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REGGEONS AS ELECTRIC MULTIPOLES

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A B S T R A C T

A simple derivation of Lovelace's N Reggeon vertex is given by associating a Reggeon with a charge distribution on the equator of a sphere. The vertex is simply the exponential of the Coulomb energy of the charge distributions, and the coherent state parameters that describe the Reggeons are their multipole moments.

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The original analogue model first proposed by Nielsen ¹⁾ for the Veneziano amplitude can be rephrased as follows: the momentum dependent part of the Veneziano integrand is the exponential of the total Coulomb energy of a set of charges p_i located at points z_i on the equator of a sphere, excluding the self-interactions of charges with themselves. The integration is done with a suitable volume element by varying the position of the charges on the equator. This analogue picture has been extended to multiloop diagrams ^{2),3)}; in this case the underlying Riemann surface is a sphere with handles and charges are allowed to move on the surface along closed cycles that split it into two symmetrical halves ⁴⁾, each one of them being the original open surface of the analogue model ¹⁾.

This formulation of the analogue model can also be extended to more general dual theories: if the charges are allowed to move along some well defined paths other than a full equator one gets the non-planar dual amplitudes for baryons and exotics ⁵⁾. In addition, Shapiro ⁶⁾ has shown that if the charges are allowed to vary freely on the surface of the sphere one gets the Virasoro amplitude with a particular intercept. However, in these cases, the corresponding surfaces are no longer symmetric and there is no equivalent analogue on an open surface with boundaries.

The previous considerations are valid whenever the external particles are scalars (Reggeons in the ground state) and the natural question to ask oneself is: what is the equivalent picture for excited particles. The purpose of this letter is to show that the integrand of the N Reggeon vertex is the total Coulomb energy of a set of charge distributions on the equator of a sphere rather than that of point charges. We want to emphasize at this point that this idea is not new. Indeed, Sakita has made explicit use of it in his functional integral approach to dual amplitudes ⁷⁾. Our aim is mainly to show how Lovelace's formula for the N Reggeon vertex ⁸⁾ in the operator formalism can easily be understood from this point of view. The results cast considerable light on the structure of the N Reggeon vertex, establish the connection with the functional integral approach

and indicate a simple way of deriving N Reggeon vertices for more general dual theories.

Let us start by considering Lovelace's N Reggeon vertex ⁸⁾ which can be written as:

$$V_N(g) = \int d\mathbf{z}^{(N-3)} \varphi_N^{(0)}(\mathbf{z}) \cdot \prod_{i=1}^N (z_{i+1} - z_{i-1})^{-\alpha' \mu^2} \cdot e^{-\frac{\alpha'}{2} \sum_{i \neq j} \langle \alpha^{(i)} | U^{(i)} V^{(j)} | \alpha^{(j)} \rangle} \quad (1)$$

where $\varphi_N^{(0)}(\mathbf{z})$ is the zero momentum limit of the Veneziano integrand for scalar particles. This formula gives the coherent state matrix elements of the vertex, $|\alpha^{(j)}\rangle$ being an infinite component vector with components $\alpha_n^{(j)}/\sqrt{n}$. Here $\alpha_n^{(j)}$ is a coherent state parameter for the n^{th} mode of the j^{th} Reggeon if $n \neq 0$, or the momentum p_j of the Reggeon if $n=0$. Moreover, $U^{(i)} V^{(j)}$ is an infinite matrix in the space of internal excitations and zero modes ⁹⁾ ($n=0, 1, \dots$) which, as indicated, factorizes into the product of operators depending on Koba-Nielsen variables related to the Reggeons i and j ¹⁰⁾. Any infinite matrix C that occurs in Veneziano theory has an associated projective transformation $T(\xi)$ from which its matrix elements are derived according to the following rules ¹¹⁾.

$$\frac{[T(\xi)]^n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \left[\frac{a\xi + b}{c\xi + d} \right]^n = C_{n0} + \sum_{m=1}^{\infty} C_{nm} \frac{\xi^m}{\sqrt{m}} = \sum_{m=0}^{\infty} C_{nm} \frac{\xi^m}{\sqrt{m}} \quad (2)$$

valid if $n \geq 1$, and with the convention $\sqrt{0} = 1$. The remaining matrix elements are given by

$$C_{0n} = (C^T)_{n0} = \frac{1}{\sqrt{n}} \left(-\frac{c}{d} \right)^n \quad (3)$$

$$C_{\infty} = \log \left| \frac{d}{\sqrt{ad-bc}} \right| \quad (4)$$

The matrices $U^{(i)}$ and $V^{(j)}$ are defined by the associated projective transformations, which, in Lovelace's notation are written as ⁸⁾

$$U^{(i)} = \begin{bmatrix} z_{i-1} & z_i & z_{i+1} \\ 0 & \infty & 1 \end{bmatrix} \quad (5)$$

$$V^{(j)} = \begin{bmatrix} \infty & 0 & 1 \\ z_{j-1} & z_j & z_{j+1} \end{bmatrix} \quad (6)$$

Finally, in order to get the matrix elements of $U^{(i)}V^{(j)}$, either the two projective transformations are multiplied first and then Eqs. (2)-(4) are applied to obtain $(U^{(i)}V^{(j)})_{nm}$ directly, or the matrix elements of $U^{(i)}$ and $V^{(j)}$ are computed first, in which case the matrix elements of $U^{(i)}V^{(j)}$ are obtained by the product law ¹¹⁾

$$(U^{(i)}V^{(j)})_{nm} = \sum_{k=1}^{\infty} U_{nk}^{(i)} V_{km}^{(j)} + U_{n0}^{(i)} \delta_{0,m} + \delta_{0,n} V_{0m}^{(j)} \quad (7)$$

We now proceed to show that the matrix element $\langle \alpha^{(i)} | U^{(i)}V^{(j)} | \alpha^{(j)} \rangle$ is indeed the energy of two charge distributions whose multipole moments are given by the components of the coherent state vectors $|\alpha^{(j)}\rangle$. We first map the sphere to the complex plane by means of a stereographic projection in such a way that the equator on which charges move becomes the real axis. We are going to associate with each Reggeon a charge distribution $p_i(z)$, and we shall identify the Koba-Nielsen variable of the Reggeon, z_i , as the point where the charge distribution will be expanded in its multipole moments. Since multipole expansions can only be done around points inside the charge distribution, and since the Koba-Nielsen variable z_i can vary only between z_{i-1} and z_{i+1} , we conclude that the charge distribution $p_i(z)$ extends from z_{i-1} to z_{i+1} .

We now compute the Coulomb energy E_{ij} of two charge distributions $p_i(z)$ and $p_j(z)$ with total charge p_i , p_j respectively.

This problem has no solution on a closed Riemann surface like the sphere unless the two charges are equal and opposite. If $p_i \neq -p_j$ one must introduce a charge $(-p_i)$ in some point a and a charge $(-p_j)$ in a point b in such a way that the total charge on the closed surface is zero. The energy of this system is then given by

$$E_{ij} = \int_{z_{i-1}}^{z_{i+1}} dz \int_{z_{j-1}}^{z_{j+1}} dz' p_i(z) p_j(z') \log \left| \frac{z-z'}{z-a} \frac{a-b}{b-z'} \right| \quad (8)$$

Gauss theorem forces the set of all charge distributions that we have on the sphere to be such that the total charge $\sum_i p_i$ is zero (momentum conservation). Therefore, if we compute the total energy of the system by summing over all pairs and including the self-energies it turns out that the dependence on a and b cancels because the net fictitious charge introduced at these points is zero; and the only term that survives in the integrand of Eq. (8) is $\log(1-z'/z)$.

When we compare charge distributions after the multipole expansion is made we would like, of course, to have them equal if all the multipole moments are equal. This means that all charge distributions should be spread along the same standard interval on the real axis and that all multipole expansions should be done around the same standard point. We shall choose $(\infty, 0, 1)$ as our standard interval, for reasons that will be discussed at the end of this paper, and introduce a linear transformation $z = T_i(\xi)$ that maps the points $(\infty, 0, 1)$ in the ξ plane to the points (z_{i-1}, z_i, z_{i+1}) in the z plane. Notice that $T_i(\xi)$ is the projective transformation that defines the matrix $V_{nm}^{(i)}$ through Eqs. (2)-(4). We can then write the energy E_{ij} as:

$$E_{ij} = \int_{-\infty}^1 d\xi \tilde{p}_i(\xi) \int_{-\infty}^1 d\xi' \tilde{p}_j(\xi') \log \left| 1 - \frac{T_j(\xi')}{T_i(\xi)} \right| \quad (9)$$

and perform the multipole expansion, that is equivalent to expanding the logarithm in the integrand around $\xi, \xi' = 0$. Before we carry

out the expansion it is convenient to introduce the projective transformation $\tilde{T}_i(\xi) = [T_i(\xi)]^{-1}$, obtained by interchanging the two rows of $T_i(\xi)$. In Lovelace's notation ⁸⁾, $\tilde{T}_i(\xi)$ is characterized by

$$\tilde{T}_i = \begin{bmatrix} \infty & 0 & 1 \\ \frac{-1}{z_{i-1}} & \frac{-1}{z_i} & \frac{-1}{z_{i+1}} \end{bmatrix} \quad (10)$$

Then, expanding the logarithm and using Eq. (2) we get

$$-\log \left[1 - \tilde{T}_i(\xi) T_j(\xi') \right] = \sum_{k=1}^{\infty} \frac{[\tilde{T}_i(\xi)]^k}{\sqrt{k}} \frac{[T_j(\xi')]^k}{\sqrt{k}} = \sum_{n,m=0}^{\infty} \frac{\xi^n}{\sqrt{n}} \left\{ \sum_{k=1}^{\infty} U_{nk}^{(i)} V_{km}^{(j)} \right\} \frac{\xi'^m}{\sqrt{m}} \quad (11)$$

where we have used the fact that the matrix generated by the projective transformation $T_j(\xi)$ is, indeed, $V_{nm}^{(j)}$. Moreover, using Lovelace's rules for computing the transposed of $\tilde{T}_i$ in his notation ⁸⁾ we can readily recognize that $\tilde{T}_i^T = U^{(i)}$. Then, we conclude that E_{ij} can be written as

$$E_{ij} = \sum_{n,m=0}^{\infty} \frac{\alpha_n^{(i)}}{\sqrt{n}} X_{nm}^{(i,j)} \frac{\alpha_m^{(j)}}{\sqrt{m}} \quad ; \quad X_{nm}^{(i,j)} = \sum_{k=1}^{\infty} U_{nk}^{(i)} V_{km}^{(j)} \quad (12)$$

where $\alpha_n^{(i)}$ are the multipole moments of the charge distribution $\rho_i(\xi)$ and $X_{nm}^{(i,j)}$ is indeed given by the first term of Eq. (7). Notice that if diagonal terms ($i=j$) are included in Lovelace's vertex, the second and third terms in Eq. (7) also cancel.

In order to complete our proof we must now get rid of the self-energies of the charge distributions. Strictly speaking, the integrand of the N Reggeon vertex must be defined as:

$$I = e^{-\frac{\alpha'}{2} \sum_{i,j} E_{ij}} / e^{-\frac{\alpha'}{2} \sum_i E_{ii}} \quad (13)$$

At this stage we remind the reader that the total Coulomb energy we are considering is unique and invariant under projective transformations, that map the Riemann sphere onto itself, only when self-interactions are included. The denominator of Eq. (13) is not unique, but we shall prove that there is a unique way of computing E_{ii} if one demands that projective invariance be maintained, and that this gives precisely the remaining terms in the N Reggeon vertex.

In order to compute only the diagonal terms E_{ii} we must use the full Eq. (8) because the a, b dependence will not be cancelled in this case. We choose $a=b$ and discard the infinite self-energy of the two charges at this point, that is to say, the $\log |1 - \frac{b}{a}|$ in the integrand of Eq. (8). Then, E_{ii} is given by

$$E_{ii} = \int_{-\infty}^1 d\xi \int_{-\infty}^1 d\xi' \tilde{p}_i(\xi) \tilde{p}_i(\xi') \log \left| 1 - \frac{T_i(\xi')}{T_i(\xi)} \right| - p_i \int_{-\infty}^1 d\xi' \tilde{p}_i(\xi') \log \left| 1 - \frac{T_i(\xi')}{a} \right| - p_i \int_{-\infty}^1 d\xi \tilde{p}_i(\xi) \log \left| 1 - \frac{a}{T_i(\xi)} \right| \quad (14)$$

The first term of the right-hand side of Eq. (14) will just cancel the diagonal terms in the numerator of Eq. (13). The contribution of the second term can be obtained by expanding

$$- \log \left[1 - \frac{T_i(\xi)}{a} \right] = \sum_{m=0}^{\infty} \left\{ \sum_{k=1}^{\infty} \frac{a^{-k}}{\sqrt{k}} V_{km}^{(i)} \right\} \frac{\xi^m}{\sqrt{m}} \quad (15)$$

However, if we now choose $a = z_{i-1}$, it follows by explicit computation that $a^{-k}/\sqrt{k} = [V^{(i)}]_{ok}^{-1}$. The product law of infinite matrices given by Eq. (2) is valid for arbitrary matrices. If we apply the product law to $[V^{(i)}]_{om}^{-1} V^{(i)}_{om} = I_{om} = 0$ we get an identity that allows us to carry out the k summation in Eq. (15),

$$\sum_{k=1}^{\infty} [V^{(i)}]_{ok}^{-1} V_{km}^{(i)} = - V_{om}^{(i)} \quad (16)$$

Moreover, if the third term of Eq. (14) is expanded in the same way and use is made of the identity

$$\frac{z_{i-1}^k}{\sqrt{k}} = [U^{(i)-1}]_{k0}$$

it follows at once that, with our choice of a , the denominator of Eq. (13), besides cancelling the diagonal terms of the numerator, adds a contribution of the form

$$e^{\frac{\alpha'}{2}} \sum_{i=1}^N \sum_{n=0}^{\infty} \left[\frac{\alpha_n^{(i)}}{\sqrt{n}} U_{n0}^{(i)} p_i + p_i V_{0n}^{(i)} \frac{\alpha_n^{(i)}}{\sqrt{n}} \right] \quad (17)$$

which is indeed identical to the contribution of the second and third terms of Eq. (7) after momentum conservation has been used.

Our choice of a is not as arbitrary as it seems: it is the unique choice that renders $\sum_i E_{ii}$ a projective invariant. An alternative way of understanding this point is as follows: the projective invariance of the N Reggeon vertex means in this case that for fixed values of the multipole moments $\alpha_n^{(i)}$, that simply define the charge distribution, each individual matrix element $[U^{(i)} V^{(j)}]_{nm}$ should be invariant when all z 's are mapped by the same transformation. From Lovelace's proof of projective invariance⁸⁾ it follows that this will be the case only when the matrix elements are defined as in Eq. (7), because this product law guarantees that the matrix elements computed in this way are the same as the ones obtained by first multiplying the projective transformations that generate each individual matrix¹¹⁾. It is clear that the extra terms in the product law occur only when there are charges involved, and that a term of the form

$$\sum_{i,j} \frac{\alpha_n^{(i)}}{\sqrt{n}} \left\{ \sum_{k=1}^{\infty} U_{nk}^{(i)} V_{k0}^{(j)} \right\} p_j$$

will be projective invariant only if $\sum_j p_j = 0$. Projective invariance is therefore destroyed for this term when the diagonal terms are dropped, but is restored when the term

$$\sum_{i=1}^N \frac{\alpha_n^{(i)}}{\sqrt{n}} U_{n0}^{(i)} p_i$$

is added. This is analogous to the usual trick of combining the Koba-

Nielsen integrand $\prod_{i \neq j} |z_i - z_j|^{-p_i \cdot p_j}$ with a measure factor $\prod |z_i - z_{i-1}|^{-\mu^2}$ to restore projective invariance.

We would like at this stage to comment on the choice of the standard interval $(\infty, 0, 1)$. It is quite clear that any standard interval will do, and that we would simply find different matrices and different multipole moments. The relevance of this particular standard interval is apparent only when the factorization properties of the vertex are considered, that is to say, when two Reggeons of different vertices are sewn together through a twisted propagator to obtain a bigger vertex⁸⁾ or when Reggeons of the same vertex are sewn to obtain a multiloop amplitude^{2),3),12)}. When two Reggeons are sewn with a twisted propagator as given in the multiperipheral configuration^{2),3),8)} the standard interval $(\infty, 0, 1)$ is crucial to get the Chan variable of the propagator in the correct range $0 \leq x \leq 1$. A change of the standard interval would imply a change in the sewing prescription. Since the multiperipheral configuration is defined by three consecutive Koba-Nielsen variables being at $(0, 1, \infty)$ we can think of the $U^{(i)}$ and $V^{(i)}$ matrices as a sort of "overlap matrices" between the multiperipheral and a more general configuration.

The ideas presented here can be extended to the multiloop case, in which the underlying Riemann surface is not a sphere but a sphere with handles. Moreover, what is also clear is the meaning of factorizing a Reggeon out of a set of scalars: one simply considers the set of charges associated with the scalars as a charge distribution and expands it in multipole moments around some point that will become the Koba-Nielsen variable of the new Reggeon. Finally, we want to comment on the connection with the functional integral approach. Sakita's prescription for sewing Reggeons⁷⁾ is to carry out a functional integral over the charge distribution $\tilde{\rho}(\xi)$ that describe the Reggeons. This is the same as sewing Reggeons by integrating over coherent state parameters, because multipole moments are simply Fourier components $\alpha_n = \int \tilde{\rho}(\xi) \xi^n d\xi$ of the charge distribution and the Gaussian functional integral over the function $\tilde{\rho}(\xi)$ is the same as the infinite dimensional Gaussian integral over its Fourier components. In both cases the measure of integration is determined by unitarity.

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