BACKWARD DISPERSION RELATIONS FOR $\pi N \rightarrow \pi \Delta$
SCATTERING AND $N \Delta \gamma$ COUPLINGS

J. Baacke

Freie Universität Berlin, Berlin

and

H. Kleinert ^{*)}

CERN - Geneva

A B S T R A C T

Using unsubtracted dispersion relations for the invariant amplitudes of the process $\pi N \rightarrow \pi \Delta$ in forward and backward direction we derive sum rules relating t channel to s and u channel resonances. We saturate these sum rules by ϱ , N, and Δ and obtain for the ϱ coupling constants

$$C_3 m = \frac{5}{13} \quad ; \quad C_4 m^2 = C_5 m^2 = \frac{5}{413}$$

where m is the average mass of N and Δ . Assuming vector meson dominance of the electromagnetic vertex $N \Delta \gamma$ these become the values of the Gourdin-Salin coupling constants in reasonable agreement with experiment.

^{*)} Permanent address: Institut für Theoretische Physik, Freie Universität Berlin, Berlin.

The assumption that the t channel cut of backward dispersion relations of pion nucleon scattering can be dominated by a ρ meson for $I_t = 1$ and that the coupling constants of ρ can be related to the electromagnetic couplings of the nucleons via the vector meson dominance hypothesis has led to three successful sum rules for κ^V , the anomalous isovector magnetic moment¹⁾. These sum rules emerge by equating the value of the amplitudes $A^{(-)}$, $B^{(-)}$ and $A'^{(-)}$ (the t channel helicity flip amplitude) at threshold obtained from a dispersion relation at $\theta_s = \pi$ with the value obtained from a forward dispersion relation and saturating the integrals with sharp resonances. We have found that a similar procedure applied to the corresponding invariant amplitudes of $\pi N - \pi \Delta$ scattering leads to good sum rules for the magnetic, electric, and Coulomb multipoles of the $N\Delta\gamma$ vertex. The kinematics of these sum rules are, in general, quite involved. The calculation becomes, however, very simple if one works at zero pion mass, is content with a saturation by means of N and Δ only, and assume, moreover, that the masses of these particles are degenerate.

Let the invariant amplitudes^{*)} be defined by

$$\langle \pi(q') \Delta(p') | T | \pi(q) N(p) \rangle \equiv \bar{u}^\mu(p') [(A_1^{ba} + Q B_1^{ba}) Q_\mu + (A_2^{ba} + Q B_2^{ba}) K^\mu] u(p) \quad (1)$$

with

$$Q \equiv \frac{1}{2}(q+q') \quad K \equiv \frac{1}{2}(q-q') \quad (2)$$

*) Our normalization is $\langle \pi(q') | \pi(q) \rangle = 2q_0 (2\pi)^3 \delta^3(q'-q)$

and $\langle N(p') | N(p) \rangle = \frac{p_0}{M} (2\pi)^3 \delta^3(p'-p)$

while T is defined by $S \equiv 1 - i(2\pi)^4 \delta^4(p_f - p_i) T$

and

$$A_1^{ba} = A_1^{(-)} I_{(-)}^{ba} + A_1^{(+)} I_{(+)}^{ba} \quad (3)$$

etc., where $I_{(+)}$ and $I_{(-)}$ are the isospin matrices of $I_t = 1$ and $I_t = 2$, Ref. 2).

$$I_{(\pm)}^{ba} = \frac{1}{2} (\delta^{cb} \tau^a \pm \delta^{ca} \tau^b) \quad (4)$$

Then we find that A and B can be expressed in terms of the s channel helicity amplitudes

$$T_{\lambda', \lambda}^{(s)} = \cos \frac{-\lambda' + \lambda}{2} \Theta_s \sin \frac{-\lambda' - \lambda}{2} \Theta_s T_{\lambda', \lambda}^{(s)} \quad (5)$$

in the form

$$\begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} = \mathcal{M}(s, \Theta_s) \begin{pmatrix} T_{3/2, 1/2}^{(s)} \\ \frac{1}{\sqrt{3}} T_{1/2, 1/2}^{(s)} \\ \frac{1}{\sqrt{3}} T_{-1/2, 1/2}^{(s)} \\ T_{-3/2, 1/2}^{(s)} \end{pmatrix} \quad (6)$$

where $\mathcal{M}$ is given in forward direction by ($W \equiv \sqrt{s}$)

$$\mathcal{M}(s,0) = -\frac{2\sqrt{2}mW}{(s-m^2)^2} \left\{ \begin{array}{cccc} m & 0 & m & 0 \\ -m & 0 & -m & -2W \\ 0 & \frac{m}{W} & 0 & 0 \\ 2 & -\frac{m}{W} & 0 & 0 \end{array} \right\} \quad (7)$$

while for backward angle $\theta_s = \pi$ we find

$$\mathcal{M}(s,\pi) = -\frac{2\sqrt{2}mW}{(s-m^2)^2} \left\{ \begin{array}{cccc} 0 & 0 & m & -\frac{s+m^2}{W} \\ 0 & 0 & -m & -\frac{s-m^2}{W} \\ \frac{s+m^2}{s} & \frac{m}{W} & 0 & \frac{m}{W} \\ \frac{s-m^2}{s} & -\frac{m}{W} & 0 & -\frac{m}{W} \end{array} \right\} \quad (8)$$

Here m has to be taken as the average mass of nucleon and Δ resonance.

Consider the $I_t = 1$ amplitudes. From the definition (1) we see that $A_1^{(-)}, B_2^{(-)}$ are even functions in $s-u$ while $A_2^{(-)}$ and $B_1^{(-)}$ are odd. As usual, we divide the factor $2/(s-u)$ out of $A_2^{(-)}$ and $B_1^{(-)}$ in order to obtain functions of $(s-u)^2$ and t only. Applying the Regge pole hypothesis we find that

$$A_1^{(-)}, \frac{2}{s-u} A_2^{(-)}, \frac{2}{s-u} B_1^{(-)}, B_2^{(-)} \quad (9)$$

behave asymptotically as

$$s^{\alpha_3-1}, s^{\alpha_3-1}, s^{\alpha_3-3}, s^{\alpha_3-1}$$

in forward direction where α_s is the intercept of the s trajectory (≈ 0.5). Hence, we can write an unsubtracted dispersion relation in s for the threshold point $s = m^2$, $t = 0$ *)

In the backward direction the same amplitudes have only dynamic singularities in the variable t . To see this, we just note that $(s-u)^2 = t(t-4m^2)$, such that no kinematic singularity comes from the variable $(s-u)^2$. The baryon trajectory with $\alpha_B \approx 0.19$ governs the asymptotic behaviour of $A_1^{(-)} - A_2^{(-)}$, $B_1^{(-)} - B_2^{(-)}$, $A_1^{(-)} + A_2^{(-)}$ and $B_1^{(-)} + B_2^{(-)}$ by

$$t^{\alpha_B - 3/2}, t^{\alpha_B - 3/2}, t^{\alpha_B - 1/2}, t^{\alpha_B - 1/2}$$

Hence, another set of unsubtracted dispersion relations can be written in the variable t for the threshold point $t = 0$.

Let us calculate the imaginary parts of $\hat{T}_{\lambda'\lambda}^{(s)}$ at a baryon pole. We find it most convenient to use PCAC and state the result in terms of the collinear invariant matrix elements of the axial current at $q^2 = (p' - p)^2 = 0$ between baryons of helicity λ

$$X_a(\lambda)_{\beta\alpha} \equiv \frac{2\sqrt{m_p m_N}}{p_0 + |p|} \langle p'\lambda\beta | A_a^{0+3} | p\lambda\alpha \rangle \quad (10)$$

which satisfy $SU(2) \times SU(2)$ algebra ³⁾

$$[X_b(\lambda), X_a(\lambda)] = i \epsilon_{bac} T_c \quad (11)$$

*) In this letter, we shall not discuss the more stringent superconvergence relations obeyed by $2/(s-u)B_1^{(-)}$.

One finds for a baryon resonance of mass m_Y and spin J

$$\text{Im } \frac{T(s)}{\lambda\lambda} = -\frac{\pi}{4m_F^2} \delta(s-m_Y^2) (m^2-m_Y^2)^2 [X_b(\lambda) X_a(\lambda)] \hat{d}_{\lambda\lambda'}^J(\theta) \quad (12)$$

where the square bracket denotes antisymmetrization with respect to the pion indices b and a and $\hat{d}_{\lambda\lambda'}$ is defined in terms of the standard $d_{\lambda\lambda'}(\theta)$ functions analogously to (5). As a consequence, the baryon resonance contributes to the unsubtracted dispersion relation of

$$\mathcal{A}^{ba} \equiv \begin{pmatrix} A_1^{(-)} \\ \frac{2}{s-u} A_2^{(-)} \\ \frac{2}{s-u} B_1^{(-)} \\ B_2^{(-)} \end{pmatrix} \quad \mathbb{I}^{ba}_{(-)}$$

in forward direction

$$\mathcal{A}_Y^{ba}(s=m^2, t=0)_f = \mathcal{M}(m_Y^2, 0) \begin{cases} (m_Y^2-m^2)^{\frac{J+1}{2}} \sqrt{(J-\frac{1}{2})} [X_b(\frac{3}{2}) X_a(\frac{1}{2})] \\ \sqrt{3} [X_b(\frac{1}{2}) X_a(\frac{1}{2})] \\ -n\sqrt{3} [X_b(\frac{1}{2}) X_a(\frac{1}{2})] \\ (m_Y^2-m^2)^n \sqrt{\frac{J+1}{2}} \sqrt{(J+\frac{1}{2})(J-\frac{1}{2})} [X_b(\frac{3}{2}) X_a(\frac{1}{2})] \end{cases} \quad (13)$$

while for the backward direction it gives

$$\mathcal{A}_Y^{ba}(s=m^2, t=0)_b = -u \mathcal{M}(m_Y^2, \pi) \begin{cases} \frac{m_Y^4-m^4}{2m_Y^2} n^{\frac{J+1}{2}} \sqrt{(J+\frac{1}{2})(J-\frac{1}{2})} [X_b(\frac{3}{2}) X_a(\frac{1}{2})] \\ -n\sqrt{3} [X_b(\frac{1}{2}) X_a(\frac{1}{2})] \\ \sqrt{3} [X_b(\frac{1}{2}) X_a(\frac{1}{2})] \\ \frac{m_Y^4-m^4}{2m_Y^2} \sqrt{(J+\frac{1}{2})(J-\frac{1}{2})} [X_b(\frac{3}{2}) X_a(\frac{1}{2})] \end{cases} \quad (14)$$

Here η is the parity of the intermediate state while n denotes the normality $n \equiv \eta (-)^{J-\frac{1}{2}}$ (*). The threshold value obtained by subtracting all backward contributions (14) from all forward ones (13) should be equal to the t channel cut of the backward amplitude

$$A_t^{ba} = \sum_s A_{sf}^{ba} - \sum_s A_{sb}^{ba} \quad (15)$$

Let us assume that this cut is dominated by a ρ meson. We introduce the ρ couplings to N and Δ according to Gourdin and Salin ⁴⁾ by defining the matrix elements of the ρ current (**)

$$\begin{aligned} \langle \Delta(p') \chi | j_\rho^\mu | N(p) \chi \rangle &\equiv 2\gamma_\rho \bar{u}(p') [C_3 (\not{q} \gamma_\nu^\mu - q_\nu \gamma^\mu) \\ &+ C_4 (q\rho' \gamma_\nu^\mu - q_\nu \rho'^\mu) + C_5 (q\rho q_\nu^\mu - q_\nu \rho^\mu)] u(p) \end{aligned} \quad (16)$$

Then the unsubtracted t channel contribution is given by (***)

$$A_t^{(-)}(s=m^2, t=0) = i \frac{1}{F_\pi} \begin{pmatrix} -2m C_3 + \frac{m^2}{2} (C_4 - C_5) \\ -(C_4 + C_5) \\ 0 \\ -2C_3 \end{pmatrix} \quad (17)$$

*) Also we have used the property of $\chi(\lambda)$ ³⁾.

$$\chi_a(\lambda)_{\rho\kappa} = -n_\rho n_\kappa \chi_a(-\lambda)_{\rho\kappa}$$

**) In our convention $2\gamma_\rho = g_{\rho\pi\pi} \approx 2.5$.

***) We have used the KSFR relation $\gamma_\rho^2 = (m_\rho^2 / 8F_\pi^2)$ to eliminate γ_ρ .

Inserting (13), (14) and (17) into (15), we can write down explicitly our sum rule for the $N\Delta\gamma$ coupling constants C_3, C_4, C_5 . We shall not do so here, however, but shall proceed directly to the approximation of saturating this sum rule with a nucleon and a Δ only. In this case, the sum rule becomes simply ^{*)}

$$A_t^{ba}(s=k^2, t=0) = \frac{1}{F_\pi^2} \frac{\sqrt{6}}{2m} \left\{ \begin{pmatrix} -m \\ 0 \\ 0 \\ -1 \end{pmatrix} [X_b(\frac{1}{2}) X_a(\frac{1}{2})]_N + \begin{pmatrix} 5m \\ \frac{3}{2}m \\ -\frac{1}{2}m \\ 5 \end{pmatrix} [X_b(\frac{1}{2}) X_a(\frac{1}{2})]_\Delta \right\} \quad (18)$$

Introduce the coupling constants $G_{\beta\alpha}$ of a pion to particles β and α ^{**)} . Then, we can write for the nucleon intermediate state

$$[X_b(\frac{1}{2}) X_a(\frac{1}{2})]_N = \frac{\sqrt{3}}{2} I_{(-)}^{ba} G_{\Delta N} G_{NN} \quad (19)$$

while for the Δ we have

$$[X_b(\frac{1}{2}) X_a(\frac{1}{2})]_\Delta = -\frac{\sqrt{3}}{2} I_{(-)}^{ba} 5 G_{\Delta\Delta} G_{\Delta N} \quad (20)$$

*) The $X_{\Delta\Delta}^{(3/2)}$ matrix element does not occur due to the helicity condition for elastic matrix elements

$$X_{\alpha\alpha}(\lambda)/X_{\alpha\alpha}(\lambda) = \lambda/\lambda$$

which follows directly from the definition (10).

**) For the normalization of $G_{\beta\alpha}$ see Ref. 5).

Thus we find

$$A_t^{(-)} = \frac{1}{F_\pi^2} \frac{3}{2\sqrt{2}m} \begin{cases} -m (G_{\Delta N} G_{NN} + 25 G_{\Delta\Delta} G_{\Delta N}) \\ -\frac{1}{m} \frac{15}{2} G_{\Delta\Delta} G_{\Delta N} \\ \frac{1}{m} \frac{15}{2} G_{\Delta\Delta} G_{\Delta N} \\ -(G_{\Delta N} G_{NN} + 25 G_{\Delta\Delta} G_{\Delta N}) \end{cases} \quad (21)$$

We have to insert values for the coupling constants. Within the spirit of our approximation we shall not use the experimental values but those which are determined from the $SU(2) \times SU(2)$ algebra (11) for the $N\Delta$ system only. These coupling constants satisfy the algebraic relations ^{*)}

$$G_{NN}^2 - G_{N\Delta}^2 = 1 \quad (22)$$

$$G_{\Delta\Delta}^2 + \frac{1}{2} G_{\Delta N}^2 = 1 \quad (23)$$

$$G_{\Delta N} G_{NN} - 5 G_{\Delta\Delta} G_{\Delta N} = 0 \quad (24)$$

They are solved by

$$(G_{NN}, G_{N\Delta}, G_{\Delta\Delta}) = \left(\frac{5}{3}, \frac{4}{3}, 1\right).$$

^{*)} Obviously, these are the Adler-Weisberg sum rules for πN , $\pi\Delta$ and $\pi N \rightarrow \pi\Delta$ scattering.

With these values $\mathcal{A}_s^{(-)}$ becomes

$$\mathcal{A}_t^{(-)} = -\frac{20}{12 F_\pi^2} \left\{ \begin{array}{c} 1 \\ 1/4 m^2 \\ -1/4 m^2 \\ 1/m \end{array} \right\} \quad (25)$$

Comparing with (17) we obtain our final result

$$C_3 m = \frac{5}{13}, \quad C_4 m^2 = C_5 m^2 = \frac{1}{4} \frac{5}{13} \quad (26)$$

The sum rule for $B_1^{(-)}$ cannot be fulfilled since the nucleon does not contribute and, at least, one resonance of spin $\geq 3/2$ is needed to obtain the required zero. The $\Delta N \mathfrak{g}$ coupling constants (26) can be compared with experiment if one assumes that the electromagnetic current of the $N \Delta$ transition is dominated by a $\mathfrak{g}$ meson in the form

$$j_\mathfrak{g}^\mu = -\frac{m_\mathfrak{g}^2 + \square}{m_\mathfrak{g}^2} 2\gamma_3 j_{em}^\mu \quad (27)$$

In this case, our C 's become directly the Gourdin-Salin electromagnetic coupling constants. They have been measured to be ^{4),6)}

$$C_3 m \approx 2.3, \quad C_4 m^2 \approx C_5 m^2 \approx .27 \quad (28)$$

We see that the agreement with the theoretical value in (26) is as good as the approximation involved.

A lot more algebraic effort is necessary to lift the degeneracy of N and Δ and to include higher resonances. The discussion of such a calculation will be presented elsewhere.

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