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ON TWO-PARTICLE CORRELATIONS IN HIGH ENERGY

PRODUCTION EXPERIMENTS

A. Bassetto, M. Toller
CERN - Geneva

and

L. Sertorio
Istituto Nazionale di Fisica Nucleare
Sezione di Roma, Roma
and
CERN - Geneva

A B S T R A C T

One-particle and two-particle inclusive distributions are considered in the light of future production experiments at CERN Intersecting Storage Rings machine. We discuss the general properties of those models which are based on factorized approximations to the production amplitudes and are in agreement with some general hypotheses like the Feynman scaling law. First the role of phase space arguments is clarified and the importance of detecting correlations in high energy production experiments as a hint for understanding the underlying dynamics is emphasized. Then two peculiar (and, in a sense, complementary) kinds of correlation in two-particle inclusive distributions are examined: the one coming from multiperipheral dynamics with Lorentz pole exchanges, the other due to "direct channel" resonances. Some general properties of inclusive distributions are expressed in terms of quantities which are suitable for direct measurement.

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1. INTRODUCTION

We are concerned in this paper with a discussion on hadron interactions with restriction to a kind of problems as those which can be investigated by means of a machine like the CERN intersecting storage rings. These problems can be divided into two classes: the study of elastic and quasi-elastic processes and the study of many-particle production reactions. Though these two kind of phenomena are connected by unitarity, their treatment requires rather different theoretical concepts. We limit our investigation to many-particle production.

In the situation we are considering, where two hadrons collide with a c.m.s. energy of the order of 50 GeV, an average yield of secondary particles around 15 is expected. This feature gives rise to severe difficulties, both of a practical and conceptual nature, in the kinematical analysis of the events, and suggests, at least as a first step, to perform inclusive measurements ¹⁾, in which only a few particles are detected. The inclusive distribution functions have also the advantage of being directly connected with the simplest results of many dynamical models. Of course, by increasing the number of the observed particles one can gradually approach the fully detailed information coming from exclusive experiments.

We base our treatment on the one and two-particle inclusive distributions. As we shall see, the one-particle distribution is connected mainly with very general properties of the interaction. Our main purpose is to stress that deeper and at present unknown dynamical effects can be detected by means of two-particle inclusive measurements. We shall also see that, if certain general ideas are correct, considerable simplifications are expected in the pionization region, where the c.m.s. momenta of the observed particles are a small fraction of their maximum value. These simplifications would facilitate both the experiments and their theoretical interpretation. We remark that a colliding beam machine is particularly suited for measurements in this kinematical region, as the pionization particles are produced well separated in angle and with small momenta.

The general concepts ²⁾, which form the basis of any treatment of the above-mentioned phenomena, are suggested by accelerator experiments at lower energies and by cosmic ray measurements, which have unfortunately a poor statistics. Two features are well established: the average transverse momentum is small and nearly energy independent and the total average multiplicity is a slowly increasing function of the total energy. These features are usually associated with the idea of peripheralism.

Some other general concepts are less clearly established by the existing data ^{3),4)}, but, if confirmed by future experiments, could complete in an essential way the picture provided by the two features mentioned above. We are thinking of the two related hypotheses of Feynman scaling ¹⁾ and limiting fragmentation ⁵⁾. For a general review of these subjects, we refer to the recent paper by G. Ranft and J. Ranft ⁶⁾. Still keeping in mind that the future experiments could disprove these assumptions, we use them as one of the starting points of our considerations.

We remark that all these general hypotheses can be formulated in terms of the one-particle inclusive distribution alone (besides the total cross-section) and provide by themselves a rough description of its shape. On the other hand, most sufficiently flexible dynamical models can incorporate these general requirements. It follows that, in order to discriminate between models and to reach a deeper dynamical insight, we have to deal with at least two-particle inclusive distributions.

In order to reach some indication of what is expected to occur in future experiments, one has to rely on the existing theoretical models. In this paper we shall be concerned only with models which are based on some explicit approximated expression for the many-particle transition matrix element. The most ambitious approaches of this kind are the generalized Veneziano models ⁷⁾⁻⁹⁾. For phenomenological purposes, at least at the present stage, it may be preferable to use more flexible and simple models. We consider in

detail those models which are based on an approximation of the transition matrix element by means of a product of functions each one depending on a small number of kinematical variables. In Section 3 we shall distinguish these models according to the nature of these kinematical variables into P factorized models, whose prototype is the uncorrelated jet model ¹⁰⁾, and Q factorized models, that are essentially the multiperipheral models ¹¹⁾⁻¹³⁾.

From a theoretical point of view, the Q factorized models are more satisfactory, since they can embody in a natural way some general principles, as crossing and analyticity. Nevertheless, in Section 4 we discuss in some detail an uncorrelated jet model. As this model, by definition, does not contain any dynamical correlation among final particles, it is suitable for the analysis of the kinematical correlations due to the total four-momentum conservation ¹⁴⁾. With regard to one-particle distributions, one realizes that the effect of four-momentum conservation is negligible in the pionization region. In the fragmentation region, where the longitudinal momentum of the observed particle is a relevant fraction of its maximum value, this effect is probably the most relevant one. However, when we consider the inclusive distribution for two particles lying in two different fragmentation regions, we show that the four-momentum conservation in the high energy limit, does not induce correlations between the two particles. A strong correlation arises when the two particles are in the same fragmentation region. These considerations explain why the pionization region is convenient for the study of genuine dynamical correlations.

We notice that the strong decrease of the amplitude for large transverse momenta has to be assumed as an input in both the P factorized and the Q factorized models. The Feynman scaling law and the logarithmic increase of the average multiplicity have also to be introduced as an input in the P factorized models, but, on the contrary, they are nearly unavoidable consequences of the Q factorized models, as shown by ABFST ^{11),12)} in their original papers on multiperipherism. As shown in Section 6, the multiperipheral

dynamics suggests, through the concept of Regge pole, some general non-trivial properties of the two and more-particle inclusive distributions, which generalize in a natural way the Feynman scaling law.

We think that to verify these general features should be the first task of high energy production experiments. If confirmed, these laws could provide a powerful framework for the search and the understanding of more specific features.

The fine structure effects are expected to be due to nearby singularities both in the sub-energies (resonances) and in the momentum transfers. Notice that these singularities generate also, through the integration on the unobserved particles, the Regge behaviour as a collective effect. We discuss in Section 7 the effect of resonances on the two-particle distribution in the pionization region. A simple estimate shows that the density of the produced particles in momentum space is so large that we expect important interference effects between resonances and background and also overlap between resonances in different sub-energies. In this situation the most promising approaches seem to be the ones based on a consistent use of the duality concepts. An approach of this kind, which is beyond the aims of the present paper, will be needed as soon as precise and detailed two and more particle distributions are available.

2. NOTATIONS

We consider the reaction

where $A, B, C_1, \dots, C_\gamma$ represent particles of different kinds. We indicate by $[n]$ the sequence $n_1, \dots, n_\gamma$. The cross-section for this reaction is given by

$$\begin{aligned} \sigma_{[n]}(s) &= (2\pi)^{4-3n} \frac{1}{2} \left[T(s, M_A^2, M_B^2) \right]^{-1/2} \cdot \\ &\cdot \prod_{i=1}^{\gamma} (n_i!)^{-1} \int \delta^4(P_A + P_B - \sum_{i=1}^{\gamma} P_{\tau i}) \cdot \\ &\cdot \left| M_{[n]}(P_1, \dots, P_{\gamma n_\gamma}, P_A, P_B) \right|^2 \prod_{i=1}^{\gamma} \frac{d^3 P_{\tau i}}{2 P_{\tau i}^0}. \end{aligned} \quad (2.2)$$

We adopt the usual notations

$$\begin{cases} s = (P_A + P_B)^2, \\ P_\alpha^0 = [P_\alpha^2 + M_\alpha^2]^{1/2}, \end{cases} \quad (2.3)$$

$$T(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac, \quad (2.4)$$

$$n = \sum_{i=1}^{\gamma} n_i. \quad (2.5)$$

We define the one-particle differential cross-section

$$d^3 \sigma_{[n]}^\alpha = F_{[n]}^\alpha(P_\alpha, P_A, P_B) \frac{d^3 P_\alpha}{2 P_\alpha^0} \quad (2.6)$$

for finding a particle of the kind α with momentum contained in the infinitesimal volume $d^3 \underline{p}_\alpha$. The connection with the invariant amplitude M is given by

$$F_{[n]}^\alpha(p_\alpha, p_A, p_B) = \frac{1}{2} \left[T(s, M_A^2, M_B^2) \right]^{-1/2} \quad (2.7)$$

$$\cdot G_{[n]}^\alpha(p_\alpha, p_A, p_B) ,$$

$$G_{[n]}^\alpha(p_{\alpha_1}, p_A, p_B) = (2\pi)^{4-3n} n_\alpha \prod_{i=1}^{\gamma} (n_i!)^{-1} .$$

$$\cdot \int \delta^4(p_A + p_B - \sum_{i=1}^{\gamma} p_{\alpha_i}) \left| M_{[n]}(p_1, \dots, p_{\gamma n_\gamma}, p_A, p_B) \right|^2 \prod_{i=1}^{\gamma} \frac{d^3 p_{\alpha_i}}{2 p_{\alpha_i}^0} , \quad (2.8)$$

where the product $\prod_{i=1}^{\gamma}$ is extended to all the final momenta, except the momentum p_{α_1} of the observed particle. We put $F_{[n]}^\alpha = 0$ when $n_\alpha = 0$.

The total production cross-section is given by

$$\sigma_I(s) = \sum_{[n]} \sigma_{[n]}(s) , \quad (2.9)$$

where the sum is extended to all the final multiplicities $[n]$ with $n \geq 3$ permitted by the conservation laws. Throughout this paper we do not treat elastic and quasi-elastic processes, since their study involves quite different concepts from the ones we develop here.

The inclusive one-particle distribution function is given by

$$F^\alpha(p_\alpha, p_A, p_B) = \sum_{[n]} F_{[n]}^\alpha(p_\alpha, p_A, p_B) , \quad (2.10)$$

where the sum has the same meaning as above. With this normalization we have

$$\int F^{\alpha}(P_{\alpha}, P_A, P_B) \frac{d^3 P_{\alpha}}{2 P_{\alpha}^0} = \sigma_I < n_{\alpha} > , \quad (2.11)$$

where $< n_{\alpha} >$ means the average number of produced particles of the kind α at the given total energy s , in inelastic events.

In the following we shall be concerned also with the inclusive distributions of two particles of the kinds α and β . The corresponding definitions are

$$d^6 \sigma_{[n]}^{\alpha\beta} = F_{[n]}^{\alpha\beta}(P_{\alpha}, P_{\beta}, P_A, P_B) \frac{d^3 P_{\alpha}}{2 P_{\alpha}^0} \frac{d^3 P_{\beta}}{2 P_{\beta}^0} , \quad (2.12)$$

$$F_{[n]}^{\alpha\beta}(P_{\alpha}, P_{\beta}, P_A, P_B) = \frac{1}{2} [T(s, M_A^2, M_B^2)]^{-1/2} \cdot C_{[n]}^{\alpha\beta}(P_{\alpha}, P_{\beta}, P_A, P_B) , \quad (2.13)$$

$$C_{[n]}^{\alpha\beta}(P_{\alpha_1}, P_{\beta_1}, P_A, P_B) = (2\pi)^{4-3n} n_{\alpha} n_{\beta} \prod_{\tau=1}^{\gamma} (n_{\tau}!)^{-1} \cdot$$

$$\int \delta^4(P_A + P_B - \sum_{\tau=1}^{\gamma} P_{\tau i}) \left| M_{[n]}(P_{11}, \dots, P_{\gamma n_{\gamma}}, P_A, P_B) \right|^2 \prod_{\tau=1}^{\gamma} \frac{d^3 P_{\tau i}}{2 P_{\tau i}^0} . \quad (2.14)$$

The meaning of $\prod_{\tau=1}^{\gamma}$ is clear. If $\alpha = \beta$ we have to perform in Eq. (2.14) the substitutions $P_{\beta_1} \rightarrow P_{\alpha_2}$, $n_{\alpha} n_{\beta} \rightarrow n_{\alpha} (n_{\alpha} - 1)$. The inclusive two-particle distribution is

$$F^{\alpha\beta}(P_{\alpha}, P_{\beta}, P_A, P_B) = \sum_{[n]} F_{[n]}^{\alpha\beta}(P_{\alpha}, P_{\beta}, P_A, P_B) . \quad (2.15)$$

The analogous of Eq. (2.11) is

$$\int F^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B) \frac{d^3 P_\alpha}{2 P_\alpha^0} \frac{d^3 P_\beta}{2 P_\beta^0} = \sigma_I \left[\langle n_\alpha n_\beta \rangle - \langle n_\alpha \rangle \delta_{\alpha\beta} \right]. \quad (2.16)$$

It is useful to introduce new variables by means of the formulae

$$\begin{cases} P_A = (M_A \cosh \xi_A, 0, 0, M_A \sinh \xi_A) \\ P_B = (M_B \cosh \xi_B, 0, 0, M_B \sinh \xi_B) \end{cases}, \quad (2.17)$$

$$P_\alpha = (\sqrt{p_\alpha^2 + M_\alpha^2} \cosh \xi_\alpha, p_\alpha \cos \varphi_\alpha, p_\alpha \sin \varphi_\alpha, \sqrt{p_\alpha^2 + M_\alpha^2} \sinh \xi_\alpha), \quad (2.18)$$

and similar equations for the other final momenta. The variables

$$p_\alpha = \sqrt{(P_\alpha^x)^2 + (P_\alpha^y)^2} = 2 \left[T(s, M_A^2, M_B^2) \right]^{-1/2} \left| \begin{pmatrix} (P_A, P_A) & (P_A, P_B) & (P_A, P_\alpha) \\ (P_B, P_A) & (P_B, P_B) & (P_B, P_\alpha) \\ (P_\alpha, P_A) & (P_\alpha, P_B) & (P_\alpha, P_\alpha) \end{pmatrix} \right|^{1/2} \quad (2.19)$$

are the transverse momenta, while the variables ξ_A , ξ_B and ξ_α are "longitudinal rapidities" ¹⁾.

The advantage of these variables is due to the fact that a boost along the z axis is obtained by adding the same constant to all the rapidities, while a rotation around the z axis results in the addition of a constant to all the angles φ_α . It follows that

all the physically relevant quantities depend only on differences of the kind

$$\begin{cases} \xi_{\alpha} - \xi_A = \xi_{\alpha A} \\ \xi_{\beta} - \xi_{\alpha} = \xi_{\beta\alpha} \\ \varphi_{\beta} - \varphi_{\alpha} = \varphi_{\beta\alpha} \end{cases} \quad (2.20)$$

In particular we have

$$\begin{cases} s = M_A^2 + M_B^2 + 2 M_A M_B \cosh \xi_{BA} \\ \xi_{BA} = \xi_B - \xi_A \end{cases} \quad (2.21)$$

The one-particle invariant volume element in momentum space in the new variables takes the form

$$\frac{d^3 p_{\alpha}}{2 p_{\alpha}^0} = \frac{1}{2} p_{\alpha} dp_{\alpha} d\varphi_{\alpha} d\xi_{\alpha} \quad (2.22)$$

If we indicate by ξ_{α}^* the parameter ξ_{α} measured in the centre-of-mass system, the kinematically allowed values of ξ_{α}^* and p_{α} are given by the inequality

$$2\sqrt{s} \sqrt{p_{\alpha}^2 + M_{\alpha}^2} \cosh \xi_{\alpha}^* \leq s - M^2 + M_{\alpha}^2 \quad (2.23)$$

where M is the sum of the masses of the unobserved final particles. For instance, if we measure pions in a proton-proton reaction, the minimum value of M is twice the nucleon mass. Comparing with Eq. (2.21) we see that for large s we have

$$|\xi_{\alpha}^*| \leq \frac{1}{2} \xi_{BA} + \log \sqrt{\frac{M_A M_B}{M_{\alpha}^2 + p_{\alpha}^2}} \quad (2.24)$$

As for large s we have in the centre-of-mass system

$$\begin{cases} \xi_B^* = \frac{1}{2} \xi_{BA} + \frac{1}{2} \log \frac{M_A}{M_B} \\ -\xi_A^* = \frac{1}{2} \xi_{BA} + \frac{1}{2} \log \frac{M_B}{M_A} \end{cases}, \quad (2.25)$$

we can write the invariant inequalities

$$\begin{cases} \xi_{\alpha A} \geq \log \frac{\sqrt{p_\alpha^2 + M_\alpha^2}}{M_A} \\ \xi_{B\alpha} \geq \log \frac{\sqrt{p_\alpha^2 + M_\alpha^2}}{M_B} \end{cases}. \quad (2.26)$$

If two particles are detected, the analogous of Eq. (2.23)

is

$$\begin{aligned} & M^2 + 2\sqrt{s} \left(\sqrt{p_\alpha^2 + M_\alpha^2} \cosh \xi_\alpha^* + \sqrt{p_\beta^2 + M_\beta^2} \cosh \xi_\beta^* \right) + \\ & + 2p_\alpha p_\beta \cos \varphi_{\beta\alpha} \leq s + M_\alpha^2 + M_\beta^2 + \\ & + 2\sqrt{p_\alpha^2 + M_\alpha^2} \sqrt{p_\beta^2 + M_\beta^2} \cosh \xi_{\beta\alpha} \end{aligned} \quad (2.27)$$

In the large s limit Eq. (2.27) takes the following factorized form (in the c.m.s.)

$$\begin{cases} \sqrt{p_\alpha^2 + M_\alpha^2} \exp \xi_\alpha^* + \sqrt{p_\beta^2 + M_\beta^2} \exp \xi_\beta^* = P_\alpha^0 + P_\alpha^z + P_\beta^0 + P_\beta^z \leq \sqrt{s}, \\ \sqrt{p_\alpha^2 + M_\alpha^2} \exp(-\xi_\alpha^*) + \sqrt{p_\beta^2 + M_\beta^2} \exp(-\xi_\beta^*) = P_\alpha^0 - P_\alpha^z + P_\beta^0 - P_\beta^z \leq \sqrt{s} \end{cases}. \quad (2.28)$$

The kinematically accessible region in the plane of the variables

ξ_α^* and ξ_β^* is given in Fig. 1.

For large total energy, the one-particle phase space defined by Eq. (2.23) is usually divided into three regions:

- a) the pionization region where p_α is small (say smaller than 1 GeV/c) and both $\xi_{\alpha A}$ and $\xi_{B\alpha}$ are large (say larger than some units);
- b) the two fragmentation regions where p_α is small and $\xi_{\alpha A}$ (respectively $\xi_{B\alpha}$) is not large;
- c) the large transverse momentum region where p_α is large.

The first two regions are much more populated than the third one and we shall restrict to them in our investigations. A fragmentation region differs from the pionization region because it is expected to be sensitive to the properties of the corresponding incoming particle and to the constraints of four-momentum conservation.

Of course, one has to keep in mind that the separation between these regions is not sharp and uniquely defined. In fact, the pionization region is characterized by the validity of certain asymptotic approximations and, therefore, its size depends on the required accuracy.

From the definition we see that the length of the pionization region in the rapidity scale is some units smaller than ξ_{BA} . This parameter is given in the Table for some values of the incident momentum. We see that at the energies of the available experimental data one can hardly speak of a pionization region.

It is customary to introduce the ratio (we work in the c.m.s.)

$$X_\alpha = \frac{2 P_\alpha^z}{\sqrt{s}} = 2 \sqrt{\frac{P_\alpha^2 + M_\alpha^2}{s}} \sinh \xi_\alpha^* \quad (2.29)$$

For large $|\xi_\alpha^*|$ we have

$$\begin{cases} x_\alpha \simeq - \frac{\sqrt{p_\alpha^2 + M_\alpha^2}}{M_A} \exp[-\xi_{\alpha A}] , & \xi_\alpha^* < 0 , \\ x_\alpha \simeq \frac{\sqrt{p_\alpha^2 + M_\alpha^2}}{M_B} \exp[-\xi_{B\alpha}] , & \xi_\alpha^* > 0 . \end{cases} \quad (2.30)$$

From these formulae we see that, if we use the variable x , the pionization region is confined in a small neighbourhood of the origin. The advantage of the variable ξ_α is that this small interval is amplified in such a way that the details of the pionization region can be examined.

The Feynman scaling law is

$$F^\alpha(p_\alpha, p_A, p_B) \xrightarrow{s \rightarrow \infty} \mathcal{Y}_\alpha(x_\alpha, p_\alpha) . \quad (2.31)$$

If the function $\mathcal{Y}_\alpha(x_\alpha, p_\alpha)$ is continuous at $x_\alpha = 0$, in the pionization region, this assumption says that the one-particle inclusive distribution depends only on p_α . Comparing with Eq. (2.30) we see that at high energy in the fragmentation regions the one-particle distribution depends only on p_α and $\xi_{\alpha A}$ (respectively p_α and $\xi_{B\alpha}$). This is just the limiting fragmentation hypothesis³⁾.

When we shall study the effect of resonances on two-particle distributions, it will be useful to adopt variables related to the motion of the centre-of-mass of the two particles. Then we write the sum of their four-momenta in the form

$$\begin{aligned} P_\alpha + P_\beta = & \left(\sqrt{p_z^2 + v_z^2} \cosh \xi_z , p_z \cos \varphi_z , \right. \\ & \left. , p_z \sin \varphi_z , \sqrt{p_z^2 + v_z^2} \sinh \xi_z \right) , \end{aligned} \quad (2.32)$$

or also

$$P_{\alpha} + P_{\beta} = L \left(a_z(\xi_z) u_z(\varphi_z) a_x(\eta_z) \right) (v_z, 0, 0, 0), \quad (2.33)$$

where $L(a)$ is a 4×4 Lorentz matrix, corresponding to the element a of the Lorentz group [or of its covering group $SL(2C)$]. $a_z(\xi)$ is a boost of rapidity ξ along the z axis and $u_z(\varphi)$ is a rotation of an angle φ around the z axis. We have put

$$\sinh \eta_z = \frac{p_z}{v_z}. \quad (2.34)$$

We assume that the particles α and β have the same mass $M_{\alpha} = M_{\beta}$. If we introduce the angles θ and ψ which determine the direction of the momentum of the particle α in the centre-of-mass of the two particles, we have

$$P_{\alpha} = L \left(a_z(\xi_z) u_z(\varphi_z) a_x(\eta_z) \right) \left(\frac{v_z}{2}, \sqrt{\frac{v_z^2}{4} - M_{\alpha}^2} \sin \theta \cos \psi, \sqrt{\frac{v_z^2}{4} - M_{\alpha}^2} \sin \theta \sin \psi, \sqrt{\frac{v_z^2}{4} - M_{\alpha}^2} \cos \theta \right). \quad (2.35)$$

P_{β} is obtained changing the sign of $\sqrt{\frac{v_z^2}{4} - M_{\alpha}^2}$.

Comparing Eq. (2.35) with Eq. (2.18) we get the relations

$$\begin{cases} v_z^2 = 2 M_{\alpha}^2 + 2 \sqrt{p_{\alpha}^2 + M_{\alpha}^2} \sqrt{p_{\beta}^2 + M_{\alpha}^2} \cosh \xi_{\beta\alpha} - 2 p_{\alpha} p_{\beta} \cos \varphi_{\beta\alpha}, \\ p_z^2 = p_{\alpha}^2 + p_{\beta}^2 + 2 p_{\alpha} p_{\beta} \cos \varphi_{\beta\alpha}, \\ (\cos \theta)^2 = 1 + \frac{(p_{\alpha}^2 - p_{\beta}^2)^2}{(p_z^2 + v_z^2)(v_z^2 - 4 M_{\alpha}^2)} + \frac{p_z^2 - 2 p_{\alpha}^2 - 2 p_{\beta}^2}{v_z^2 - 4 M_{\alpha}^2}. \end{cases} \quad (2.36)$$

The two-particle volume element in momentum space in terms of these variables takes the form

$$\frac{d^3 P_\alpha}{2 P_\alpha^0} \frac{d^3 P_\beta}{2 P_\beta^0} = \frac{1}{8} \sqrt{v_z^2 - 4M_\alpha^2} dv_z p_z dp_z \cdot \quad (2.37)$$

$$\cdot d\xi_z d\varphi_z d(\cos\theta) d\psi \cdot$$

3. FACTORIZED MODELS, P FACTORIZATION AND Q FACTORIZATION

Several reasonable approaches to the phenomenology of many-particle amplitudes start by approximating the invariant amplitude M , which depends on a very large number of variables, by means of a given combination of functions of a limited number of variables.

Following the heuristic idea of maximum simplicity it makes sense to put

$$M_{[n]}(P_{11}, \dots, P_{n_y}, P_A, P_B) = c \prod_{\tau i} f_{\tau}(P_{\tau i}, P_A, P_B). \quad (3.1)$$

Whenever, as in the present paper, one is concerned in quantities which depend only on the modulus of M , we can use the less restrictive assumption

$$\left| M_{[n]}(P_{11}, \dots, P_{n_y}, P_A, P_B) \right|^2 = c^2 \prod_{\tau i} N_{\tau}(P_{\tau i}, P_A, P_B) (2\pi)^3. \quad (3.2)$$

The most relevant feature that one has to take into account is the fast decrease of the cross-section with increasing transverse momenta. This suggests to take as a first approximation

$$N_{\tau}(P_{\tau}, P_A, P_B) = N_{\tau}(p_{\tau}), \quad (3.3)$$

where $N_{\tau}(p_{\tau})$ is a fast decreasing function of the transverse momentum defined in Eq. (2.19).

The fast decrease in the transverse momenta, which is the only ingredient of this model, is just a phenomenological input. Its theoretical explanation is perhaps the most challenging problem of hadron dynamics.

These are examples of uncorrelated jet models ¹⁰⁾. They have the great advantage of simplicity and, as far as we are concerned with one-particle distributions and average properties, they are as good as other more sophisticated models. Of course, if we

use the more drastic approximation (3.3), we cannot reproduce the important feature of the outgoing nucleons whose longitudinal momenta tend to be near to the momenta of the incoming protons. This feature can be taken into account modifying the assumption (3.3) only in the fragmentation regions. As we shall indicate in detail in the next Section, these modifications do not affect certain general predictions regarding the pionization region, in particular the Eq. (3.4).

A characteristic feature of the above approximation is that in the pionization region, where the effect of four-momentum conservation is not important, the two-particle distribution factorizes as follows (see next Section):

$$\frac{F^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B)}{\sigma_I} = \frac{F^\alpha(P_\alpha, P_A, P_B)}{\sigma_I} \frac{F^\beta(P_\beta, P_A, P_B)}{\sigma_I} \quad (3.4)$$

When this equation is satisfied, by definition we say that the two particles are uncorrelated. Otherwise, we say that there is a two-particle correlation. For instance, the presence of resonances would violate Eq. (3.4) at least for low values of the invariant mass

$$(P_\alpha + P_\beta)^2 = \nu^2 \quad (3.5)$$

(see Section 7). Other correlation effects arise as a consequence of a large class of multiperipheral models (see Section 6).

Since we are interested in the study of correlations, we have to improve the approximation (3.1). The next factorized approximation would be (assuming that all the particles are of the same kind)

$$M_n(P_1, \dots, P_n, P_A, P_B) = c \prod_{\substack{i,j \\ i > j}} f(P_i, P_j, P_A, P_B) \quad (3.6)$$

Models of this kind, where the factors contain, besides P_A and P_B , a small number of four-momenta P_i , will be called P factorized models. From a phenomenological point of view they are the most simple and natural factorized models, since the connection of the single factors with experimentally measurable quantities is rather direct, at least in the pionization region.

However, these models are not completely satisfactory from a theoretical point of view, due to the special role assigned to the momenta P_A and P_B of the incoming particles. This is a feature that does not agree with the idea of crossing, according to which the incoming momenta should be treated on the same footing as the outgoing ones. For this reason the more theoretically oriented models prefer a different kind of factorization, which we call Q factorization.

For the largest part of the events one can find a frame of reference in which all the components of the momenta perpendicular to the z axis are small. These are the peripheral events; the other events will not be considered. Of course, this frame will practically coincide with a frame in which the momenta $\underline{P}_A$ and $\underline{P}_B$ are directed along the z axis. We have preferred to start from a crossing invariant definition.

Then, one orders the particles according to the increasing rapidity along the z axis. The rapidities $\{A$ and $\{B$ of the incoming particles lie necessarily at the extremities of the rapidity scale¹⁵⁾. One defines the four-momentum transfers

$$Q_i = P_A - \sum_{j=1}^i P_j, \quad i = 1, \dots, n-1. \quad (3.7)$$

The simplest Q factorized model (considering only one kind of particles) has the form

$$M_n(P_1, \dots, P_n, P_A, P_B) = c \prod_{i=1}^{n-1} g(Q_i^2). \quad (3.8)$$

The next approximation is

$$M_n(p_1, \dots, p_n, p_A, p_B) = g_A(p_A, Q_1) g(Q_1, Q_2) \dots g_B(Q_{n-1}, p_B) \quad (3.9)$$

and so on.

We see that these are just the models which have the property called "short range correlation" by Chew, Low and Goldberger¹³⁾. Following these authors we call "range of correlation" the number of four-momenta Q_i contained in each factor. Motivations for Q factorized models of higher range are provided by multi-Regge models with or without dependence on the vertex-angles¹³⁾ and by the possibility that the produced particles are grouped into clusters of relatively low invariant masses¹⁶⁾. All these models from a conceptual point of view are only slight generalizations of the multiperipheral models studied extensively since 1962 by ABFST^{11), 12)}.

Also in this case the fundamental feature of the cut-off on the transverse momenta is practically imposed as an input condition, that is the strong decrease of the functions g in their arguments Q_i^2 .

In a Q factorized model the connection between the factor functions and the k particle inclusive distributions is not so direct as for a P factorized model. We shall show in Section 5 that, as far as one-particle distributions are concerned, the Q factorized models give predictions which can also be reproduced by the simplest P factorized models. However, as we shall see in Section 6, even the simplest Q factorized models of the kind (3.8) provide non-trivial two-particle correlations.

The most appealing and challenging research program, however, would be a reasonable detailed derivation of the functions g in Eq. (3.8) or (3.9) starting from the experimental knowledge of the inclusive distributions.

4. THE SIMPLEST P FACTORIZED MODEL

Now, we want to study some consequences of the assumption stated in Eq. (3.2). Within this model and without further assumptions, we give the general formulae for the inelastic cross-sections and one and two-particle distributions. We make use of the approach introduced by Lurçat and Mazur¹⁷⁾ and recently developed by other authors^{18),19)}.

Our main purpose in this Section is to study the correlations induced by four-momentum conservations and to show that they are negligible in some kinematical configurations. These properties should not be interpreted as predictions, but rather as a term of comparison. Deviations of the experimental results from such formulae have to be ascribed to definite dynamical effects. Some examples of these effects will be discussed in the successive Sections.

It is useful to introduce the function of the four-vector
R

$$\phi_{[n]}(R, P_A, P_B) = \int \delta^4(R - \sum_{zi} P_{zi}) \cdot \prod_{zi} \left[N_z(P_{zi}, P_A, P_B) \frac{d^3 P_{zi}}{2 P_{zi}^0} \right] \quad (4.1)$$

In the following we shall always work in the centre-of-mass system with the z axis along the direction of the incoming particle B. Then, the explicit dependence of all the functions on P_A and P_B can be replaced by the variable s.

In terms of this function from Eqs. (3.2), (2.2), (2.7), (2.8), (2.13) and (2.14) we obtain

$$\sigma_{[n]}(s) = (2\pi)^4 \frac{c^2}{2} [T(s, M_A^2, M_B^2)]^{-1/2} \cdot \prod_{\tau=1}^{\gamma} (n_{\tau}!)^{-1} \cdot \phi_{[n]}(P_A + P_B, s), \quad (4.2)$$

$$F_{[n]}^{\alpha}(P_{\alpha}, P_A, P_B) = (2\pi)^4 \frac{c^2}{2} [T(s, M_A^2, M_B^2)]^{-1/2} \cdot n_{\alpha} \prod_{\tau=1}^{\gamma} (n_{\tau}!)^{-1} \phi_{[n-1]}(P_A + P_B - P_{\alpha}, s) N_{\alpha}(P_{\alpha}, s), \quad (4.3)$$

$$F_{[n]}^{\alpha\beta}(P_{\alpha}, P_{\beta}, P_A, P_B) = (2\pi)^4 \frac{c^2}{2} [T(s, M_A^2, M_B^2)]^{-1/2} \cdot n_{\alpha} n_{\beta} \prod_{\tau=1}^{\gamma} (n_{\tau}!)^{-1} \phi_{[n-2]}(P_A + P_B - P_{\alpha} - P_{\beta}, s) N_{\alpha}(P_{\alpha}, s) N_{\beta}(P_{\beta}, s). \quad (4.4)$$

We have indicated by $[n-1]$ the sequence $[n]$ where n_{α} has been replaced by $n_{\alpha} - 1$. $[n-2]$ is defined in a similar way. As in Eq. (2.14), when the two particles are of the same kind the factor $n_{\alpha} n_{\beta}$ has to be replaced by $n_{\alpha} (n_{\alpha} - 1)$.

In order to obtain the inclusive distributions and the total inelastic cross-section we define the function

$$\phi(R, s) = \sum'_{[n]} \prod_{\tau=1}^{\gamma} (n_{\tau}!)^{-1} \phi_{[n]}(R, s), \quad (4.5)$$

where the sum is extended to all the configurations $[n]$ with $n \geq 3$ allowed by the conservation laws²⁰⁾. Then from Eqs. (4.2)-(4.4) we obtain

$$\sigma_I(s) = (2\pi)^4 \frac{c^2}{2} [T(s, M_A^2, M_B^2)]^{-1/2} \phi(P_A + P_B, s), \quad (4.6)$$

$$\frac{F^\alpha(P_\alpha, P_A, P_B)}{\sigma_I(s)} = N_\alpha(P_\alpha, s) \frac{\phi(P_A + P_B - P_\alpha, s)}{\phi(P_A + P_B, s)}, \quad (4.7)$$

$$\frac{F^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B)}{\sigma_I(s)} = N_\alpha(P_\alpha, s) N_\beta(P_\beta, s) \frac{\phi(P_A + P_B - P_\alpha - P_\beta, s)}{\phi(P_A + P_B, s)} \quad (4.8)$$

Equations (4.7) and (4.8) show transparently an important general feature of the model, namely, the factorization of the normalized distribution functions into the dynamical input (the functions N_α and N_β) and a factor which takes into account the influence of the unobserved particles through four-momentum conservation. Notice also that the last factor in Eq. (4.8) explicitly exhibits some correlations between the particles α and β . These correlations have more a kinematical origin rather than a dynamical one¹⁴⁾.

As we shall show [Eqs. (4.44) and (4.45)], when all the observed particles are in the pionization region, the last factor in Eqs (4.7) and (4.8) is very near to one, so that Eq. (3.4) follows and any experimentally observed deviation from this factorized behaviour is an indication of a genuine dynamical effect.

In order to further develop these concepts, following Lurçat and Mazur¹⁷⁾ we consider the Fourier transform

$$\tilde{\phi}_{[n]}(V, s) = \int d^4R \exp[i(V, R)] \phi_{[n]}(R, s) \quad (4.9)$$

As the function $\phi_{[n]}(R, s)$ vanishes for negative R^0 , $\tilde{\phi}_{[n]}(V, s)$ is analytic for $\text{Im } V^0 > 0$. Then, one has immediately

$$\tilde{\phi}_{[n]}(V, s) = \prod_z [\tilde{\phi}_z(V, s)]^{n_z}, \quad (4.10)$$

where

$$\tilde{\phi}_z(V, s) = \int \exp[i(V, P_z)] N_z(P_z, s) \frac{d^3 P_z}{2 P_z^0} \quad (4.11)$$

The usual procedure^{18),19)} is to invert first approximately the Fourier transformation (4.9) and then to perform the sum (4.5). A simpler and more easily justifiable procedure is first to perform the sum and then to invert the Fourier transform

$$\tilde{\phi}(V, s) = \sum_{[n]} \prod_z \frac{\delta}{(n_z!)} [\tilde{\phi}_z(V, s)]^{n_z} \quad (4.12)$$

This sum can be performed explicitly in many interesting cases in terms of exponentials, or of modified Bessel functions if charge conservation is taken into account.

Let us consider first the simpler case in which Eq. (3.3) holds. Then, the arguments P_A and P_B which appear in the above defined functions have the only role of defining the z axis. Therefore, the explicit dependence of $\phi(R, s)$ on s drops out and, as everything is invariant with respect to boosts along the z axis and rotations around the z axis we have

$$\phi(R, s) = \phi(z_L, z) \theta(R^0) \quad (4.13)$$

$$\tilde{\phi}(V, s) = \tilde{\phi}(v_L, v) \quad (4.14)$$

where

$$r_L = \sqrt{(R^0)^2 - (R^z)^2}, \quad r = \sqrt{(R^x)^2 + (R^y)^2}, \quad (4.15)$$

$$v_L = \sqrt{(V^0)^2 - (V^z)^2}, \quad \text{Im } v_L > 0, \quad v = \sqrt{(V^x)^2 + (V^y)^2}. \quad (4.16)$$

The function in the right-hand side of Eq. (4.14) is analytic for $\text{Im } v_L > 0$.

Then, the Fourier transforms (4.9) and (4.11) can be simplified by means of the formulae (7.12.9) and (7.12.19) of H.T.F. ²¹⁾ and we get, putting $v_L = iz$,

$$\tilde{\phi}(iz, v) = 4\pi \int_0^\infty r dr \int_0^\infty (vr) \int_0^\infty r_L dr_L K_0(zr_L) \phi(r_L, r), \quad (4.17)$$

$$\tilde{\phi}_r(iz, v) = 2\pi \int_0^\infty p dp \int_0^\infty (vp) K_0(z\sqrt{p^2 + M_\pi^2}) N_r(p). \quad (4.18)$$

The inverse of the transformation (4.17) can be obtained by means of the formulae (8.1.1) and (10.1.1) of T.I.T. ²²⁾ and is

$$\phi(r_L, r) = \frac{1}{4\pi^2 i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} z dz \int_0^\infty v dv \int_0^\infty (zv) \tilde{\phi}(iz, v). \quad (4.19)$$

The function $\phi(r_L, r)$ is completely determined by Eqs. (4.12), (4.18) and (4.19). A similar procedure can be followed for the functions $\phi_{[n]}^{(r_L, r)}$.

If we consider only chargeless pions, the sum (4.12) can be performed and we get

$$\begin{aligned} \tilde{\phi}(v_L, v) &= \exp \left[\tilde{\phi}_\pi(v_L, v) \right] - \frac{1}{2} \left[\tilde{\phi}_\pi(v_L, v) \right]^2 \\ &- \tilde{\phi}_\pi(v_L, v) - 1 \end{aligned} \quad (4.20)$$

Now we want to compute the inversion integral (4.19) in the asymptotic limit of large r_L . This behaviour depends essentially on the behaviour of the function (4.20) for small v_L . We shall also show during the calculation that only small values of v are relevant. Then, we consider Eq. (4.18) and we use the following approximation [Eq. (7.2.38) of H.T.F. ²¹⁾]

$$K_0(z) = -\log \frac{Cz}{2} + O(z^2 \log z), \quad C = e^\gamma = 1.7810\dots \quad (4.21)$$

We obtain for small z and v

$$\tilde{\phi}_\tau(iz, v) \simeq -2h_\tau \log(zM_\tau) \cdot \left[1 - \frac{v^2}{4\Lambda_\tau^2} \right] + 2\lambda_\tau h_\tau, \quad (4.22)$$

where

$$h_\tau = \pi \int N_\tau(p) p dp, \quad (4.23)$$

$$\Lambda_\tau^{-2} h_\tau = \pi \int N_\tau(p) p^3 dp, \quad (4.24)$$

$$\lambda_\tau h_\tau = -\pi \int N_\tau(p) \log \left[\frac{C}{2} \sqrt{1 + \frac{p^2}{M_\tau^2}} \right] p dp. \quad (4.25)$$

The leading term in Eq. (4.20) is

$$\tilde{\phi}(iz, v) \simeq (zM_\pi)^{-2h_\pi} \left[1 - \frac{v^2}{4\Lambda_\pi^2} \right] \cdot e^{2\lambda_\pi h_\pi} \quad (4.26)$$

We substitute in Eq. (4.19) and we perform an integration by means of Eq. (10.2.1) of T.I.T. ²²⁾. We get ²³⁾

$$\phi(z_L, z) \simeq e^{2\lambda_\pi h_\pi} (\pi z_L^2)^{-1} \int_0^\infty f_0(zv) v dv \cdot \left(\frac{z_L}{2M_\pi}\right)^{2h_\pi} [1 - v^2/4\Lambda_\pi^2] \cdot \left[\Gamma(h_\pi [1 - v^2/4\Lambda_\pi^2]) \right]^{-2} \quad (4.27)$$

We remark that, when r_L becomes large, the power term in the integrand is more and more peaked around the origin. This fact justifies our small v approximations and permits us to put $v=0$ in the argument of the Γ function. Then the integral can be performed by means of Eq. (8.6.10) of T.I.T. ²²⁾ and we get

$$\phi(z_L, z) \underset{z_L \rightarrow \infty}{\simeq} e^{2\lambda_\pi h_\pi} \cdot \frac{\Lambda_\pi^2}{4\pi h_\pi M_\pi^2} \cdot \left[\Gamma(h_\pi) \right]^{-2} \cdot \left(\frac{z_L}{2M_\pi}\right)^{2h_\pi-2} \cdot \left[\log \frac{z_L}{2M_\pi} \right]^{-1} \cdot \exp \left[- \frac{z^2 \Lambda_\pi^2}{2h_\pi \log \frac{z_L}{2M_\pi}} \right] \quad (4.28)$$

Now we show how this equation is modified if we take into account the two protons in the final state. The corresponding factors N_N , as we have already said cannot be assumed to satisfy Eq. (3.3). We put instead (we treat the two nucleons as different kinds of particles, as their wavefunctions do not overlap)

$$\begin{cases} N_N(p_N, s) = \eta(p_N, \xi_N - \xi_A) \\ N_{N'}(p_{N'}, s) = \eta(p_{N'}, \xi_B - \xi_{N'}) \end{cases} \quad (4.29)$$

The function η has to account for the experimental fact that the outgoing nucleons have rapidities close to the rapidities of the corresponding incoming protons. For our treatment we need only the property that it is fast decreasing and strongly peaked around the origin in both arguments.

We introduce Eq. (4.29) into Eq. (4.11) and we observe that for large values of

$$-\xi_A^* = \xi_B^* = \log \frac{\sqrt{s}}{M_N}, \quad (4.30)$$

also $|\xi_N^*|$ is large and we can put

$$\cosh \xi_N^* \simeq |\sinh \xi_N^*| \simeq \frac{1}{2} \exp |\xi_N^*|. \quad (4.31)$$

Then, we get

$$\begin{aligned} \tilde{\phi}_N(V, s) &= \int \exp \left[\frac{i}{2} e^{|\xi_N^*|} \sqrt{p_N^2 + M_N^2} (V^0 + V^2) - i V^x p_N^x - \right. \\ &\quad \left. - i V^y p_N^y \right] \eta(p_N, \xi_N^* - \xi_A^*) d p_N^x d p_N^y \frac{1}{2} d \xi_N^* = \\ &= \psi(\sqrt{s} (V^0 + V^2), v) \end{aligned} \quad (4.32)$$

and in a similar way

$$\tilde{\phi}_{N'}(V, s) = \psi(\sqrt{s} (V^0 - V^2), v). \quad (4.33)$$

This scaling property is the only ingredient of our reasoning.

In order to take into account the presence of the two nucleons, we have to multiply the function (4.20) by the two factors (4.32) and (4.33). Also in this case we keep only the leading term (4.26) and, introducing the new variables

$$\begin{cases} \hat{V}^{\pm} = \sqrt{s} (V^0 \pm V^z) \\ s v_L^2 = \hat{V}^+ \hat{V}^- \end{cases}, \quad (4.34)$$

we get

$$\begin{aligned} \tilde{\phi}(V, s) \simeq & \left[\frac{-s}{M_\pi^2 \hat{V}^+ \hat{V}^-} \right]^{h_\pi (1 - v^2/4\Lambda_\pi^2)} \cdot e^{2\lambda_\pi h_\pi} \\ & \cdot \psi(\hat{V}^+, v) \psi(\hat{V}^-, v) \end{aligned} \quad (4.35)$$

If we define also

$$\hat{R}^{\pm} = \frac{R^0 \pm R^z}{\sqrt{s}}, \quad (4.36)$$

and we remark that in the limit of large s the first factor in Eq. (4.35) is strongly peaked at $v=0$, we can partially perform the inverse Fourier transform and we get in the limit of large s and for fixed r , $\hat{R}^+$ and $\hat{R}^-$

$$\phi(R, s) \simeq \left(\frac{s}{M_\pi^2} \right)^{h_\pi - 1} \left(\log \frac{s}{M_\pi^2} \right)^{-1} \hat{\phi}(\hat{R}^+) \hat{\phi}(\hat{R}^-), \quad (4.37)$$

where

$$\begin{aligned} \hat{\phi}(\hat{R}^+) = & \left[e^{2\lambda_\pi h_\pi} \frac{\Lambda_\pi^2}{8\pi^3 h_\pi M_\pi^2} \right]^{1/2} \cdot \int (-i \hat{V}^-)^{-h_\pi} \\ & \cdot \psi(\hat{V}^-, 0) \exp \left[-\frac{i}{2} \hat{V}^- \hat{R}^+ \right] d\hat{V}^- \end{aligned} \quad (4.38)$$

In order to avoid the singularity at the origin in this integral we have to give to $\hat{V}^-$ a small positive imaginary part.

Remark that the function (4.28) satisfies an asymptotic formula of the kind (4.37) with

$$\hat{\phi}(\hat{R}^+) = \left[e^{2\lambda_{\pi} h_{\pi}} \frac{\Lambda_{\pi}^2}{2\pi h_{\pi} M_{\pi}^2} \right]^{1/2} \left[r(h_{\pi}) \right]^{-1} \left(\frac{\hat{R}^+}{2} \right)^{h_{\pi}-1} \quad (4.39)$$

The behaviour (4.37) can be obtained also under the more general hypothesis that the functions $N_{\tau}(P_{\tau}, s)$ satisfy the Feynman scaling law. For instance, one could add to the pion factor function $N_{\pi}(p_{\pi})$ a correction term in the fragmentation region of the form (4.29). One can also consider particles of different kinds, of course replacing the quantity h_{π} by

$$h = \sum_{\tau} h_{\tau} \quad (4.40)$$

From Eq. (4.6) we get the inelastic cross-section

$$\sigma_I(s) = (2\pi)^4 \frac{c^2}{2s} \left(\frac{s}{M_{\pi}^2} \right)^{h_{\pi}-1} \left(\log \frac{s}{M_{\pi}^2} \right)^{-1} \left[\hat{\phi}(1) \right]^2 \quad (4.41)$$

We see that, in order to get a nearly constant high energy behaviour, we have to assume ¹⁸⁾ $h_{\pi} = 2$ or, if several kinds of particles are produced

$$h = 2 \quad (4.42)$$

We consider now the one and two-particle distributions given by Eqs. (4.7) and (4.8). It is easy to show that in the pionization region the quantities

$$\frac{P_{\alpha}^0}{\sqrt{s}}, \quad \frac{P_{\alpha}^z}{\sqrt{s}} \quad (4.43)$$

are small. Hence, from Eq. (4.37) we get

$$\frac{\phi(P_A + P_B - P_{\alpha}, s)}{\phi(P_A + P_B, s)} \approx \frac{\hat{\phi}\left(1 - \frac{P_{\alpha}^0 + P_{\alpha}^z}{\sqrt{s}}\right) \hat{\phi}\left(1 - \frac{P_{\alpha}^0 - P_{\alpha}^z}{\sqrt{s}}\right)}{[\hat{\phi}(1)]^2} \sim 1. \quad (4.44)$$

In a similar way, if both the particles α and β are in the pionization region, we get

$$\begin{aligned} & \frac{\phi(P_A + P_B - P_{\alpha} - P_{\beta}, s)}{\phi(P_A + P_B, s)} = \\ & \approx \frac{\hat{\phi}\left(1 - \frac{P_{\alpha}^0 + P_{\beta}^0 + P_{\alpha}^z + P_{\beta}^z}{\sqrt{s}}\right) \hat{\phi}\left(1 - \frac{P_{\alpha}^0 + P_{\beta}^0 - P_{\alpha}^z - P_{\beta}^z}{\sqrt{s}}\right)}{[\hat{\phi}(1)]^2} \sim 1. \end{aligned} \quad (4.45)$$

As it has already been explained, this means that in the pionization region the four-momentum conservation does not influence the inclusive distributions. When only one of the particles, say the particle α , is in the pionization region, we see that the factor (4.45) does not depend appreciably on P_{α} . In this case, the four-momentum conservation can affect the distribution of the particle β , but it does not induce correlations between α and β .

Finally, we examine the case in which both particles are in the fragmentation regions. If they are both in the same fragmentation regions, they will certainly be correlated by four-momentum conservation. If they are in two different fragmentation regions,

we have

$$\frac{P_{\alpha}^2}{\sqrt{s}} \approx - \frac{P_{\alpha}^0}{\sqrt{s}} , \quad \frac{P_{\beta}^2}{\sqrt{s}} \approx \frac{P_{\beta}^0}{\sqrt{s}} , \quad (4.46)$$

and we see immediately that the expression (4.45) factorizes in two parts depending separately on P_{α} and P_{β} , so that it does not induce correlations.

As these are asymptotic formulae, at finite energy there are corrections which can be estimated and, if necessary, taken into account. We shall deal with this problem in a forthcoming paper.

From the above discussion we see that the quantities defined in Eqs. (4.23) and (4.24) have a direct physical meaning. The quantity h_{τ} is the average number of particles of the kind τ produced in the pionization region per inelastic event and per unit of rapidity. As we shall better see in Section 8, the choice (4.42) is not too far from the experimental findings. The quantity Λ_{τ}^{-2} is the average of the squared transverse momentum of the particles of the kind τ in the pionization region.

5. ONE-PARTICLE DISTRIBUTIONS IN MULTIPERIPHERAL DYNAMICS

Now, following the classification of Section 3, we consider the Q factorized models, that is, essentially, the multiperipheral models. The inclusive one-particle distributions within these models have been investigated several years ago by ABFST^{11),12)} and have been reconsidered recently by several authors²⁴⁾⁻²⁶⁾. The simplest approach is based on Eq. (2.8) where the function M is approximated by a Q factorized expression.

A different but strictly related approach has been developed recently by Mueller²⁷⁾. He does not consider directly the n particle production amplitude, but he writes the one-particle inclusive distribution as a crossed absorptive part of a six-particle amplitude which is approximated by means of a two-reggeon model. This approximation can be derived from a multiperipheral hypothesis, but different approaches are also possible. Of course, the method can be extended to two-particle inclusive distributions.

We follow the approach directly based on the multiperipheral model, which permits a more direct physical insight. As already said in Section 3, the Q factorized models are more suitable than the P factorized ones for taking into account the general properties of the S matrix. In particular, important analyticity properties can be incorporated, giving rise to a reasonable determination of the phase, which permit to reconstruct, through unitarity the properties of the diffraction peak. As we need only the modulus of the amplitude, we are not concerned with these aspects of the model.

Another simplification which appears in the problems we are considering is the symmetry of the equations with respect to the Lorentz group $O(3,1)$. This symmetry and the cut on the transferred momenta are the sources of all the general and most interesting consequences of the model.

These main features of the model appear already in the simplest version based on Eq. (3.8) and we shall consider it in detail. For the mathematical techniques needed for the treatment of models with more complicated correlations, see Refs. 28), 29). For simplicity we assume that all the initial and the final particles are of the same kind.

Following the prescriptions given in Section 3, we order the particles according to increasing longitudinal rapidity; it is convenient to label them as shown in Fig. 2, which shows also the definition of the transferred four-momenta Q_i and Q'_i . Then, according to Eq. (3.8) we have

$$|M_n|^2 = c^2 \Delta(u'_1) \dots \Delta(u'_r) \Delta(u_2) \dots \Delta(u_1), \quad (5.1)$$

where

$$\begin{cases} u_i = - (Q_i)^2, & u'_i = - (Q'_i)^2, \\ \Delta(u) = |g(u)|^2 \end{cases} \quad (5.2)$$

(5.3)

Now, we make the usual assumption that the product which appears in the right-hand side of Eq. (5.1) is small outside the region where the rapidities have the right order, i.e., the region where Eq. (5.1) is assumed to hold. This assumption is based on the strong decrease of $\Delta(u)$ with increasing u . Then $|M_n|^2$ is given by the sum of $n!$ terms obtained from the right-hand side of Eq. (5.1) by permuting the final momenta in all the possible ways.

Substituting in Eq. (2.8) and summing over $n \geq 3$, we get after some manipulations

$$G_2^{\alpha}(P_{\alpha}, P_A, P_B) = 2\pi \int A(Q - P_{\alpha}, P_B) \cdot \Delta((Q - P_{\alpha})^2) \cdot \Delta(Q^2) \cdot A(Q, P_A) d^4 Q + 2\pi c \Delta((P_{\alpha} - P_B)^2) \cdot \hat{A}(P_{\alpha} - P_B, P_A) + 2\pi c \Delta((P_{\alpha} - P_A)^2) \cdot \hat{A}(P_{\alpha} - P_A, P_B),$$

where A is a solution of the multiperipheral integral equation ^(5.4)₃₀₎

$$A(Q, P_A) = (2\pi)^{-3} c \delta[(P_A - Q)^2 - M^2] \theta(P_A^0 - Q^0) + (2\pi)^{-3} \int \delta[(Q' - Q)^2 - M^2] \theta(Q'^0 - Q^0) \Delta((Q')^2) A(Q', P_A) d^4 Q' \quad (5.5)$$

and $\hat{A}$ is the second term in the right-hand side of Eq. (5.5), i.e., the non-singular part of A.

In Eq. (5.4) the first term in the right-hand side refers to particles produced in the middle of the multiperipheral chain, while the other two terms refer to particles produced at one of the two extreme vertices. From the fast decrease property of the function Δ we see that these two terms are appreciable only in the respective fragmentation regions.

It is well known that Eq. (5.5) can be partial-wave analyzed using its symmetry under the Lorentz group ³¹⁾⁻³²⁾. The expansion formula to be used is the following

$$A(Q, P_A) = -i \int_{c-i\infty}^{c+i\infty} \alpha(\lambda, u) \frac{\exp(\lambda \sigma)}{\sqrt{\pi} \cosh \sigma} d\lambda, \quad (5.6)$$

where

$$\begin{cases} u = -Q^2, \\ \sinh \sigma = - \frac{(Q, P_A)}{M \sqrt{u}} \end{cases} \quad (5.7)$$

The function $a(\lambda, u)$, which is the solution of the projected integral equation, has poles in the complex variable λ which are called Lorentz poles and determine the asymptotic behaviour of the function $A(Q, P_A)$ for large values of σ .

Now we substitute the Eq. (5.6) into Eq. (5.4). As we are studying the pionization region we can drop the contributions of the extreme vertices. The details of the calculation are given in the Appendix A and the result is

$$Q^*(P_\alpha, P_A, P_B) = \int_{c-i\infty}^{c+i\infty} d\lambda \int_{c-i\infty}^{c+i\infty} d\lambda' \sum_{j'm} \mathcal{D}_{j'm}^{0\lambda} (a_{\alpha A}) \cdot F_j(\lambda, \lambda') \cdot \mathcal{D}_{j'm}^{0\lambda'} (a_{B\alpha}) \quad (5.8)$$

The functions $F_j(\lambda', \lambda)$ are given by integrals containing the function $a(\lambda, u)$. We have indicated by $\mathcal{D}_{j'm}^{M\lambda}(a)$ the matrix elements of the irreducible representations of the Lorentz group^{33),34)}. $a_{B\alpha}$ is an element of the Lorentz group which connects the rest systems of the particle B and of the observed particle; $a_{\alpha A}$ is defined in a similar way. As in Ref. 26), we use the choice

$$\begin{cases} a_{B\alpha} = a_x(-\eta_\alpha) a_z(\xi_{B\alpha}) u_z(-\varphi_\alpha), \\ a_{\alpha A} = u_z(\varphi_\alpha) a_z(\xi_{\alpha A}) a_x(\eta_\alpha) \end{cases} \quad (5.9)$$

where

$$\sinh \eta_\alpha = \frac{\pi_\alpha}{M_\alpha} \quad (5.10)$$

The other variables are defined in Section 2.

We remark again that Eq. (5.8) is much more general than the simple model we have used to derive it. In this model the leading singularity of the function $F_j(\lambda, \lambda')$ is a pole with factorized residue in both of its variables for $\lambda = a+1$, $\lambda' = a+1$ with $a > -1$. In practice we have $a \simeq 1$ (Pomeranchuk pole) ³⁵⁾. Then we can write the double Lorentz pole contribution

$$G_2^\alpha(P_\alpha, P_A, P_B) \simeq \sum_{jm} \tilde{p}^A \mathcal{O}_{00jm}^{0,a+1}(a_{\alpha A}) \tilde{p}_j^\alpha \mathcal{O}_{jm00}^{0,a+1}(a_{B\alpha}) \tilde{p}^B, \quad (5.11)$$

which dominates when both $\{B\alpha}$ and $\{\alpha A\}$ are large. The coefficients $\tilde{p}^A \tilde{p}_j^\alpha \tilde{p}^B$ are proportional to the residues of $F_j(\lambda, \lambda')$.

We remember that the matrix elements of the representation of the Lorentz group have the form

$$\mathcal{O}_{jmj'm'}^{M\lambda}(u a_z(\xi|u')) = \sum_{m''} R_{mm''}^j(u) d_{m''j'j'}^{M\lambda}(\xi) R_{m''m'}^{j'}(u'), \quad (5.12)$$

where u and u' are rotations and $R_{mm'}^j(u)$ is the usual rotation matrix. We also have ^{33), 36)} for large ξ

$$d_{mjj'}^{M\lambda}(\xi) = c_j^{M|\lambda} c_{j'}^{M|\lambda} \exp[(\lambda-1)\xi] \delta_{M,-m} + O(\exp(\lambda-2)\xi), \quad (5.13)$$

$[\text{Re } \lambda > 0],$

where the coefficients $c_j^{M|\lambda}$ are known.

We can also write the elements (5.9) in the form

$$\begin{cases} a_{B\alpha} = u_y(-\chi_\alpha) a_z(\zeta_{B\alpha}) u_y(\chi_B) u_z(-\varphi_\alpha), \\ a_{\alpha A} = u_z(\varphi_\alpha) u_y(\chi_A) a_z(\zeta_{\alpha A}) u_y(-\chi_\alpha). \end{cases} \quad (5.14)$$

A simple computation shows that for large values of $\{B\alpha}$ and $\{\alpha A\}$ we have

$$\left\{ \begin{array}{l} \exp \zeta_{B\alpha} \simeq \exp \xi_{B\alpha} \cosh \eta_{\alpha}, \\ \tanh \chi_{\alpha} \simeq \sinh \eta_{\alpha}, \\ \exp \zeta_{\alpha A} \simeq \exp \xi_{\alpha A} \cosh \eta_{\alpha}, \\ \chi_A \simeq \chi_B \simeq 0 \end{array} \right. \quad (5.15)$$

If we substitute Eq. (5.14) into the Lorentz pole contribution (5.11), we get using Eqs. (5.12) and (5.13)

$$\begin{aligned} G^{\alpha}(P_{\alpha}, P_A, P_B) &\simeq \sum_{jm} p^A \exp(a \zeta_{\alpha A}) R_{om}^j(u_y(-\chi_{\alpha})) \cdot \\ &\cdot p_j^{\alpha} \cdot R_{mo}^j(u_y(-\chi_{\alpha})) \exp(a \zeta_{B\alpha}) p^B \simeq \\ &\simeq p^A p^B \exp(a \xi_{BA}) (\cosh \eta_{\alpha})^{2a} \sum_j p_j^{\alpha} P_j(\cos 2\chi_{\alpha}). \end{aligned} \quad (5.16)$$

We have included the coefficients $c_j^{|\mathbb{M}|\lambda}$ in the residues p_j^{α} , and we have used the well-known formula

$$\begin{aligned} \sum_m R_{om}^j(u_y(-\chi_{\alpha})) R_{mo}^j(u_y(-\chi_{\alpha})) &= R_{oo}^j(u_y(-2\chi_{\alpha})) = \\ &= P_j(\cos 2\chi_{\alpha}) \end{aligned} \quad (5.17)$$

First we remark that, as the Legendre polynomials form a complete set of functions, the sum over j in Eq. (5.16) represents an arbitrary function of $\cos 2\chi_{\alpha}$ or of p_{α} . Of course, a particular model specifies the coefficients p_j^{α} and, therefore, this function, but we are interested mainly in the predictions of the multiperipheral assumptions which do not depend on the details of the model.

Then, we remark that the total cross-section has the high energy behaviour

$$\sigma_I \simeq \frac{1}{2} s^{-1} p^A p^B \exp(a \xi_{BA}) \quad (5.18)$$

In conclusion, using Eq. (2.7), we obtain from Eq. (5.16)

$$\frac{F^\alpha(p_\alpha, p_A, p_B)}{\sigma_I} \simeq \frac{F^\alpha(p_\alpha)}{\sigma_I} = N_\alpha(p_\alpha), \quad \begin{cases} \xi_{B\alpha} \gg 1, \\ \xi_{\alpha A} \gg 1. \end{cases} \quad (5.19)$$

This well-known formula, which holds in the pionization region, coincides with Eq. (4.7), taking into account Eq. (4.44). However, the function $N_\alpha(p_\alpha)$, which in Section 4 was the input of the model, here is the result of a complicated calculation.

We see that, if $\xi_{B\alpha}$ and $\xi_{\alpha A}$ are sufficiently large, the one-particle distribution does not depend on them and, therefore, it is also independent of the total energy. As the factors p^A and p^B cancel out, we see that it is also independent of the kind of the incoming particles. In conclusion, $N_\alpha(p_\alpha)$ is a universal function which depends only on the kind of particle observed and on the transverse momentum p_α .

If $\xi_{B\alpha}$ and (or) $\xi_{\alpha A}$ are not very large, we can introduce the corrections due to non-leading Lorentz poles. As the intercept of these poles is at $a \simeq 0.5$, we expect corrections of the order of $\exp(-\frac{1}{2} \xi_{B\alpha})$ and of $\exp(-\frac{1}{2} \xi_{\alpha B})$. These corrections become important near the fragmentation regions.

In a fragmentation region the double Lorentz pole expansion is not allowed and one has to use a single Lorentz pole expansion. Then, using a procedure analogous to the preceding one we get ²⁴⁾⁻²⁶⁾

$$\frac{F^\alpha(p_\alpha, p_A, p_B)}{\sigma_I} \simeq f_A(\xi_{\alpha A}, p_\alpha), \quad \xi_{B\alpha} \gg 1. \quad (5.20)$$

This function does not depend on the kind of the particle B.

A more careful analysis should take into account parity conservation and the possibility of the contribution of Lorentz poles with different quantum numbers ³⁸⁾ σ and M . However, the same argument used for the total cross-section formula ³⁸⁾ shows that only Lorentz poles with $\sigma = 1$ and $M = 0$ can contribute if the incoming particles are unpolarized.

6. TWO-PARTICLE DISTRIBUTIONS IN MULTIPERIPHERAL DYNAMICS

The arguments of the preceding section can be generalized to the case in which many particles are observed. Some details are given in Appendix B. The main results can be summarized in two general laws. The first is the well-known factorization law, which, applied to inclusive distributions takes the following form.

- i) We consider a k particle inclusive distribution in the region defined by

$$\left\{ \begin{array}{l} \xi_j - \xi_A \gg 1 \\ \xi_j - \xi_i \gg 1 \\ \xi_B - \xi_i \gg 1 \\ \text{whenever } i \leq k' < j \\ p_i \text{ small for } i=1, \dots, k \end{array} \right. , \quad (6.1)$$

where k' is an integer between 0 and k . Then, we have

$$\begin{aligned} \sigma_I^{-1} F^{\alpha_1 \dots \alpha_k} (p_1, \dots, p_k, p_A, p_B) &\simeq \\ &\simeq \mathcal{F}_A(\xi_{1A}, p_1, \varphi_1, \dots, \xi_{k'A}, p_{k'}, \varphi_{k'}) \cdot \mathcal{F}_B(\xi_{B, k'+1}, p_{k'+1}, \\ &\varphi_{k'+1}, \dots, \xi_{Bk}, p_k, \varphi_k) = \sigma_I^{-2} F^{\alpha_1 \dots \alpha_{k'}} (p_1, \dots, p_{k'}, p_A, p_B) F^{\alpha_{k'+1} \dots \alpha_k} (p_{k'+1}, \dots, p_k, p_A, p_B) \end{aligned} \quad (6.2)$$

This means that two groups of particles well separated in the rapidity scale are uncorrelated. Moreover, each factor is independent of the rapidity and of the kind of one of the incoming particles and, therefore, it is independent of the total energy.

This is a general formulation of the limiting fragmentation hypothesis. Equation (5.20) is the simplest example of this general law. Another simple example is given by the equation

$$\begin{aligned} & \sigma_I^{-1} F^{\alpha\beta}(p_\alpha, p_\beta, p_A, p_B) \simeq \\ & \simeq \sigma_I^{-2} F^\alpha(p_\alpha, p_A, p_B) F^\beta(p_\beta, p_A, p_B), \quad |\xi_{\beta\alpha}| \gg 1. \end{aligned} \quad (6.3)$$

The second general consequence of multiperipherality is summarized in the following "invariance law" or "generalized scaling law":

- ii) If all the k observed particles are in the pionization region, the k particle inclusive distribution function divided by σ_I is independent of the total energy and of the kind of the incoming particles. We measure the inclusive distribution function in system of reference such that the incoming particles move along the z axis. Then, this distribution function is invariant with respect to the subgroup Π of the Lorentz group generated by the boosts along the z axis, the rotations around the z axis, the inversion of the z co-ordinate and the inversion of the x co-ordinate. It is also invariant with respect to the whole internal symmetry group. Of course, the invariance with respect to an element of Π is valid only if it does not carry the rapidities outside the pionization region.

Equation (5.19) is the simplest example of this law. It is clear that this law, if confirmed experimentally, greatly facilitates the analysis of the pionization phenomena. For instance, the two-particle distribution function in the pionization region should depend only on the variables $p_\alpha, p_\beta, |\xi_{\beta\alpha}|$ and $|\varphi_{\beta\alpha}|$ (see Section 2) or, equivalently, on the variables $v_r, p_r, \sin \theta$ and $|\psi|$.

When the observed particles are two charged pions, the invariance under charge conjugation gives the relations (valid in the pionization region)

$$\begin{cases} F^{\pi^+ \pi^+} = F^{\pi^- \pi^-} \\ F^{\pi^+ \pi^-} = F^{\pi^- \pi^+} \end{cases} \quad (6.4)$$

Multiperipheral dynamics can give some further important informations on the two-particle inclusive distribution when also the parameter $\xi_{\beta\alpha}$ is large ²⁶⁾. We assume

$$\xi_{\alpha A} \gg 1, \quad \xi_{B\beta} \gg 1, \quad \xi_{\beta\alpha} \gg 1 \quad (6.5)$$

As we are interested in the pionization region, we consider only the leading terms in $\xi_{\alpha A}$ and $\xi_{B\beta}$, but we consider also the first non-leading term in $\xi_{\beta\alpha}$.

As in the preceding Section, we introduce the Lorentz transformations which connect the various rest frames:

$$\begin{cases} a_{\alpha A} = u_z(\varphi_\alpha) a_z(\xi_{\alpha A}) a_x(\eta_\alpha) \\ a_{\beta\alpha} = a_x(-\eta_\alpha) u_z(\varphi_{\beta\alpha}) a_z(\xi_{\beta\alpha}) a_x(\eta_\beta) \\ a_{B\beta} = a_x(-\eta_\beta) a_z(\xi_{B\beta}) u_z(-\varphi_\beta) \end{cases} \quad (6.6)$$

The straightforward generalization of Eq. (5.11) is the triple Lorentz pole contribution

$$\begin{aligned} \mathcal{C}_2^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B) &\simeq \sum_{j m j' m'} \tilde{\rho}^A \mathcal{D}_{00 j m}^{0, \alpha+1}(a_{\alpha A}) \cdot \\ &\cdot \tilde{\rho}_j^\alpha \cdot \mathcal{D}_{j m j' m'}^{0, \alpha+1}(a_{\beta\alpha}) \tilde{\rho}_{j'}^\beta \mathcal{D}_{j' m' 00}^{0, \alpha+1}(a_{B\beta}) \tilde{\rho}^B \end{aligned} \quad (6.7)$$

In this formula the central Lorentz pole is assumed to have quantum numbers $\sigma = 1$ and $M=0$. However, in this case, also poles with $M=1,2,\dots$ can contribute. The corresponding formula is (taking into account parity conservation)

$$G^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B) \simeq \sum_{q=\pm 1} \sum_{j m j' m'} \tilde{p}^A \mathcal{O}_{00 j m}^{0, a+1}(a_{\alpha A}) \cdot \tilde{p}_j^\alpha \mathcal{O}_{j m j' m'}^{q M, a+1}(a_{\beta \alpha}) \tilde{p}_{j'}^\beta \mathcal{O}_{j' m' 00}^{0, a+1}(a_{B\beta}) \tilde{p}^B. \quad (6.8)$$

Of course, the quantities a' , $\tilde{p}_j^\alpha$ and $\tilde{p}_j^\beta$ depend on the pole exchanged in central position. Poles with $M=0$ and $\sigma = -1$ cannot contribute.

Now we proceed as in the preceding Section. If the conditions (6.4) are satisfied, we can write

$$\begin{cases} a_{\alpha A} \simeq u_z(\varphi_\alpha) a_z(z_{\alpha A}) u_y(-\chi_\alpha), \\ a_{\beta \alpha} \simeq u_y(-\chi_\alpha) u_z(\varphi_{\beta \alpha}) a_z(z_{\beta \alpha}) u_y(-\chi_\beta), \\ a_{B\beta} \simeq u_y(-\chi_\beta) a_z(z_{B\beta}) u_z(-\varphi_\beta), \end{cases} \quad (6.9)$$

where

$$\begin{cases} \exp \zeta_{\alpha A} = \exp \xi_{\alpha A} \cosh \eta_\alpha, \\ \exp \zeta_{\beta \alpha} = \exp \xi_{\beta \alpha} \cosh \eta_\alpha \cosh \eta_\beta, \\ \exp \zeta_{B\beta} = \exp \xi_{B\beta} \cosh \eta_\beta, \end{cases} \quad \begin{cases} \operatorname{tg} \chi_\alpha = \sinh \eta_\alpha = \frac{p_\alpha}{M_\alpha}, \\ \operatorname{tg} \chi_\beta = \sinh \eta_\beta = \frac{p_\beta}{M_\beta}. \end{cases} \quad (6.10)$$

Using these formulae and Eqs. (5.12) and (5.13), our Lorentz pole contributions can be written in the form

$$G^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B) \simeq \exp(a \xi_{\alpha A} + a' \xi_{\beta\alpha} + a \xi_{B\beta}) \cdot (\cosh \eta_\alpha)^{a+a'} (\cosh \eta_\beta)^{a+a'} p^A p^B \sum_j p_j^\alpha P_j (\omega \chi_\alpha) \sum_{j'} p_{j'}^\beta P_{j'} (\omega \chi_\beta) \quad (6.11)$$

for $M=0$ and

$$G^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B) \simeq \exp(a \xi_{\alpha A} + a' \xi_{\beta\alpha} + a \xi_{B\beta}) \cdot (\cosh \eta_\alpha)^{a+a'} (\cosh \eta_\beta)^{a+a'} p^A p^B 2 \cos(M \varphi_{\beta\alpha}) \cdot \sum_j p_j^\alpha R_{0M}^j(u_\gamma(-2\chi_\alpha)) \cdot \sum_{j'} p_{j'}^\beta R_{M0}^{j'}(u_\gamma(-2\chi_\beta)) \quad (6.12)$$

for $M>0$.

Comparing with Eq. (5.18) for the total cross-section we have for the leading term with $M=0$, $a=a'$:

$$\frac{1}{\sigma_I} F^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B) \simeq N^\alpha(p_\alpha) N^\beta(p_\beta) \cdot \begin{cases} \xi_{B\beta} \gg 1 \\ \xi_{\beta\alpha} \gg 1 \\ \xi_{\alpha A} \gg 1 \end{cases} \quad (6.13)$$

This is just a consequence of Eqs. (6.3) and (5.19).

If $\xi_{\beta\alpha}$ is not large, we have to introduce correction terms. From the general invariance law stated at the beginning of the Section we have in the pionization region

$$\sigma_I^{-1} F^{\alpha\beta}(p_\alpha, p_\beta, p_A, p_B) \simeq N^\alpha(p_\alpha) N^\beta(p_\beta) + \\ + H^{\alpha\beta}(p_\alpha, p_\beta, \xi_{\beta\alpha}, \varphi_{\beta\alpha}) , \quad \begin{cases} \xi_{B\beta} \gg 1 \\ \xi_{\alpha A} \gg 1 \end{cases} . \quad (6.14)$$

The function H can be expanded asymptotically in terms of non-leading poles. If for each value of M we consider a pole with intercept a'_M , from Eqs. (6.11) and (6.12) we obtain

$$H^{\alpha\beta}(p_\alpha, p_\beta, \xi_{\beta\alpha}, \varphi_{\beta\alpha}) = \sum_M \exp[(a'_M - a)/\xi_{\beta\alpha}] \cdot \\ \cdot \cos(M \varphi_{\beta\alpha}) b_M^\alpha(p_\alpha) b_M^\beta(p_\beta) + O(\exp(-|\xi_{\beta\alpha}|)) . \quad (6.15)$$

As this expression has to be real, it is clear that, considering a given pole, the functions $b_M^\alpha(p)$ have to be either real or imaginary and the choice has to be independent of α . A full use of factorization shows that for $M=0$ the real choice has to be used for poles which give a positive contribution to proton-proton total cross-sections. For example, the P' pole is expected to be of the real type, while the ρ and the ω poles are expected to be of the imaginary type.

One can separate the contributions of the different values of M introducing the Fourier transforms

$$\tilde{H}^{\alpha\beta, M}(p_\alpha, p_\beta, \xi_{\beta\alpha}) = \frac{1}{\pi} \int_0^\pi \cos(M \varphi_{\beta\alpha}) H^{\alpha\beta}(p_\alpha, p_\beta, \xi_{\beta\alpha}, \varphi_{\beta\alpha}) d\varphi_{\beta\alpha} . \quad (6.16)$$

Also, the internal quantum numbers of the exchanged Lorentz poles have an important meaning. As the leading Pomeranchuk pole has the vacuum quantum numbers, in the approximation of Eq. (6.13)

the internal quantum numbers of the particles α and β are completely uncorrelated. Only the exchange of non-vacuum Lorentz poles can introduce correlations of the internal quantum numbers in the approximation of Eqs. (6.14) and (6.15).

If we observe two charged pions, it is useful to consider the two functions

$$\begin{cases} F^0 = \frac{1}{2} (F^{\pi^+\pi^-} + F^{\pi^+\pi^+}) , \\ F^1 = \frac{1}{2} (F^{\pi^+\pi^-} - F^{\pi^+\pi^+}) , \end{cases} \quad (6.17)$$

[see Eq. (6.4)]. Then, it is easy to show that isoscalar poles contribute to F^0 and isovector poles contribute to F^1 .

The same procedure can be applied to the functions (6.16) obtaining the functions $\tilde{H}^{0M}$ and $\tilde{H}^{1M}$. Then, $\tilde{H}^{00}$ and $\tilde{H}^{10}$ for large $|\xi_{\beta\alpha}|$ should be dominated by the P' and ρ poles respectively and should decrease as $\exp(-\frac{1}{2}|\xi_{\beta\alpha}|)$. If, as it has been suggested^{39),40)}, the pole ρ' has $M=1$, it should contribute to $\tilde{H}^{11}$.

If the statistics of the experiments are not sufficiently good, one can get an idea of the importance of these different contributions considering the partially integrated normalized distributions

$$K^{\alpha\beta, M}(\xi_{\beta\alpha}) = \pi^2 \int \tilde{H}^{\alpha\beta, M}(p_\alpha, p_\beta, \xi_{\alpha\beta}) p_\alpha dp_\alpha p_\beta dp_\beta . \quad (6.18)$$

We remark that the two-particle inclusive distribution represents a unique opportunity to investigate singularities with arbitrary M quantum number in the Lorentz λ plane.

7. TWO-PARTICLE CORRELATIONS DUE TO RESONANCES

The simplest kind of correlation is provided by the resonances. In the pionization region, a two-particle resonance affects the two-particle distribution $F^{\alpha\beta}$ enhancing it in the region where the invariant mass v_r defined in Eq. (2.36) is near the resonance mass.

A naïve procedure to describe the effect of a resonance is to treat separately its production and its decay. As far as its production is concerned, it is assimilated to a stable particle. This procedure is justified if the peak in the two-particle distribution is much higher than the underlying background, as in this case one can disregard the interference between the decay products of the resonance and differently produced particles. When this condition is fulfilled one can define an inclusive distribution function for the resonance itself and consistently subtract the contribution of the decay products from the respective one and two-particle inclusive distributions.

Since we know that in the pionization region the produced particles are mostly pions, the most important effect of this kind is due to the ρ meson. We want to see whether the naïve procedure can be consistently applied to this case. The most favourable situation would be the one in which all the pions produced are decay products of ρ mesons.

We consider in detail neutral ρ mesons. Their distribution is given by

$$d^4\sigma = F^{\rho}(p_p, p_A, p_B) d^4p_{\rho} \quad (7.1)$$

In the pionization region we have, if we accept the general scheme described in the preceding Sections,

$$\sigma_I^{-1} F^P(P, P_A, P_B) = N_P(p_P, v_P) \quad (7.2)$$

Remark that this function depends also on the mass of the resonant system. It does not depend on the charge of the meson.

Equation (4.23) has to be replaced by

$$h_P = \pi \int N_P(p_P, v_P) p_P dp_P d(v_P^2) \quad (7.3)$$

This quantity is the average number of ρ^0 mesons produced in the pionization region per inelastic event and per unit of rapidity. According to our assumptions, it is also equal to 1/6 of the number of pions (with any charge) produced in the same conditions. From the estimate (8.4) we get $hg \simeq 0.25$, disregarding particles different from pions.

We shall use the explicit formula

$$N_P(p_P, v_P) = \frac{2\pi^{-2} h_P \Lambda_P^2 M_P \Gamma_P e^{-\Lambda_P^2 p_P^2}}{(v_P^2 - M_P^2)^2 + M_P^2 \Gamma_P^2} \cdot \left[\frac{M_P^2}{v_P^2} \frac{v_P^2 - 4M_\pi^2}{M_P^2 - 4M_\pi^2} \right]^{3/2} \quad (7.4)$$

Equation (7.3) is satisfied in the approximation of small width.

The distribution of the momenta of the pions coming from a decay of a ρ meson of four-momentum P is given by

$$8\delta^4(P - P_{\pi^+} - P_{\pi^-}) \left[1 - \frac{4M_\pi^2}{v_P^2} \right]^{-1/2} Y(\theta) \frac{d^3 P_{\pi^+}}{2P_{\pi^+}^0} \frac{d^3 P_{\pi^-}}{2P_{\pi^-}^0} \quad (7.5)$$

The angle θ is defined in Eq. (2.36). For unpolarized ρ meson we have

$$Y(\theta) = (4\pi)^{-1}, \quad (7.6)$$

and, if the spin projection m_z is zero, we have

$$Y(\theta) = \frac{3}{4\pi} (\cos\theta)^2. \quad (7.7)$$

Putting together Eqs. (7.4), (7.5) we get the one-pion distribution

$$N_{\pi^+}(p_{\pi^+}) = 2 \int N_\rho(p_\rho, v_\rho) \cdot 8 \left[1 - \frac{4M_\pi^2}{v_\rho^2} \right]^{-1/2} \cdot Y(\theta) \cdot \frac{d^3 p_{\pi^-}}{2 p_{\pi^-}^0}. \quad (7.8)$$

The factor 2 in front of the right-hand side of this formula takes into account the pions coming from the decay of a charged ρ .

Integrating this formula by means of Eqs. (2.37), (4.23) and (7.3) we get the expected result

$$h_{\pi^+} = h_{\pi^-} = h_\pi = \pi \int N_{\pi^+}(p_{\pi^+}) p_{\pi^+} dp_{\pi^+} = 2 h_\rho \quad (7.9)$$

The normalized one-pion distribution in Eq. (7.8) is plotted in Fig. 3 assuming $\Lambda_\rho^2 = 4(\text{GeV}/c)^{-2}$.

Neglecting correlations between ρ mesons, the two-pion distribution is

$$\sigma_I^{-1} F^{\pi^+\pi^-}(p_{\pi^+}, p_{\pi^-}, p_A, p_B) = N_\rho(p_\rho, v_\rho) \cdot 8 \cdot \left[1 - \frac{4M_\pi^2}{v_\rho^2} \right]^{-1/2} \cdot Y(\theta) + N_{\pi^+}(p_{\pi^+}) N_{\pi^-}(p_{\pi^-}) \quad (7.10)$$

The last term represents the background of uncorrelated pions coming from two different ρ mesons.

The ratio R between the first and the second term in Eq. (7.10), that is the ratio between the resonant contribution and the background, is given in Fig. 4 for some interesting values of the parameters, always such that $v_\rho = M_\rho$, i.e., on the resonance peak. We see that this ratio is in general of the order of one. It is smaller if, as probable, not all the pions come from the decay of ρ mesons. Therefore, in most cases the condition which permits to assimilate the ρ resonance to a stable particle is not fulfilled.

Nevertheless, we think that the naïve treatment can give some qualitative indication about the possible resonance effects. In Fig. 5 we plot the functions [see Eq. (6.18)]

$$\begin{cases} h_\rho^{-1} K^{\pi^+\pi^-,0}(\xi_{\pi^+\pi^-}) \\ h_\rho^{-1} K^{\pi^+\pi^-,1}(\xi_{\pi^+\pi^-}) \end{cases}, \quad (7.11)$$

computed with the naïve model assuming unpolarized ρ mesons and the same quantities for fully polarized ρ mesons. We see that the correlation effects are small with respect to the uncorrelated background $(h_\pi)^2 h_\rho^{-1}$, which, under our assumptions, is ~ 1 .

These calculations do not take into account interference effects which, as we have said, are probably relevant in these conditions.

The estimates of Fig. 4 show that we are in a characteristic strong coupling situation. The situation in which a given pion can simultaneously resonate with several other different pions occurs rather often. As a consequence, we are faced with problems of double counting and the concept of duality becomes essential. We conclude that a detailed treatment of resonances in the pionization region is possible only within the framework of a consistent unitarized dual model.

8. AVERAGE MULTIPLICITIES AND OTHER AVERAGE QUANTITIES

In the preceding Sections we have examined only one and two-particle inclusive distributions. Now we show that many other measurable quantities can be expressed in terms of these functions.

First of all we see that from Eqs. (2.11) and (2.16) we get the average multiplicities $\langle n_\alpha \rangle$ and the corresponding variance $\langle n_\alpha^2 \rangle - \langle n_\alpha \rangle^2$. From Eq. (2.11) we have

$$\langle n_\alpha \rangle = \xi_{BA} \pi \int_0^\infty N_\alpha(p_\alpha) p_\alpha dp_\alpha + \pi \iint d\xi_\alpha p_\alpha dp_\alpha \cdot$$

$$\cdot \left[\sigma_I^{-1} F^\alpha(p_\alpha, p_A, p_B) - N_\alpha(p_\alpha) \theta(\xi_{\alpha A}) \theta(\xi_{\alpha B}) \right].$$

(8.1)

Assuming the Feynman scaling law [see Eq. (5.19)], we see that the second integral is confined to the fragmentation regions and tends to a finite limit for high energy, so that we have

$$\langle n_\alpha \rangle \simeq h_\alpha \log \frac{s}{c_\alpha}.$$

(8.2)

We note that c_α can depend on the properties of the incoming particles, while h_α [see Eq. (4.23)] does not.

If we sum over α we get

$$\langle n \rangle \simeq h \log \frac{s}{c},$$

(8.3)

where h is given by Eq. (4.40). Cosmic ray experiments suggest the value ^{3), 41)}

$$h = 1.5$$

(8.4)

Using Eq. (2.16) we get

$$\langle n_\alpha^2 \rangle - \langle n_\alpha \rangle^2 = \langle n_\alpha \rangle + R_{\alpha\alpha} , \quad (8.5)$$

$$\langle n_\alpha n_\beta \rangle - \langle n_\alpha \rangle \langle n_\beta \rangle = R_{\alpha\beta} , \quad \alpha \neq \beta , \quad (8.6)$$

and, as a consequence,

$$\langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle + \sum_{\alpha\beta} R_{\alpha\beta} , \quad (8.7)$$

where we have introduced the general definition

$$R_{\alpha\beta} = \iint \frac{d^3 p_\alpha}{2 p_\alpha^0} \frac{d^3 p_\beta}{2 p_\beta^0} \cdot \left[\sigma_I^{-1} F^{\alpha\beta}(p_\alpha, p_\beta, p_A, p_B) - \sigma_I^{-2} F^\alpha(p_\alpha, p_A, p_B) F^\beta(p_\beta, p_A, p_B) \right]. \quad (8.8)$$

This integral vanishes in absence of two-particle correlations. According to Eq. (6.3) in this formula the integrand function vanishes if $|\xi_\alpha - \xi_\beta|$ is large. Moreover, when both ξ_α and ξ_β are in the pionization region, the integrand depends only on their difference. Then, from Eq. (6.14) we easily see that

$$R_{\alpha\beta} \simeq \frac{\pi}{2} \xi_{BA} \int H^{\alpha\beta}(p_\alpha, p_\beta, \xi_{\alpha\beta}, \varphi_{\alpha\beta}) p_\alpha dp_\alpha p_\beta dp_\beta d\xi_{\alpha\beta} d\varphi_{\alpha\beta} . \quad (8.9)$$

We have disregarded a term which tends to a finite limit for large energy. This expression is proportional to $\log s$.

In order to get a rough estimate of this integral, we may try to extrapolate the asymptotic formula (6.15) at smaller values of $\{\alpha\beta\}$. We see that only Lorentz poles with $M=0$ can contribute. If we consider the sum in Eq. (8.7), we realize that only poles with vacuum internal quantum numbers survive. As far as the pole P' is concerned, we see from the discussion of Section 6 that its contribution is positive. Then, from Eq. (8.7) we have that, as an effect of two-particle correlations, the variance is larger than expected from Poisson distribution. It is clear that correlations due to resonances have a similar effect. These considerations are in agreement with experiments ⁴¹⁾ which show that the two terms in the right-hand side of Eq. (8.7) are of the same order of magnitude.

It is useful to generalize the preceding considerations to treat quantities more directly connected with experiments. We consider a particle detector with an efficiency $W_\alpha(P_\alpha)$ which depends on the kind α of the particle and on its four-momentum. We assume that this apparatus can detect many particles simultaneously and give for each event their total number n_W . Then, we have

$$\langle n_W \rangle = \sum_{\alpha} \int \sigma_I^{-1} F^{\alpha}(P_{\alpha}, P_A, P_B) W_{\alpha}(P_{\alpha}) \frac{d^3 \underline{P}_{\alpha}}{2 P_{\alpha}^0} , \quad (8.10)$$

$$\begin{aligned} \langle n_W^2 \rangle - \langle n_W \rangle^2 &= \langle n_W \rangle + \sum_{\alpha\beta} \iint \frac{d^3 \underline{P}_{\alpha}}{2 P_{\alpha}^0} \frac{d^3 \underline{P}_{\beta}}{2 P_{\beta}^0} \cdot \\ &\cdot W_{\alpha}(P_{\alpha}) W_{\beta}(P_{\beta}) \left[\sigma_I^{-1} F^{\alpha\beta}(P_{\alpha}, P_{\beta}, P_A, P_B) - \sigma_I^{-2} \cdot \right. \\ &\cdot \left. F^{\alpha}(P_{\alpha}, P_A, P_B) F^{\beta}(P_{\beta}, P_A, P_B) \right] \end{aligned} \quad (8.11)$$

More interesting experiments can be done using also a second detector with efficiency function $W'_\alpha(P_\alpha)$. We assume that

$$W_\alpha(P_\alpha) \cdot W'_\alpha(P_\alpha) = 0, \quad (8.12)$$

that means that the two detectors do not overlap. Then, we can perform a correlation experiment and measure the quantity

$$\begin{aligned} \langle n_w n_{w'} \rangle - \langle n_w \rangle \langle n_{w'} \rangle &= \sum_{\alpha\beta} \iint \frac{d^3P_\alpha}{2P_\alpha^0} \frac{d^3P_\beta}{2P_\beta^0} \cdot \\ &\cdot W_\alpha(P_\alpha) W'_\beta(P_\beta) \cdot \left[\sigma_I^{-1} F^{\alpha\beta}(P_\alpha, P_\beta, P_A, P_B) - \right. \\ &\left. - \sigma_I^{-2} F^\alpha(P_\alpha, P_A, P_B) F^\beta(P_\beta, P_A, P_B) \right]. \end{aligned} \quad (8.13)$$

If the ideas described in the preceding Sections are right and if the regions where W and W' respectively do not vanish are sufficiently separated in the rapidity scale, this quantity is expected to be small.

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APPENDIX A

THE LORENTZ EXPANSION OF THE ONE-PARTICLE DISTRIBUTION

We put

$$\begin{cases} Q = -\sqrt{u} \hat{Q} , \\ \hat{Q}^2 = -1 , \\ \hat{Q} = (\sinh \sigma, \cosh \sigma \underline{\hat{Q}}) . \end{cases} \quad (A.1)$$

The diagonalization of the multiperipheral equation (5.5) is performed projecting on the harmonic functions on the one-sheet hyperboloid in Minkowski space. The functions which are needed have the form ^{42),43)}

$$B_{jm}^{\lambda}(\hat{Q}) = b_j^{\lambda}(\sigma) Y_{jm}(\hat{Q}) , \quad (A.2)$$

where $Y_{jm}(\hat{Q})$ are spherical harmonics and

$$b_j^{\lambda}(\sigma) = i^j \left[\frac{\lambda \Gamma(j+1-\lambda)}{\sin(\pi\lambda) \cdot \Gamma(j+1+\lambda)} \right]^{\frac{1}{2}} (\cosh \sigma)^{-1} P_j^{\lambda}(\tanh \sigma) . \quad (A.3)$$

These functions form a basis for a representation of the Lorentz group, in the sense that we have ⁴⁴⁾

$$B_{jm}^{\lambda}(L(a)\hat{Q}) = \sum_{j'm'} \mathcal{D}_{jmj'm'}^{\sigma\lambda}(a) B_{j'm'}^{\lambda}(\hat{Q}) . \quad (A.4)$$

In the frame where $P_A = (M, 0, 0, 0)$ the function $A(Q, P_A)$ is spherically symmetric and, therefore, only the harmonic

$$B_{00}^{\lambda}(\hat{Q}) = \frac{1}{\sqrt{4\pi}} b_0^{\lambda}(\sigma) = \frac{1}{\sqrt{4\pi}} \frac{\exp(\lambda\sigma)}{\sqrt{\pi} \cosh \sigma} \quad (\text{A.5})$$

appears in the expansion (5.6).

In the frame in which

$$\begin{cases} P_{\alpha} = (M, 0, 0, 0) \\ P_A = L(a_{\alpha A}^{-1})(M, 0, 0, 0) \\ P_B = L(a_{B\alpha})(M, 0, 0, 0) \end{cases} ,$$

(A.6)

from Eqs. (5.6) and (A.4) we have

$$A(Q, P_A) = -i\sqrt{4\pi} \int_{c-i\infty}^{c+i\infty} a(\lambda, u) \sum_{j'm} \mathcal{D}_{00j'm}^{0\lambda}(a_{\alpha A}) B_{j'm}^{\lambda}(\hat{Q}) d\lambda, \quad (\text{A.7})$$

$$\begin{aligned} A(Q', P_B) &= -i\sqrt{4\pi} \int_{c-i\infty}^{c+i\infty} a(\lambda', u') \sum_{j'm'} \mathcal{D}_{00j'm'}^{0\lambda'}(a_{B\alpha}^{-1}) B_{j'm'}^{\lambda'}(\hat{Q}') d\lambda' = \\ &= -i\sqrt{4\pi} \int_{c-i\infty}^{c+i\infty} a(\lambda', u') \sum_{j'm'} \mathcal{D}_{j'-m', 0, 0}^{0\lambda'}(a_{B\alpha}) (-1)^{j'-m'} B_{j'm'}^{\lambda'}(\hat{Q}') d\lambda'. \end{aligned} \quad (\text{A.8})$$

In the last step we have used a symmetry property of the $\mathcal{D}$ functions³⁶⁾. We have put

$$Q' = P_{\alpha} - Q, \quad (\text{A.9})$$

and we have used for this four-vector notations similar to those of Eq. (A.1).

We substitute Eqs. (A.7) and (A.8) into Eq. (5.4). Using the formula

$$(-1)^{j'-m'} \int y_{j',-m'}(-\hat{\underline{Q}}) y_{j,m}(\hat{\underline{Q}}) d\Omega = \delta_{jj'} \delta_{mm'}, \quad (\text{A.10})$$

we get

$$\begin{aligned} & -8\pi^2 \int d^4Q \Delta(u) \Delta(u') Q(\lambda, u) Q(\lambda', u') \\ & \cdot B_{jm}^\lambda(\hat{\underline{Q}}) (-1)^{j'+m'} B_{j',-m'}^{\lambda'}(\hat{\underline{Q}}') = \delta_{jj'} \delta_{mm'} F_j(\lambda, \lambda'). \end{aligned} \quad (\text{A.11})$$

With this definition of $F_j(\lambda, \lambda')$, Eq. (5.8) follows immediately.

Using the independent variables u and u' , we have

$$\begin{aligned} F_j(\lambda, \lambda') &= -\frac{\pi^2}{M^2} \iint \Delta(u) \Delta(u') Q(\lambda, u) Q(\lambda', u') \\ &\cdot b_j^\lambda(\sigma) b_j^{\lambda'}(\sigma') [T(-u, -u', M^2)]^{1/2} du du', \end{aligned} \quad (\text{A.12})$$

where

$$\begin{cases} \sinh \sigma = -\frac{u' - u + M^2}{2M\sqrt{u}}, \\ \sinh \sigma' = -\frac{u - u' + M^2}{2M\sqrt{u'}}. \end{cases}$$

(A.13)

APPENDIX B

THE INVARIANCE GROUP OF PIONIZATION

In this Appendix we deduce from a Lorentz pole expansion the "generalized scaling law" mentioned in Section 6. It is possible that this law has a larger validity than expected from the arguments given below, which are based on multiperipheral dynamics.

Extending the arguments given in Section 5 and in the Appendix A, one can show that when all the observed particles are in the pionization region, the inclusive distribution can be approximated by a double Lorentz pole contribution. In order to write this contribution, we adapt the formalism developed in Refs. 45), 46). Besides the rest systems S_A and S_B of the incoming particles, we introduce two other systems S_C and S . S_C is obtained from S_A (respectively S_B) by means of the boost $a_z(\xi_{CA})$ [respectively $a_z(\xi_{BC})$] and is such that in this system the total four-momentum of the observed particles is small. The system S is a centre-of-mass system of the observed particles and its orientation is determined by means of some rule by the momenta $P_1, \dots, P_k$ of the observed particles [for details, see Refs. 45), 46)]. S is connected to S_C by means of the transformation $\hat{a}$. It is clear that the four-momenta of the observed particles are uniquely determined by ξ_{CA} , $\hat{a}$, and by a set of invariants built by means of these four-vectors alone which we shall collectively indicate by the symbol I .

It is sufficient to prove the invariance under the group Π of the inclusive distribution measured in the system S_C . If we use the variables ξ_{CA} , $\hat{a}$, I , the effect of a transformation $b \in \Pi$, is to replace $\hat{a}$ by $b\hat{a}$.

We approximate both the inclusive distribution and the total inelastic cross-section by the leading Lorentz pole contribution and we get, generalizing Eqs. (5.11) and (5.18)

$$\begin{aligned}
 & \sigma_I^{-1} F^{\alpha_1 \dots \alpha_k} (P_1, \dots, P_k, P_A, P_B) \simeq \\
 & \simeq \sum_{j_A m_A j_B m_B} \rho_{j_A m_A}^A \mathcal{D}_{j_A m_A j_B m_B}^{0, a+1} (a_z(\xi_{CA}) \hat{a}) \cdot \\
 & \cdot \rho_{j_B m_B}^B (I) \mathcal{D}_{j_B m_B}^{0, a+1} (\hat{a}^{-1} a_z(\xi_{BC})) \rho_{j_B m_B}^B \\
 & \cdot \left[\sum_{j_A m_A j_B m_B} \rho_{j_A m_A}^A \mathcal{D}_{j_A m_A j_B m_B}^{0, a+1} (a_z(\xi_{CA} + \xi_{BC})) \rho_{j_B m_B}^B \right]^{-1}.
 \end{aligned}
 \tag{B.1}$$

We have considered polarized incoming particles, as otherwise the invariance with respect to rotations around the z axis is trivial. This generalization has required the introduction of the indices j_A, m_A, j_B, m_B .

The limit we are considering is the one in which ξ_{CA} and ξ_{BC} go to infinity. Then, we can use the representation property and the asymptotic equation (5.13) and we get after some manipulations

$$\begin{aligned}
 & F^{\alpha_1 \dots \alpha_k} (P_1, \dots, P_k, P_A, P_B) \sigma_I^{-1} \simeq \\
 & \sum_{j_m j'_m j'' j'''} C_{j''}^{0, a+1} \mathcal{D}_{j'' j_m}^{0, a+1} (\hat{a}) \rho_{j_m j'_m} (I) \cdot \\
 & \cdot \mathcal{D}_{j'_m j'' j'''}^{0, a+1} (\hat{a}^{-1}) C_{j'''}^{0, a+1} = \lim_{z \rightarrow \infty} \exp(-a_z) \cdot \\
 & \cdot \sum_{j'_m j_m} \mathcal{D}_{j'_m j_m}^{0, a+1} (\hat{a}^{-1} a_z(z) \hat{a}) \rho_{j_m j'_m} (I).
 \end{aligned}
 \tag{B.2}$$

First of all we remark that any dependence on the properties of the incoming particles and on the total energy has disappeared. Then, we note that the replacement $\hat{a} \rightarrow b\hat{a}$ does not influence this expression if b commutes with all the boosts $a_z(\zeta)$. This proves our result for all the elements of Π apart from those containing a reflection of the z co-ordinate.

In order to treat this last case, imagine that the reaction is initiated by two identical pions. Then, the invariance under reflection of the z co-ordinate is trivial. Since the quantity (B.2) does not depend on the incoming particles, the invariance is proven in general.

The invariance of the inclusive distribution under the internal symmetry group, follows immediately from the assumption that the leading pole has the vacuum quantum numbers.

T A B L E

Lab. Incident momentum (GeV/c)	s (GeV) ²	ξ_{BA}
28	56	4.1
70	133	5.0
500	940	7.0
1670	3136	8.2

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FIGURE CAPTIONS

Figure 1:

Kinematically accessible region in the plane of the rapidities $\{ \alpha \}^*$ and $\{ \beta \}^*$ of two observed pions for $s = 3136 \text{ (GeV)}^2$. Multiperipheral dynamics indicates that correlations are important only in the regions A, A' and B. Correlations due to four-momentum conservation are restricted roughly to the regions A and A'.

Figure 2:

Definition of variables in the multiperipheral configuration.

Figure 3:

Normalized distribution in the transverse momentum of pions coming from the decay of a ρ meson in the pionization region. The continuous line refers to completely polarized ρ mesons and the dashed line to unpolarized ones [see Eq. (7.8)].

Figure 4:

Ratio between the ρ resonant peak and the background in the two-pion inclusive distribution in the pionization region, according to the simplified picture we are considering. Continuous and dashed lines refer respectively to the completely polarized and to the unpolarized case.

Figure 5:

Distributions in relative rapidities of pairs of pions coming from the decay of a ρ meson, in the pionization region. These distributions are integrated on the transverse momenta and Fourier analyzed with respect to the azimuthal angle [see Eqs. (6.18) and (7.11)]. Continuous and dashed lines refer to the completely polarized and to the unpolarized case respectively.

FIG.1

FIG.2

FIG.3

FIG.5