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ZEROS AND SPECTRUM OF INFINITELY
SUPERCONVERGENT AMPLITUDES

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A B S T R A C T

We study a class of meromorphic superconvergent amplitudes with the Veneziano formula as a particular case, whose pattern of real zeros we relate to the spectrum of intermediate states.

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From the success of the duality concept in finite energy sum rules ¹⁾ together with the experimental trend of resonances belonging to linearly rising Regge trajectories, it has been more and more widely accepted in current literature that suitable meromorphic functions could be reasonable approximations to physical scattering amplitudes. Moreover, the possibility of neglecting exotic states has led people to impose infinite superconvergence relations when a channel with exotic quantum numbers is involved ²⁾. All these requirements are indeed fulfilled by the Veneziano formula ³⁾, that is the most popular and, perhaps, simplest solution. However, it should be useful to get more general expressions both for phenomenological purposes where often a larger flexibility is requested and for a thorough understanding of the problem.

Some works in this direction have already appeared ⁴⁾: the scattering amplitude can be written as an absolutely convergent series of Veneziano terms

$$A(s, t) = \sum_{n=1}^{\infty} \pi_n(u) B(n-\alpha_s, n-\alpha_t) \quad (1)$$

π_n being a polynomial of degree at most n . But this representation is too general for practical purposes: in particular, it is met by functions that neither do exhibit a correct spectrum of intermediate states, nor are Regge-behaved.

In this letter, after sketching a derivation of Eq. (1) together with the corresponding expression for the continued partial wave amplitude in the exotic channel, we face the task of constructing superconvergent amplitudes with given patterns of real zeros and a reasonable spectrum of intermediate states. This seems to become an interesting subject in strong interactions phenomenology ⁵⁾.

We shall expound the connection between real zeros and resonances in a class of amplitudes having the Veneziano formula as a particular case.

In the following we are confined to the elastic pion-pion scattering amplitude with isospin in the u channel $I_u = 2$, but most of our conclusions will be true for a general invariant amplitude with crossing symmetry between the s and t channels and exotic quantum numbers in the u one. We assume the pion with a vanishing mass, our unit of mass being such that $m_\rho^2 = \frac{1}{2}$; we also define $2\nu = s-t$. The degenerate ρ -f trajectory will be $\alpha_s = s + \frac{1}{2}$.

From the results in Refs. 2) and 4), the amplitude can be written as a Mellin transform

$$A(\nu, u) = \int_0^\infty z^{\frac{u-1}{2} - \nu} \psi(z, u) dz \quad (2)$$

where $\psi(z, u)$ is the generating function of the residues. We can further factorize $\psi(z, u)$ in the following way

$$\psi(z, u) = (1+z)^{-1-u} H(z, u) \quad (3)$$

and, owing to crossing symmetry

$$H(z, u) = H\left(\frac{1}{z}, u\right) \quad (4)$$

H turns out to be analytic inside and outside the open unit circle as a function of z ; for $|z| = 1$ it has a priori an arbitrary number and kind of singularities, but at $z = 1$ it is continuous, although not necessarily analytic ⁴⁾.

We normalize our amplitude by putting

$$H(0, u) = u \quad (5)$$

according to the Adler self-consistency condition ⁶⁾. The definition

$$h \left[\frac{4z}{(1+z)^2}, u \right] = H(z, u) \quad (6)$$

and the absolutely and uniformly convergent expansion in the closed unit disc $\left| 4z/(1+z)^2 \right| \leq 1$ ⁴⁾

$$h \left[\frac{4z}{(1+z)^2}, u \right] = \sum_{n=1}^{\infty} \pi_n(u) \left[\frac{z}{(1+z)^2} \right]^{n-1} \quad (7)$$

when introduced into Eq. (2), lead us to Eq. (1).

We can also evaluate the continued partial wave amplitude in the exotic channel

$$a_2^+(l, u) = \frac{1}{2\pi} \left(\frac{4}{u} \right)^{l+1} \int_0^{\infty} Q_l \left(1 + \frac{2x}{u} \right) dx \left[A_s(u, x) + A_t(u, x) \right] \quad (8)$$

$A_{s,t}$ being the s, t absorptive parts of the $I_u = 2$ invariant amplitude. From Eqs. (1) and (7) and the representation

$$Q_l(z) = \sqrt{\frac{\pi}{2}} \int_0^{\infty} \frac{dx}{\sqrt{x}} I_{l+1/2}(x) e^{-xz} \quad (9)$$

we get the general expression

$$a_2^+(l, u) = - \sqrt{\frac{\pi}{2}} \left(\frac{4}{u} \right)^{l+1} \int_0^{\infty} \frac{dx}{\sqrt{x}} I_{l+1/2}(x) \cdot$$

$$\cdot \left[2 \cosh \frac{x}{u} \right]^{-u-1} h \left[\left(\cosh \frac{x}{u} \right)^{-2}, u \right] \quad (10)$$

As a by-product we see that suitable choices of h can avoid wrong signature nonsense poles; indeed, the requirement that the function $h(w,u)$ vanishes at $w = 1$ together with its derivatives of all orders, does not imply $h \equiv 0$, since h needs not be analytic there ^{4),7)}.

We come now to our point, the connection between zeros and poles of the invariant amplitude. If we first impose to the general expression (1) a grid of linear zeros at non-positive integer values of u , we find the Veneziano formula as unique solution. Indeed, from the condition

$$A(\gamma, -n) \equiv 0 \quad n = 0, 1, \dots \quad (11)$$

identically in γ , and from Eq. (1) we get

$$\begin{aligned} \pi_n(-2j) &= 0 & n \geq j+1 \\ \pi_n(-2j-1) &= 0 & n \geq j+2 \end{aligned} \quad \begin{aligned} j &= 0, 1, \dots \end{aligned} \quad (12)$$

and, since π_n is a polynomial of degree at most n , it follows

$$\pi_1(u) = u \quad \pi_n(u) = 0 \quad n \geq 2 \quad (13)$$

Let us now generalize this case by imposing an alternating grid of linear zeros

$$A(\nu, -n) \equiv 0 \quad n = 0, 2, \dots \quad (14)$$

according to phenomenological suggestions^{8),9)}. These zeros are of course not enough to avoid the double poles; the function we shall consider in a moment will automatically exhibit further zeros, not controlled by Eq. (14), on which, in principle, information could be obtained from phenomenology. Here, we shall pursue the opposite attitude and try to get them from conditions on the spectrum of the amplitude. Equation (14), when imposed onto expression (1), entails

$$\pi_n(-2j) = 0 \quad n \geq j+1 \quad j = 0, 1, \dots \quad (15)$$

that is

$$\pi_n(u) = \beta_n \frac{\Gamma\left(n + \frac{u}{2}\right)}{\Gamma\left(\frac{u}{2}\right)} \quad (16)$$

the arbitrariness at this stage being reflected into the real normalization constants β_n . On simplicity grounds, in this letter we confine ourselves in discussing the choice

$$\beta_n = 2 \frac{(4\lambda)^{n-1}}{(n-1)!} \quad (17)$$

leading to the amplitude

$$A(s, t, \lambda) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} {}_2F_1\left(1-\alpha_s, 1-\alpha_t; 1-\frac{\alpha_s+\alpha_t}{2}; \lambda\right) \quad (18)$$

The Π_n generating function [see Eq. (7)] turns out to be

$$h(w, u, \lambda) = u (1 - \lambda w)^{-1 - \frac{u}{2}} \quad (19)$$

The fixed u superconvergence requirement imposes

$$0 \leq \lambda < 1 \quad (20)$$

since h must be analytic in the cut w plane from $w_0 \geq 1$ to infinity and continuous at $w = 1$. We notice in particular that, when $\lambda = 1$, exotic poles are produced through the divergence of the series and the amplitude one gets

$$A(s, t, 1) = \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)} \cdot \frac{\frac{\alpha_s - \alpha_t}{2 - \alpha_s - \alpha_t} \sin \frac{\pi}{2} (\alpha_s - \alpha_t)}{\sin \frac{\pi}{2} (\alpha_s + \alpha_t)} \quad (21)$$

is of the kind considered in Ref. 8) *). Moreover, since we are confined to an elastic case, to get rid of ghosts on the mother trajectory, λ must be less than $\frac{1}{2}$. In the limiting case $\lambda = \frac{1}{2}$, the amplitude

$$A(s, t, \frac{1}{2}) = \frac{\Gamma\left(1 - \frac{\alpha_s}{2}\right) \Gamma\left(\frac{1 - \alpha_t}{2}\right)}{\Gamma\left(\frac{1 - \alpha_s - \alpha_t}{2}\right)} + \frac{\Gamma\left(1 - \frac{\alpha_t}{2}\right) \Gamma\left(\frac{1 - \alpha_s}{2}\right)}{\Gamma\left(\frac{1 - \alpha_s - \alpha_t}{2}\right)} \quad (22)$$

does not reggeize correctly, the higher spin states being decoupled. To sharpen further the range of λ one should probably take care of the positivity condition at the daughter level, if one believes in the daughter structure as predicted in a generalized Veneziano model. The precise paths of the zeros in Eq. (18) can only be evaluated by a computer. A qualitative picture can, however, be obtained in a simple way and is reported in Figs. 1, 2, and 3.

Figure 1 shows the well-known pattern of zeros of the Veneziano amplitude, Eq. (18), with $\lambda = 0$. Figure 2 corresponds to the case $\lambda = \frac{1}{2}$: we notice that the odd zeros are strongly distorted far from the s - t double spectral region, where they are forced to cross the double poles. But, in this region, the situation is not essentially different from the Veneziano case. This could be important for phenomenology: would, in fact, drastic

*) The elastic pion amplitude in Eq. (21) exhibits also a tachyon in the exotic channel at $u = -1$. However, if it is continued in one of the external masses and is used to describe the annihilation $\bar{p}n \rightarrow 3\pi$ at rest, the kinematics give approximately $s+t+u = 3$ and the first exotic pole occurs at $u = 2$ and $J = 0$. This amplitude with linear alternating zeros (see Fig. 3) should present the advantage, with respect to the one in Ref. 8), of predicting the annihilation through a 0^- state; the presence, however, of an exotic state with mass around 1.5 GeV and $J = 0$ is difficult to be accepted.

variations in the positions of the zeros be requested, some of the conditions we imposed (the absence of exotic states for instance) should be released. We report in Fig. 3 the case with $\lambda = 1$, just to show the changes one could get.

To summarize, we have studied a class of meromorphic superconvergent amplitudes with a reasonable spectrum of intermediate states and different patterns of zeros. We are aware that the problem is far from being exhausted and hope this note will stimulate further work on this subject.

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R E F E R E N C E S

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FIGURE CAPTIONS

Figure 1 : Zeros (dashed lines) and poles (solid lines) of the elastic pion-pion Veneziano amplitude ($\lambda = 0$).

Figure 2 : Zeros (dashed lines) and poles (solid lines) of the amplitude with $\lambda = \frac{1}{2}$.

Figure 3 : Zeros (dashed lines) and poles (solid lines) of the amplitude with $\lambda = 1$.

FIG1

FIG.2

FIG. 3