[11, 13] already.

We work on an arbitrary symplectic manifold \((\mathcal{M}_{2N}, \omega)\). By definition [14] the symplectic 2–form

\[
\omega = \frac{1}{2} \omega_{ab}(\phi) d\phi^a \wedge d\phi^b
\]  

(2.1)
is closed, \(d\omega = 0\), and nondegenerate, i.e., the matrix \((\omega_{ab})\) possesses an inverse \((\omega^{ab})\):

\[
\omega^{ab} \omega^{bc} = \delta^c_a
\]  

(2.2)

We may choose Darboux local coordinates \(\phi^a = (p^k, q^k), k = 1, \cdots, N\) with respect to which \(\omega_{ab}\) is constant:

\[
(\omega_{ab}) = \begin{pmatrix}
0 & +1 \\
-1 & 0
\end{pmatrix}
\]  

(2.3)
The inverse of this matrix is precisely (1.2) which appeared in the Clifford algebra. A globally hamiltonian vector field \(h = h^a \partial_a\), \(\partial_a \equiv \partial/\partial \phi^a\), can be expressed in terms of a generating function \(H : \mathcal{M}_{2N} \to \mathbb{R}\) according to

\[
h^a = \omega^{ab} \partial_b H
\]  

(2.4)
The flow generated by \(h\) leaves the form \(\omega\) invariant, i.e., the (ordinary) Lie–derivative of \(\omega\) with respect to \(h\) vanishes:

\[
\ell_h \omega_{ab} \equiv h^c \partial_c \omega_{ab} + \partial_a h^c \omega_{cb} + \partial_b h^c \omega_{ac} = 0
\]  

(2.5)

Quite generally, the Lie–derivative for any tensor field \(\chi\) on an arbitrary \(n\)–dimensional manifold \(\mathcal{M}_n\) is of the form [15]

\[
\ell_v \chi = v^a \partial_a \chi - \partial_b v^a G^b_a \chi
\]  

(2.6)

Here \(v = v^a \partial_a\) is an arbitrary vector field, and the matrices \(G^b_a\) are the generators of \(GL(n, \mathbb{R})\) in the representation to which \(\chi\) belongs. They obey the Lie algebra relations

\[
[G^b_a, G^c_d] = \delta^b_d G^c_a - \delta^c_d G^b_a
\]  

(2.7)

Lie–derivatives form a closed algebra under commutation:

\[
[\ell_{v_1}, \ell_{v_2}] = \ell_{[v_1,v_2]}
\]  

(2.8)
Here
\[ [v_1, v_2]^a \equiv v_1^b \partial_b v_2^a - v_2^b \partial_b v_1^a \] (2.9)
denotes the Lie–bracket. The Lie–derivative is a covariant differentiation which needs no connection for its definition; if \( \chi \) transforms as a tensor under general coordinate transformation, so does \( \ell_v \chi \).

For spinors the situation is more complicated. Under general coordinate transformations they behave like scalars, \( \delta_C \psi = v^a \partial_a \psi \), but they transform non-trivially under local frame rotations: \( \delta_F \psi = -\frac{i}{2} \kappa_{ab} \Sigma^{ab} \psi \). (The generators \( \Sigma^{ab} \) and parameters \( \kappa_{ab} \) are antisymmetric in \( a \) and \( b \) for \( O(n) \) and symmetric for \( Sp(2N) \).)

One can try to construct a Lie–derivative for \( \psi \) by combining a general coordinate transformation with an appropriate frame rotation, and to express the \( G^{a}_{b} \)'s in eq.(2.6) in terms of the generators \( \Sigma^{ab} \). In general this is possible only for a restricted class of vector fields.

On **Riemannian manifolds** the construction can be performed only if \( v \) is a Killing vector field. In this case \( \Sigma^{ab} \) and \( G^{b}_{a} \) are related by a contraction with the metric, which has a vanishing Lie–derivative. As a consequence, the resulting \( \ell_v \) has all the formal properties of a Lie–derivative [15].

On **symplectic manifolds** a spinorial Lie–derivative can be defined for a much larger (infinite dimensional, in fact) space of vector fields, namely for all globally hamiltonian vector fields. For them, \( \ell_v \omega = 0 \), and we may use \( \omega_{ab} \) and \( \omega^{ab} \) in order to relate \( G^{b}_{a} \) to \( \Sigma^{ab} \). This is particularly clear in Darboux coordinates. If we set
\[ \Sigma^{ab} = i \left( G^{a}_{c} \omega^{cb} + G^{b}_{c} \omega^{ca} \right) \] (2.10)
it can be checked that the algebras (2.7) and (1.7) for \( G^{a}_{b} \) and \( \Sigma^{ab} \), respectively, become equivalent. Now we specialize
\[ \ell_h = h^a \partial_a - \partial_b h^a G^{b}_{a} \] (2.11)
for the hamiltonian vector field (2.4) and use (2.10):
\[ \ell_H \equiv \ell_h = h^a(\phi) \partial_a + \frac{i}{2} \partial_a \partial_b H(\phi) \Sigma^{ab} \] (2.12)
This equation defines a Lie–derivative for fields transforming in an arbitrary representation of the algebra (1.7), for spinors in particular. Usually we deal with
tensor products of the vector and the spinor representation and their respective duals. Then we have explicitly
\[
\ell_H \chi^a_{b\ldots y} = h^c \partial_c \chi^a_{b\ldots y} - \partial_c h^a \chi^c_{b\ldots y} + \partial_b h^c \chi^a_{c\ldots y} \\
+ \frac{i}{2} \partial_c \partial_d H \left( \Sigma^{cd} \chi^a_{b\ldots y} - \frac{i}{2} \partial_c \partial_d H \chi^a_{c\ldots y} \right) + \ldots
\]  
(2.13)

Here \( \Sigma^{ab} \) has now the more concrete meaning of the generators (1.6) written in terms of the \( \gamma \)-matrices (1.10). It is easy to check that (2.13) defines a covariant derivation. If we replace all partial derivatives in \( \ell_H \) by covariant derivatives with respect to a symplectic connection (see below) all terms involving the connection are seen to cancel. Moreover, as a consequence of (2.8), the \( \ell_H \)'s form a representation of the infinite–dimensional Lie algebra of symplectic diffeomorphisms:
\[
[\ell_{H_1}, \ell_{H_2}] = -\ell_{\{H_1, H_2\}}
\]  
(2.14)

Here
\[
\{H_1, H_2\} = \partial_a H_1 \omega^{ab} \partial_b H_2
\]  
(2.15)

denotes the Poisson bracket.

Sometimes the bra–ket notation is more convenient than the component notation. For \( \psi^x = \langle x | \psi \rangle_\phi \), say,
\[
\ell_H \psi^x_\phi = h^a \partial_a \psi^x_\phi + \frac{i}{2} \partial_a \partial_b H \Sigma^{ab} \psi^x_\phi
\]  
(2.16)

The Lie–derivative of the operator field (1.20) reads likewise
\[
\ell_H A(\phi) = h^a \partial_a A(\phi) + \frac{i}{2} \partial_a \partial_b H \left[ \Sigma^{ab}, A(\phi) \right]
\]  
(2.17)

Applying (2.13) to \( \gamma^a_{b\ldots y} \) we find that the \( \gamma \)-matrices and the \( \hat{\varphi}^a \)'s have vanishing Lie–derivatives:
\[
\ell_H \gamma^a = 0 \quad , \quad \ell_H \hat{\varphi}^a = 0
\]  
(2.18)

Let us now turn to the physical interpretation of \( \ell_H \). Given the vector field \( h^a \), we can introduce a one–particle dynamics on \( \mathcal{M}_{2N} \) by virtue of Hamilton’s equation of motion (the dot denotes the time derivative):
\[
\dot{\phi}^a(t) = h^a (\phi(t))
\]  
(2.19)
The equivalent many–particle description is provided by Liouville’s equation \( \partial_t \rho = \{ H, \rho \} = -\hbar \partial_a \rho \) for the probability density \( \rho \equiv \rho(\phi; t) \). In standard symplectic geometry one generalizes Liouville’s equation to arbitrary tensor fields \( \chi \), whose evolution along the hamiltonian flow involves the ordinary Lie–derivative: \( \partial_t \chi = -\ell_H \chi \). It is therefore very natural to investigate the analogous time–evolution for more general fields \( \chi \equiv (\chi^a_{\bar{a} \cdots \bar{x} \cdots}) \),

\[
\partial_t \chi(\phi; t) = \ell_H \chi(\phi; t) \tag{2.20}
\]

with \( \ell_H \) given by (2.13)\(^3\). For example,

\[
-\partial_t \psi^x(\phi; t) = h^a \partial_a \psi^x(\phi; t) + \frac{i}{2} \partial_a \partial_b H(\phi) \left( \Sigma^{ab} \right)^x_y \psi^y(\phi; t) \tag{2.21}
\]

Up to now we considered metaplectic spinors which are defined at each point \( \phi \) of \( \mathcal{M}_{2N} \). Equally important are spinors which are defined only along a certain curve in phase–space. In a slight abuse of language we shall refer to them as ”world–line spinors”. We are particularly interested in the case where the curve is given by some solution \( \Phi_{cl}^a(t) \) of Hamilton’s equation (2.19). Let \( \eta(t) \) be a world–line spinor along this classical trajectory, i.e., for different times \( \eta(t) \) lives in different Hilbert spaces:

\[ \eta(t) \in \mathcal{V}_{\Phi_{cl}(t)}. \]

Given a world–line spinor, we can define a singular spinor field by writing

\[ \psi^x(\phi; t) = \eta^x(t) \delta(\phi - \Phi_{cl}(t)) \tag{2.22} \]

We demand that \( \psi^x \) satisfies the evolution equation (2.21). This entails the following equation of motion for \( \eta^x \) and its dual \( \tilde{\eta}_x = (\eta^x)^* \):

\[
\partial_t \eta^x(t) = -\frac{i}{2} \partial_a \partial_b H(\Phi_{cl}(t)) \left( \Sigma^{ab} \right)^x_y \eta^y(t) \\
\partial_t \tilde{\eta}_x(t) = -\frac{i}{2} \partial_a \partial_b H(\Phi_{cl}(t)) \tilde{\eta}_b(t) \left( \Sigma^{ab} \right)^y_x \tag{2.23}
\]

There exists an interesting relation between (2.23) and the corresponding equation of motion for a world–line vector field \( c^a(t) \):

\[ \partial_t c^a(t) = \partial_b h^a(\Phi_{cl}(t)) c^b(t) \tag{2.24} \]

\(^3\)See refs.[16] and [11] for a path–integral solution of eq.(2.20) in the case of \( p \)-forms and of antisymmetric \((0, s)\)–multispinors, respectively. The latter path–integral describes a kind of topological field theory which was used [13] to detect obstructions for the existence of metaplectic spin structures on \( \mathcal{M}_{2N} \) [17, 18].
In this case \( c(t) \in T_{\Phi(t)}\mathcal{M}_{2N} \) lives in the tangent spaces along the classical path. With the ansatz
\[
V^a(\phi; t) = c^a(t) \delta(\phi - \Phi_{\text{cl}}(t))
\] (2.25)
eq(2.24) is indeed equivalent to \(-\partial_t V^a = \hbar V^a\). Eq.(2.24) is precisely Jacobi's equation which governs small classical fluctuations about the trajectory \( \Phi_{\text{cl}}(t) \). In fact, if we write \( \Phi_{\text{cl}}'(t) = \Phi_{\text{cl}}(t) + c(t) \) and require that, to first order in \( c \), also \( \Phi_{\text{cl}}' \) solves Hamilton's equation, then the "Jacobi field" \( c(t) \) must obey (2.24). The world–line spinors have the remarkable property of being a kind of "square root" of the Jacobi–fields. If \( \eta \) and \( \bar{\eta} \) transform in the spinor representation of \( Sp(2N) \) and its dual, respectively, it is clear that \( \bar{\eta} \gamma^a \eta \equiv (2/\hbar)^{1/2} \bar{\eta} \tilde{\varphi}^a \eta \) transforms as a vector. Furthermore, if one sets
\[
c^a(t) = \bar{\eta}(t) \tilde{\varphi}^a \eta(t)
\] (2.26)
and uses the equation of motion for \( \eta \) and \( \bar{\eta} \), eq.(2.23), then it follows that (2.26) is a solution of Jacobi's equation (2.24). This fact finds a natural interpretation in the context of the semiclassical quantization which we discuss next.

We consider a quantum system with phase–space \( \mathcal{M}_{2N} \) and classical Hamiltonian \( H(\phi^a) = H(p^k, q^k) \). The probability amplitude for a transition between two points in configuration space, \( q_1^k \) and \( q_2^k \), is given by the path–integral
\[
\langle q_2, t_2 | q_1, t_1 \rangle_H = \int \mathcal{D}p(t) \int \mathcal{D}q(t) \exp \left[ \frac{i}{\hbar} \int_{t_1}^{t_2} dt \left\{ p^k \dot{q}^k - H(p, q) \right\} \right]
\] (2.27)
subject to the boundary conditions \( q(t_{1,2}) = q_{1,2} \). Let us shift the variable of integration \( \phi(t) \equiv (p(t), q(t)) \) by an arbitrary solution of Hamilton’s equation, \( \Phi_{\text{cl}}(t) \equiv (p_{\text{cl}}(t), q_{\text{cl}}(t)) \):
\[
\phi^a(t) = \Phi_{\text{cl}}^a(t) + \varphi^a(t) \quad , \quad \varphi^a(t) \equiv \left( \pi^k(t), x^k(t) \right)
\] (2.28)
Inserting this shift on the RHS of (2.27) and using \( \dot{\Phi}_{\text{cl}}^a = \hbar \Phi^a(\Phi_{\text{cl}}) \) we obtain
\[
\langle q_2, t_2 | q_1, t_1 \rangle_H = \exp \left[ \frac{i}{\hbar} \left\{ S_{\text{cl}} + p_{\text{cl}}^k(t_2) x_2^k - p_{\text{cl}}^k(t_1) x_1^k \right\} \right] \langle x_2, t_2 | x_1, t_1 \rangle_H
\] (2.29)
\[4\]For simplicity we assume here that \( \mathcal{M}_{2N} = \mathbb{R}^{2N} \) is the symplectic plane which can be covered by a single chart of Darboux coordinates \( \phi^a \equiv (p^k, q^k), \ a = 1 \cdots 2N, \ k = 1 \cdots N. \)
Here \( S_{cl} \equiv \int dt \left\{ p_{cl}^i q_{cl}^i - H(p_{cl}, q_{cl}) \right\} \) is the action along the classical trajectory; furthermore we defined the shifted path–integral

\[
\langle x_2, t_2 | x_1, t_1 \rangle_H \equiv \int D\pi(t) \int Dx(t) \exp \left[ \frac{i}{\hbar} \int_{t_1}^{t_2} dt \left\{ \pi^k \dot{x}^k - H(\phi^a; \Phi_{cl}^a) \right\} \right]
\]

with its Hamiltonian

\[
H(\phi; \Phi_{cl}) \equiv H(\Phi_{cl} + \phi) - \phi^a \partial_a H(\Phi_{cl}) - H(\Phi_{cl})
\] (2.31)

and the boundary values

\[
x_{1,2} \equiv q_{1,2} - q_{cl}(t_{1,2})
\] (2.32)

Note that the shifted path–integral (2.30) is a solution of the Schrödinger equation

\[
[i\hbar \partial_t - H(\phi^a; \Phi_{cl}^a)] \langle x, t | x_1, t_1 \rangle_H = 0
\] (2.33)

The operators \( \hat{\phi}^a \equiv (\hat{\pi}^k, \hat{x}^k) \) result from the ordinary canonical quantization of the \( \phi^a \)-degrees of freedom; they satisfy the canonical commutation relations

\[
[\hat{\phi}^a, \hat{\phi}^b] = i\hbar \omega^{ab}.
\]

In eq.(2.33) their representation in terms of multiplication and derivative operators is employed: \( \hat{x}^k = x^k \), \( \hat{\pi}^k = -i\hbar \partial / \partial x^k \).

Up to this point, no approximation has been made. In a traditional semi-classical calculation one would expand about a classical trajectory connecting \( q_1 \) and \( q_2 \) for which \( x_1 = x_2 = 0 \) therefore, and one would expand \( H(\phi; \Phi_{cl}) \) with respect to the fluctuation \( \phi \); to lowest order, only the quadratic term is kept:

\[
H(\phi; \Phi_{cl}) = \frac{1}{2} \partial_a \partial_b H(\Phi_{cl}) \phi^a \phi^b + O(\phi^3)
\] (2.34)

Here we shall not assume that \( x_1 \) and \( x_2 \) are zero exactly but only that the transition amplitude is dominated by a classical trajectory which passes near \( q_1 \) and \( q_2 \) at \( t = t_1 \) and \( t = t_2 \), respectively.

It is an important observation that the operator version of the approximated Hamiltonian (2.35) lies in the Lie algebra of \( Mp(2N) \). In fact, the operators \( \hat{\phi}^a \) which appear naturally in the operatorial formulation of the quantum mechanics defined by the shifted path–integral (2.30) can be identified with the \( \hat{\phi} ^a \)’s which were introduced in section 1 as a realization of the metaplectic \( \gamma \)–matrices. They

\(^5\)Here and in the following we assume that all operators are Weyl–ordered and that the path–integrals are discretized corresponding (mid–point rule).
are related to the generators of $Mp(2N)$ by eq.(1.15). Hence the Hamiltonian operator which governs the $\varphi$--dynamics in the quadratic approximation reads

$$
\mathcal{H}(\hat{\varphi}; \Phi_{cl}) = \frac{1}{2} \partial_a \partial_b H(\Phi_{cl}) \hat{\varphi}^a \hat{\varphi}^b + \cdots
= \frac{\hbar}{2} \partial_a \partial_b H(\Phi_{cl}) \Sigma^{ab} + \cdots
$$

Given the matrix elements $\langle x, t | x_1, t_1 \rangle_{\mathcal{H}}$ we can fix $x_1$ and $t_1$ and define a world–line spinor along $\Phi_{cl}(t)$ by

$$
\eta^x(t) \equiv \langle x | \eta(t) \rangle_{\Phi_{cl}(t)} \equiv \langle x, t | x_1, t_1 \rangle_{\mathcal{H}}
$$

For every fixed time $t$, $\eta(t)$ lives in the fiber $\mathcal{V}_{\Phi_{cl}(t)}$. The justification for calling the transition matrix element a world–line spinor is that with the semiclassical Hamiltonian (2.35) the Schrödinger equation (2.33) is exactly the same as the original equation of motion of world–line spinors, eq.(2.23).

To summarize: For globally hamiltonian vector fields the notion of a Lie–derivative and the corresponding transport along the hamiltonian flow can be generalized from tensor to metaplectic spinor fields. For spinor fields which have support only along classical trajectories, this transport induces a well–defined equation of motion for the ”world–line” spinors. Semiclassical wave functions, describing quantum fluctuations about classical trajectories, are found to be world–line spinors in this sense.

Thus we see that semiclassical quantum mechanics is most naturally formulated in terms of a family of Hilbert spaces along the classical trajectory, $\mathcal{V}_{\Phi_{cl}(t)}$. In the full quantum theory we integrate over all paths $\phi(t)$, and generically classical trajectories do not play any preferred rôle. Therefore it is plausible to conjecture that the generalization of the above picture to full–fledged quantum mechanics will involve a family of Hilbert spaces $\mathcal{V}_\phi$, $\phi \in \mathcal{M}_{2N}$, with a copy of $\mathcal{V}$ at all points of phase–space. Before we can investigate this question we have to introduce the notion of a spin connection.
3 Spin Connections

3.1 General hermitian spin connection

We saw that sections through the Hilbert bundle are locally given by functions
\( \psi^x(\phi) = \langle x|\psi \rangle \). Let us find a covariant derivative of the form
\[
\nabla_a|\psi\rangle = \partial_a|\psi\rangle + \frac{i}{\hbar} \Gamma_a(\phi)|\psi\rangle
\]
(3.1)

For the dual spinor we set
\[
\nabla_a\phi\langle \chi| = (\nabla_a|\psi\rangle)_\dagger \tag{3.2}
\]
where the adjoint is with respect to the inner product of \( \mathcal{V}_\phi \). A priori, \( \{\Gamma_a(\phi), a = 1, \cdots, 2N\} \) is a set of \( 2N \) arbitrary operators on \( \mathcal{V}_\phi \). We require that for all \( |\psi\rangle \in \mathcal{V}_\phi \) and \( \phi\langle \chi| \in \mathcal{V}_\phi^* \)
\[
\nabla_a\phi\langle \chi|\psi\rangle = \partial_a\phi\langle \chi|\psi\rangle \tag{3.3}
\]

Then eqs.(3.1), (3.2) and the Leibniz rule for \( \nabla_a \) imply that the \( \Gamma_a \)'s must be hermitian:
\[
\Gamma_a(\phi) = \Gamma_a(\phi)\dagger \tag{3.4}
\]

For the time being we do not impose any further conditions on \( \Gamma_a \). Hence the Lie algebra of the structure group (gauge group) is defined to be the space \( \mathcal{G} \) of all hermitian operators on \( \mathcal{V} \). The infinite dimensional gauge group \( G \) is the group of all unitary operators on \( \mathcal{V} \). It has \( Mp(2N) \) as a finite dimensional subgroup. By using \( \mathcal{G} \)–valued connections we generalize the idea of "local frame rotations" in a way appropriate for Hilbert spaces; here "frame" means a basis of \( \mathcal{V} \). In all fibers \( \mathcal{V}_\phi \) we may perform independent changes of their bases by means of a gauge transformation \( U : \mathcal{M}_{2N} \rightarrow G, \phi \mapsto U(\phi) \). It acts according to
\[
|\psi\rangle'_\phi = U(\phi)|\psi\rangle \tag{3.5}
\]

From the covariance of \( \nabla_a \equiv \nabla_a(\Gamma) \),
\[
\nabla_a(\Gamma') = U(\phi) \nabla(\Gamma) U(\phi)^{-1} \tag{3.6}
\]
we obtain the transformation law of \( \Gamma_a \):
\[
\Gamma_a'(\phi) = U(\phi) \Gamma_a(\phi) U(\phi)^{-1} + i\hbar \partial_a U(\phi) U(\phi)^{-1} \tag{3.7}
\]
For an infinitesimal gauge transformation $U(\phi) = 1 - i\varepsilon(\phi)/\hbar$, $\varepsilon = \varepsilon^\dagger$, one has

$$
\delta_F|\psi\rangle_\phi = -\frac{i}{\hbar} \varepsilon |\psi\rangle_\phi
$$

$$
\delta_F \Gamma_a(\phi) = \partial_a \varepsilon + \frac{i}{\hbar} [\Gamma_a, \varepsilon]
$$

(3.8)

The local frame rotations are Yang–Mills-type gauge transformations which are unrelated to coordinate transformations on $\mathcal{M}_{2N}$. Under a coordinate transformation $\delta_C \phi^a = -h^a(\phi)$ the spinor transforms as a scalar and $\Gamma_a$ as a vector:

$$
\delta_C|\psi\rangle_\phi = h^a \partial_a |\psi\rangle_\phi
$$

$$
\delta_C \Gamma_a(\phi) = h^b \partial_b \Gamma_a + \partial_a h^b \Gamma_b
$$

(3.9)

If we write the components of $\Gamma_a$ with respect to an arbitrary basis $\{|x\rangle\}$ of $\mathcal{V}$ in the form

$$
\Gamma_a(\phi)_{xy} = \langle x|\Gamma_a(\phi)|y\rangle
$$

(3.10)

then eq.(3.1) becomes

$$
\nabla_a \psi_{x\ldots y} = \partial_a \psi_{x\ldots y} + \frac{i}{\hbar} \Gamma_a(\phi)_{xy} \psi_{x\ldots y} - \frac{i}{\hbar} \chi_{x\ldots z} \Gamma_a(\phi)_{zy} + \ldots
$$

(3.11)

More generally we can consider $(r,s)$ multispinors $\chi^{x_1\ldots x_r}_{y_1\ldots y_s}$ which respond to a gauge transformation as

$$
\chi^{x_1\ldots x_r}_{y_1\ldots y_s}(\phi)' = U(\phi)^{\kappa}_{\nu} U^\dagger(\phi)_{y_1}^{\nu} \cdots \chi^{x_1\ldots x_r}_{y_1\ldots y_s}(\phi)
$$

(3.12)

Their covariant derivative reads

$$
\nabla_a \chi^{x_1\ldots x_r}_{y_1\ldots y_s} = \partial_a \chi^{x_1\ldots x_r}_{y_1\ldots y_s} + \frac{i}{\hbar} \Gamma_a(\phi)^{\kappa}_{\nu} \chi^{x_1\ldots x_r}_{y_1\ldots y_s} - \frac{i}{\hbar} \chi^{x_1\ldots x_r}_{y_1\ldots y_s} \partial_a(\phi)^{\kappa}_{y_1} + \ldots
$$

(3.13)

For an $(1,1)$–multispinor $A(\phi)^{x}_{y}$, interpreted as an operator on $\mathcal{V}_\phi$, one has for instance

$$
\nabla_a A = \partial_a A + \frac{i}{\hbar} [\Gamma_a, A]
$$

(3.14)

Hence the second equation of (3.8) is simply $\delta_F \Gamma_a = \nabla_a \varepsilon$.

Furthermore, $\Gamma_a$ gives rise to an exterior derivative $\nabla = d\phi^a \nabla_a$ of $(r,s)$–spinor–valued differential forms

$$
F = F_{a_1\cdots a_r}(\phi) \, d\phi^{a_1} \cdots d\phi^{a_r}
$$

(3.15)
which is covariant with respect to both Yang–Mills gauge transformations and
diffeomorphisms of $\mathcal{M}_{2N}$. The components $F_{a_1 \cdots a_p}$ are multispinor fields. We
define

\[ \nabla F_{y^{x \cdots}} = \nabla F_{y^{x \cdots}, a_1 \cdots a_p}(\phi) \, d\phi^{a_1} \cdots d\phi^{a_p} \]  

with the derivative of the components given by (3.13). For an operator–valued
$p$–form, say,

\[ \nabla A = dA + \frac{i}{\hbar} [\Gamma, A] \]  

Here $d \equiv d\phi^a \partial_a$ is the usual exterior derivative and

\[ \Gamma \equiv \Gamma_a(\phi)d\phi^a \]  

denotes the connection 1–form. In equations such as (3.17) the square brackets
denote the graded commutator

\[ [A, B] = AB - (-1)^{|A||B|} BA \]  

with $[A] = +1$ ($[A] = -1$) if the rank of $A$ as a differential form is even (odd).
The product $AB$ is a combination of operator multiplication and wedge product.
For any two operator–valued differential forms $A$ and $B$ one has the product rule

\[ \nabla(AB) = (\nabla A)B + (-1)^{|A|} A\nabla B \]  

The curvature of $\Gamma_a$ is given by the hermitian operators $\Omega_{ab} = -i\hbar[\nabla_a, \nabla_b]$, or explicitly,

\[ \Omega_{ab} = \partial_a \Gamma^b - \partial_b \Gamma^a + \frac{i}{\hbar} [\Gamma_a, \Gamma_b] \]  

with the components

\[ \Omega_{ab}^{x y} = \partial_a \Gamma^x_{b} - \partial_b \Gamma^x_{a} + \frac{i}{\hbar} \left( \Gamma^x_{a z} \Gamma^z_{b y} - \Gamma^z_{b z} \Gamma^x_{a y} \right) \]  

The curvature 2–form

\[ \Omega = \frac{i}{2} \Omega_{ab}(\phi) \, d\phi^a d\phi^b \]  

reads in terms of $\Gamma$:

\[ \Omega = d\Gamma + \frac{i}{\hbar} \Gamma^2 \]  

Using this representation together with (3.17) it is easy to show that $\Omega$ satisfies
Bianchi’s identity

\[ \nabla \Omega \equiv d\Omega + \frac{i}{\hbar} [\Gamma, \Omega] = 0 \]  

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and that for any operator–valued $p$–form $A$

$$\nabla^2 A = \frac{i}{\hbar} [\Omega, A]$$

(3.26)

Under the gauge transformation (3.5),

$$\Omega'_{ab}(\phi) = U(\phi) \Omega_{ab}(\phi) U(\phi)$$

(3.27)

### 3.2 Symplectic connections

Next we introduce a special class of spin connections which assume values in the Lie algebra of the most important subgroup of $G$, namely $Mp(2\mathbb{N})$.

While in most parts of this paper Darboux local coordinates are used, we shall employ a generic system of local coordinates $\phi^\alpha$ in this subsection. Hence $\omega_{ab}(\phi)$ will not be given by the constant canonical matrix (2.3) in general. For clarity, the latter is denoted

$$\left( \omega^\alpha_{\alpha\beta} \right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(3.28)

here. We introduce a basis of vielbein fields $e_\alpha = e_\alpha^a \partial_a$ and of dual 1–forms

$$e^\alpha = e^\alpha_a d\phi^a, \alpha = 1, \cdots, 2\mathbb{N}.$$ Their components satisfy

$$e_\alpha^a e_\beta^a = \delta_\alpha^\beta, e_\alpha^a e_\alpha^b = \delta_a^b$$

(3.29)

In this subsection we make a notational distinction between coordinate indices $a, b, c, \cdots$ and frame indices $\alpha, \beta, \gamma, \cdots$. We identify the vielbeins with the transformation which brings $\omega_{ab}(\phi)$ to the skew–diagonal form $\omega^\alpha_{\alpha\beta}$:

$$\omega_{ab}(\phi) = e_\alpha^a(\phi)e_\beta^b(\phi) \omega^\alpha_{\alpha\beta}$$

(3.30)

Clearly $\omega_{ab}$ and $\omega^\alpha_{\alpha\beta}$ are analogous to the metric on a curved space and on flat space, respectively. The group $Sp(2\mathbb{N})$ acts on the frame indices with matrices $S^\alpha_{\beta}$ preserving $\omega$: $\omega_{\alpha\beta} S^\alpha_{\gamma} S^\beta_{\delta} = \omega_{\gamma\delta}$. An infinitesimal local $Sp(2\mathbb{N})$–rotation reads

$$S^\alpha_{\beta}(\phi) = \delta^\alpha_{\beta} + \omega^\alpha_{\alpha\gamma} \kappa_{\gamma\beta}(\phi)$$

(3.31)

with symmetric parameters $\kappa_{\alpha\beta} = \kappa_{\beta\alpha}$. (Recall that the matrices in $Sp(2\mathbb{N})$ are products of $\omega^\alpha_{\alpha\beta}$ with some symmetric matrix [9]. If $G^\alpha_{\beta}$ is any generator of $Sp(2\mathbb{N})$ in its defining representation, than $\omega^\alpha_{\alpha\gamma} G^\gamma_{\beta}$ is symmetric in $\alpha$ and $\beta$. )
Let us introduce a symplectic connection on the tensor bundle over $M_{2N}$. By definition [19], this is a torsion–free connection $\Gamma^c_{ab} = \Gamma^c_{ba}$ which preserves the symplectic structure:

$$\nabla_c \omega_{ab} = \partial_c \omega_{ab} - \Gamma^d_{ca} \omega_{db} - \Gamma^d_{cb} \omega_{ad} = 0 \quad (3.32)$$

We introduce a related connection $\Gamma^\alpha_{a\beta}$ on the frame bundle in such a way that the (total) covariant derivative of the vielbein vanishes:

$$\nabla_a e^\alpha_b = \partial_a e^\alpha_b - \Gamma^c_{ab} e^\alpha_c + \Gamma^\alpha_{a\beta} e^\beta_b = 0 \quad (3.33)$$

This fixes $\Gamma^\alpha_{a\beta}$ in terms of $\Gamma^{b}_{ac}$:

$$\Gamma^\alpha_{a\beta} = e^\alpha_c \Gamma^c_{ab} e^\beta_b - e^\beta_b \partial_a e^\alpha_b \quad (3.34)$$

Since both $\omega_{ab}$ and $e^\alpha_a$ have vanishing covariant derivative, eq.(3.30) implies that

$$\nabla_a \hat{\omega}_{\alpha\beta} = - \left( \hat{\omega}_{\gamma\beta} \Gamma^\gamma_{a\alpha} + \hat{\omega}_{\alpha\gamma} \Gamma^\gamma_{a\beta} \right) = 0 \quad (3.35)$$

This equation tells us that the matrix $\Gamma_a \equiv (\Gamma^\alpha_{a\beta})$ is an element of $sp(2N)$. As a consequence, the coefficients

$$\Gamma_{a\alpha\beta} \equiv - \hat{\omega}_{\alpha\gamma} \Gamma^\gamma_{a\beta} \quad (3.36)$$

have the symmetry $\Gamma_{a\alpha\beta} = \Gamma_{a\beta\alpha}$.

The $Mp(2N)$–covariant derivative of arbitrary tensorial objects $\chi^{\ldots\gamma\ldots}_{y\ldots\beta\ldots}$ is of the form $\nabla_a = \partial_a + \frac{i}{\hbar} \Gamma_a$ with

$$\Gamma_a = \frac{\hbar}{2} \Gamma_{a\alpha\beta} \Sigma^{\alpha\beta} \quad (3.37)$$

Here $\Sigma^{\alpha\beta}$ can be in any representation of $Sp(2N)$ or $Mp(2N)$. The $\Sigma$’s acting on the spinor indices are given by (1.15); the generators in the vector representation and its dual are such [11] that each upper frame index contributes a term with a left–matrix action of $\Gamma_a = (\Gamma^\alpha_{a\beta})$, and each lower index a term where $-\Gamma_a$ acts from the right. Restricting ourselves to spinors, eq.(3.37) is a special example of the spin connection which we introduced in the previous subsection. It assumes values in the Lie algebra $mp(2N)$ rather than the full algebra $G$ of all hermitian operators.
Under a local frame rotation a spinor transforms as

$$\delta_F \psi^x = -\frac{i}{2} \kappa_{\alpha\beta}(\phi) \Sigma^{\alpha\beta} \psi^y$$

(3.38)

and arbitrary multispinors obey (3.12) with $U(\phi) = 1 - i\varepsilon(\phi)/\hbar$ where

$$\varepsilon(\phi) \equiv \frac{\hbar}{2} \kappa_{\alpha\beta}(\phi) \Sigma^{\alpha\beta}$$

(3.39)

This entails that the $\gamma$–matrices and the $Mp(2N)$–generators do not change at all if the transformations acting on the vector indices are included: $\delta_F \gamma^{ax}_{\ y} = 0$, $\delta_F \Sigma^{\alpha\beta}_{\ y} = 0$. From (3.34), (3.36) and the transformation law for the vielbeins it follows that

$$\delta_F \Gamma_{a\alpha\beta} = \partial_a \kappa_{\alpha\beta} + \kappa_{\alpha\gamma} \bar{\omega}^\gamma_{\delta} \Gamma_{a\beta\delta} + \kappa_{\beta\gamma} \bar{\omega}^\gamma_{\delta} \Gamma_{a\alpha\delta}$$

(3.40)

The change of the spin connection is $\delta_F \Gamma_a = \frac{\hbar}{2} (\delta_F \Gamma_{a\alpha\beta}) \Sigma^{\alpha\beta}$. It is easy to check that this coincides precisely with the Yang–Mills gauge transformation (3.8) if the generator $\varepsilon$ is given by (3.39).

Under a coordinate transformation on $\mathcal{M}_{2N}$, $\Gamma^c_{ab}$ transforms like a Christoffel symbol and $\Gamma_{a\alpha\beta}$ as a vector. Hence the operator (3.37) satisfies (3.9).

Up to this point the discussion has a certain similarity with the theory of spinors on curved space–times. Now we come to a special feature of symplectic geometry not shared by Riemannian geometry. Typically, general coordinate transformations can be used to transform a metric $g_{\mu\nu}$ to the flat metric $\eta_{\mu\nu}$ in a single point only. In the symplectic case, the situation is much more favorable: according to Darboux’s theorem there exist local coordinates with respect to which $\omega_{ab} = \bar{\omega}_{ab} = \text{const}$ can be achieved in an extended domain. Using such Darboux local coordinates $\phi^a$, the symplectic structure $\omega_{ab}$ is of the canonical form (3.28) at all points of $\mathcal{M}_{2N}$ covered by the chart to which the $\phi^a$’s belong.

If $\omega_{ab} = \bar{\omega}_{ab}$ then eq.(3.30) tells us that the vielbein coefficients must be chosen such that $[e^a_a] \in Sp(2N)$. The simplest and most natural choice is

$$e^a_a = \delta^a_a, \quad e^a_a = \delta^a_a$$

(3.41)

In the following we shall always employ this specific vielbein when we use Darboux coordinates. No distinction between frame and coordinate indices is necessary then, and we shall use the uniform notation $a, b, c, \cdots$ in either case. From now on, and as in the previous sections, $\omega_{ab}$ stands for the constant canonical matrix.
Furthermore, eq.(3.34) shows that for the vielbein (3.41) the connection coefficients $\Gamma_{\alpha\beta}^{a}$ and $\Gamma_{\alpha\beta}^{c}$ become identical. Thus, in Darboux coordinates,

$$\Gamma_{abc} = -\omega_{ad} \Gamma_{dc}$$

(3.42)

and

$$\Gamma_{a} = \frac{h}{2} \Gamma_{abc} \Sigma^{bc}$$

(3.43)

Using eq.(3.32) together with the fact that the connection has vanishing torsion shows that $\Gamma_{abc}$ is symmetric in all three indices: $\Gamma_{abc} = \Gamma_{(abc)}$. In Darboux coordinates, the $Mp(2N)$–covariant derivative reads explicitly:

$$\nabla_{a} \chi_{x\cdots b\cdots y\cdots c\cdots} = \partial_{a} \chi_{x\cdots b\cdots y\cdots c\cdots} + \frac{1}{2} \Gamma_{ade} \left( \Sigma_{de}^{x\cdots} \chi_{z\cdots b\cdots y\cdots c\cdots} \right) - \chi_{x\cdots b\cdots y\cdots c\cdots} \left( \Sigma_{de}^{z\cdots} \right)$$

$$+ \frac{1}{2} \Gamma_{bce} \chi_{x\cdots e\cdots y\cdots c\cdots} - \frac{1}{2} \Gamma_{ace} \chi_{x\cdots b\cdots y\cdots e\cdots} + \cdots$$

(3.44)

In particular, $\nabla_{a} \gamma^{b} = 0$ and $\nabla_{a} \Sigma^{bc} = 0$. The curvature of $\nabla_{a}$ is given by

$$\Omega_{ab} = \frac{h}{2} R_{abcd} \Sigma^{cd}$$

(3.45)

The tensor $R_{abcd}$ is antisymmetric in $a$ and $b$ but symmetric in $c$ and $d$. Thanks to the simple choice of the vielbein, it is directly related to the Riemann tensor:

$$R_{ab}^{\phantom{ab}c} = -\omega^{ce} R_{abcd}$$

(3.46)

In our conventions,

$$R_{ab}^{\phantom{ab}c} = \partial_{a} \Gamma_{bd}^{c} - \partial_{b} \Gamma_{ad}^{c} + \Gamma_{ae}^{c} \Gamma_{bd}^{e} - \Gamma_{be}^{c} \Gamma_{ad}^{e}$$

(3.47)

It is an important question which subgroup of the group of diffeomorphisms and local frame rotations is left unbroken after fixing the vielbein (3.41). From the general transformation laws

$$\delta_{C} e_{a}^{\alpha} = h^{b} \partial_{b} e_{a}^{\alpha} + \partial_{a} h^{b} e_{b}^{\alpha}$$

$$\delta_{F} e_{a}^{\alpha} = \omega^{\alpha \beta \gamma} \kappa_{\beta \gamma} e_{a}^{\gamma}$$

(3.48)

it is clear that $e_{a}^{\alpha} = \delta_{a}^{\alpha}$ is not stable under either of these transformations. However, if both the vector field $h^{a}$ and the parameters $\kappa_{ab}$ are defined in terms of the same generating function $H(\phi)$ by

$$h^{a} = \omega^{ab} \partial_{b} H, \quad \kappa_{ab} = -\partial_{a} \partial_{b} H$$

(3.49)
then the combined transformation \( \delta_{CF} \equiv \delta_C + \delta_F \) leaves the vielbein (3.41) invariant:

\[
\delta_{CF} e^a_\alpha = 0 \quad \text{for} \quad e^a_\alpha = \delta^a_\alpha
\]  

(3.50)

Therefore the stability group of the fixed vielbein, under which the theory continues to be covariant, is the group of symplectic diffeomorphisms of \( \mathcal{M}_{2N} \), accompanied by appropriate frame rotations which are completely determined by the generating function of the diffeomorphism.

From (3.9) and (3.38) we find the action of \( \delta_{CF} \) on spinors:

\[
\delta_{CF} \psi_\phi = h^a \partial_a \psi_\phi + i \frac{\bar{\partial}}{\partial} H \Sigma^{ab} |\psi_\phi \rangle
\]  

(3.51)

Remarkably enough, this equals precisely the Lie derivative \( \ell_H |\psi_\phi \rangle \) of eq.(2.12).

For arbitrary multispinors, the continuous frame rotations which occur during the Lie–dragging governed by \( -\partial_t \chi = \ell_H \chi = \delta_{CF} \chi \) are precisely those which are needed in order to preserve the canonical form of the vielbein.

### 3.3 Parallel transport

Let us fix a generic \( G \)-valued connection \( \Gamma_a \) and an arbitrary smooth, oriented curve on \( \mathcal{M}_{2N} \), \( C = C(\phi_2, \phi_1) \), connecting the points \( \phi_2 \) and \( \phi_1 \). We choose a parametrization \( \phi^a(s), s \in [0, 1] \), with \( \phi^a(0) = \phi^a_1 \) and \( \phi^a(1) = \phi^a_2 \). Let us pick a vector \( |\psi_\phi \rangle \), which lives in the fiber \( \mathcal{V}_{\phi_1} \) located at the initial point of the trajectory. We can parallel transport this state to all Hilbert spaces \( \mathcal{V}_{\phi(s)} \) along the curve \( \phi^a(s) \). The family \( |\psi_\phi \rangle \in \mathcal{V}_{\phi(s)} \) of parallel Hilbert space vectors satisfies \( \dot{\phi}^a \nabla_a |\psi_\phi \rangle = 0 \) or

\[
\frac{d}{ds} |\psi_\phi \rangle = \frac{i}{\hbar} \dot{\phi}^a(s) \Gamma_a (\phi(s)) |\psi_\phi \rangle
\]  

(3.52)

where the dot means \( d/ds \). (We avoid the notation \( t \) for the parameter here, because in general \( \phi^a(s) \) is not related to a hamiltonian flow.) Eq.(3.52) can be solved formally in terms of a path–ordered exponential,

\[
|\psi_\phi \rangle = V \left[ C (\phi(s), \phi_1) \right] |\psi_\phi_1 \rangle
\]  

(3.53)

with ( \( P \) denotes the path ordering operator)

\[
V \left[ C (\phi(s), \phi_1) \right] = P \exp \left[ -\frac{i}{\hbar} \int_0^s ds' \dot{\phi}^a(s') \Gamma_a (\phi(s')) \right]
\]  

(3.54)
In particular, $|\psi_{\phi_2}\rangle = V[C(\phi_2, \phi_1)]|\psi_{\phi_1}\rangle$. The vector $|\psi_{\phi_2}\rangle$ depends on $C$ in general. The obvious generalization for multispinors is

$$\chi_C(\phi_2)_{\sigma}^{\cdots} = V[C(\phi_2, \phi_1)]_{\nu}^{\sigma} V^\dagger[C(\phi_2, \phi_1)]_{\nu}^{w} \cdots \chi(\phi_1)_{\nu}^{w} \cdots$$ (3.55)

We write $\chi_C(\phi_2)$ for the transported multispinor in order to indicate its path dependence. If $\Gamma_a$ is a flat connection, $\Omega_{ab}(\Gamma) = 0$, the parallel transport is path–independent and every vector in the "reference Hilbert space" at $\phi_1$ gives rise to a spinor field over the entire phase–space: $|\psi\rangle = V[C(\phi, \phi_1)]|\psi_{\phi_1}\rangle$. It is "covariantly constant": $\nabla_a |\psi\rangle_{\phi} = 0$.

Because the connection is a hermitian operator, the parallel transport operator $V$ is unitary, and inner products are preserved under parallel transport. One easily verifies that if $\Gamma_a$ is gauge transformed according to (3.7), $V$ changes as

$$V[C(\phi_2, \phi_1)]' = U(\phi_2) V[C(\phi_2, \phi_1)] \ U(\phi_2) \dagger$$ (3.56)

As a consequence, bilinears of the type $\phi_2 \langle \chi | V[C(\phi_2, \phi_1)] | \psi \rangle_{\phi_1}$ are invariant under local frame rotations.

## 4 Split Symmetry and Exact Quantum Mechanics

In the following we return to the question raised at the end of section 2: given the fact that semiclassical quantum mechanics has a natural interpretation in terms of world–line spinors, what is the corresponding many–Hilbert space description of full–fledged, exact quantum mechanics? Rather than using the standard single–Hilbert space formulation as a guide line we postulate an independent physical principle, invariance under the background–quantum split symmetry, and show a posteriori that this is precisely equivalent to the ordinary quantization rules.

### 4.1 The background–quantum split symmetry

In classical mechanics, a solution of Hamilton’s equation infinitesimally close to a solution $\phi_{\text{cl}}^a(t)$ is given $\phi_{\text{cl}}^a(t) + c^a(t)$ where $c^a(t)$ is a Jacobi field. In semiclassical quantum mechanics, the $\phi_a^m$–expectation value for a particle in the vicinity of $\phi_{\text{cl}}^a$
reads $\phi_0^a(t) + \eta(t)\hat{\varphi}^a\eta(t)$. Let us consider the situation at a fixed time $t$ when the particle is at the point $\phi_0^a(t) \equiv \phi^a$. In classical mechanics we parametrize a point $\Phi^a$ which is infinitesimally close to $\phi^a$ according to $\Phi^a = \phi^a + \xi^a$ where $\xi \in T_{\phi}\mathcal{M}_{2N}$ is a vector in the tangent space at $\phi$, and in semiclassical quantum mechanics one parametrizes $\Phi^a = \phi^a + \phi\langle \psi|\hat{\varphi}^a|\psi\rangle_\phi$ in terms of a Hilbert space vector $|\psi\rangle_\phi \in \mathcal{V}_\phi$.

In a fully quantum mechanical situation a particle is not forced to stay in the vicinity of any classical trajectory and therefore $\phi\langle \psi|\hat{\varphi}^a|\psi\rangle_\phi$ is not small relative to the classical contribution $\phi^a$. Thus, in order to understand exact quantum mechanics in our framework, we need a tool to handle situations when $\Phi^a$ and $\phi^a$ are not close to each other.

In the classical case there exists a familiar method of parametrizing points $\Phi^a$ in a (non–infinitesimal) neighborhood of $\phi^a$ in terms of vectors $\xi \in T_{\phi}\mathcal{M}$. It is based on geodesics and is at the heart of the Riemann normal coordinate expansion [20, 21, 22]. Let $\Gamma_{\hat{a}\hat{b}}$ be an arbitrary connection on $\mathcal{M}_{2N}$, and let $\Phi^a(s)$ denote the solution of the pertinent geodesic equation with the initial point $\Phi^a(s = 0) = \phi^a$ and the initial velocity $\dot{\Phi}^a(s = 0) = \xi^a$. We set $\Phi^a \equiv \Phi^a(1)$ for the point which is visited by the trajectory at the "time" $s = 1$. In this manner one can establish a diffeomorphism between a ball in $T_{\phi}\mathcal{M}_{2N}$, centered at $\xi = 0$, and a neighborhood of $\phi$ in $\mathcal{M}_{2N}$. Points $\Phi$ in this neighborhood are characterized by the initial velocity $\xi$ of the geodesic connecting them to the reference point $\phi$.

From the geodesic equation one finds explicitly:

$$\Phi^a = \phi^a + \xi^a - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma_{\hat{b}_1\ldots\hat{b}_n}^a(\phi) \xi^{b_1} \cdot \cdot \cdot \xi^{b_n}$$

(4.1)

$\Gamma_{\hat{b}_1\ldots\hat{b}_n}^a(\phi) \equiv \bar{\nabla}_{(\hat{b}_1} \cdots \bar{\nabla}_{\hat{b}_{n-2}} \Gamma_{\hat{b}_{n-1}\hat{b}_n)}^a(\phi)$

The "covariant derivative" $\bar{\nabla}^a$ is to be taken with respect to the lower indices only. When applied to a flat space with $\Gamma_{\hat{a}\hat{b}} = 0$, eq.(4.1) collapses to

$$\Phi^a = \phi^a + \xi^a$$

(4.2)

and the vectors $\xi$ in a single tangent space are sufficient to parametrize the entire manifold. ($\xi$ is not infinitesimal in (4.2)!!)

The parametrization (4.1) is subject to a rather restrictive consistency condition. In the simple example (4.2) it is obvious that for an arbitrary vector $\varepsilon^a$ the pairs $(\phi^a, \xi^a)$ and $(\phi^a + \varepsilon^a, \xi^a - \varepsilon^a)$, respectively, describe the same point $\Phi^a$.
Stated differently, $Φ^a$ is invariant under the split symmetry $δ_S φ^a = ε^a$, $δ_S ξ^a = −ε^a$.

This is a rather trivial symmetry transformation for a flat space, but in the general case it involves a complicated nonlinear transformation [22]:

\[
δ_S φ^a = ε^a \\
δ_S ξ^a = F^a_b (φ, ξ) ε^b
\]

(4.3)

The functions $F^a_b$ must be determined from the requirement that $δ_S Φ^a = 0$.

We start the investigation of the quantum case with the example of the flat phase-space $M_{2N} = IR^{2N}$. In this case

\[
Φ^a = φ^a + φ ⟨ψ| \hat{ϕ}^a|ψ⟩_φ
\]

(4.4)

provides a globally valid parametrization of phase–space in terms of Hilbert space vectors. (The expectation value in (4.4) is not infinitesimal.) With the reference point $φ^a$ kept fixed, every $|ψ⟩_φ ∈ V_φ$ gives rise to a well–defined point $Φ^a$. Conversely, every $Φ^a$ can be represented in the form (4.4), but there are infinitely many Hilbert space vectors which achieve this. Let us ask how $|ψ⟩_φ$ must change in order to compensate for an infinitesimal change of the reference point, $φ^a → φ^a + δφ^a$.

We make the following ansatz for the ”background–quantum split symmetry”

\[
δ_S φ^a = ε^a \\
δ_S |ψ⟩_φ = −i \hbar ε^a \tilde{Γ}_a (φ) |ψ⟩_φ
\]

(4.5)

and determine the operators $\tilde{Γ}^a$ in such a way that $δ_S Φ^a = 0$. Applying (4.5) to (4.4) yields the condition

\[
[\tilde{Γ}_a, \hat{ϕ}^b] = i \hbar δ^b_a
\]

(4.6)

if one assumes that $|ψ⟩_φ$ is normalized to unity. Its general solution reads

\[
\tilde{Γ}_a (φ) = ω_{ab} \hat{ϕ}^b + α_a (φ)
\]

(4.7)

where $α_a$ is an arbitrary $c$–number valued one–form. Thus we managed to associate to $φ^a$, regarded as a classical observable, a family of operators (labeled by the points of phase–space)

\[
O_φ^a (φ) = φ^a + \hat{ϕ}^a
\]

(4.8)

such that $φ ⟨ψ|O_φ^a (φ)|ψ⟩_φ$ is independent of $φ$ provided the states $|ψ⟩_φ$ at different points $φ$ are related by (4.5).
Returning now to an arbitrary curved phase–space, we expect that the RHS of eq.(4.8) has to be modified by terms involving products of $\hat{\phi}'s$. Clearly eq.(4.8) is a kind of quantum analogue of (4.2), and the rôle of $\xi^a$ is played by the operators $\hat{\phi}^a$ now. Rather than looking at $\phi^a$ only, we shall consider a general classical observable, i.e., a real–valued function $f = f(\phi)$, and associate a family of operators

$$O_f(\phi) = f(\phi) + \Delta O_f(\phi)$$

(4.9)
to it such that $\phi \langle \psi | O_f(\phi) | \psi \rangle_\phi$ is independent of $\phi$. In (4.9), $\Delta O_f(\phi)$ stands for a power–series in $\hat{\phi}'s$ which starts with a linear term.

Neither the form of $O_f(\phi)$ nor the prescription for transporting the $|\psi\rangle's$ is known a priori. We shall keep the ansatz (4.5) for the split symmetry also in the general case, but (4.7) will have to be generalized. Assume $\tilde{\Gamma}_a$ is given, then we can transport any vector $|\psi\rangle_\phi$ from $\phi$ to $\phi + \delta \phi$:

$$\delta S |\psi\rangle_\phi \equiv |\psi\rangle_{\phi + \delta \phi} - |\psi\rangle_\phi = \varepsilon^a \partial_a |\psi\rangle_\phi + O(\varepsilon^2)$$

(4.10)
This means that the function $\phi \mapsto |\psi\rangle_\phi$ is constrained by the condition

$$i \hbar \partial_a |\psi\rangle_\phi = \tilde{\Gamma}_a(\phi) |\psi\rangle_\phi$$

(4.11)
If we perform local frame rotations of the form (3.5) both in $V_\phi$ and in $V_{\phi + \delta \phi}$, we observe that the operators $\tilde{\Gamma}_a$ transform precisely according to (3.7). This means that $\tilde{\Gamma}_a$ is a spin connection of the type introduced in section 3.1. Eq.(4.11) tells us that the spinor field $\psi^x(\phi) = \langle x | \psi \rangle_\phi$ should be "covariantly constant" with respect to $\tilde{\Gamma}_a$: $D_a |\psi\rangle_\phi = 0$. We shall use the notation

$$D_a \equiv \partial_a + i \frac{\hbar}{\imath} \tilde{\Gamma}_a \equiv \nabla_a(\tilde{\Gamma})$$

(4.12)
for the covariant derivative formed from $\tilde{\Gamma}_a$. The split transformation $\delta S$ is nothing but an infinitesimal parallel transport from $\phi$ to $\phi + \delta \phi$. If we require invariance under the split symmetry (4.5), i.e., that $\phi \langle \psi | O_f(\phi) | \psi \rangle_\phi$ is independent of $\phi$, it follows that $O_f$, too, must be covariantly constant (or a "flat section"):

$$\partial_a O_f + \frac{\hbar}{\imath} \left[ \tilde{\Gamma}_a, O_f \right] \equiv D_a O_f = 0$$

(4.13)
A necessary condition for the integrability of this equation is that $D[a] D_b] O_f = 0$, or

$$\left[ \Omega_{ab}(\tilde{\Gamma}), O_f(\phi) \right] = 0$$

(4.14)
We conclude that $\tilde{\Gamma}$ must be an abelian connection. By definition [19], an abelian connection has a curvature which commutes with all other operators. This happens if $\Omega_{ab}(\tilde{\Gamma})$ equals a $c$-number valued 2-form times the unit operator on $\mathcal{V}$: $\Omega_{ab}(\tilde{\Gamma}) = \beta_{ab}(\phi)$. As a consequence, the resulting parallel transport operator $V[\mathcal{C}]$ for a closed curve $\mathcal{C}$ is simply a phase factor times the unit operator, i.e., the holonomy group of $\tilde{\Gamma}$ is $U(1)$ rather than the full group $G$. For $\mathcal{C}$ an infinitesimal parallelogram, for instance, one has $V[\mathcal{C}] = 1 - \frac{i}{\hbar} \beta_{ab} d\phi^a d\phi^b$.

The analogous integrability condition of the equation for states, $D_a |\psi\rangle_\phi = 0$, is more restrictive. It requires the connection to be flat: $\Omega_{ab}(\tilde{\Gamma}) = 0$. If this condition is satisfied, we can parallel transport a state $|\psi\rangle_\phi$ around a closed loop $\mathcal{C}$ and, at least locally, we are guaranteed to get back precisely the original state after having completed the circuit. If, instead, the connection is not flat but only abelian, $|\psi\rangle_\phi$ will in general pick up a phase factor during the excursion around $\mathcal{C}$. For our purposes this is equally acceptable, because the phase cancels in $\langle \psi | O_f(\phi) | \psi \rangle_\phi$. Thus, in order to make sure that this expectation value is independent of $\phi$, we shall only require that $\tilde{\Gamma}_a$ is an abelian connection, and relax the consistency condition at the level of $|\psi\rangle_\phi$ to a "consistency up to a phase". In the next section we shall see that an abelian connection can be found for any phase-space $\mathcal{M}_{2N}$.

For flat phase-space, $\tilde{\Gamma}_a$ is given in eq.(4.7). Its curvature 2-form reads

$$\Omega(\tilde{\Gamma}) = \omega + d\alpha$$

As it should be, this connection is abelian: both the symplectic 2-form and $d\alpha$ are proportional to the unit operator in $\mathcal{V}$. Flat space is special in that $\omega$ is globally exact, i.e., $\omega = d\theta$ where the symplectic potential $\theta = \theta_a d\phi^a$ has the components

$$\theta_a(\phi) = - \frac{1}{2} \omega_{ab} \phi^b$$

Hence setting $\alpha = -\theta$ gives us a flat connection:

$$\tilde{\Gamma}_a(\phi) = \omega_{ab} \left( \phi^b + \frac{1}{2} \phi^b \right)$$

This situation is not typical though. In the next section we shall employ an algorithm which, in a "canonical" way, provides us with an abelian connection with curvature $\Omega(\tilde{\Gamma}) = \omega$. We could try to modify $\tilde{\Gamma}$ by adding a $c$-number valued 1-form $\alpha$; this leads to the curvature $\Omega(\tilde{\Gamma} + \alpha) = \omega + d\alpha$. But generically
ω cannot be written as $d\theta$ in terms of a globally well defined 1–form $\theta$. Therefore it is not possible then to obtain a flat connection by setting $\alpha = -\theta$.

To summarize: To every classical observable $f$ and to every point $\phi$ of phase–space we have associated an operator $O_f(\phi) : \mathcal{V}_\phi \to \mathcal{V}_\phi$. In each fiber $\mathcal{V}_\phi$ we can compute expectation values of the form

$$\langle \psi | O_f(\phi) | \psi \rangle_{\phi} = f(\phi) + \phi \langle \psi | \Delta O_f(\phi) | \psi \rangle_{\phi}$$

We have postulated that this expectation value should be the same in all fibers, i.e., independent of $\phi$. This requires $O_f(\phi)$, regarded as a function of $\phi$, to be covariantly constant with respect to an arbitrary abelian connection $\tilde{\Gamma}$ and $|\psi\rangle_{\phi}$ to be covariantly constant up to a phase. Changing the point $\phi$ in (4.18) means changing the individual contributions of the classical ”background” term $f(\phi)$ and of the quantum mechanical $\Delta O_f$–expectation value. The position independence of their sum is what we mean by ”background–quantum split symmetry”.

### 4.2 Abelian connections

The implementation of the split symmetry involves two steps: first, find an abelian connection $\tilde{\Gamma}$, and second, construct multispinor fields (states, operators, ...) which are covariantly constant with respect to $\tilde{\Gamma}$, possibly modulo a phase. We shall employ a method which was proposed by Fedosov [23, 19] and which allows for an iterative construction of $\tilde{\Gamma}$ on any symplectic manifold.

We start by fixing an arbitrary symplectic connection. We write $\Gamma$ for this connection and $\nabla_a \equiv \nabla_a(\Gamma)$ for its covariant derivative, and continue to use the notation $\tilde{\Gamma}$ and $D_a \equiv \nabla_a(\tilde{\Gamma})$ for the abelian connection and its covariant derivative. We make the ansatz

$$\tilde{\Gamma} = \Gamma + \lambda$$

where $\lambda = \lambda_a(\phi)\, d\phi^a$ is some globally defined 1–form. While $\Gamma \in mp(2N)$, $\lambda$ may assume values in the larger algebra $\mathcal{G}$. The curvature forms of $\Gamma$ and $\tilde{\Gamma}$ are related by

$$\Omega(\tilde{\Gamma}) = \Omega(\Gamma) + \nabla\lambda + \frac{i}{\hbar} \lambda^2$$

It will turn out convenient to separate off from $\lambda$ the flat space solution (4.7):

$$\lambda \equiv \omega_{ab} \hat{\phi}^b\, d\phi^a + r$$
Inserting (4.21) into (4.20) one finds after some calculation which makes essential use of the fact that $\Gamma$ is symplectic:

$$\Omega(\tilde{\Gamma}) = \omega + \Omega(\Gamma) - \delta r + \nabla r + \frac{i}{\hbar} r^2$$

(4.22)

Here we introduced the linear map $\delta$. It acts on operator–valued differential forms $F$ according to

$$\delta F = d\phi^a \hat{\partial}_a F$$

(4.23)

with

$$\hat{\partial}_a F = -\frac{i}{\hbar} \omega_{ab} [\hat{\varphi}^b, F]$$

(4.24)

At this point some technical remarks are in order. The connection $\tilde{\Gamma}$ assumes values in the Lie algebra $\mathcal{G}$ consisting of all hermitian operators on $\mathcal{V}$. More precisely, we define $\mathcal{G}$ to be generated by completely symmetrized products of $\hat{\varphi}$’s:

$$\hat{\varphi}^{(a_1 \ldots \varphi^{a_n})}, \quad n = 0, 1, 2, 3, \ldots$$

(4.25)

These operators are Weyl–ordered. Their commutator algebra is a (generalization of the) $W_{\infty}$–algebra [5, 6]. The quadratic generators $\hat{\varphi}^{(a\varphi^b)} = \hbar \Sigma^{ab}$ form the subalgebra of $mp(2N)$.

When applied to a symmetrized monomial (4.25), $\hat{\partial}_a$ acts formally like a differentiation $\partial/\partial \hat{\varphi}^a$, i.e., if $g(\varphi)$ is an analytic $c$–number function and we associate the operator

$$g(\tilde{\varphi}) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{a_1} \ldots \partial_{a_n} g(0) \hat{\varphi}^{(a_1 \ldots \varphi^{a_n})}$$

(4.26)

to it, then $\hat{\partial}_a g(\tilde{\varphi}) = (\partial g/\partial \varphi^a)(\tilde{\varphi})$. The operation $\hat{\partial}_a$ has to be carefully distinguished from the partial derivative $\partial_a \equiv \partial/\partial \phi^a$.

Frequently we shall deal with monomials of the form

$$\tilde{F}_{mn} \equiv \hbar^m \hat{\varphi}^{(a_1 \ldots \varphi^{a_n})}$$

(4.27)

An important concept is their degree which is defined as $\deg(\tilde{F}_{mn}) = 2m + n$, i.e., $\deg(\hbar) = 2$ and $\deg(\hat{\varphi}^a) = 1$. For a product of two monomials (4.27), the degree is additive:

$$\deg(F_1 F_2) = \deg(F_1) + \deg(F_2)$$

(4.28)

A priori, $F_1 F_2$ is not totally symmetrized if $F_1$ and $F_2$ are, but it can be expanded in the basis (4.25) by repeatedly applying the canonical commutation relations
(1.9). Each time (1.9) is used, two $\hat{\phi}$'s are removed and one factor of $\hbar$ is added, thus leaving the degree unchanged. Hence all terms generated in the process of Weyl ordering $F_1F_2$ have the same degree, $\deg(F_1) + \deg(F_2)$.

In the spirit of a semiclassical expansion, the degree is the total power of $\hbar^{1/2}$ contained in a given monomial. In fact, the $\hat{\phi}$'s satisfy the commutation relation (1.9) which involves an explicit factor of $\hbar$; hence $\hat{\phi}^a$ itself is of order $\hbar^{1/2}$. The $\gamma$–matrices satisfy an $\hbar$–independent Clifford algebra instead, and are of degree zero therefore. Reexpressing $\hat{F}_{mn}$ in terms of $\gamma$–matrices, the degree coincides with the explicit power of $\hbar^{1/2}$:

$$\hat{F}_{mn} = 2^{-n/2} \hbar^{(2m+n)/2} \gamma^{(a_1 \cdots \gamma^{a_n})}$$ (4.29)

Let us return to eq.(4.22). It is clear that if we manage to find an operator valued 1–form $r \equiv r_a d\phi^a$ in such a way that

$$\delta r = \Omega(\Gamma) + \nabla r + \frac{i}{\hbar} r^2$$ (4.30)

then

$$\tilde{\Gamma}_a = \Gamma_a + \omega_{ab} \hat{\phi}^b + r_a$$ (4.31)

is an abelian connection with

$$\Omega_{ab}(\tilde{\Gamma}) = \omega_{ab}$$ (4.32)

It is here that the work of Fedosov [23, 19] comes in. He showed that (4.30) can be solved on any symplectic manifold and he developed an iterative procedure for calculating $r$. If one wants to compute $r$ only up to terms of some fixed degree only a finite number of iterations is needed. Some details of this method are given in the Appendix. Here we only quote the first two terms of the result:

$$r(\phi, \hat{\phi}) = -\frac{1}{8} R_{abcd} \hat{\phi}^b \hat{\phi}^c \hat{\phi}^d d\phi^a + \frac{1}{36} \nabla_b R_{acde} \hat{\phi}^a \hat{\phi}^b \hat{\phi}^c \hat{\phi}^d d\phi^e + O(5)$$ (4.33)

$R \equiv \Omega(\Gamma)$ is the curvature of the symplectic connection, and "$O(5)$" stands for terms of degree 5 and higher.

Our second task is the construction of covariantly constant fields $\chi$, in particular of the operators $\mathcal{O}_f(\phi)$. Given $\tilde{\Gamma}$, Fedosov’s method can also be used to solve the equation $D_a \mathcal{O}_f = 0$ by iteration. For any function $f$, this equation has
a unique solution if one requires that the term in $O_f(\phi)$ proportional to the unit operator is precisely $f(\phi)$. In the appendix we show that

$$O_f(\phi) = f(\phi) + \hat{\varphi}^a \nabla_a f + \frac{1}{2} \hat{\varphi}^{(a} \hat{\varphi}^{b)} \nabla_a \nabla_b f + \frac{1}{6} \hat{\varphi}^{(a} \hat{\varphi}^{b} \hat{\varphi}^{c)} \nabla_a \nabla_b \nabla_c f$$

$$+ \frac{1}{24} R_{abcd} \omega^{be} \partial_c f \hat{\varphi}^{(a} \hat{\varphi}^{c} \hat{\varphi}^{d)} + O(4)$$

(4.34)

As an example, let us consider the observable $f(\phi) = \phi^a$ for a fixed. With the notation $\nabla_a$ for the "covariant derivative" which acts on lower indices only, one finds

$$O_{\phi^a}(\phi) = \phi^a + \hat{\varphi}^a - \frac{1}{2} \Gamma^a_{bc} \hat{\varphi}^{(b} \hat{\varphi}^{c)} - \frac{1}{3} \nabla_a (\Gamma^a_{bc}) \hat{\varphi}^{(b} \hat{\varphi}^{c) \hat{\varphi}^{d)}$$

$$+ \frac{1}{24} \omega^{ab} R_{bcde} \hat{\varphi}^{(c} \hat{\varphi}^{d} \hat{\varphi}^{e)} + O(4)$$

(4.35)

This expansion assumes a particularly simple form if one uses normal Darboux coordinates. Every symplectic connection defines an essentially unique system of Darboux local coordinates, the normal Darboux coordinates [19], with the property that at a given point $\phi_0$ both $\Gamma_{abc} = 0$ and $\partial_{(e_1} \partial_{e_2} \cdots \partial_{e_n} \Gamma_{abc)} = 0$ for all $n = 1, 2, \cdots$. From these equations one can deduce that $\partial_a \Gamma_{bcd}(\phi_0) = \frac{3}{4} R_{a(bcd)}(\phi_0)$, and similar relations for higher partial derivatives of $\Gamma_{abc}$. In terms of normal Darboux coordinates centered at $\phi_0 = \phi$, eq.(4.35) becomes

$$O_{\phi^a}(\phi) = \phi^a + \hat{\varphi}^a - \frac{1}{2} \omega^{ab} R_{bcde} \hat{\varphi}^{(c} \hat{\varphi}^{d} \hat{\varphi}^{e)} + O(4)$$

$$= \phi^a + \hat{\varphi}^a - \frac{1}{6} \nabla_a (\Gamma_{cd}) \hat{\varphi}^{(b} \hat{\varphi}^{c) \hat{\varphi}^{d)} + O(4)$$

(4.36)

This is precisely what one obtains from the classical expansion (4.1) if one replaces $\Phi^a \to O_{\phi^a}$ and $\xi^a \to \hat{\varphi}^a$! This correspondence will not persist at the higher orders, though, since eventually the iteration will produce terms with explicit powers of $\hbar$.

4.3 Recovering exact quantum mechanics

The split symmetry requires all fields $O_f(\phi)$ to be covariantly constant with respect to an abelian connection. As the curvature $\Omega(\tilde{\Gamma}) = \omega$ is proportional to the unit operator, the holonomy group of $\tilde{\Gamma}$, generated by the operators $V[C] =$

\footnote{For an interpretation of Fedosov’s work in terms of a ”quantum deformed exponential map” see ref.[24].}
\[ P \exp \left( -\frac{i}{\hbar} \int \Gamma \right) \] for closed curves \( \mathcal{C} \), is reduced to a \( U(1) \) subgroup of \( G \). The parallel transport around an infinitesimal parallelogram yields \( V[\mathcal{C}] = 1 - \frac{i}{\hbar} \omega_{ab} d\phi^a d\phi^b \); for arbitrary loops this integrates to \( V[\mathcal{C}] = \exp \left( -\frac{i}{\hbar} \int_{\mathcal{D}} \omega \right) \) where \( \mathcal{D} \) is an arbitrary surface in \( M_{2N} \) bounded by \( \mathcal{C} \), \( \partial \mathcal{D} = \mathcal{C} \). To begin with, let us focus on a single chart of an atlas covering \( M_{2N} \). Since \( \omega \) is closed, locally there exists an 1–form \( \theta \) such that \( \omega = d\theta \) and \( V[\mathcal{C}] = \exp \left( -\frac{i}{\hbar} \int_{\mathcal{C}} \theta \right) \). Let us suppose \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are two open curves with the same endpoints. Then \( \mathcal{C}_1 - \mathcal{C}_2 \) is closed and

\[ V[\mathcal{C}_1] V[\mathcal{C}_2]^\dagger = \exp \left( -\frac{i}{\hbar} \int_{\mathcal{C}_1 - \mathcal{C}_2} \theta \right) \] (4.37)

is a pure phase factor. Therefore, if one defines

\[ V[\mathcal{C}(\phi, \phi_0)] \equiv \exp \left( -\frac{i}{\hbar} \int_{\mathcal{C}(\phi, \phi_0)} \theta \right) \tau(\phi, \phi_0) \] (4.38)

then the unitary operator \( \tau \) depends only on the two points but not on \( \mathcal{C} \). We can parallel transport an arbitrary \((r, s)\)–spinor from \( \phi_0 \) to \( \phi \), and the only path dependence will reside in an overall phase factor:

\[ \chi_{\mathcal{C}}(\phi)_{\stackrel{\mathcal{I}_1}{\cdots} \mathcal{I}_r} = \exp \left[ -\frac{i}{\hbar} (r - s) \int_{\mathcal{C}(\phi, \phi_0)} \theta \right] \times \tau(\phi, \phi_0)_{\mathcal{I}_1} \cdots \chi(\phi_0)_{\mathcal{I}_1} \cdots \tau^\dagger(\phi, \phi_0)_{\mathcal{I}_1} \cdots \] (4.39)

For \( r = s \) the phase factor cancels, and in particular the parallel transport of operators \((r = s = 1)\) is path independent. The most important example are the "local observables" \( O_f \):

\[ O_f(\phi) = \tau(\phi, \phi_0) O_f(\phi_0) \tau(\phi, \phi_0)^{-1} \] (4.40)

For states,

\[ |\psi_\mathcal{C}\rangle = \exp \left( -\frac{i}{\hbar} \int_{\mathcal{C}(\phi, \phi_0)} \theta \right) \tau(\phi, \phi_0) |\psi\rangle_{\phi_0} \] (4.41)

Given the operator \( O_f(\phi_0) \) at a fixed point \( \phi_0 \), eq.(4.40) defines a \((1, 1)\)–spinor field on the entire phase–space. Because of the path–dependent exponential, this is not the case for the vectors (4.41). However, \( |\psi\rangle_{\phi_0} \) gives rise to well defined spinor fields over a special class of submanifolds in \( M_{2N} \). In fact, every \( 2N \)–dimensional symplectic manifold has certain \( N \)–dimensional submanifolds \( \mathcal{K} \),
the Lagrangian manifolds \([10, 9]\), with the property that for every path \(C\) in \(\mathcal{K}\) the integral \(\int_C \theta\) is invariant under smooth deformations of \(C\) which stay within \(\mathcal{K}\). Lagrangian manifolds can be thought of as the image of configuration space under some canonical transformation. If \(\phi_0 \in \mathcal{K}\), the parallel transport is locally integrable, i.e., \((4.41)\) defines a single–valued spinor field \(|\psi\rangle_\phi\) on \(\mathcal{K}\) in a neighborhood of \(\phi_0\). Globally, complications might arise if two paths connecting \(\phi\) to \(\phi_0\) form a closed curve in \(\mathcal{K}\) which cannot be smoothly contracted to a point while remaining on \(\mathcal{K}\). This leads to topological quantization conditions \([10]\) which we shall not discuss here.

As we pointed out at the beginning of this section, our aim is to show that semiclassical quantum mechanics plus invariance under the background–quantum split symmetry implies exact quantum mechanics. Eqs.(4.40) and (4.41) are the main ingredients for establishing this equivalence. These two equations show that the physical contents which is encoded in the local physical states and operators related to the infinitely many Hilbert spaces \(\mathcal{V}_\phi\) can be described by the states and operators belonging to one single ”reference” Hilbert space. We choose this reference Hilbert space to be the fiber \(\mathcal{V}_\phi\) at \(\phi = \phi_0\) with \(\phi_0\) an arbitrary but fixed point of \(\mathcal{M}_{2N}\). Since the operators and states at any other point are related to those at \(\phi_0\) by \((4.40)\) and \((4.41)\), it does not matter at which point transition matrix elements or expectation values of physical observables are computed:

\[
\phi\langle \chi | \psi_C \rangle_\phi = \phi_0 \langle \chi | \psi \rangle_{\phi_0} \\
\phi\langle \chi_C | O_f(\phi) | \psi_C \rangle_\phi = \phi_0 \langle \chi | O_f(\phi_0) | \psi \rangle_{\phi_0}
\]

Note that the LHS of these equations is actually independent of \(C\) because the path–dependent phase factor cancels.

In the standard formulation of quantum mechanics, the states of a physical system are described by the vectors of a single Hilbert space \(\mathcal{V}_{\text{QM}}\) and observables by hermitian operators on \(\mathcal{V}_{\text{QM}}\). The discussion above suggests identifying \(\mathcal{V}_{\text{QM}}\) with the reference Hilbert space at \(\phi_0\),

\[
\mathcal{V}_{\text{QM}} \cong \mathcal{V}_{\phi_0}
\]

and the quantum mechanical state vectors and observables with the vectors \(|\psi\rangle_{\phi_0} \in \mathcal{V}_{\phi_0}\) and operators \(O_f(\phi_0) : \mathcal{V}_{\phi_0} \to \mathcal{V}_{\phi_0}\), respectively. Let us make this identification more precise.
At the semiclassical level, the time evolution of multispinor fields $\chi(\phi; t)$ is governed by the Lie–derivative $\ell_H$. It remains to be understood how this type of dynamics ("Lie–transport along the Hamiltonian flow") is related to the conventional Schrödinger or Heisenberg equation on $V_{QM}$. The unifying dynamical principle which, on the one hand, reduces to the $\ell_H$–dynamics in the semiclassical limit and on the other hand implies the ordinary Schrödinger equation when all Hilbert spaces are identified is a time–dependent local frame rotation with the generator $O_H(\phi)$. Let $\chi(\phi; t)$ be an arbitrary, not necessarily covariant constant multispinor satisfying the equation of motion

$$-\partial_t \chi(\phi; t) = \mathcal{L}_H(\phi) \chi(\phi; t)$$

with

$$\mathcal{L}_H(\phi) \chi(\phi)^{x_1 \ldots x_r}_{y_1 \ldots y_s} = \frac{i}{\hbar} \sum_{m=1}^{r} O_H(\phi)^{x_m}_{y_m} \chi(\phi)^{x_1 \ldots x_r}_{y_1 \ldots y_s} - \frac{i}{\hbar} \sum_{m=1}^{s} \chi(\phi)^{x_1 \ldots x_r}_{y_1 \ldots y_s} O_H(\phi)^{y_m}_{z_m}$$

This equation of motion is the most general dynamical law, based upon a local frame rotation, which is compatible with the background–quantum split symmetry. We have to make sure that if $D_a \chi = 0$ is satisfied initially it still holds true at any later time. This is the case if the generator is annihilated by $D_a$, i.e., if it is of the form $O_H(\phi)$ for some function $H$.

Let us determine the semiclassical limit of the operator (4.45), $\mathcal{L}^{scl}_H$. Due to the explicit factors of $\hbar$ in (4.45) this amounts to disregarding all terms of positive degree: $\mathcal{L}_H = \mathcal{L}^{scl}_H + O(1)$. As a consequence, terms of degree 3 and higher may be omitted from $O_H(\phi)$ and only the first three terms on the RHS of (4.34) are relevant. Writing out the covariant derivatives and using (1.5) one finds (with $h^a \equiv \omega^{ab} \partial_b H$):

$$\frac{i}{\hbar} O_H(\phi) = \frac{i}{\hbar} H(\phi) + \frac{i}{2} \partial_a \partial_b H(\phi) \Sigma^{ab} - \frac{i}{\hbar} h^a \left[ \omega_{ab} \tilde{\varphi}^b + \frac{\hbar}{2} \Gamma_{abc} \Sigma^{bc} \right] + O(1)$$

It is interesting that the terms in the square brackets are exactly the degree–2 approximation of $\tilde{\Gamma}_a$. In fact, because $r_a$ starts with a term of order 3, eq.(4.31) with (3.43) reduces to

$$\tilde{\Gamma}_a = \omega_{ab} \varphi^b + \frac{\hbar}{2} \Gamma_{abc} \Sigma^{bc} + O(3)$$

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We interpret the connection–term in (4.46) as \( d_a^\Gamma = D_a - \partial_a \). Thus,

\[
\mathcal{L}_H^\text{cl} \chi(\phi)_{\hat{y}_1 \ldots \hat{y}_s} = \left[ h^a \partial_a - h^a D_a + \frac{i}{\hbar} (r - s) H \right] \chi_{\hat{y}_1 \ldots \hat{y}_s} \\
+ \frac{i}{2} \partial_a \partial_b H \left[ (\Sigma^{ab})_z^{x_1} \chi_z^{x_1 \ldots x_r} - \chi_z^{x_1 \ldots x_r} (\Sigma^{ab})_z^{y_1} \pm \cdots \right]
\]

(4.48)

Comparison with (2.13) shows that \( \mathcal{L}_H^\text{cl} \) is closely related to the Lie–derivative \( \ell_H \):

\[
\mathcal{L}_H^\text{cl} \chi = \ell_H \chi - h^a D_a \chi + \frac{i}{\hbar} (r - s) H \chi
\]

(4.49)

Obviously, \( \mathcal{L}_H^\text{cl} \) and \( \ell_H \) differ by terms involving the classical Hamiltonian and \( D_a \chi \). For multispinors with \( r = s \), e.g., for operators \( (r = s = 1) \), the \( H \)–term is absent. But also for other multispinors it can be removed from the equation of motion (4.44), (4.49) in a trivial way: the field \( \chi' \) defined by

\[
\chi(\phi; t) \equiv \exp \left[ -\frac{i}{\hbar} (r - s) H(\phi) t \right] \chi'(\phi; t)
\]

(4.50)

satisfies the same equation without the \( H \)–term. For \( (1, 0) \)–spinors, eq.(4.50) contains a generalization of the familiar dynamical phase \( \exp(-iEt/\hbar) \). The equivalence of \( \mathcal{L}_H^\text{cl} \) and \( \ell_H \) becomes complete when they are applied to covariantly constant fields. Then \( D_a \chi = 0 \) and \( \mathcal{L}_H \chi = \ell_H \chi + O(1) \).

It is quite remarkable how this equality comes about. In contradistinction to \( \ell_H \), the operator \( \mathcal{L}_H \) effects a pure frame rotation and no diffeomorphism. In fact, the partial derivatives cancel between the first and the second term of (4.49). It is only because the condition \( D_a \chi = 0 \) couples displacements in \( \mathcal{M}_{2N} \) to rotations in \( \mathcal{V}_\phi \) that \( \mathcal{L}_H \) becomes effectively equivalent to a Lie derivative.

Under an infinitesimal frame rotation with parameters \( \varepsilon \), the connection changes by an amount \( \delta_F \hat{\Gamma}_a = D_a \varepsilon \). In the case at hand, \( \varepsilon \equiv O_H(\phi) t \) is covariantly constant, and the connection is invariant therefore: \( \delta_F \hat{\Gamma}_a = 0 \).

The most important property of the equation of motion (4.44), (4.45) is its covariance with respect to the split symmetry. It can be solved in terms of the local evolution operators

\[
U(\phi; t) = \exp \left[ -\frac{i}{\hbar} O_H(\phi) t \right]
\]

(4.51)

They have the property

\[
U(\phi; t) \tau(\phi, \phi_0) = \tau(\phi, \phi_0) U(\phi_0, t)
\]

(4.52)
Applying this equation to a vector $|\psi(t = 0)\rangle_{\phi_0}$, say, we see that it does not matter whether we first parallel-transport this vector from $\phi_0$ to $\phi$ and time-evolve it there using $U(\phi; t)$ or if we first time-evolve at $\phi_0$, using $U(\phi_0; t)$, and parallel-transport afterwards:

$$|\psi(t)\rangle_{\phi} = U(\phi; t)|\psi(0)\rangle_{\phi} = V[C(\phi, \phi_0)]|\psi(t)\rangle_{\phi_0}$$

(4.53)

The vectors $|\psi(t)\rangle_{\phi}$ satisfy the local Schrödinger equation

$$i\hbar \partial_t |\psi(t)\rangle_{\phi} = \mathcal{O}_H(\phi)|\psi(t)\rangle_{\phi}$$

(4.54)

For different points $\phi$, these equations are related by (4.40), (4.41) and are equivalent therefore. In particular, they are all equivalent to the Schrödinger equation in the reference Hilbert space $\mathcal{V}_{\phi_0} \cong \mathcal{V}_{QM}$. Comparing (4.54) to the ordinary Schrödinger equation on $\mathcal{V}_{QM}$, $i\hbar \partial_t |\psi\rangle = \hat{H}|\psi\rangle$, this means that we may identify $|\psi\rangle \equiv |\psi\rangle_{\phi_0}$ and

$$\hat{H} \equiv \mathcal{O}_H(\phi_0)$$

(4.55)

Eq.(4.55) is a consistent generalization of the naive "$H(\phi)$ becomes $H(\hat{\phi})$"-rule appropriate for general phase-spaces. In the next section we shall see that indeed, for a flat phase-space, $\mathcal{O}_H(\phi_0 = 0) = H(\hat{\phi})$.

An analogous discussion applies to arbitrary $(r, s)$-multispinors. For an operator $\mathcal{O}_\rho$, for instance, eq.(4.44) reads

$$i\hbar \partial_t \mathcal{O}_\rho(t)(\phi) = \left[\mathcal{O}_H(\phi), \mathcal{O}_\rho(t)(\phi)\right]$$

(4.56)

This is a local version of von Neumann's equation. At $\phi = \phi_0$ it coincides with the usual von Neumann equation for the ordinary statistical operator $\mathcal{O}_\rho(\phi_0) \equiv \hat{\rho}$. We are using the Schrödinger picture here, i.e., the entire time dependence resides in the state vectors (pure states) or in the statistical operator (mixed states). The operators $\mathcal{O}_f(\phi)$ representing observables are time independent.

Up to now our discussion of the spin bundles mostly dealt with their local properties. Typically, in order to guarantee their existence globally, certain topological quantization conditions must be fulfilled. Let $\{\mathcal{P}_\alpha\}$ be a contractible open covering of phase-space, $\mathcal{M}_{2N} = \bigcup_\alpha \mathcal{P}_\alpha$, and let $\theta_\alpha$, $\Gamma_\alpha$ and $\psi_\alpha$ locally represent a symplectic potential, a $G$-valued connection and an arbitrary section through

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the spin bundle on the patch $\mathcal{P}_\alpha$. On the intersections $\mathcal{P}_\alpha \cap \mathcal{P}_\beta$, the symplectic potentials are related by an abelian gauge transformation, $\theta(\alpha) = \theta(\beta) + d\lambda_{\alpha\beta}$ and $\Gamma(\alpha)$ is obtained from $\Gamma(\beta)$ by a Yang–Mills gauge transformation (3.7) with a $G$–valued transition function $U \equiv U_{\alpha\beta}$. Likewise, $\psi(\alpha) = t_{\alpha\beta}U_{\alpha\beta}\psi(\beta)$ with $t_{\alpha\beta} \equiv \exp(-i\lambda_{\alpha\beta}/\hbar)$.

When we parallel transport a spinor from a point in $\mathcal{P}_\alpha$ to a point in $\mathcal{P}_\beta$, eq.(4.41) is to be understood in the sense that the transition functions $t_{\alpha\beta}$ and $U_{\alpha\beta}$ are applied to the spinor at an arbitrary point in $\mathcal{P}_\alpha \cap \mathcal{P}_\beta$ which is visited by the curve $C$.

If we ignore the transition functions $U_{\alpha\beta}$ for a moment (i.e., assume that $U_{\alpha\beta} = 1$ for all $\alpha, \beta$), the 1–forms $\theta(\alpha)$ and the parameters $\lambda_{\alpha\beta}$ define a line bundle over $\mathcal{M}_{2N}$ with curvature $\omega$. This is precisely the Kostant–Souriau prequantum bundle [10] which is familiar from geometric quantization. A simple Dirac–type argument shows that this bundle exists provided $\int_\Sigma \omega/(2\pi\hbar)$ is integer for any 2–cycle $\Sigma$, i.e., if $[\omega/(2\pi\hbar)] \in H^2(\mathcal{M}_{2N}, \mathbb{Z})$. Inequivalent bundles are classified by the cohomology class of the 2–form $c_1 = \omega/(2\pi\hbar)$, the first Chern class. It would be interesting to have an analogous classification of the bundles with structure group $G$, i.e., with the $U_{\alpha\beta}$’s retained. No general results are available in this situation, but in ref.[25] the important special case when the structure group can be reduced to $[Sp(2N) \times U(1)]/\mathbb{Z}_2$ has been discussed. One finds that the above quantization condition is modified by the ”metaplectic anomaly” involving the curvature of the bundle of unitary frames. In ref.[13] we investigated a similar anomaly within our present framework (at the semiclassical level). It can be used to detect topological obstructions for the existence of a metaplectic spin structure on $\mathcal{M}_{2N}$ and is closely related to the Maslov index.

5 Flat Phase–Space

In this section we continue our investigation for the special case of a flat phase–space. Besides being very important in its own right, it illustrates the general ideas of the previous sections in a rather explicit form.
5.1 $\tilde{\Gamma}_a$ from the path–integral

From now on we identify phase–space with the symplectic plane $(\mathbb{R}^{2N}, \omega)$. We use global Darboux coordinates $\phi^a = (p^k, q^k)$ with respect to which $\omega_{ab}$ assumes the standard form (2.3). As before, we employ the notation $\varphi^a = (\hat{x}^k, \hat{\pi}^k)$ for the Heisenberg operators in the fiber. For the symplectic connection we choose the trivial one, $\Gamma_{abc} = 0$. Hence $R_{abcd} = 0$ and eq.(4.30) is solved by $r_a = 0$ therefore. Thus the connection (4.31) boils down to

$$\tilde{\Gamma}_a = \omega_{ab} \varphi^b \quad (5.1)$$

This connection is independent of $\phi$. Its curvature $\omega_{ab}$ is entirely due to the commutator terms in $\Omega_{ab}$, which shows quite clearly the non–abelian nature of the underlying gauge group. The most general admissible connection for flat space was given in (4.7). Fedosov’s subsidiary condition (see the Appendix) amounts to the choice $\alpha_a = 0$, which we shall adopt henceforth. The covariant derivative

$$\frac{D}{D\phi^a} \equiv D_a = \partial_a + \frac{i}{\hbar} \omega_{ab} \varphi^b \quad (5.2)$$

decomposes according to

$$\frac{D}{Dq^k} = \frac{\partial}{\partial q^k} - i \frac{\hat{\pi}^k}{\hbar} = \frac{\partial}{\partial q^k} - \frac{\partial}{\partial x^k} \quad (5.3)$$

$$\frac{D}{Dp^k} = \frac{\partial}{\partial p^k} + i \frac{\hat{x}^k}{\hbar} = \frac{\partial}{\partial p^k} + i \frac{x^k}{\hbar} \quad (5.4)$$

The second equality in eqs.(5.3) and (5.4), respectively, holds in the representation in which $\hat{x}^k$ is diagonal.

Let us return to the path–integral of section 2. In eq.(2.29) we wrote down an exact formula which relates two different path–integrals which differ by a shift of the variable of integration. If it was not for the nontrivial boundary conditions, such a shift of a dummy variable would not have any effect. The exponential in eq.(2.29) is a consequence of these nontrivial boundary conditions. The matrix element $\langle q_2, t_2|q_1, t_1 \rangle_H$ is manifestly independent of the shift, $\Phi_{cl}(t)$. Hence the $\Phi_{cl}$–dependence of $\langle x_2, t_2|x_1, t_1 \rangle_H$ cancels precisely that of the exponential.

Let us consider eq.(2.29) for two different trajectories, $\Phi_{cl}(t)$ and $\Phi_{cl}(t) + \delta \phi(t)$. Both of them are classical solutions, and $\delta \phi \equiv (\delta p_{cl}, \delta q_{cl})$ solves Jacobi’s
equation (2.24) therefore. We would like to eliminate \( \langle q_2, t_2 | q_1, t_1 \rangle_H \) by subtracting these two equations. This makes it necessary to combine the change of the trajectory with a corresponding change of the arguments of the matrix element: \( \delta x_{1,2} = -\delta q(t_{1,2}) \). Then, by eq.(2.32), both equations refer to the same values of \( q_1 \) and \( q_2 \), and their difference reads

\[
\langle x_2 - \delta q_{cl}(t_2), t_2 | x_1 - \delta q_{cl}(t_1), t_1 \rangle_{H(\Phi_{cl} + \delta \phi)} - \langle x_2, t_2 | x_1, t_1 \rangle_{H(\Phi_{cl})}
\]

\[
= -\frac{i}{\hbar} \delta [S_{cl} + p_{cl}(t_2) x_2 - p_{cl}(t_1) x_1] \langle x_2, t_2 | x_1, t_1 \rangle_{H(\Phi_{cl})}
\]

(5.5)

In the last line of (5.5) we used (2.32) and \( \delta S_{cl} = p_{cl}(t_2) \delta q_{cl}(t_2) - p_{cl}(t_1) \delta q_{cl}(t_1) \).

The variation of the matrix element due to the change of the trajectory alone, i.e., at fixed \( x_{1,2} \), is given by

\[
\delta \langle x_2, t_2 | x_1, t_1 \rangle_{H} = \left[ \delta q_{cl}(t_2) \frac{\partial}{\partial x_2} + \delta q_{cl}(t_1) \frac{\partial}{\partial x_1} - \frac{i}{\hbar} (x_2 \delta p_{cl}(t_2) - x_1 \delta p_{cl}(t_1)) \right] \langle x_2, t_2 | x_1, t_1 \rangle_{H}
\]

(5.6)

This is an exact result; no semiclassical expansion has been invoked.

There exists an alternative derivation of eq.(5.6) which makes no reference to the initial path–integral before the shift. If one varies the classical background trajectory directly in the path–integral (2.30), performs a compensating transformation on the integration variables and uses Jacobi’s equation for \( \delta \phi \) in order to eliminate the terms linear in \( \varphi^a \), one reproduces precisely eq.(5.6).

If one regards the matrix element as a function of the variables referring to the final point, eq.(5.6) shows that

\[
\left[ \frac{\partial}{\partial q^k(t_2)} - \frac{\partial}{\partial x_2^k} \right] \langle x_2, t_2 | x_1, t_1 \rangle_{H} = 0
\]

(5.7)

\[
\left[ \frac{\partial}{\partial p^k(t_2)} + \frac{i}{\hbar} x_2^k \right] \langle x_2, t_2 | x_1, t_1 \rangle_{H} = 0
\]

(5.8)

The interpretation of these equations is as follows. If we allow \( \delta \phi(t) \) to run through a complete set of Jacobi fields, the trajectories \( \Phi_{cl}(t) + \delta \phi(t) \) connect an infinitesimal neighborhood of \( q_1 \) to a corresponding neighborhood of \( q_2 \). Fixing
\( \delta p_{cl}(t_2) \) and \( \delta q_{cl}(t_2) \) means picking a specific trajectory. Comparing eqs. (5.7) and (5.8) to eqs. (5.3) and (5.4) we see that the transition matrix element responds to a change of the classical background trajectory in precisely such a way that 
\[ \psi_{x}(p(t_2), q(t_2)) \equiv \langle x, t_2 | x_1, t_1 \rangle_{H} \]
is covariantly constant with respect to the abelian connection.

This calculation provides us with an independent derivation of the connection \( \tilde{\Gamma}_a \) and makes it explicit that conventional quantum mechanics "knows" which connection we have to use in the Hilbert bundle approach.

### 5.2 The operators \( O_f \) and Moyal’s equation

On a flat phase–space the covariant derivative for operators reads \( D_a = \partial_a - \hat{\partial}_a \) and the corresponding exterior differential for operator–valued differential forms is \( D = d - \delta \). The covariantly constant fields \( O_f \) satisfy

\[
\partial_a O_f + \frac{i}{\hbar} \omega_{ab} [\hat{\varphi}^b, O_f] = 0 \quad (5.9)
\]

A simple calculation or eq.(A.35) from the Appendix show that for every analytic \( f \) eq.(5.9) is solved by

\[
O_f(\phi) = f(\phi + \hat{\varphi}) \quad (5.10)
\]

This operator is understood to be Weyl–ordered, i.e., it is defined by the following expansion in terms of symmetrized operator products:

\[
O_f(\phi) = \sum_{m=0}^{\infty} \frac{1}{m!} \partial_{a_1} \cdots \partial_{a_m} f(\phi) \hat{\varphi}^{(a_1} \cdots \hat{\varphi}^{a_m)} \equiv \exp (\hat{\varphi}^a \partial_a) f(\phi) \quad (5.11)
\]

As it should be, the lowest order term of this power series is the classical observable \( f(\phi) \).

Let us evaluate the product of two \( O_f \)'s. With the notation \( \partial_a^{(1,2)} \equiv \partial/\partial \phi_{1,2}^a \) we have

\[
O_f(\phi)O_g(\phi) = \exp \left( \hat{\varphi}^a \partial_a^{(1)} \right) \exp \left( \hat{\varphi}^b \partial_b^{(2)} \right) f(\phi_1)g(\phi_2) \bigg|_{\phi_{1,2}=\phi} \]

\[
= \exp \left[ \hat{\varphi}^a \left( \partial_a^{(1)} + \partial_a^{(2)} \right) \right] \exp \left[ \frac{i}{\hbar} \omega_{ab}^{(1)} \omega^{ab} \partial_b^{(2)} \right] f(\phi_1)g(\phi_2) \bigg|_{\phi_{1,2}=\phi} \quad (5.12)
\]
where the Baker–Campbell–Hausdorff formula was used. In order to evaluate the coincidence limit we employ the identity

$$G(\partial_a^{(1)} + \partial_a^{(2)}) F(\phi_1, \phi_2)\big|_{\phi_{1,2} = \phi} = G(\partial_a) F(\phi, \phi)$$

(5.13)

which holds true for arbitrary analytic functions $F$ and $G$. Hence

$$\mathcal{O}_f(\phi) \mathcal{O}_g(\phi) = \exp[\tilde{\phi}^a \partial_a] (f * g)(\phi)$$

(5.14)

or

$$\mathcal{O}_f(\phi) \mathcal{O}_g(\phi) = \mathcal{O}_{f \ast g}(\phi)$$

(5.15)

with the product

$$(f * g)(\phi) \equiv f(\phi) \exp\left[\frac{i\hbar}{2} \tilde{\omega}_{ab} \tilde{\partial}_a \tilde{\partial}_b\right] g(\phi)$$

(5.16)

As a consequence of eq.(5.15), the quantum observables form a closed commutator algebra

$$[\mathcal{O}_f(\phi), \mathcal{O}_g(\phi)] = i\hbar \{f, g\}_M(\phi)$$

(5.17)

with the structure constants given by the Moyal bracket $\{f, g\}_M \equiv (f * g - g * f)/i\hbar = \{f, g\} + O(\hbar^2)$.

Thus we have made contact with the familiar phase–space formulation of quantum mechanics: (5.16) is precisely the ”star product” appearing in the Weyl–Wigner–Moyal symbol calculus [26, 27] which underlies the deformation theory approach to quantization. The star product is an associative but noncommutative product on the space of smooth functions over phase–space. It is the image of the operator product under the so–called symbol map which establishes a one–to–one correspondence between operators on $V_{QM}$ and $c$–number functions over $\mathcal{M}_{2N}$.

The star product (5.16) is appropriate if $\mathcal{M}_{2N}$ is a flat symplectic space. It is an important question if and how it can be generalized to an arbitrary symplectic manifold [28]. Fedosov [23] has shown that a star product indeed exists for any symplectic manifold and he devised an iterative method to determine it. Within our present framework, his idea is as follows. Given $\mathcal{O}_{f \ast g}$, one can recover $f * g$ by extracting the term proportional to the unit operator. Therefore, for an arbitrary curved phase–space, eq.(5.15) can be used as the definition of a star product. Usually no closed–form solution for the $\mathcal{O}_f$’s is available then, but it is always possible to insert the series expansion (4.34) on the LHS of (5.15) and to construct
the star product iteratively. As this method has already been discussed in refs.[19, 23] we shall not go into any details here. (See also refs.[24, 29].)

In the Appendix we use a symbol calculus in order to represent operators $A(\phi, \hat{\varphi})$ in terms of symbols $A(\phi, y)$. Here $y$ is a coordinate on a flat auxiliary phase–space which can be identified with the tangent space at $\phi$ equipped with the constant symplectic structure. Accidentally, the star product (5.16) for flat space has the same structure as the $\circ$–product (A.2). For a general $M_{2N}$ the $\star$–product becomes more complicated, but as the $\circ$–product refers to the flat tangent spaces $(T_\phi M_{2N}, \omega)$ it is always given by (A.2).

In section 4.3 we mentioned that the dynamics of mixed states is governed by the local von Neumann equation (4.56). A priori this is an operatorial equation for the generalized statistical operator $O_\rho(\phi)$, but it can be converted to an equivalent $c$–number equation for the function $\rho(\phi; t)$. If we use (5.17) in (4.56) and project on the term proportional to the unit operator it follows that

$$\partial_t \rho(\phi; t) = \{H, \rho(t)\}_M$$  \hspace{1cm} (5.18)

Conversely, (5.18) implies (4.56) because both $O_H$ and $O_\rho$ are of the type (5.10).

Remarkably enough, eq.(5.18) is precisely Moyal’s equation [9] for the time evolution of the pseudodensity $\rho(\phi; t)$, i.e., for the Weyl symbol of the ordinary statistical operator $\hat{\rho}$ on $\mathcal{V}_{QM}$. This fact can be understood as follows. The fiberwise symbol calculus of the appendix represents operators acting on $\mathcal{V}_\phi$ in terms of $c$–number functions over the flat ”auxiliary phase–space” $(T_\phi M_{2N}, \omega)$. This construction applies in particular at the reference point $\phi = \phi_0$ so that operators on $\mathcal{V}_{QM}$ are changed into functions over the tangent space of $\phi_0$. As we are dealing with a flat phase–space here, we may identify $T_{\phi_0} M_{2N}$ with the manifold $M_{2N} = \mathbb{R}^{2N}$ itself and regard symbols such as $[\text{symb}(\hat{\varphi}^a)](\phi) = \phi^a$ as functions $M_{2N} \rightarrow \mathbb{R}$. As a result, we recover exactly the traditional symbol calculus which relates operators on $\mathcal{V}_{QM}$ to functions over the classical phase–space. For the statistical operator $\hat{\rho} = O_\rho(\phi_0)$ we have in particular

$$[\text{symb}\{\hat{\rho}\}](\phi) = [\text{symb}\{\rho(\phi_0 + \hat{\varphi})\}](\phi) = \rho(\phi_0 + \phi)$$  \hspace{1cm} (5.19)

It is convenient to identify the reference point with the origin $\phi_0^a = 0$. In this case the Weyl symbol of the ordinary statistical operator equals precisely the function $\rho$
which characterizes the local operator $O_\rho(\hat{\phi})$, and eq.(5.18) has its usual meaning therefore.

With the choice $\phi^a_0 = 0$ the conventional quantum observables are

$$\hat{f} \equiv O_f(0) = f(\hat{\phi})$$

(5.20)

They are obtained from the classical observables $f(\phi)$ by the usual substitution $\phi \to \hat{\phi}$ together with the Weyl ordering prescription.

### 5.3 The parallel transport operators

For the connection (5.1), the parallel transport operator $V(s) \equiv V[C(\phi(s), \phi_0)]$ of (3.54) satisfies the differential equation

$$i\hbar \frac{d}{ds} V(s) = \dot{\phi}^a(s) \omega_{ab} \hat{\phi}^b V(s)$$

(5.21)

It can be solved explicitly in terms of the Weyl operators [9]

$$T(\phi) \equiv \exp \left( \frac{i}{\hbar} \phi^a \omega_{ab} \phi^b \right)$$

(5.22)

They provide a projective representation of the translations on the symplectic plane. We shall need the following properties:

$$T(\phi)^\dagger \hat{\phi}^a T(\phi) = \hat{\phi}^a + \phi^a$$

(5.23)

$$T(\phi)^\dagger = T(-\phi)^{-1} = T(-\phi)$$

(5.24)

$$T(\phi_1) T(\phi_2) = \exp \left[ \frac{i}{\hbar} \phi_1^a \omega_{ab} \phi_2^b \right] T(\phi_1 + \phi_2)$$

(5.25)

$$\partial_a T(\phi) = \frac{i}{\hbar} \omega_{ab} \left[ \hat{\phi}^b - \frac{1}{2} \phi^b \right] T(\phi)$$

(5.26)

Using these formulae it is easy to verify that the solution of (5.21) with $V(0) = 1$ is given by

$$V(s) = \exp \left[ \frac{i}{\hbar} \int_0^s ds' \dot{\phi}^a(s') \omega_{ab} \phi^b(s') \right] T(\phi(s))^\dagger T(\phi(0))$$

(5.27)

Setting $s = 1$ and $\phi(0) = \phi_0$, $\phi(1) = \phi$, eq.(5.27) is of the form (4.38) with the path-independent part

$$\tau(\phi, \phi_0) = T(\phi)^\dagger T(\phi_0)$$

(5.28)
The path–dependent phase factor in eq.(4.38) contains the symplectic potential (4.16). Incidentally, if we solve the parallel transport equation for the flat connection (4.17) rather than the abelian one with curvature $\omega$, this phase factor is absent and one has $V[C(\phi, \phi_0)] = T(\phi)^\dagger T(\phi_0)$.

For convenience we identify $V_{QM}$ with the Hilbert space at the origin of the symplectic plane, $\phi_0 = 0$, and we set $\tau(\phi) \equiv \tau(\phi, 0)$. We pick some conventional quantum mechanical wave function $\Psi(x)$ in $V_{QM}$ and interpret it as a spinor in the fiber at $\phi = 0$: $\psi(0)^x \equiv \Psi(x)$. This wave function can be parallel transported to any point of phase–space: $\psi_C(\phi)^x = V[C(\phi, 0)]^y_x \psi(0)^y$. Using the position–space matrix elements of the Weyl–operators [9] one finds for $\phi \equiv (p, q)$:

$$
\tau(p, q)^x_y \psi(0)^y = \exp \left[ -\frac{i}{\hbar} p \left( x + \frac{1}{2} q \right) \right] \psi(0)^{x+q} 
$$

As we mentioned already in section 4.3, any vector in the reference Hilbert space can be consistently extended to a spinor field on an arbitrary Lagrangian submanifold $K$ which contains $\phi_0$. The most important example of such a submanifold is the configuration space $K = \{(p = 0, q) | q \in \mathbb{R}^N\}$. For every path $C$ which connects $\phi_0 = 0$ to the point $\phi = (0, q)$ and stays in the plane $p = 0$ one has $\int_C \theta = 0$. Hence, when transported to $\phi$, the wave function reads

$$
\psi(0, q)^x = \Psi(q + x) 
$$

This defines a covariantly constant spinor field on $K$. In this simple example the meaning of the background–quantum split symmetry is particularly clear. The field (5.30) depends only on the invariant combination $q + x$, which is annihilated by (5.3). In the Hilbert space located at $(0, q)$, the ”quantum” part of the argument of $\Psi$, $x$, is augmented by the ”background” piece $q$. But also the observables to be used at $(0, q)$ are different, $O_f(0, q) = f(\hat{\pi}, q + \hat{x})$, and it does not matter therefore in which Hilbert space expectation values or matrix elements are calculated.

### 5.4 Time evolution and bilinear covariants

In order to write down the universal equation of motion (4.44), (4.45) one has to know the generator $O_H$ of local frame rotations. On flat phase–space we have the closed–form solution

$$
O_H(\phi) = H(\phi + \hat{\phi}) 
$$
In order to describe the relation of our approach to the path–integral quantization, it is helpful to pull out the first two terms of its power series in $\hat{\phi}$:

$$H(\hat{\phi}; \phi) \equiv H(\phi + \hat{\phi}) - \partial_a H(\phi) \hat{\phi}^a - H(\phi) \quad (5.32)$$

Now we insert (5.31) with (5.32) into (4.45) and rewrite the terms linear in $\hat{\phi}$ by a trick similar to the one which we used in section 4.3 in the investigation of the semiclassical limit. For a single upper spinor index, say,

$$\partial_a H(\phi) \hat{\phi}^a = i \hbar (h^a D_a - h^a \partial_a) \quad (5.33)$$

As a result, eq.(4.45) assumes the following suggestive form:

$$L_H(\phi) \chi(\phi)_{x_1 \cdots x_r}^{y_1 \cdots y_s} = \left[ h^a \partial_a - h^a D_a + \frac{i}{\hbar} (r - s) H \right] \chi_{x_1 \cdots x_r}^{y_1 \cdots y_s}
+ i \hbar \mathcal{H}(\hat{\phi}; \phi)^{x_1 \cdots x_r} \chi_{y_1 \cdots y_s}^{x_1 \cdots x_r} \mathcal{H}(\hat{\phi}; \phi)^{y_1 \cdots y_s} \quad (5.34)$$

While this equation is exact, its structure is similar to the semiclassical $L_{scl}^H$ of (4.48). In fact, for the special case of $M_{2N}$ flat, eq.(5.34) can be thought of as the correct all–order generalization of the metaplectic "Lie–derivative". In general, the operator

$$\mathcal{H}(\hat{\phi}; \phi) = \sum_{m=2}^{\infty} \frac{1}{m!} \partial_{a_1} \cdots \partial_{a_m} H(\phi) \hat{\phi}^{(a_1 \cdots a_m)} \quad (5.35)$$

contains terms of arbitrarily high degree. Only in the semiclassical limit it becomes an element of $mp(2N)$,

$$\mathcal{H} = \frac{\hbar}{2} \partial_a \partial_b H(\phi) \Sigma^{ab} + O(3) \quad (5.36)$$

and (5.34) reduces to (4.48) then. Applying $L_H$ to covariantly constant fields so that $D_a \chi = 0$ and disregarding the term proportional to $(r - s)$ which produces a phase only, eq.(5.34) is reminiscent of a Lie–derivative. However, we emphasize that $L_H(\phi)$ generates only a local frame rotation but no diffeomorphism on $M_{2N}$. The $h^a \partial_a$–term in $L_H(\phi)$ originates from the condition $D_a \chi = 0$ which allows us to trade shifts on the base manifold for transformations in the fibers.

In order to better understand the properties of $L_H$ and $\ell_H$ it is quite instructive to look at the following bilinears formed from two arbitrary spinors $\psi$ and
χ:

\[ E(\phi) = \phi \langle \chi | \psi \rangle_{\phi} \]
\[ T^a(\phi) = \phi \langle \chi | \gamma^a | \psi \rangle_{\phi} \]  
\[ R^{ab}(\phi) = \phi \langle \chi | \Sigma^{ab} | \psi \rangle_{\phi} \]  

These bilinears are related to the generators of the group \( IMp(2N) \), the semidirect product of \( Mp(2N) \) with the Weyl group \([11]\). Their metaplectic Lie–derivative (2.13) is easily worked out \([11]\):

\[ \ell_H E = h^a \partial_a E \]  
\[ \ell_H T^a = h^b \partial_b T^a - \partial_b h^a T^b \]  
\[ \ell_H R^{ab} = h^c \partial_c R^{ab} - \partial_c h^a R^{cb} - \partial_c h^b R^{ac} \]

Obviously, \( E, T^a \) and \( R^{ab} \) transform as a scalar, a vector and a symmetric tensor, respectively. On the other hand, applying \( L_H \) yields\(^7\)

\[ \mathcal{L}_H E(\phi) = 0 \]  
\[ \mathcal{L}_H T^a(\phi) = -\sqrt{2/h} \phi \langle \chi | h^a (\phi + \tilde{\varphi}) | \psi \rangle_{\phi} = -\sqrt{2/h} h^a E - \partial_b h^a T^b + O(2) \]  
\[ \mathcal{L}_H R^{ab}(\phi) = -h^{-1} \phi \langle \chi | \tilde{\varphi}^{(a} h^{b)} (\phi + \tilde{\varphi}) + h^{[a} (\phi + \tilde{\varphi}) \tilde{\varphi}^{b]} | \psi \rangle_{\phi} = -\sqrt{2/h} h^{[a} T^{b]} - \partial_c h^a R^{cb} - \partial_c h^b R^{ac} + O(3) \]

Several lessons can be learned from these results. First, \( \ell_H \) contains the generators of \( Mp(2N) \) which are quadratic in \( \tilde{\varphi} \). Therefore \( \ell_H \) preserves the number of \( \gamma \)-matrices and generates a linear transformation of the bilinears. This is not the case for \( \mathcal{L}_H \) which is related to the \( W_\infty \)-type Lie algebra \( \mathcal{G} \) spanned by all monomials of the form \( \tilde{\varphi}^{(a_1 \cdots \tilde{\varphi}^{a_n})} \). If one expands \( h(\phi + \tilde{\varphi}) \) in eq.(5.42) one obtains terms

\(^7\)Here we generalize the definition of \( \mathcal{L}_H \) to tensorial quantities with coordinate or frame indices in such a way that \( \mathcal{L}_H \gamma^{ax}_y = 0 \). For a c-number vector \( v^a(\phi) \), say, one needs the nonlinear transformation \( \mathcal{L}_H v^a(\phi) = -h^a(\phi + v) \). This is precisely as it should be, because we may identify the flat phase space with the "reference tangent space" at \( \phi = 0 \). Therefore \( v^a \equiv v^a(\phi = 0) \) can be identified with the coordinate \( \phi^a \) on \( M_{2N} \). Hence the above nonlinear transformation yields the equation of motion \( \partial_t v^a = h^a(v) \), which is nothing but Hamilton’s equation for \( v^a \equiv \phi^a \).
of arbitrarily high degree. Second, we observe that for generic spinor fields $\psi$ and $\chi$ even the terms in $L_H$ of lowest degree do not agree with $\ell_H$. In fact, $L_H = \ell_H + O(1)$ is expected to hold only if the dynamical phase cancels (which is the case here) and if the fields involved are covariantly constant. Let us assume therefore that $|\psi\rangle_\phi$ and $\phi\langle\chi|$ are obtained by parallel transporting $|\psi\rangle_0$ and $0\langle\chi|$ from the reference point $\phi = 0$ to $\phi$. The path–dependent phase cancels from the bilinears, and using $\tau(\phi, 0) = T(-\phi)$ in (4.41) one finds that their functional form is highly constrained:

\[ E(\phi) = E(0) \]
\[ T^a(\phi) = T^a(0) - \sqrt{2/\hbar} \phi^a E(0) \]
\[ R^{ab}(\phi) = R^{ab}(0) - \sqrt{2/\hbar} \phi^{(a} T^{b)}(0) + \hbar^{-1} \phi^a \phi^b E(0) \]  

Using (5.44) in the semiclassical expansions of (5.41)–(5.43) it is easy to check that the terms of nonpositive degree generated by $L_H$ coincide precisely with those of $\ell_H$, eqs.(5.38)–(5.40). For instance, the fact that $E$ is strictly constant for covariantly constant spinors reconciles (5.38) with (5.41).

To close with, let us look at the all–order dynamics of the world–line spinors. They provide a natural link between our approach and standard path–integral quantization also beyond the semiclassical limit. We shall make contact with the exact shifted path–integral (2.30). To this end we consider an arbitrary classical trajectory $\Phi_{cl}(t)$ which, at time $t = 0$, starts at the point $\phi_0$. We identify the fiber at the initial point with the quantum mechanical Hilbert space, $V_{\phi_0} \cong V_{QM}$. Furthermore, we pick some time–dependent state vector $\psi(\phi_0; t)$ in $V_{QM}$; it evolves according to $i\hbar \partial_t \psi(\phi_0; t) = \mathcal{O}_H(\phi_0) \psi(\phi_0; t)$. For every fixed time $t$, we parallel transport this state along the classical trajectory from $\phi_0$ to $\Phi_{cl}(t)$:

\[ \psi(\Phi_{cl}(t); t) = V \left[ C \left( \Phi_{cl}(t), \phi_0 \right) \right] \psi(\phi_0; t) \]
\[ = V \left[ C \left( \Phi_{cl}(t), \phi_0 \right) \right] U(\phi_0; t) \psi(\phi_0; 0) \]  

In this manner we construct a spinor field along $\Phi_{cl}$: $\eta(t) \equiv \psi(\Phi_{cl}(t); t) \in V_{\Phi_{cl}(t)}$. Taking the time derivative of (5.45) one obtains

\[ i\hbar \partial_t \eta = \left[ \mathcal{O}_H(\Phi_{cl}) - \partial_a H(\Phi_{cl}) \hat{\phi}^a \right] \eta \]  

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where the term linear in $\hat{\phi}$ results from differentiating the parallel transport operator. Thus, since $H$ is constant along $\Phi_{cl}$,

$$i\hbar \partial_t \eta = [\mathcal{H}(\hat{\phi}; \Phi_{cl}) + H(\phi_0)] \eta$$  \hspace{1cm} (5.47)$$

Comparing this equation of motion to eq.(2.33) we see that, apart from a trivial phase, the world line spinor $\eta$ is precisely the same thing as the matrix element represented by the shifted path integral of section 2:

$$\eta^x(t) = \exp \left( -\frac{i}{\hbar} H(\phi_0)t \right) \langle x, t | x_0, 0 \rangle_H$$  \hspace{1cm} (5.48)$$

In establishing this result we have understood the all–order generalization of the semiclassical world–line spinors and their relation to the background–quantum split symmetry. Both in the construction above and in the path–integral derivation of (2.33), a classical trajectory was introduced as a ”background”, but nevertheless we were always dealing with full–fledged quantum mechanics.

## 6 Discussion and Conclusion

In this paper, we have uncovered a hidden gauge theory structure of quantum mechanics which is not visible in its conventional formulations. When this structure is made manifest, one realizes that the conceptual framework of quantum theory shows a remarkable similarity with a gauge theory, in particular with general relativity.

We constructed a Yang–Mills–type theory on the phase–space of an arbitrary hamiltonian system. A local Hilbert space was associated to each point of phase–space, and ”matter fields” were introduced which assume values in these spaces. We referred to them as ”metaplectic spinors” because under $Sp(2N)$ they transform in its spinor, or metaplectic, representation. The underlying gauge group $G$ is infinite–dimensional though. It consists of all unitary frame rotations in the fiber $\mathcal{V}$.

We discussed in detail how conventional quantum mechanics can be reformulated within this general setting, and we showed that this reformulation provides a new way of understanding the structure of quantum theory and the transition from classical to quantum mechanics. It turned out that the rules of canonical
quantization can be replaced by two new "postulates" with a much clearer group theoretical and geometrical interpretation.

The first postulate is of a purely group theoretical nature and contains the transition from classical mechanics to semiclassical quantum mechanics. In the classical context, we are dealing with tensor fields over phase–space. Under local frame rotations they transform in some product of the vector representation of $Sp(2N)$. The first postulate says that we have to replace the vector representation of $Sp(2N)$ by the spinor representation of its double covering $Mp(2N)$ and to use multispinors rather than tensors for the description of physical states and observables. We motivated this rule by applying it to the classical Jacobi fields; the resulting world–line spinors are precisely the semiclassical wave functions.

For nonlinear systems we need a second rule which tells us how to recover the exact quantum theory from its semiclassical approximation. The second postulate is that the multispinor fields must be covariantly constant (up to a phase possibly) with respect to an arbitrary abelian $G$–valued connection $\tilde{\Gamma}$. This flatness condition means that the states and operators related to the different local Hilbert spaces can all be identified by virtue of the parallel transport defined by $\tilde{\Gamma}$. In this manner, the equivalence of the Hilbert bundle approach and the standard one–Hilbert space formulation of quantum mechanics is established.

It is one of our main results that the second postulate is equivalent to a very simple but deep symmetry principle: invariance under the background–quantum split symmetry. We visualize the exact quantum theory as the result of consistently sewing together an infinity of local quantum theories, one at each point of phase–space. Classically, if a particle sits at the point $\phi$, the value of the observable $f$ is $f(\phi)$. We assume that the quantum correction to this value is locally determined by the expectation value of a certain operator $\Delta\mathcal{O}_f(\phi)$ with respect to a vector in $\mathcal{V}_\phi$: $\langle \hat{f} \rangle \equiv f(\phi) + \langle \psi | \Delta\mathcal{O}_f(\phi) | \psi \rangle_\phi$. When applied to each point $\phi$ separately, this prescription raises the following consistency problem. For $f(\phi) = \phi^a$ and flat space, say, we would like to interpret $\langle \hat{\phi}^a \rangle = \phi^a + \langle \psi | \hat{\phi}^a | \psi \rangle_\phi$ as the exact quantum mechanical expectation value of the position in phase–space. Clearly this is possible only if $\langle \hat{\phi}^a \rangle$ is independent of the point $\phi$. Going to a new point $\bar{\phi}$, the new state $|\psi\rangle_{\bar{\phi}}$ must be such that the resulting change of the $\hat{\phi}$–expectation value cancels the difference $\phi^a - \bar{\phi}^a$. The postulate of the background–quantum
split symmetry means that, more generally, the value of \( \langle \hat{f} \rangle \) should be independent of \( \phi \), i.e., that all the local quantum theories agree on the expectation value of the quantum observables \( O_f = f + \Delta O_f \). The form of these operators is fixed by the condition \( \lim_{\hbar \to 0} O_f(\phi) = f(\phi) \) together with the split symmetry; it implies that they must be covariantly constant with respect to an abelian connection.

The physical motivation for the second postulate comes from semiclassical considerations again. Even if not in practice, at least from a conceptional point of view, our quantization program first constructs a semiclassical approximation of quantum mechanics as a link between the classical and the exact quantum theory. Roughly speaking, the idea is to recover the exact theory from the totality of the semiclassical wave functions calculated for all classical trajectories. Let us assume we know the solutions \( \eta(t) \) of the semiclassical Schrödinger equation (2.23) or (2.33) for all classical paths \( \Phi(t) \). Heuristically, the wave function \( \eta_1(t) \) belonging to some trajectory \( \Phi_1(t) \) provides a reasonable approximation to the complete theory within a tubular neighborhood of \( \Phi_1(t) \). Within this neighborhood, the neglected nonlinearities should be irrelevant. Let us suppose there is a nearby classical path \( \Phi_2(t) \) such that the tubular neighborhood within which its wave function \( \eta_2(t) \) is valid overlaps with the one of \( \Phi_1(t) \). Thus there is a region in phase–space where both semiclassical expansions apply. The expectation value of the ”position”, say, is given by \( \Phi_1 + \eta_1(t) \hat{\varphi} \eta_1(t) \) according to the first, and by \( \Phi_2(t) + \eta_2(t) \hat{\varphi} \eta_2(t) \) according to the second one. These values must coincide if both expansions are approximations to the same exact theory. If we look at this situation at a fixed instant of time, we are back to the above picture of local quantum theories at a point, and it is clear that the necessary consistency requirement is precisely that \( \Phi(t) \) and \( \eta(t) \) must be connected by the split symmetry.

Furthermore, from a dynamical point of view, the ”current” \( \eta_1(t) \hat{\varphi} \eta(t) \) has the same properties as the classical Jacobi field \( c^a(t) \). This yields a simple interpretation of the first postulate as well: we have to take the ”square root” of the Jacobi field, \( c^a \rightarrow \eta_0 \hat{\varphi} \eta \). This substitution converts the position of a classical particle propagating near \( \Phi(t) \), i.e., \( \Phi^a(t) + c^a(t) \), to the corresponding expectation value in semiclassical quantum mechanics. By virtue of the split symmetry, the ”quantum Jacobi fields” \( \eta(t) \) can be glued together consistently to yield fields \( \psi^x(\phi; t) \). Their value at an arbitrary reference point can be identified with the conventional wave function: \( \Psi(x; t) \equiv \psi^x(\phi_0; t) \).
It is amusing to compare the process of gluing together local semiclassical expansions to the transition from special to general relativity. The tubular neighborhoods surrounding $\Phi(t)$ correspond to the freely falling "Einstein elevators" within which the laws of special relativistic physics are a good approximation. If one tries to consistently patch up the observations made in different such elevators, not necessarily close to each other, one needs a connection, or a parallel transport, and thus starts feeling the curvature of space–time. Special–relativistic physics corresponds here to the local semiclassical approximation of quantum mechanics which, too, can be formulated without a connection. Both in general relativity and in our Hilbert bundle approach the connection is the essential tool for "globalizing" local physics. However, because of the strong constraints coming from the split symmetry, this connection is not dynamical in the case of quantum mechanics.

In many of our derivations and "thought experiments", classical solutions played a central rôle. We emphasize, however, that in order to actually quantize a given system along the lines proposed here no knowledge of these solutions and their moduli space is necessary. All one has to do is to find an abelian connection and to construct covariantly constant sections, which is similar to the method of ref. [25]. From a purely pragmatic point of view, the subdivision of the quantization process according to the two postulates is unnecessary.

Our main emphasis was on gaining a better understanding of what it means to “quantize” a hamiltonian system. While the familiar rules of canonical quantization can be formulated quite easily, their origin is rather obscure still. We replaced them by two postulates which in our opinion are much more natural and easier to understand intuitively.

The first postulate involves nothing but changing the representation of the "Lorentz group" appropriate for phase–space. Comparing this to the situation in space–time, we are perfectly familiar with the idea that besides the spin-1 representation (photons, gluons, ...) also the spin-$\frac{1}{2}$ representation (electrons, quarks, ...) of the Lorentz group is realized in nature. Hence it appears quite natural that also at the more fundamental level of quantum theory in general nature takes advantage of spinor representations. Moreover, the second postulate is formulated in the same language of classical differential geometry as general relativity or Yang–Mills theory. It is our hope, therefore, that the approach proposed here
will help for instance in understanding the interrelation of gravity and quantum theory on a geometric basis.

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Appendix

This appendix serves two purposes: First, we show how the operators on $V_\phi$ are represented in a fiberwise Weyl–symbol calculus, and second, we follow Fedosov [23, 19] and use this method in order to solve eq.(4.30) for $r$ and to establish eq.(4.34).

For the time being we consider a single copy of the typical fiber $V$. We represent operators $\hat{f} : V \rightarrow V$ by their Weyl symbols [27] $f = \text{symb}(\hat{f})$. They are functions of the auxiliary variable $y \equiv (y^a) \in \mathbb{R}^{2N}$: $f(y) = \text{symb}(\hat{f})(y)$. The precise definition of the symbol map can be found in refs.[27, 9]. Here we only mention some properties which will be needed later on. Under the symbol map, the operator product is mapped onto the $\circ$-product,

$$\text{symb}(\hat{f} \hat{g}) = \text{symb}(\hat{f}) \circ \text{symb}(\hat{g})$$ (A.1)

which reads explicitly

$$(f \circ g)(y) = f(y) \exp \left[ \frac{i}{\hbar} \frac{\partial}{\partial y^a} \omega^{ab} \frac{\partial}{\partial y^b} \right] g(y)$$ (A.2)

This is precisely the familiar "star product" for functions over the symplectic vector space $(\mathbb{R}^{2N}, \omega)$. This auxiliary phase–space should be carefully distinguished from the actual phase–space of the physical system under consideration, $M_{2N}$. (We reserve the conventional notation "$\ast$" for the star–product referring to $M_{2N}$.)

The Weyl symbol has the property $[\text{symb}(\hat{\varphi}^a)](y) = y^a$ from which it follows that

$$\left[ \text{symb}(\hat{\varphi}^a \hat{\varphi}^b) \right](y) = y^a \circ y^b = y^a y^b + i \frac{\hbar}{2} \omega^{ab}$$ (A.3)

Upon symmetrization, the $\omega^{ab}$–term vanishes, and $y^a y^b$ is found to be the symbol of $\hat{\varphi}^{(a} \hat{\varphi}^{b)}$. This result generalizes to arbitrary symmetrized products of $\hat{\varphi}$‘s:

$$\left[ \text{symb} \left( \hat{\varphi}^{(a_1} \cdots \hat{\varphi}^{a_n)} \right) \right](y) = y^{a_1} y^{a_2} \cdots y^{a_n}$$ (A.4)

Now we return to the bundle of local Hilbert spaces $V_\phi$ and represent the operators $A(\phi) : V_\phi \rightarrow V_\phi$ at all points $\phi$ by their symbols. More generally, we consider operator–valued $k$–forms of the type

$$A_{kl}(\phi, \hat{\varphi}, d\phi) = A_{a_1 \cdots a_l b_1 \cdots b_k}(\phi) \hat{\varphi}^{(a_1} \cdots \hat{\varphi}^{a_l)} d\phi^{b_1} \cdots d\phi^{b_k}$$ (A.5)
Their Weyl symbols read

\[ A_{kl}(\phi, y, d\phi) = A_{a_1 \ldots a_l b_1 \ldots b_k}(\phi) y^{a_1} \cdots y^{a_l} d\phi^{b_1} \cdots d\phi^{b_k} \quad (A.6) \]

We define the following linear operations on the space of symbols \( A \equiv \sum_{kl} A_{kl} \).

The projectors \( P_0 \) and \( P_{00} \):

\[ (P_0 A)(\phi, y, d\phi) = A(\phi, 0, d\phi) = A_0(\phi, d\phi) \quad (A.7) \]

\[ (P_{00} A)(\phi, y, d\phi) = A(\phi, 0, 0) = A_{00}(\phi) \quad (A.8) \]

The number operators \( \mathcal{N}_{d\phi}, \mathcal{N}_y \) and \( \mathcal{N} \):

\[ \mathcal{N}_{d\phi} = d\phi^a i(\partial_a) \quad (A.9) \]

\[ \mathcal{N}_y = dy^a \frac{\partial}{\partial y^a} \quad (A.10) \]

\[ \mathcal{N} = \mathcal{N}_{d\phi} + \mathcal{N}_y \quad (A.11) \]

The nilpotent maps \( \delta, \delta^* \) and \( \delta^{-1} \):

\[ \delta = d\phi^a \frac{\partial}{\partial y^a} \quad (A.12) \]

\[ \delta^* = y^a i(\partial_a) \quad (A.13) \]

\[ \delta^{-1} = \delta^* \mathcal{N}^{-1}(1 - P_{00}) \quad (A.14) \]

Note that (A.12) is nothing but eqs.(4.23), (4.24) in symbol language. In eq.(A.9), \( i(\partial_a) \) denotes the contraction with the vector field \( \partial_a \). It satisfies \( [i(\partial_a), d\phi^b] = \delta^b_a \), which shows that \( \mathcal{N}_{d\phi} A_{kl} = kA_{kl} \). Likewise, \( \mathcal{N}_y A_{kl} = lA_{kl} \). It is easy to see that

\[ [\delta, \mathcal{N}] = 0 \quad , \quad [\delta^*, \mathcal{N}] = 0 \quad (A.15) \]

and

\[ \delta P_{00} = P_{00} \delta = 0 \]

\[ \delta^* P_{00} = P_{00} \delta^* = 0 \quad (A.16) \]

The operators \( \delta, \delta^* \) and \( \mathcal{N} \) constitute a kind of supersymmetry algebra:

\[ \delta^2 = 0 \quad , \quad (\delta^*)^2 = 0 \]

\[ \delta \delta^* + \delta^* \delta = \mathcal{N} \quad (A.17) \]
It follows from (A.15), (A.16) and (A.17) that

\[ \delta \delta^{-1} + \delta^{-1} \delta + P_{00} = 1 \]  
(A.18)

and that \( \delta^{-1} \) is nilpotent, too: \( (\delta^{-1})^2 = 0 \). Eq. (A.18) is an important identity. It implies that any operator valued differential form \( A \) admits the "Hodge decomposition"

\[ A = A_{00} + \delta \delta^{-1} A + \delta^{-1} \delta A \]  
(A.19)

This decomposition is the main tool for solving eq. (4.30),

\[ \delta r = R + \nabla r + \frac{i}{\hbar} r^2 , \quad R \equiv \Omega(\Gamma) \]  
(A.20)

Here \( r_a(\phi, y) \) are the symbols of the operators \( r_a(\phi, \bar{\varphi}) \); since the latter are Weyl ordered by definition, their symbols are obtained by simply replacing \( \bar{\varphi}^a \to y^a \) everywhere. Let us apply (A.19) to \( r \equiv r_a(\phi, y) d\phi^a \):

\[ r = r_{00} + \delta \delta^{-1} r + \delta^{-1} \delta r \]  
(A.21)

Clearly, \( r_{00} = 0 \) because \( r \) is a 1–form. A priori eq. (A.20) admits many different solutions. For our program of implementing the background–quantum split symmetry it is necessary to know only one particular solution and not all of them. In order to make the solution unique, one may impose a subsidiary condition. A very convenient choice is

\[ \delta^{-1} r = 0 \quad \text{or} \quad y^a r_a(\phi, y) = 0 \]  
(A.22)

Assuming, as always, an analytic dependence on \( y^a \), this implies that

\[ r_a(\phi, 0) = 0 \quad \text{and} \quad \partial_{(b_1}^a \cdots \partial_{b_n)}^a r_a(\phi, 0) = 0 \]  
(A.23)

This means that in particular \( r \) contains no term proportional to the unit operator. For this choice, eq. (A.21) becomes \( r = \delta^{-1} \delta r \) or

\[ r = \delta^{-1} R + \delta^{-1} \left[ \nabla r + \frac{i}{\hbar} r^2 \right] \]  
(A.24)

The crucial observation is that the operator acting on \( r \) on the RHS of this equation increases the degree. (The degree is defined as in section 4.2 with \( y^a \) playing the rôle of \( \bar{\varphi}^a \) now, i.e., \( \deg(y^a) = 1 \).) In fact, \( \delta^{-1} \) increases the degree by one
unit, and since \( \text{deg}(\Gamma_a) = 2 \) for a symplectic connection, \( \nabla \) preserves the degree. This suggests the possibility of solving eq.(A.24) by an iteration

\[
r^{(n+1)} = \delta^{-1} R + \delta^{-1} \left[ \nabla r^{(n)} + \frac{i}{\hbar} \left( r^{(n)} \right)^2 \right]
\]

(A.25)

for \( n = 0, 1, 2, \cdots \) with the initial condition \( r^{(0)} = 0 \). It can be proven [23] that, for \( n \to \infty \), \( r^{(n)} \) converges indeed towards the solution of (A.24). The first few iterates are

\[
\begin{align*}
  r^{(0)} &= 0 \\
  r^{(1)} &= \delta^{-1} R \\
  r^{(2)} &= \delta^{-1} R + \delta^{-1} \nabla (\delta^{-1} R) + O(5) \\
  r^{(3)} &= r^{(2)} + O(5)
\end{align*}
\]

(A.26)

Here we observe a property of this iteration which holds true in general: in order to obtain all terms up to some fixed degree, only a finite number of iteration steps is needed. If we are interested only in the terms of degree 4 or less, two iterations are sufficient, i.e., \( r = r^{(2)} + O(5) \). Therefore, with (note that \( \nabla_y y = 0 \))

\[
\begin{align*}
  R &= \frac{1}{4} R_{abcd} y^c y^d d\phi^a d\phi^b \\
  \delta^{-1} R &= -\frac{1}{8} R_{abcd} y^b y^c y^d d\phi^a \\
  \nabla (\delta^{-1} R) &= -\frac{1}{8} (\nabla_e R_{abcd}) y^b y^c y^d d\phi^e d\phi^a \\
  \delta^{-1} \nabla (\delta^{-1} R) &= \frac{1}{40} (\nabla_{b} R_{acde}) y^a y^b y^c y^d d\phi^e
\end{align*}
\]

(A.27)

one arrives at eq.(4.33) given in the main text.

A similar method can be used in order to determine the operators \( \mathcal{O}_f(\phi) \) from

\[
0 = D\mathcal{O}_f(\phi) \equiv d\mathcal{O}_f(\phi) + \frac{i}{\hbar} \left[ \tilde{\Gamma}, \mathcal{O}_f(\phi) \right] \equiv (\nabla - \delta) \mathcal{O}_f(\phi) + \frac{i}{\hbar} \left[ r, \mathcal{O}_f(\phi) \right]
\]

(A.28)

The notation \( \mathcal{O}_f(\phi) \) is used also for the symbols and the correspondence \( \tilde{\varphi}^a \leftrightarrow y^a \) is understood. The square brackets in (A.28) denote the commutator with respect to \( \circ \)-multiplication. We used that

\[
\tilde{\Gamma}_a = \Gamma_a + \omega_{ab} \tilde{\varphi}^b + r_a
\]

(A.29)
which implies \[ D = \nabla - \delta + \frac{i}{\hbar} [r, \cdot] \]. In section 4 we explained that \( \mathcal{O}_f(\phi) \) is a zero-form operator whose term proportional to the unit operator is given by the classical observable \( f(\phi) \). Hence the projections (A.7) and (A.8) yield

\[
P_{00} \mathcal{O}_f(\phi) = P_0 \mathcal{O}_f(\phi) = f(\phi)
\]

(A.30)

Since \( \mathcal{O}_f \) is a zero-form, \( \delta^{-1} \mathcal{O}_f \) vanishes identically, and the Hodge decomposition of \( \mathcal{O}_f \) reads therefore \( \mathcal{O}_f = f + \delta^{-1} \delta \mathcal{O}_f \). Thus, every solution of \( D \mathcal{O}_f = 0 \) satisfies

\[
\mathcal{O}_f = f + \delta^{-1} (D + \delta) \mathcal{O}_f
\]

(A.31)

This equation can be solved by the iteration [23]

\[
\mathcal{O}^{(n+1)}_f = f + \delta^{-1} (D + \delta) \mathcal{O}^{(n)}_f, \quad n = 0, 1, 2, \ldots
\]

(A.32)

with \( \mathcal{O}^{(0)}_f = f \). Again, the operator \( \delta^{-1} (D + \delta) \) increases the degree because \( r \) contains only terms of degree 3 and higher. In order to calculate all terms of a fixed degree, only a finite number of iterations is needed. Given \( f(\phi) \), the solution of (A.28) is unique; no subsidiary condition must be imposed. For practical calculations the component form of (A.32) is more useful:

\[
\mathcal{O}^{(n+1)}_f = f + N^{-1}_y y^a \nabla_a \mathcal{O}^{(n)}_f + \frac{i}{\hbar} N^{-1}_y y^a [r_a, \mathcal{O}^{(n)}_f]
\]

(A.33)

After three iterations, all terms of degree 3 and less are stable already: \( \mathcal{O}_f = \mathcal{O}^{(3)}_f + O(4) \). Using the explicit form of \( r \), eq.(4.33), one easily arrives at eq.(4.34).

For a flat phase-space with \( \Gamma^c_{ab} = 0 \) and \( r = 0 \) the recurrence relation (A.33) can be solved exactly:

\[
\mathcal{O}^{(n)}_f(\phi) = \sum_{m=0}^{n} \frac{1}{m!} (y^a \partial_a)^m f(\phi)
\]

(A.34)

Thus, for \( n \to \infty \),

\[
\mathcal{O}_f(\phi) = f(\phi + y)
\]

(A.35)

This important special case is investigated further in section 5.
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