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DISPERSIVE APPROACH TO OFF-MASS-SHELL $\pi\pi$ AMPLITUDES

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A B S T R A C T

It is shown that $\pi\pi$ amplitudes with one pion off its mass shell are determined by a set of linear integral equations in terms of physical amplitudes once off-shell absorptive parts are known in the inelastic region. We show how this can be used in practice to obtain off-shell amplitudes in the low energy region. The arbitrariness of off-shell extensions is discussed; it is suppressed in the soft pion limit owing to Adler's self-consistency condition. Applications of our method to Adler-Weisberger type relations, and to off-shell effects in Chew-Low extrapolation experiments are discussed.

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It has recently been shown that fixed momentum transfer dispersion relations can be used in order to express the implications of crossing on partial wave amplitudes in the physical region ^{1),2)}. In $\pi\pi$ scattering, owing to the identity of the three crossed channels, each partial wave amplitude is expressed as an integral over absorptive parts in the physical region and Roy has argued that, by making use of unitarity, such relations can also be considered as non-linear singular integral equations defining the elastic amplitude, once absorptive parts are given in the inelastic region ¹⁾. In this note we want to show that similar relations can be obtained in cases where the unitarity relation becomes linear - off-mass-shell amplitudes or weak production processes such as $\gamma\pi \rightarrow \pi\pi$, - and, therefore, result in linear integral equations defining these amplitudes in terms of the physical ones. We will concentrate on off-mass-shell $\pi\pi$ amplitudes; we have in mind two important applications, a correct evaluation of the soft pion "cross-sections" which enter Adler-Weisberger relations ^{3),4)}, and the estimation of off-shell dependence of $\pi\pi$ amplitudes in Chew-Low extrapolation experiments.

In what follows, we shall consider the case of neutral pions, $\pi^0\pi^0$, the extension of the method to the general case with isospin is straightforward ^{1),2)}. We set the physical pion mass equal to unity, the $\pi^0\pi^0$ total amplitude with one pion off its mass shell is represented on Fig. 1; we denote it by $F(s,t,m^2)$, $A(s,t,m^2)$ is its s channel absorptive part and $f_\ell(s,m^2)$ its partial wave projections. Since only one pion is off-shell, we have Bose symmetry in all three channels and hence full crossing symmetry of $F(s,t,m^2)$ under s, t, u exchange. The relation between the scattering angle θ in the s channel and t is given by

$$\cos \theta = (2t + s - m^2 - 3)/4pq \quad (1)$$

where the initial and final momenta p and q are

$$p = \left\{ [s^2 - 2s(m^2+1) + (m^2-1)^2] / 4s \right\}^{1/2} \quad (2a)$$

$$q = [(s-4)/4]^{1/2} \quad (2b)$$

Our basic assumptions are the following.

- a) We assume Mandelstam analyticity ⁵⁾, i.e., that $F(s,t,m^2)$ has the same analyticity properties as Feynman graphs an example of which is given on Fig. 2.
- b) We assume that the number of subtractions is the same as for the on-shell amplitude; two subtractions are sufficient.

For clarity, we shall first derive our equations in the unsubtracted case and then show what modifications are due to subtractions.

A. Unsubtracted case

The fixed t dispersion relations can be written as

$$F(s, t, m^2) = \frac{1}{\pi} \int_4^\infty ds' \left[\frac{1}{s' - s} + \frac{1}{s' - u} \right] A(s', t, m^2)$$

$$u = 3 + m^2 - s - t \quad (3)$$

and, owing to t - u symmetry, the partial wave amplitudes are obtained as

$$f_\ell(s, m^2) = \int_0^1 d(\cos \theta) P_\ell(\cos \theta) F(s, t, m^2) \quad (4)$$

In Eq. (3) we can replace $A(s', t, m^2)$ by its Legendre expansion

$$A(s', t, m^2) = \sum_{\ell' \text{ even}} (2\ell' + 1) \operatorname{Im} f_{\ell'}(s', m^2) P_{\ell'}(\cos \theta') \quad (5)$$

where $\cos \theta'$ is obtained by replacing s by s' in Eqs. (1) and (2). The convergence domain of this expansion in the $\cos \theta'$ plane is an ellipse with foci at ± 1 and whose semi-major axis is obtained by replacing t in Eq. (1) by its value at the boundary of the double spectral region. This boundary is easily obtained, from the Feynman graph (2), as the envelope of the two curves

$$t = 4 + \frac{8(9 - m^2)}{s' - 16} \quad t = 16 + \frac{8(9 - m^2)}{s' - 4}$$

$$(6)$$

We denote the resulting ellipse by $E_L(s')$. In Eqs. (3)-(5) $\cos \theta'$ is related to $\cos \theta$ by

$$\cos \theta' = \frac{p' q}{p' q'} \cos \theta + \frac{s' - s}{4 p' q'} \quad (7)$$

If it lies in $E_L(s')$, we obtain the following well-defined relation among partial wave amplitudes

$$f_\ell(s, m^2) = \frac{1}{\pi} \int_4^\infty ds' \sum_{\ell'=0}^\infty K_\ell^{\ell'}(s, s', m^2) \operatorname{Im} f_{\ell'}(s', m^2) \quad (8)$$

where

$$K_{\ell}^{\ell'}(s, s', m^2) = (2\ell' + 1) \int_0^1 d(\cos\theta) P_{\ell}(\cos\theta) P_{\ell'}(\cos\theta') \left[\frac{1}{s' - s} + \frac{1}{s' - u} \right] \quad (9)$$

The convergence of the ℓ' summation in (8) is as follows.

- a) We must have $m^2 < 9$, as expected.
- b) For $m^2 \leq 1$, $s \geq 4$, the phase space factor in Eq. (7) is real and we obtain from the envelope of ellipses $E_L(s')$

$$s \leq 37 - m^2 + 4 \sqrt{8(9 - m^2)} \quad (10)$$

In the soft pion limit $m^2 = 0$, we have $s \leq 37 + 24\sqrt{2} \sim 71$.

- c) For $m^2 \leq 1$, $s < 4$, pq becomes complex and blows up at $s = 0$. Therefore, there is a region around $s = 0$ where the ℓ' summation diverges. In what follows, we shall only be interested in $s < 4$ in connection with Adler's self consistency condition, and it is easy to check that for $m^2 = 0$, Eq. (8) is well defined at $s = 1$.

We now come to our most important observation. In the elastic region $4 \leq s \leq 16$ one can show on general grounds that the unitarity relation for off-shell partial wave amplitudes is a linear relation with the on-shell amplitudes

$$Im f_{\ell}(s, m^2) = \sqrt{\frac{s-4}{s}} f_{\ell}(s, m^2) f_{\ell}^*(s, 1) \quad (11)$$

where $f_{\ell}(s, 1)$ is a physical partial wave amplitude. Upon insertion of Eq. (11) into Eq. (8), we can formulate the following exact statement.

Once the off-shell absorptive parts are known in the inelastic region, the off-shell amplitude in the elastic region $4 \leq s \leq 16$ is determined by a linear integral equation in terms of the physical amplitude.

This integral equation is of the Omnès type ⁶⁾. We note that the singular part of the kernel comes from the $(s' - s)^{-1}$ term in Eq. (9), it is only present for $\ell' = \ell$. Our equation is slightly more complicated than the usual Omnès equation since the kernel also contains a regular part; it can nevertheless be cast in a Fredholm form by standard techniques ⁶⁾. We insist that since the contribution of left-hand singularities is given explicitly in terms of absorptive parts in Eq. (8), the unitarity relation (11) is used in all crossed channels, and the off-shell extension of left-hand cut contributions is treated consistently. For practical purposes, we will argue later on that Eq. (11) can be used up to 1 GeV since inelasticity seems weak below

this energy experimentally; above 1 GeV we may use $\text{Im} f_\ell(s, m^2) \sim \text{Im} f_\ell(s, 1)$ assuming external mass differences can be neglected at high energies. Such approximations allow to compute low energy off-shell amplitudes directly in terms of physical quantities.

B. Subtracted case

Turning to a more realistic situation, we write the twice subtracted fixed t dispersion relations as

$$F(s, t, m^2) = g(t, m^2) + \frac{1}{\pi} \int_4^\infty \frac{ds'}{(s' - s_0)^2} \left[\frac{(s - s_0)^2}{s' - s} + \frac{(t - s_0)^2}{s' - t} \right] A(s', t, m^2) \quad (12)$$

where s_0 is some given subtraction point, which may depend on m^2 . The t dependence of the subtraction function $g(t, m^2)$ can be explicitated as follows: write Eq. (12) for $s = s_0$ and use the crossing symmetry $F(s_0, t, m^2) = F(t, s_0, m^2)$ to get

$$\begin{aligned} g(t, m^2) = & \varphi(m^2) + \frac{1}{\pi} \int_4^\infty \frac{ds'}{(s' - s_0)^2} \left[\frac{(t - s_0)^2}{s' - t} + \frac{(3 + m^2 - 2s_0 - t)^2}{s' + s_0 + t - 3 - m^2} \right] A(s', s_0, m^2) \\ & - \frac{1}{\pi} \int_4^\infty \frac{ds'}{(s' - s_0)^2} \frac{(3 + m^2 - 2s_0 - t)^2}{(s' + s_0 + t - 3 - m^2)} A(s', t, m^2) \end{aligned} \quad (13)$$

where

$$\varphi(m^2) = g(s_0, m^2) \quad (14)$$

The function $\varphi(m^2)$ is arbitrary except that its on-shell value $\varphi(1)$ is known and can be related to the s wave scattering length ^{1), 2)}. This is equivalent to the arbitrariness of off-shell extensions in quantum field theory, where only the physical S matrix elements are independent of the choice of the interpolating field ⁷⁾.

Inserting Eq. (13) into (12) and using the same arguments as above we obtain

$$f_\ell(s, m^2) = \varphi(m^2) \delta_{\ell 0} + \frac{1}{\pi} \int_4^\infty ds' \sum_{\ell'} \tilde{K}_\ell^{\ell'}(s, s', m^2) \text{Im} f_{\ell'}(s', m^2) \quad (15)$$

We note that the only restriction we place on s_0 is that the partial wave expansion of $A(s', s_0, m^2)$ in (13) converges. The kernels $\tilde{K}_\ell^{\ell'}(s, s', m^2)$ in Eq. (15) are slightly more complicated than Eq. (9). The convergence region of Eq. (15) is the same as for Eq. (8). We see also that by using the unitarity relation (11), we convert these equations into linear integral equations for $f_\ell(s, m^2)$, which, however, will depend on the arbitrary quantity $\varphi(m^2)$.

C. Adler's self consistency condition

The arbitrariness of $\varphi(m^2)$ can be reduced by making use of Adler's condition ⁸⁾ which states that the $\pi\pi$ amplitude vanishes at $s=t=u=1$, i.e., $F(1,1,0)=0$. This determines $\varphi(0)$, and, therefore, off-shell amplitudes with one massless pion are unambiguous. We note that owing to analyticity properties in t , the threshold behaviour of off-shell partial wave amplitudes can be established ⁹⁾; we have for $0 \leq m^2 \leq 1$

$$f_\ell(s, m^2) \underset{s \rightarrow (m+1)^2}{\sim} (pq)^\ell \quad (16)$$

Hence, at Adler's point, $s=t=u=1$, all partial waves except the s wave vanish because of kinematical factors. Adler's condition, therefore, constrains the s wave only. In the above unsubtracted case, this would result in a constraint on physical amplitudes, through the integral equation for $f_0(s, 0)$.

D. Applications

The simplest way to use Eq. (15) in practice is as follows: a) since inelasticity seems weak in the $\pi\pi$ system below 1 GeV, one may use elastic unitarity (11) up to that energy; b) above, we may approximate off-shell absorptive parts by physical ones [note that in the subtracted case, high energy contributions are anyhow very weak owing to rapid convergence of s' integration in (15), as pointed out in Ref. 2)]. Thus we can compute $f_\ell(s, m^2)$ in terms of physical quantities if we have a prescription for $\varphi(m^2)$.

1) Adler's sum rule

Adler's sum rule for $\pi\pi$ scattering is ³⁾

$$\frac{1}{f_\pi^2} = \frac{1}{\pi} \int_4^\infty \frac{ds}{s-1} [\sigma_0^{+-}(s) - \sigma_0^{++}(s)] \quad (17)$$

where $f_\pi \sim \frac{2}{5}m_\pi$ is the pion decay constant, and σ_0^{+-} (σ_0^{++}) the $\pi^+\pi^-$ ($\pi^+\pi^+$) total cross-section with one massless pion. In the elastic region, these cross-sections are simply related to the moduli squared of amplitudes with one zero mass pion [see Fig. 3)]. We recall that a) if s wave $\pi\pi$ scattering lengths are small, as

suggested by current algebra, one needs a broad $I=0$ s wave resonance in order to saturate the sum rule (17), and b) asymptotic contributions are not negligible in (17) and comparable to the ρ resonance contribution¹⁰⁾. Our method allows an exact calculation of the low energy contribution in Eq. (17), and therefore provides an accurate way to relate current algebra parameters and physical quantities, by comparison with the on-shell analogue of Eq. (17)

$$2a_0 - 5a_2 = \frac{3}{4\pi^2} \int_4^\infty \frac{ds}{\sqrt{s(s-4)}} [\sigma^{+-}(s) - \sigma^{++}(s)] \quad (18)$$

We think that the combination of $\pi\pi$ scattering lengths $2a_0 - 5a_2$ should be somewhat larger than Weinberg's value $2a_0 - 5a_2 = 3/(4\pi f_\pi^2) \sim 0.54 m_\pi^{-1}$. The application of our method to the Adler-Weisberger relation in πN is in principle feasible²⁾ although much more difficult technically.

2) Off-shell behaviour of $\pi\pi$ amplitudes

For $m^2 \neq 0$ we must make an assumption about $\varphi(m^2)$. The smoothness assumption usually made in conjunction with PCAC consists in choosing a linear form

$$\varphi(m^2) = a + b m^2 \quad (19)$$

where a and b are known since we know $\varphi(1)$ and $\varphi(0)$. Under this assumption we can determine the off-shell dependence of $\pi\pi$ amplitudes entering in Chew-Low extrapolation experiments. The problem of Adler zeros in momentum transfer in such reactions has been considered in recent years¹¹⁾, but essentially in the Weinberg linear approximation, and our method provides a way of implementing unitarity in such calculations in a consistent way. We insist on the fact that Eq. (19) is an approximation which can only be reasonable for small values of m^2 ; it is unambiguous at $m^2 = 0$ only.

To conclude, we believe that the present discussion clarifies the problem of off-mass-shell pions. Our main result is that it is possible to compute off-shell amplitudes directly in terms of on-shell ones, i.e., experimental quantities. In the zero mass pion case this is done unambiguously; otherwise we must rely on the smoothness assumption.

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$$\begin{aligned}
 p_1^2 &= m^2 \\
 p_2^2 &= p_3^2 = p_4^2 = 1 \\
 s &= (p_1 + p_2)^2 \\
 t &= (p_1 - p_3)^2 \\
 u &= (p_1 - p_4)^2
 \end{aligned}$$

FIG.1

FIG. 2

FIG. 3