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Particle Creation and Vacuum Polarization of Nonconformal Scalar Field Near the Isotropic Cosmological Singularity

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Abstract

The effects of particle creation and vacuum polarization in external gravitational field offer one possibility to attack certain problems of classical cosmology such as the occurrence of particle horizons.

We calculate the vacuum expectation value of the stress-energy tensor of a nonconformal scalar field in a form that is suitable for studying the back reaction effects in cosmology. The gravitational background metric is of Robertson-Walker type. By exploiting the early time approximation it proves possible to represent the result as an explicit functional of the scale factor. Its properties are discussed and the conformal anomaly is correctly reproduced. The density of created particles is also calculated.

The energy density is probed for the particular class of degree-type scale factors which are relevant in Friedmann cosmology. Except for the square root expansion law it is found to depend sensitively on the curvature coupling coefficient. The new contributions can become large compared to the previously known conformal contributions and may significantly influence the initial gravitational field.

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1 Introduction

According to the standard cosmological model [1] our universe was born around 15 milliard years ago in the hot big bang, a singular state in space and time. When approaching this initial singularity the temperature grows beyond all limits thereby being responsible for the name “hot“ big bang. The existence of a state of complete thermodynamic equilibrium is a key assumption in the construction of the standard cosmological model. At some very early stage of the cosmic evolution ($T \geq 10^{11}$ K) all kinds of matter and radiation are supposed to be in thermal equilibrium, mainly via pair creation and annihilation processes. Then, as the cosmic expansion causes the temperature to decrease, not all the reactions can maintain thermal equilibrium. As a consequence, different sorts of matter and radiation subsequently drop out of equilibrium and decouple. The 2,7 K cosmic black body radiation discovered in 1965 by Penzias and Wilson [2] is the most prominent witness of that era. The nucleosynthesis in the early universe provides even more evidence in favour of the hot big bang scenario. The predicted abundances of the light elements agree very well with those from observation. Comprehensive accounts of the standard cosmological model can be found, e.g., in the refs. [1, 3].

Nevertheless, despite its remarkable success, the standard cosmological model suffers from severe problems that could not consistently be resolved to date. The inevitable appearance of particle horizons [4] and a missing explanation of the isotropy of the cosmic expansion are among the major drawbacks of the classical cosmological model. The existence of particle horizons results in a universe that in the past necessarily had to consist of many causally disconnected regions. This fact naturally raises the question of how a state of thermodynamic equilibrium could have come about under such circumstances. The isotropy of the cosmic expansion enters the model as a postulate, namely the Cosmological Principle [1]. From the observational side, this postulate is well justified by the high isotropy of the cosmic microwave background ($\Delta T/T \propto 10^{-5}$). On the other hand, the descriptive framework of General Relativity allows for a much wider class of solutions in the vicinity of the initial singularity. It therefore remains unclear why our universe could reach such a highly symmetric evolutionary stage.

Confronted with these difficulties, one is naturally led to ask for physical processes in the early universe which could bring about the desired effects. Particle creation and vacuum polarization in strong gravitational field offer

one possibility to cope with cosmological problems of this kind [5, 6]. Over the last thirty years, this idea has been the object of intense investigations. Recently, quantum effects in the early universe have also been studied in connection with inflationary models [7] and the discussion has been extended to include interacting matter fields [8]. Several results have been obtained so far. It turns out that the mechanism of particle creation and vacuum polarization is able to avoid particle horizons and even singularities [9, 10, 11, 12, 13]. Likewise, a sufficiently rapid isotropization of the cosmic expansion can be achieved [14, 15, 16, 17]. Although these results are very promising they rely on various restrictions such as conformal coupling ($\xi = 1/6$) [10, 15], massless fields [16] or quasiclassical approximations for the density of created particles [14].

The effects of vacuum polarization and particle creation are expected to become relevant at times between the Planck time ($t_p = \sqrt{G\hbar/c^5} \sim 5 \cdot 10^{-44} s$) and the Compton time of the matter field. In this region the quantum aspects of the matter field are essential. In contrast, the gravitational field can still be considered as a classical field. In this so called semiclassical theory the back reaction of the matter on the gravitational field is accounted for by the expectation value of the stress-energy tensor of the matter field which acts as a source in the modified Einstein equations [18, 19]

$$G_{ik} = -8\pi G \langle T_{ik} \rangle. \quad (1)$$

In the framework of the semiclassical approach one encounters both, conceptual and technical problems. On the fundamental side, for example, a general principle for choosing the quantum state of the matter is not at hand and one has to deal with this question in the particular model under consideration. Moreover, since a consistent quantum theory of gravity is absent the precise status of the semiclassical equations has not yet been clarified completely [20] (and refs. therein). From the technical point of view, the renormalized expectation value of the stress-energy tensor of the matter field represents a complicated, nonlocal functional of the background metric. Its actual calculation turns out to be a tremendously difficult task. The necessity to make approximations becomes obvious.

The primary aim of the present paper is the calculation of the renormalized stress-energy tensor (SET) of a quantized scalar field in a gravitational background of Robertson-Walker type. We wish to obtain it in a form that is most suitable for studying cosmological implications of the back reaction.

More specifically, we derive a representation of the vacuum expectation value of the SET as an explicit functional of the scale factor of the metric. This explicit result is achieved with the help of the early time approximation. As discussed below, the early time approximation is very well adjusted to the conditions in the early universe. An additional advantage of this approach is reflected by the fact that the mass of the matter field as well as the curvature coupling coefficient ξ can be kept arbitrary so that the full effects can be taken into account. Indeed, the energy density is found to depend sensitively on the coupling coefficient ξ when the cosmologically important degree-type scale factors are investigated.

The outline of the paper follows. In sec. 2 we briefly recapitulate the formalism of quantized scalar fields in isotropic gravitational background [18, 19]. We derive formal expressions for the vacuum–vacuum matrix element of the SET and the density of created particles in a new parametrization. For this reason, the issue of renormalization of the SET is reconsidered in sec. 3. As expected, the subtracted terms are shown to coincide with the known ones [19]. In sec. 4 we carefully elaborate the early time approximation. With its help the vacuum expectation value of the SET as well as the created particle density are calculated explicitly. Their properties are discussed in some detail. We show, for instance, that the conformal anomaly of nonconformal scalar field is correctly reproduced. We also comment on the scaling properties of the renormalized SET. In sec. 5 the results are probed for particular scale factors, mainly the ones of degree-type which occur in Friedmann cosmology. As already quoted above, the energy density changes drastically when the coupling coefficient is relaxed from its conformal value. We conclude with a discussion of the results in sec. 6. The use of the early time approximation in a typical calculation is illustrated in the appendix.

If not otherwise indicated we use units such that $\hbar = c = 1$.

2 Particle density and the matrix elements of the stress–energy–tensor

The nonconformal, complex scalar field satisfies the equation

$$(\nabla_i \nabla^i + m^2 + \xi R) \varphi(x) = 0, \quad (2)$$

where R denotes the scalar curvature of the spacetime and ξ is the curvature coupling coefficient. We permit it to take arbitrary values, $\xi \in (-\infty, \infty)$, in-

cluding the particular cases $\xi = 0$ (minimal coupling) and $\xi = 1/6$ (conformal coupling).

The metric of the spatially homogeneous, isotropic spacetime under consideration is given by

$$ds^2 = g_{ik} dx^i dx^k = dt^2 - a^2(t) dl^2, \quad (3)$$

where dl is the line element of a 3-space of constant curvature $\kappa = -1, 0, 1$:

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta = d\chi^2 + f^2(\chi) (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (4)$$

Here, the function $f(\chi)$ is $\sinh \chi$, χ and $\sin \chi$ for $\kappa = -1$ (open model), $\kappa = 0$ (quasi-Euclidean model) and $\kappa = +1$ (closed model), respectively.

The scalar curvature R in the metric (3) can be expressed as

$$R = \frac{6}{a^2} \left(\frac{a''}{a} + \kappa \right), \quad (5)$$

where the prime denotes here and below the derivative with respect to the conformal time $\eta = \int dt/a$.

After separation of variables (for details see [19])

$$\varphi = \frac{1}{\sqrt{2}} \frac{1}{a(\eta)} g_J(\eta) \Phi_J(\mathbf{x}), \quad (6)$$

where $\Phi_J(\mathbf{x})$, $J = (\lambda, l, m)$, are the eigenfunctions of the Laplace–Beltrami operator of the 3-space of constant curvature, (2) takes the oscillatory form

$$g_J''(\eta) + \Omega^2(\eta) g_J(\eta) = 0. \quad (7)$$

The frequency $\Omega(\eta)$ is given by

$$\Omega^2(\eta) = \omega^2(\eta) - q(\eta), \quad (8)$$

where the abbreviations

$$\omega^2(\eta) = \lambda^2 + m^2 a^2(\eta), \quad q(\eta) = (\Delta\xi) a^2 R, \quad \Delta\xi = \frac{1}{6} - \xi \quad (9)$$

have been introduced. In the cases $\kappa = -1, 0$ the quantum numbers take the values $0 \leq \lambda < \infty$; $l = 0, 1, 2, \dots$; $m = -l, -l+1, \dots, l$. For $\kappa = +1$ one has

$\lambda = 0, 1, 2, \dots$ and $l = 0, 1, \dots, \lambda - 1$ whereas m varies in the same range as before.

In order to keep the discussion as general as possible we do not fix the values of the spacetime parameters $a(\eta)$ and $q(\eta)$ at the initial moment η_0 (which will be suggested below to correspond to $t_0 \gtrsim t_p = \sqrt{G}$, where G is the gravitational constant). They can be chosen freely:

$$a(\eta_0) = a_0, \quad q(\eta_0) = q_0. \quad (10)$$

By inspection of (7) we suggest to consider the quantity Ω as a natural dimensionless one-particle energy (the physical one-particle energy being Ω/a). This is in contrast to the formalism developed in [6, 19, 21] where the quantity ω (or its generalization for the anisotropic case) plays the role of an one-particle energy. It will shortly become clear that this definition corresponds to the concept of adiabatic particles [5]. Since the quantity Ω depends only on the quantum number λ , we will write hereafter g_λ instead of g_J .

We specify a complete orthonormal set of solutions to (2) by imposing the following initial conditions on the function g_λ

$$g_\lambda(\eta_0) = \Omega^{-\frac{1}{2}}(\eta_0) \quad g'_\lambda(\eta_0) = i\Omega(\eta_0)g_\lambda(\eta_0). \quad (11)$$

The operator of the quantized nonconformal scalar field can now be represented as the mode decomposition

$$\varphi(x) = \frac{1}{\sqrt{2a}} \int d\mu(J) \left[g_\lambda \Phi_J^*(\mathbf{x}) a_J^{(+)} + g_\lambda^* \Phi_J(\mathbf{x}) a_J^{(-)} \right], \quad (12)$$

where the $a_J^{(+)}$, $a_J^{(-)}$ are the creation operator of antiparticles and the annihilation operator of particles, respectively. They obey the usual commutation relations. The two addends of (12) correspondingly represent the positive- and negative-frequency parts of the field operator at the initial moment η_0 in the sense of the frequency Ω . The integration in (12) with the measure $d\mu(J)$ implies the integration over continuous and the summation over discrete quantum numbers.

The vacuum state of the field $\varphi(x)$ is defined by

$$a_J^{(-)}|0\rangle = a_J^{*(-)}|0\rangle = 0. \quad (13)$$

The choice of the initial conditions (11) ensures this vacuum to be an adiabatic vacuum state [18]. This can easily be seen when identically representing the solution to (7) in the form:

$$\begin{aligned} g_\lambda(\eta) &= \frac{1}{\sqrt{\Omega(\eta)}} \{ \alpha_\lambda^*(\eta) \exp[i\Theta(\eta_0, \eta)] + \beta_\lambda(\eta) \exp[-i\Theta(\eta_0, \eta)] \}, \\ g'_\lambda(\eta) &= i\sqrt{\Omega(\eta)} \{ \alpha_\lambda^*(\eta) \exp[i\Theta(\eta_0, \eta)] - \beta_\lambda(\eta) \exp[-i\Theta(\eta_0, \eta)] \}, \end{aligned} \quad (14)$$

where the notation

$$\Theta(\eta_0, \eta) = \int_{\eta_0}^{\eta} d\eta_1 \Omega(\eta_1) \quad (15)$$

has been used. As a consequence of (7) the functions $\alpha_\lambda(\eta)$ and $\beta_\lambda(\eta)$ obey the system of first order equations

$$\alpha_\lambda^{*'} = \frac{\Omega'}{2\Omega} \exp(-2i\Theta) \beta_\lambda, \quad \beta_\lambda' = \frac{\Omega'}{2\Omega} \exp(2i\Theta) \alpha_\lambda^*. \quad (16)$$

The initial conditions (11) on g_λ translate into

$$\alpha_\lambda(\eta_0) = 1, \quad \beta_\lambda(\eta_0) = 0. \quad (17)$$

From here the adiabatic nature of the vacuum state defined in (13) becomes evident.

Substituting (14) into the expansion (12) and separately collecting terms with positive and negative frequency Θ we get the creation and annihilation operators of adiabatic particles:

$$\begin{aligned} b_J^{(-)}(\eta) &= \alpha_\lambda(\eta) a_J^{(-)} + (-1)^m \beta_\lambda(\eta) a_J^{(+)}, \\ b_J^{(+)}(\eta) &= \alpha_\lambda^*(\eta) a_J^{(+)} + (-1)^m \beta_\lambda^*(\eta) a_J^{(-)}, \end{aligned} \quad (18)$$

where $\bar{J} = (\lambda, l, -m)$. Then, the spectral particle density per unit volume takes the form

$$n_J(\eta) = \langle 0 | b_J^{*(+)}(\eta) b_J^{(-)}(\eta) | 0 \rangle = |\beta_\lambda(\eta)|^2. \quad (19)$$

Integrating this expression over all momenta yields the density of created pairs per unit volume

$$n(\eta) = \frac{1}{2\pi^2 a^3} \int d\mu(\lambda) \lambda^2 |\beta_\lambda(\eta)|^2. \quad (20)$$

As it will be shown below, the density of adiabatic particles (20) defined by the initial conditions (11) is finite (compare with the number of quasiparticles defined by the method of Hamiltonian diagonalization which is infinite in the nonconformal case [22]).

Now let us calculate the vacuum matrix elements of the metrical stress-energy tensor which is defined by [18, 19]

$$T_{ik} = (1 - 2\xi)(\partial_i \varphi^* \partial_k \varphi + \partial_k \varphi^* \partial_i \varphi) - 2\xi [\varphi^* \nabla_i \nabla_k \varphi + \lambda (\nabla_i \nabla_k \varphi^*) \varphi] \quad (21)$$

$$+ (4\xi - 1)g_{ik} \partial' \varphi^* \partial_l \varphi + [(1 - 4\xi)m^2 g_{ik} - 2\xi G_{ik} - 4\xi^2 R g_{ik}] \varphi^* \varphi,$$

where G_{ik} is the Einstein tensor. We proceed with the calculation by substituting (12) into (21) and then making use of (7), (8) and (13). Some special properties of the eigenfunctions Φ_J have also to be utilized [19, 23, 24]. The result of all these manipulations may be displayed in the following form:

$$\begin{aligned} \langle 0|T_{00}|0 \rangle &= \frac{1}{\pi^2 a^2} \int d\mu(\lambda) \lambda^2 \left[\Omega S + \frac{\Omega}{2} + 3(\Delta\xi)cV \right. \\ &\quad \left. + 3(\Delta\xi)(c' + 2c^2) \frac{1}{\Omega} \left(S + \frac{1}{2}U + \frac{1}{2} \right) \right], \\ \langle 0|T_{\alpha\beta}|0 \rangle &= \frac{\gamma_{\alpha\beta}}{\pi^2 a^2} \int d\mu(\lambda) \lambda^2 \left[\frac{\lambda^2}{3\Omega} \left(S + \frac{1}{2} \right) - \frac{\Omega^2 - \lambda^2}{6\Omega} U \right. \\ &\quad \left. - (\Delta\xi)(3c' + 2\kappa) \frac{1}{\Omega} \left(S + \frac{1}{2}U + \frac{1}{2} \right) \right. \\ &\quad \left. - 2(\Delta\xi)\Omega U + 3(\Delta\xi)cV \right]. \end{aligned} \quad (22)$$

Here we have introduced the notation $c \equiv a'/a$. The three functions S , U and V are bilinear in g_λ . They are defined as:

$$\begin{aligned} S &= \frac{1}{4\Omega} (|g'_\lambda|^2 + \Omega^2 |g_\lambda|^2) - \frac{1}{2}, \\ U &= -\frac{1}{2\Omega} (|g'_\lambda|^2 - \Omega^2 |g_\lambda|^2), \quad V = -\frac{1}{2} \frac{d}{d\eta} |g_\lambda|^2. \end{aligned} \quad (23)$$

As a result of (7) the quantities S , U and V obey the following system of first-order differential equations

$$S' = \frac{\Omega'}{2\Omega} U, \quad U' = \frac{\Omega'}{\Omega} (1 + 2S) - 2\Omega V, \quad V' = 2\Omega U. \quad (24)$$

Due to (11) they are subject to the initial conditions

$$S(\eta_0) = U(\eta_0) = V(\eta_0) = 0. \quad (25)$$

In order to determine the large momentum behavior of the solutions S , U and V it proves convenient to convert the equations (24) together with the initial conditions (25) to an equivalent system of Volterra integral equations

$$\begin{aligned} U &= \int_{\eta_0}^{\eta} d\eta_1 w(\eta_1) [1 + 2S(\eta_1)] \cos [2\Theta(\eta_1, \eta)], \\ V &= \int_{\eta_0}^{\eta} d\eta_1 w(\eta_1) [1 + 2S(\eta_1)] \sin [2\Theta(\eta_1, \eta)], \\ S &= \frac{1}{2} \int_{\eta_0}^{\eta} d\eta_1 w(\eta_1) \int_{\eta_0}^{\eta_1} d\eta_2 w(\eta_2) [1 + 2S(\eta_2)] \cos [2\Theta(\eta_2, \eta_1)] \end{aligned} \quad (26)$$

with $w \equiv \Omega'/\Omega$. The quantity Θ has been defined in (15). By making use of (14) and (23) it is possible to express the functions S , U and V in terms of α_λ and β_λ :

$$S = |\beta_\lambda|^2, \quad U = 2\text{Re}(\alpha_\lambda \beta_\lambda e^{-2i\Theta}), \quad V = -2\text{Im}(\alpha_\lambda \beta_\lambda e^{-2i\Theta}). \quad (27)$$

The expressions (22) for the vacuum expectation value of the SET evidently contain divergencies. The required renormalization procedure will be discussed in the next section.

3 Renormalization

The vacuum expectation value of the SET (22) diverges at the upper integration limit like Λ^4 , Λ^2 and $\ln \Lambda$, Λ being the cut-off momentum. Hence, it needs to be renormalized. In the last few decades, the renormalization of the SET in curved spacetime has been studied intensively and many powerful techniques have been developed. Reviews of this subject can be found, e.g., in the monographs [18, 19]. Despite the fact that the renormalization of the SET has become a standard procedure we feel that it is appropriate to discuss this issue here for the following reason. In the quantization procedure outlined above we have chosen Ω as the adiabatic frequency. This led us to the new representation (22) of the vacuum matrix elements of the SET in

terms of the quantities S , U and V . Therefore, the necessary subtraction terms should be calculated here in this particular representation. For this purpose, we will rely on the so called n-wave regularization [6] that has been shown to be equivalent to adiabatic regularization.

According to the n-wave procedure, the integrands of (22) have to be expanded in inverse powers of ω . Then the first three terms of this expansion should be subtracted from the formal expressions (22). Finally, the procedure yields the renormalized expectation value of the SET:

$$\langle T_{ik} \rangle_{ren} = \langle 0|T_{ik}|0 \rangle - \sum_{l=0}^2 T_{ik}(l). \quad (28)$$

The explicit form of the subtracted terms $T_{ik}(l)$ may be displayed as follows:

$$T_{00}(0) = \frac{1}{2\pi^2 a^2} \int_0^\infty d\lambda \lambda^2 \omega, \quad T_{\alpha\beta}(0) = \frac{\gamma_{\alpha\beta}}{6\pi^2 a^2} \int_0^\infty d\lambda \frac{\lambda^4}{\omega}, \quad (29)$$

(first subtraction)

$$\begin{aligned} T_{00}(1) &= \frac{1}{\pi^2 a^2} \int_0^\infty d\lambda \lambda^2 \left[\omega S_2 + 3c(\Delta\xi) V_1 + \frac{3}{2\omega} (\Delta\xi) (c^2 - \kappa) \right], \\ T_{\alpha\beta}(1) &= \frac{\gamma_{\alpha\beta}}{\pi^2 a^2} \int_0^\infty d\lambda \lambda^2 \left\{ \frac{1}{3} \left[\frac{\lambda^2}{\omega} S_2 - \frac{m^2 a^2}{2\omega} \left(U_2 + \frac{q}{2\omega^2} \right) \right] \right. \\ &\quad \left. + (\Delta\xi) \left[\frac{1}{2\omega} (3c^2 - 2\xi a^2 R + \kappa) + 3c V_1 - 2\omega \left(U_2 + \frac{q}{2\omega^2} \right) \right] \right\}, \end{aligned} \quad (30)$$

(second subtraction)

$$\begin{aligned} T_{00}(2) &= \frac{1}{\pi^2 a^2} \int_0^\infty d\lambda \lambda^2 \left\{ \omega \left(S_4 + \frac{q^2}{16\omega^4} + \frac{q}{4\omega^2} U_2 \right) \right. \\ &\quad \left. + 3(\Delta\xi) \left[c V_3 + \frac{1}{\omega} (c^2 - \kappa) \left(S_2 + \frac{1}{2} U_2 + \frac{q}{4\omega^2} \right) \right] \right\}, \\ T_{\alpha\beta}(2) &= \frac{\gamma_{\alpha\beta}}{\pi^2 a^2} \int_0^\infty d\lambda \lambda^2 \left\{ \frac{1}{3} \left[\frac{\lambda^2}{\omega} \left(S_4 + \frac{q}{4\omega^2} U_2 + \frac{q^2}{16\omega^4} \right) \right. \right. \\ &\quad \left. \left. - \frac{m^2 a^2}{2\omega} \left(U_4 + \frac{q^2}{4\omega^4} + \frac{q}{\omega^2} S_2 \right) \right] \right. \\ &\quad \left. + (\Delta\xi) \left[\frac{1}{\omega} (3c^2 - 2\xi a^2 R + \kappa) \left(S_2 + \frac{1}{2} U_2 + \frac{q}{4\omega^2} \right) \right] \right\} \end{aligned} \quad (31)$$

$$+3cV_3 - 2\omega \left(U_4 + \frac{q^2}{4\omega^4} + \frac{q}{\omega^2} S_2 \right) \Big] \Big\},$$

(third subtraction). Note that the continuous measure in the subtracted expressions (29)–(31) also applies to the case of positive spatial curvature ($\kappa = +1$) reflecting the fact that the large momentum behavior is entirely determined by local properties of the spacetime [25].

In (29)–(31) we have used the notation:

$$\begin{aligned} V_1 &= \frac{1}{2}W, & U_2 &= \frac{1}{2}DW, & S_2 &= \frac{1}{16}W^2, \\ V_3 &= \frac{1}{16}W^3 - \frac{1}{8}D^2W - \frac{\omega}{4}D\left(\frac{q}{\omega^3}\right), \\ U_4 &= \frac{1}{32}DW^3 - \frac{1}{16}D^3W - \frac{1}{8}D\left(\omega D\frac{q}{\omega^3}\right) + \frac{1}{8}\frac{q}{\omega^2}DW, \\ S_4 &= \frac{3}{256}W^4 - \frac{1}{32}WD^2W + \frac{1}{64}(DW)^2 - \frac{\omega}{16}WD\left(\frac{q}{\omega^3}\right), \end{aligned} \quad (32)$$

where

$$W = \frac{\omega'}{\omega^2}, \quad D = \frac{1}{\omega} \frac{d}{d\eta}. \quad (33)$$

It may be seen with the help of (32) that the subtracted terms (29)–(31) are exactly the same as the ones previously obtained in [26, 19]. This is to be expected because one can show by making use of (23), that expressions (22) for the vacuum expectation value of the SET differ from the corresponding expressions of [26, 19] at most by the choice of the initial conditions (11). However, as the counter terms are of purely local nature, they do not depend on the initial conditions. Consequently, as it was shown in [26, 19] using dimensional regularization, the subtraction of the three expressions (29)–(31) can be interpreted as a renormalization of the cosmological constant, the gravitational constant and the coupling coefficients of the quadratic in curvature tensors in the bare gravitational action.

Performing the subtractions in (22) according to (29)–(33) we arrive at the following expression for the renormalized vacuum expectation value of the SET:

$$\langle T_{00} \rangle_{ren} = \frac{1}{\pi^2 a^2} \int_0^\infty d\lambda \lambda^2 \left\{ \Omega \left(\frac{1}{2} + S \right) - \omega \left(\frac{1}{2} - \frac{q}{4\omega^2} - \frac{q^2}{16\omega^4} \right) \right.$$

$$\begin{aligned}
& -\frac{q}{2\omega^2}S_2 + S_2 + S_4) + 3c(\Delta\xi)(V - V_1 - V_3) \\
& + 3(c' + 2c^2)(\Delta\xi)\left[\frac{1}{\Omega}\left(\frac{1}{2} + S + \frac{1}{2}U\right)\right. \\
& \left. - \frac{1}{\omega}\left(\frac{1}{2} + S_2 + \frac{1}{2}U_2 + \frac{q}{4\omega^2}\right)\right]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\langle T_{\alpha\beta} \rangle_{ren} = & \frac{\gamma_{\alpha\beta}}{\pi^2 a^2} \int_0^\infty d\lambda \lambda^2 \left\{ \frac{1}{3} \left[\frac{\lambda^2}{\Omega} \left(\frac{1}{2} + S \right) - \frac{\lambda^2}{\omega} \left(\frac{1}{2} + S_2 + \frac{q}{4\omega^2} \right. \right. \right. \\
& + S_4 + \frac{q}{2\omega^2}S_2 + \frac{3q^2}{16\omega^4} \Bigg) - \frac{m^2 a^2 - q}{2\Omega}U + \frac{m^2 a^2 - q}{2\omega}U_2 \\
& + \frac{m^2 a^2}{2\omega} \left(U_4 + \frac{q}{2\omega^2}U_2 \right) \Bigg] + 3c(\Delta\xi)(V - V_1 - V_3) \\
& - 2(\Delta\xi) \left[\Omega U - \omega \left(U_2 + U_4 - \frac{q}{2\omega^2}U_2 \right) \right] \\
& - (\Delta\xi)(3c' + 2\kappa) \left[\frac{1}{\Omega} \left(\frac{1}{2} + S + \frac{1}{2}U \right) - \frac{1}{\omega} \left(\frac{1}{2} + S_2 \right. \right. \\
& \left. \left. + \frac{1}{2}U_2 + \frac{q}{4\omega^2} \right) \right] \Bigg\}. \tag{34}
\end{aligned}$$

These expressions are rigorously valid only in the case of hyperbolic ($\kappa = -1$) and flat ($\kappa = 0$) spatial geometry because here we have used the continuous measure for the unrenormalized SET (22). Turning the summation into an integration in the case of spherical geometry ($\kappa = +1$) results in the appearance of finite, topological contributions to the SET [19, 27, 28]. When applying (34) to the spherical case $\kappa = +1$, it should be kept in mind that those topological terms have been omitted.

Further below, the vacuum SET is shown to possess the correct conformal anomaly. In order to derive it we need to know the renormalized trace of the vacuum SET:

$$\begin{aligned}
\langle T_i^i \rangle_{ren} = & \frac{1}{\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \left\{ [m^2 a^2 + 6c'(\Delta\xi)] \left[\frac{1}{2\Omega} - \frac{1}{2\omega} - \frac{q}{4\omega^3} \right. \right. \\
& + \frac{1}{\Omega} \left(S + \frac{1}{2}U \right) - \frac{1}{\omega} \left(S_2 + \frac{1}{2}U_2 \right) \Bigg] - 6c(\Delta\xi)(V - V_1 \\
& - V_3) - \frac{m^2 a^2}{\omega} \left(S_4 + \frac{1}{2}U_4 + \frac{q}{2\omega^2}S_2 + \frac{q}{4\omega^2}U_2 + \frac{3q^2}{16\omega^4} \right)
\end{aligned}$$

$$+6(\Delta\xi) \left[\Omega U - \omega \left(U_2 + U_4 - \frac{q}{2\omega^2} U_2 \right) \right] \Big\}. \quad (35)$$

In the next section, we will mainly be concerned with the question of how to simplify the expressions (34) for arbitrary scale factors $a(t)$.

4 Explicit calculation of the particle creation rate and vacuum polarization in the early time approximation

Let us summarize what has been achieved so far. With the expressions (34) we have an integral representation for the vacuum SET at our disposal. Beside the local subtraction terms, the integrands of (34) are composed of the functions S , U and V which are solutions to the system (24). Since these equations cannot be solved explicitly for arbitrary scale factors $a(t)$, the integrands of (34) are only known as implicit functions of the momentum variable λ . Hence, the momentum integration cannot be carried out. However, when attempting to solve the back reaction equation (1) in a cosmological context one wishes to access the vacuum SET in a more explicit form. In this situation one might raise the question whether the physical conditions in the early universe admit some reasonable approximations.

The application region of the semiclassical approach (matter fields are quantized, the gravitational field is still classical) ranges from the Planck time $t_p = \sqrt{G}$ to the Compton time $t_c = m^{-1}$ of the matter field. For usual particles, this region covers about twenty orders of magnitude. That means, there is a range of many orders of magnitude for the inequality

$$mt \ll 1 \quad (36)$$

to hold within the framework of the semiclassical theory. Beside (36) one more condition connected with the generically nonconformal curvature coupling of the scalar field enters the discussion. More specifically, we demand the inequality

$$\int_{t_0}^t dt_1 \sqrt{|(\Delta\xi)R(t_1)|} \ll 1 \quad (37)$$

to be satisfied. The two conditions (36), (37) constitute the early time approximation for the nonconformal scalar field. This approximation is to be

understood in the following sense. For a given pair of the parameters m and ξ the early time approximation restricts the time variable t to those values for which the inequalities (36), (37) are fulfilled. In other words, the larger m and $|1/6 - \xi|$ are the smaller t has to be chosen. If, conversely, the time scale t is fixed, the conditions (36), (37) translate into a restriction on the parameters m and $|1/6 - \xi|$. In that respect, the early time approximation contains the expansion in a small mass and a small deviation of the coupling coefficient ξ from its conformal value. In particular, the inequalities (36), (37) are trivially satisfied for a massless, conformally coupled field.

Let us now turn to the question of what the consequences of the early time approximation for the mode solutions g_λ and the vacuum SET are. For this purpose, we first consider values of the momentum for which $\lambda(\eta - \eta_0) \ll 1$ so that the quantity

$$\Theta(\eta_0, \eta) = \int_{\eta_0}^{\eta} d\eta_1 \Omega(\eta_1) \lesssim \lambda(\eta - \eta_0) + m(t - t_0) + \int_{\eta_0}^{\eta} d\eta_1 \sqrt{|q(\eta_1)|} \ll 1 \quad (38)$$

is small compared to unity as a consequence of (36), (37).

Then, solving the system (16) in zeroth order of the quantity Θ , one readily obtains the solutions for small momenta:

$$\alpha_\lambda^* = \frac{1}{2} \left(\sqrt{\frac{\Omega_0}{\Omega}} + \sqrt{\frac{\Omega}{\Omega_0}} \right), \quad \beta_\lambda = -\frac{1}{2} \left(\sqrt{\frac{\Omega_0}{\Omega}} - \sqrt{\frac{\Omega}{\Omega_0}} \right), \quad (39)$$

where $\Omega_0 = \Omega(\eta_0)$. With the help of relations (27) we get the following expressions for S , U and V :

$$S = \frac{(\Omega - \Omega_0)^2}{4\Omega\Omega_0}, \quad U = \frac{\Omega^2 - \Omega_0^2}{2\Omega\Omega_0}, \quad V = 0. \quad (40)$$

In the next step, we look at the region of large momenta

$$\lambda \gg \sqrt{m^2 a^2 - q}, \quad (41)$$

where the expression under the root is assumed to be nonnegative. In this case the solutions S , U and V are most conveniently constructed from the system of Volterra integral equations (26). The leading terms in the small parameter

$$\frac{\sqrt{m^2 a^2 - q}}{\lambda} \equiv \frac{Q}{\lambda} \ll 1 \quad (42)$$

are already contained in the first iteration which results from (26) by substituting $S \equiv 0$ into the right-hand side. Finally, the asymptotic solutions for large momenta can be represented in the form:

$$\begin{aligned} S &= \frac{1}{16\lambda^4} \int_{\eta_0}^{\eta} d\eta_1 Q^{2'}(\eta_1) \int_{\eta_0}^{\eta} d\eta_2 Q^{2'}(\eta_2) \cos 2\lambda(\eta_1 - \eta_2), \\ U &= \frac{1}{2\lambda^2} \int_{\eta_0}^{\eta} d\eta_1 Q^{2'}(\eta_1) \cos 2\lambda(\eta - \eta_1), \\ V &= \frac{1}{4\lambda^3} Q^{2'} - \frac{1}{4\lambda^3} Q_0^{2'} \cos 2\lambda(\eta - \eta_1) - \frac{1}{4\lambda^3} \int_{\eta_0}^{\eta} d\eta_1 Q^{2''}(\eta_1) \cos 2\lambda(\eta - \eta_1). \end{aligned} \quad (43)$$

Due to the conditions (36), (37) it is always possible to find a λ_0 so that the inequalities

$$\sqrt{m^2 a^2 - q} \ll \lambda_0 \ll (\eta - \eta_0)^{-1} \quad (44)$$

hold simultaneously. In other words, the asymptotic regions for small and large momenta overlap in the early time regime. In order to explore the behavior of the asymptotic solutions (40), (43) for transitional momenta λ_0 we first expand (40) up to leading order in Q/λ_0 . We find

$$S = \frac{1}{16\lambda_0^4} (Q^2 - Q_0^2)^2, \quad U = \frac{1}{2\lambda_0^2} (Q^2 - Q_0^2), \quad V = 0. \quad (45)$$

The same expressions follow from (43) in zeroth order of the small parameter $\lambda_0(\eta - \eta_0)$. So we are led to conclude that the asymptotic solutions (40), (43) join smoothly in the transition region $\lambda \sim \lambda_0$.

We are now in a position to calculate the renormalized vacuum SET for arbitrary scale factors. With the solutions (40), (43) at hand, the integrands of (34) are given as explicit functions of the momentum. Thus, the momentum integration in (34) can be performed where the solutions (40), (43) are to be used in the intervals $(0, \lambda_0)$ and (λ_0, ∞) , respectively. Moreover, the smooth joining of the asymptotics in the transition region ensures the result to be independent of λ_0 . The details of this calculation are presented in the appendix. Then, the total renormalized vacuum SET can be displayed as follows:

$$\langle T_{ik} \rangle_{ren} = \sum_{a=1}^5 \langle T_{ik}^{(a)} \rangle, \quad (46)$$

where

$$\begin{aligned}
\langle T_{ik}^{(1)} \rangle &= -\frac{m^4}{16\pi^2} \left(C + \frac{1}{4} \right) g_{ik} + \frac{m^2}{144\pi^2} \left[1 - 36(\Delta\xi) \left(C + \frac{3}{2} \right) \right] G_{ik} \\
&\quad + \frac{1}{144\pi^2} (\Delta\xi) [-1 + 18(\Delta\xi)(C + 1)] {}^{(1)}H_{ik} \\
&\quad + \frac{1}{1440\pi^2} \left(-\frac{1}{6} {}^{(1)}H_{ik} + {}^{(3)}H_{ik} \right). \tag{47}
\end{aligned}$$

Here, G_{ik} is the Einstein tensor and C denotes Euler's constant. The tensors ${}^{(1)}H_{ik}$ and ${}^{(3)}H_{ik}$ are quadratic in the curvature. Their definition can be found, e.g., in [18]. The second contribution to the vacuum SET is given by

$$\begin{aligned}
\langle T_{00}^{(2)} \rangle &= \frac{1}{4\pi^2} \left[-\frac{m^4}{4} g_{00} - (\Delta\xi) m^2 G_{00} + \frac{1}{2} (\Delta\xi)^2 {}^{(1)}H_{00} \right] \ln(ma), \\
\langle T_{\alpha\beta}^{(2)} \rangle &= \frac{1}{4\pi^2} \left[-\frac{m^4}{4} g_{\alpha\beta} - (\Delta\xi) m^2 G_{\alpha\beta} + \frac{1}{2} (\Delta\xi)^2 {}^{(1)}H_{\alpha\beta} \right] \ln(ma) \\
&\quad - \frac{\gamma_{\alpha\beta}}{12\pi^2} \left[-\frac{m^4}{4} g_{00} - (\Delta\xi) m^2 G_{00} + \frac{1}{2} (\Delta\xi)^2 {}^{(1)}H_{00} \right]. \tag{48}
\end{aligned}$$

The third addend of (46) reads:

$$\begin{aligned}
\langle T_{00}^{(3)} \rangle &= \frac{3m^2\kappa}{144\pi^2} - \frac{3\kappa^2}{720\pi^2 a^2} + \frac{\kappa}{4\pi^2} (\Delta\xi) \left[-3m^2 + \frac{1}{2a^2} (c^2 - \kappa) \right] \\
&\quad - \frac{9}{2\pi^2 a^2} (\Delta\xi)^2 c^2 (c' + c^2 + \kappa), \\
\langle T_{\alpha\beta}^{(3)} \rangle &= \gamma_{\alpha\beta} \left\{ -\frac{m^2\kappa}{144\pi^2} - \frac{\kappa^2}{720\pi^2 a^2} + \frac{\kappa}{4\pi^2} (\Delta\xi) \left[m^2 + \frac{1}{6a^2} (-2c' \right. \right. \\
&\quad \left. \left. + c^2 - \kappa) \right] + \frac{3}{2\pi^2 a^2} (\Delta\xi)^2 [c''c + 2c'^2 + c'c^2 \right. \\
&\quad \left. \left. + (c^2 + \kappa)(2c' - c^2)] \right\}. \tag{49}
\end{aligned}$$

The fourth contribution is:

$$\begin{aligned}
\langle T_{00}^{(4)} \rangle &= -\frac{Q_0^4}{64\pi^2 a^2} + \frac{Q_0^2}{16\pi^2 a^2} \left(C + \frac{1}{2} + \ln Q_0 \right) \times \\
&\quad [2m^2 a^2 + 12(\Delta\xi)(c^2 - \kappa) - Q_0^2],
\end{aligned}$$

$$\begin{aligned} \langle T_{\alpha\beta}^{(4)} \rangle &= \gamma_{\alpha\beta} \left\{ -\frac{Q_0^4}{192\pi^2 a^2} + \frac{Q_0^2}{48\pi^2 a^2} \left(C + \frac{1}{2} + \ln Q_0 \right) \times \right. \\ &\quad \left. [-2m^2 a^2 + 12(\Delta\xi) (-2c' + c^2 - \kappa) - Q_0^2] \right\}. \end{aligned} \quad (50)$$

The last, fifth contribution consists of integrals of the following form:

$$\begin{aligned} \langle T_{00}^{(5)} \rangle &= -\frac{1}{16\pi^2 a^2} \int_{\eta_0}^{\eta} d\eta_1 Q^{2'}(\eta_1) \int_{\eta_0}^{\eta} d\eta_2 Q^{2'}(\eta_2) \ln |\eta_1 - \eta_2| \\ &\quad + \frac{3}{4\pi^2 a^2} (\Delta\xi) \left[c Q_0^{2'} \ln |\eta - \eta_0| + c \int_{\eta_0}^{\eta} d\eta_1 Q^{2''}(\eta_1) \ln |\eta - \eta_1| \right. \\ &\quad \left. - (c' + 2c^2) \int_{\eta_0}^{\eta} d\eta_1 Q^{2'}(\eta_1) \ln |\eta - \eta_1| \right], \\ \langle T_{\alpha\beta}^{(5)} \rangle &= \frac{\gamma_{\alpha\beta}}{48\pi^2 a^2} \left[4m^2 a^2 \int_{\eta_0}^{\eta} d\eta_1 Q^{2'}(\eta_1) \ln |\eta - \eta_1| \right. \\ &\quad \left. - \int_{\eta_0}^{\eta} d\eta_1 Q^{2'}(\eta_1) \int_{\eta_0}^{\eta} d\eta_2 Q^{2'}(\eta_2) \ln |\eta_1 - \eta_2| \right] \\ &\quad + \frac{\gamma_{\alpha\beta}}{4\pi^2 a^2} (\Delta\xi) \left[3c Q_0^{2'} \ln |\eta - \eta_0| - \frac{Q_0^{2'}}{|\eta - \eta_0|} - Q_0^{2''} \ln |\eta - \eta_0| \right. \\ &\quad \left. - \int_{\eta_0}^{\eta} d\eta_1 Q^{2'''}(\eta_1) \ln |\eta - \eta_1| + 3c \int_{\eta_0}^{\eta} d\eta_1 Q^{2''}(\eta_1) \ln |\eta - \eta_1| \right. \\ &\quad \left. + (c' - 2c^2) \int_{\eta_0}^{\eta} d\eta_1 Q^{2'}(\eta_1) \ln |\eta - \eta_1| \right]. \end{aligned} \quad (51)$$

It has been checked by a direct calculation that the presented vacuum SET is covariantly conserved $\nabla_i \langle T^{ik} \rangle_{ren} = 0$. The conservation condition holds for every contribution to (46) separately:

$$\left(\frac{d}{d\eta} + c \right) \langle T_{00}^{(a)} \rangle + c \gamma^{\alpha\beta} \langle T_{\alpha\beta}^{(a)} \rangle = 0. \quad (52)$$

However, it should be noted that we do not distinguish between the different addends of (46) for any fundamental reason. The splitting serves us only to be able to discuss the properties of the vacuum SET more conveniently. Some remarks are in order:

The contribution $\langle T_{ik}^{(1)} \rangle$ only consists of the purely geometrical tensors g_{ik} , G_{ik} and ${}^{(1,3)}H_{ik}$. The appearance of ${}^{(3)}H_{ik}$ here is connected with the

conformal anomaly to be discussed below. In contrast, $\langle T_{ik}^{(2)} \rangle$ cannot be identified with a geometrical tensor although it is of local nature. The contributions $\langle T_{ik}^{(3,4)} \rangle$ originate from different sources. One sort of terms is generated by the nonvanishing spatial curvature κ . Other terms arise when the gravitational field is switched on nonadiabatically. In this case, the quantity $c = a'/a$ and its derivatives do not vanish initially. The last contribution $\langle T_{ik}^{(5)} \rangle$ contains all those terms of different nature.

If the expression (46) is specialized to the quasi-Euclidean case ($\kappa = 0$) and to zero initial conditions for the spacetime parameters ($a_0 = c_0 = c'_0 = c''_0 = 0$) it is found to coincide with the vacuum SET obtained in [21] with the help of a different calculational formalism. Thus, the result of [21] is a special case of (46). If, additionally, $\xi = 1/6$ is substituted into (46) we reproduce the known result for the vacuum SET of conformal scalar field [19].

We proceed with a discussion of the scaling properties of the vacuum SET (46). The logarithm in $\langle T_{ik}^{(2)} \rangle$ in (48) can identically be transformed according to

$$\ln(ma) = \ln \frac{m}{\mu} + \ln(\mu a) \quad (53)$$

with μ being an arbitrary mass scale. Then, the term proportional to $\ln(m/\mu)$,

$$\frac{1}{4\pi^2} \left[-\frac{m^4}{4} g_{ik} - (\Delta\xi) m^2 G_{ik} + \frac{1}{2} (\Delta\xi)^2 {}^{(1)}H_{ik} \right] \ln \frac{m}{\mu} \quad (54)$$

represents a local, geometrical tensor that is solely made up of g_{ik} , G_{ik} and ${}^{(1)}H_{ik}$. Hence, the removal of this term from the vacuum SET can be interpreted as a finite renormalization of the respective gravitational coupling constants. Note that the terms of $\langle T_{ik}^{(1)} \rangle$ depending on g_{ik} , G_{ik} and ${}^{(1)}H_{ik}$ can be absorbed analogously in a finite renormalization.

Thus, the renormalization scale dependence of the vacuum SET manifests itself in the logarithmic structure of the contribution $\langle T_{ik}^{(2)} \rangle$. As we will see in a moment, there is yet another way of parametrizing the dependence of the vacuum SET on the renormalization point. We first note that the vacuum SET (46) is invariant under the combined scaling transformations

$$\ln(ma) \rightarrow \ln \frac{ma}{\mu}, \quad \ln Q_0 \rightarrow \ln \frac{Q_0}{\mu}, \quad \ln |\eta_r - \eta_s| \rightarrow \ln \mu |\eta_r - \eta_s|, \quad (55)$$

where μ is now a dimensionless scale parameter. The time arguments $\eta_r - \eta_s$ are those occurring in (51). The transformation (55) causes the different contributions to the total vacuum SET to change according to

$$\langle T_{ik}^{(a)} \rangle \rightarrow \langle T_{ik}^{(a)} \rangle + \delta \langle T_{ik}^{(a)} \rangle \quad (56)$$

with

$$\begin{aligned} \delta \langle T_{ik}^{(a)} \rangle &= 0 \quad a = 1, 3 \\ \delta \langle T_{ik}^{(2)} \rangle &= \frac{1}{4\pi^2} \left[-\frac{m^4}{4} g_{ik} - (\Delta\xi) m^2 G_{ik} + \frac{1}{2} (\Delta\xi)^2 {}^{(1)}H_{ik} \right] \ln \frac{1}{\mu}, \\ \delta \langle T_{ik}^{(4)} \rangle &= \frac{Q_0^2}{16\pi^2 a^2} [2m^2 a^2 + 12(\Delta\xi)(c^2 - \kappa) - Q_0^2] \ln \frac{1}{\mu}, \\ \delta \langle T_{ik}^{(5)} \rangle &= -\delta \langle T_{ik}^{(2)} \rangle - \delta \langle T_{ik}^{(4)} \rangle. \end{aligned} \quad (57)$$

Just like the term (54), the contribution $\delta \langle T_{ik}^{(2)} \rangle$ can be removed by a finite renormalization. So we find that due to the invariance property (55) the renormalization scale dependence of the vacuum SET can alternatively be expressed by introducing a dimensionless scale parameter in $\langle T_{ik}^{(4)} \rangle$ and $\langle T_{ik}^{(5)} \rangle$ according to (55).

We are now in a position to study the particularly interesting massless limit. The logarithmic term $(1/8\pi^2)(\Delta\xi)^2 {}^{(1)}H_{ik} \ln(ma)$ in (48) seems to produce an apparent infrared divergence. However, as we have seen in the last paragraph, it is this term that carries the renormalization scale dependence of the vacuum SET. So the appearance of the mass m in this logarithm is a peculiarity of the actual subtraction scheme. Consequently, this term has to be retained in the massless limit with an arbitrary scale parameter in place of m . It expresses the fact that due to the renormalization procedure a scale dependence enters the quantum theory even if the corresponding classical theory does not possess an inherent scale [18, 20].

The so called scale anomaly [20, 29] is closely related to the renormalization scale dependence. It describes the transformation of the renormalized vacuum SET under a rescaling of the metric

$$g_{ik} \rightarrow \mu^2 g_{ik}. \quad (58)$$

With the help of (47)-(51) (and now $m = 0$) we find the following transformation law for the vacuum SET

$$\langle T_{ik} \rangle_{ren} \rightarrow \mu^{-2} \left[\langle T_{ik} \rangle_{ren} + \frac{1}{8\pi^2} (\Delta\xi)^2 {}^{(1)}H_{ik} \ln \mu \right]. \quad (59)$$

Comparing this transformation with expression (54) for $m = 0$ reveals the fact that rescaling the metric (58) has the same effect on the vacuum SET as a change of the renormalization scale. Beside the term due to nonconformal coupling in (59) there will be an additional contribution to the anomalous scaling tensor if the spacetime is not conformally flat [30]. Thus, the vacuum SET (46) possesses the correct scaling properties [20].

Another important point to be discussed in this context is the conformal trace anomaly. The conformal trace anomaly is obtained as the difference of the expectation values

$$\langle T_i^i \rangle^A = \langle T_i^i \rangle_{ren} (m \rightarrow 0) - \langle T_i^i(m \rightarrow 0) \rangle_{ren}, \quad (60)$$

where the massless limit in $\langle T_i^i \rangle_{ren} (m \rightarrow 0)$ is to be taken after the matrix element whilst in $\langle T_i^i(m \rightarrow 0) \rangle_{ren}$ the massless limit comes first. We find from (46) or by a direct calculation from (35):

$$\begin{aligned} \langle T_i^i \rangle_{ren} (m \rightarrow 0) = & \quad (61) \\ & \frac{1}{240\pi^2 a^4} \left[c''' - 4c'c^2 + 30(\Delta\xi) (c''' - 6c'c^2 - 2c'\kappa) \right. \\ & \left. - 540(\Delta\xi)^2 (c' + c^2 + \kappa)^2 \right] + \frac{1}{8\pi^2 a^4} (\Delta\xi) (c''' - 6c'c^2) \\ & - \frac{9}{2\pi^2 a^4} (\Delta\xi)^2 \left[2c''c + c'^2 + 2c'c^2 + c' (c^2 + \kappa) - 2c^2 (c^2 + \kappa) \right] \\ & - \frac{3}{4\pi^2} (\Delta\xi)^2 (C + 1 + \ln \mu a) \nabla^k \nabla_k R - \frac{3c'q_0}{4\pi^2 a^4} (\Delta\xi) (2C + 1 + \ln |q_0|) \\ & + \frac{3}{4\pi^2 a^4} (\Delta\xi) \left\{ 2cq'_0 \ln |\eta - \eta_0| - \frac{q'_0}{|\eta - \eta_0|} - q''_0 \ln |\eta - \eta_0| \right. \\ & \left. - \int_{\eta_0}^{\eta} d\eta_1 [q'''(\eta_1) - 2c(\eta)q''(\eta_1) - 2c'(\eta)q'(\eta_1)] \ln |\eta - \eta_1| \right\}. \end{aligned}$$

The scale parameter μ is that of (53). The expectation value $\langle T_i^i(m \rightarrow 0) \rangle_{ren}$ is calculated likewise. Then, we obtain for the trace anomaly

$$\begin{aligned} \langle T_i^i \rangle^A &= \langle T_i^i \rangle_{ren} (m \rightarrow 0) - \langle T_i^i(m \rightarrow 0) \rangle_{ren} \\ &= -\frac{1}{1440\pi^2} \left[R^{ik} R_{ik} - \frac{1}{3} R^2 + (30\xi - 6) \nabla^k \nabla_k R + 90(\Delta\xi)^2 R^2 \right], \end{aligned} \quad (62)$$

which coincides with the known result [31].

Before proceeding with the investigation of the vacuum SET for particular scale factors, we study the particle density defined in (20). Again, the explicit calculation is carried out using the asymptotic solutions S , U and V for the respective momentum region (details are given in the appendix). The result can be represented in the following form:

$$n(\eta) = \frac{Q}{48\pi a^3} \left[(Q^2 + Q_0^2) F\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right) - 2Q_0^2 F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) \right] \quad (63)$$

for $Q > Q_0$. Here, F denotes the hypergeometric function and the argument z is an abbreviation for $1 - Q_0^2/Q^2$. The result for the case $Q_0 > Q$ is obtained from (63) by analytical continuation which is simply achieved by interchanging the role of Q and Q_0 .

For $Q_0 = 0$ (e.g. in the quasi-Euclidean case ($\kappa = 0$) with zero initial conditions (10)) the particle density (63) simplifies to

$$n = \frac{1}{24\pi^2} [m^2 - (\Delta\xi)R]^{\frac{3}{2}}. \quad (64)$$

If, in addition, conformal coupling is imposed ($\xi = 1/6$), (64) becomes [19]

$$n = \frac{m^3}{24\pi^2}, \quad (65)$$

which is independent of time. Moreover, the particle density (65) vanishes for zero mass reflecting the well known fact that conformally trivial fields do not cause particle production [5].

However, if the curvature coupling coefficient ξ is kept arbitrary, a finite nonvanishing particle density follows from (64) also for the massless field

$$n = \frac{1}{24\pi^2 a^3} \left[(6\xi - 1) \left(\frac{a''}{a} + \kappa \right) \right]^{\frac{3}{2}}. \quad (66)$$

In the radiation dominated universe (with $\kappa = \pm 1, 0$) the scalar curvature vanishes identically $R = 0$ so that we can replace Q by ma and Q_0 by ma_0 in (63). Then, the particle density (63) reduces to (65) for $t_0 = 0$. It should also be mentioned here that the condition (37) is trivially satisfied for vanishing scalar curvature and the early time approximation becomes exact in the massless limit.

In the next section, we study the vacuum SET for particular scale factors.

5 Application to particular scale factors

It is a well known fact from Friedmann cosmology that the degree-type scale factors

$$a(t) = b_0 t^q, \quad \frac{1}{3} < q < 1, \quad b_0 > 0 \quad (67)$$

correctly describe the early stages (after the isotropization) of the evolution of the universe for an arbitrary sign of the spatial curvature. The application region of the semiclassical theory and the cosmological period of degree-type evolution overlap. Therefore, one might expect a selfconsistent solution of the semiclassical Einstein equation to exhibit at least an approximate degree-type behavior in the overlap region. For this reason it is particularly interesting to study the vacuum SET for such scale factors.

The initial moment t_0 corresponding to η_0 that has been introduced in sec. 2 will be considered here to be greater than or of the order of the Planck time

$$t_0 \gtrsim t_p = \sqrt{G}. \quad (68)$$

We would like to stress here that the vacuum SET (46) is valid for arbitrary initial conditions of the spacetime parameters (10) as well as for all values of the spatial curvature κ . In contrast, the discussion of the vacuum SET in [21] was restricted to the quasi-Euclidean case ($\kappa = 0$) and to zero initial conditions (10). Therefore, it turned out to be necessary to introduce an additional smoothing of the expansion law in the vicinity of the initial moment when considering the degree-type scale factors in [21]. Within the present formalism, the degree-type scale factors are included naturally.

In terms of the conformal time the expansion law (67) takes the form

$$a(\eta) = b_1 \eta^p, \quad \frac{1}{2} < p < \infty, \quad p = \frac{q}{1-q}, \quad b_1 = b_0^{\frac{1}{1-q}} (1-q)^{\frac{q}{1-q}}. \quad (69)$$

Due to the conservation condition (52) it will be sufficient to present only the result for the energy density. After substituting (69) into (46) and performing the calculations we arrive at the result:

$$\langle T_0^0 \rangle_{ren} = \langle T_0^0 \rangle^C + \frac{m^4}{16\pi^2} \frac{a_0^2}{a^2} \left(2 - \frac{a_0^2}{a^2} \right) \ln \frac{Q_0}{ma_0} + \frac{q^2}{8\pi^2 t^4} (\Delta\xi) \left(3 - 6q \right)$$

$$\begin{aligned}
& -\frac{\kappa t^2}{a^2} \Big) - \frac{3m^2 q^2}{4\pi^2 t^2} (\Delta\xi) \left[\ln \frac{1}{mt} - C - \frac{1}{2q} - C_1(\kappa) - D_1(T) \right] \\
& + \frac{27q^2(1-2q)}{4\pi^2 t^4} (\Delta\xi)^2 \left[\ln \frac{1}{mt} - \frac{1-2q}{12} + \frac{2}{3} - C - C_2(\kappa) \right. \\
& \left. - D_2(T) \right]. \tag{70}
\end{aligned}$$

The following notations have been introduced. The first contribution in the right-hand of (70) is the vacuum energy density for the conformal scalar field

$$\begin{aligned}
\langle T_0^0 \rangle^C &= -\frac{m^2 q^2}{48\pi^2 t^2} + \frac{q^4 + 3q^2(1-2q)}{480\pi^2 t^4} + \frac{m^4}{16\pi^2} \left\{ \ln \frac{1}{mt} + \ln(1-q) \right. \\
& \left. + \Psi \left(\frac{1+q}{1-q} \right) + \frac{1-2q}{4q} + D_0(T) \right\} \tag{71}
\end{aligned}$$

with

$$\begin{aligned}
D_0(T) &= 2 \frac{a_0^2}{a^2} \ln \left(mt_0 \left| \frac{1-T}{1-q} \right| \right) + \frac{1-q}{2q} \frac{a_0^2}{a^2} \left[F \left(1, \frac{2q}{1-q}; \frac{1+q}{1-q}; T^{-1} \right) \right. \\
& \left. - F \left(1, -\frac{2q}{1-q}; \frac{1-3q}{1-q}; T^{-1} \right) \right] + \frac{a_0^2}{a^2} \left(2C - \frac{1-2q}{q} \right) \tag{72} \\
& + \frac{a_0^4}{a^4} \left[\ln \frac{1}{mt_0} + \ln(1-q) + \Psi \left(\frac{1+q}{1-q} \right) + \frac{1-4q}{4q} + \pi \cot \frac{2\pi q}{1-q} \right],
\end{aligned}$$

where Ψ is the logarithmic derivative of the Γ -function. The argument T is an abbreviation for

$$T \equiv \left(\frac{t}{t_0} \right)^{1-q}, \quad 1 \leq T < \infty. \tag{73}$$

The term $D_0(T)$ contains the dependence of the conformal energy density (71) on the initial data. It vanishes in the limit $t_0 \rightarrow 0$ and the conformal energy density $\langle T_0^0 \rangle^C$ reduces to the known result [18, 19].

The two terms $C_{1,2}(\kappa)$ in (70) are generated by a nonvanishing spatial curvature κ . Their explicit form is given by

$$\begin{aligned}
C_1(\kappa) &= \frac{\kappa t^2}{q^2 a^2} \left(\ln \frac{ma}{Q_0} + \frac{1}{4} \frac{a_0^2}{a^2} \right), \tag{74} \\
C_2(\kappa) &= \frac{\kappa t^2}{3(1-2q)a^2} \left[\frac{\kappa t^2}{q^2 a^2} \left(\ln \frac{ma}{Q_0} + \frac{3}{4} \right) + 1 - 2 \ln \frac{ma}{Q_0} - \frac{1-2q}{2q} \frac{a_0^2 t^2}{a^2 t_0^2} \right].
\end{aligned}$$

The contributions $D_{1,2}(T)$ also depend on the initial data. They may be displayed as follows:

$$\begin{aligned}
D_1(T) = & \ln \left| \frac{1-T}{1-q} \right| + T + \frac{1}{2}T^2 + \frac{1-2q}{q}T^2 \left[1 + C + \ln \left(\frac{t_0 Q_0}{a_0} \left| \frac{1-T}{1-q} \right| \right) \right] \\
& - \frac{(1-q)^2}{2q^2} T^2 F \left(1, -\frac{2q}{1-q}; \frac{1-3q}{1-q}; T^{-1} \right) + \frac{a_0^2}{a^2} \left\{ C - \frac{1-2q}{2q} \right. \\
& \ln \left(\frac{t_0 Q_0}{a_0} \left| \frac{1-T}{1-q} \right| \right) + T + T^2 \left[\ln \left(\frac{t_0 Q_0}{a_0} \left| \frac{1-T}{1-q} \right| \right) - \ln T + C \right. \\
& \left. \left. - \frac{1}{4q} \right] + \frac{1-q}{q} T^2 \left[\ln(1-q) + \Psi \left(\frac{1+q}{1-q} \right) - 1 - \ln \frac{t_0 Q_0}{a_0} \right. \right. \\
& \left. \left. + \pi \cot \frac{2\pi q}{1-q} \right] \right\} \quad (75)
\end{aligned}$$

and

$$\begin{aligned}
D_2(T) = & \ln \left| \frac{1-T}{1-q} \right| + T - \frac{2q}{3}T^2 \left[1 + C - \frac{3}{4q} \ln \left(\frac{t_0 Q_0}{a_0} \left| \frac{1-T}{1-q} \right| \right) \right] \quad (76) \\
& + \frac{1-2q}{3}T^3 + \frac{1-2q}{3}T^4 \left[C - \ln T + \ln \left(\frac{t_0 Q_0}{a_0} \left| \frac{1-T}{1-q} \right| \right) \right].
\end{aligned}$$

The dominating contributions to the energy density (70) come from the terms $D_{1,2}(T)$ because the quantity T from (73) is large compared to unity provided that $t \gg t_0$ is satisfied

$$T \gg 1, \text{ if } t \gg t_0. \quad (77)$$

It should be kept in mind that we demanded the early time approximation to hold. So, the time t cannot exceed the initial time t_0 arbitrarily. However, with the help of (36) and (37) the time t is shown to become so large that the nonconformal contributions to (70) dominate the previously known conformal ones.

As a result, the energy density (70) changes drastically when the curvature coupling coefficient ξ is relaxed from its conformal value $1/6$. Hence, there is an essential difference between conformal and nonconformal curvature coupling for the degree-type scale factors (67).

There is one important exception, namely the square root expansion law ($q = 1/2$, $\kappa = \pm 1, 0$). When substituting $q = 1/2$ into (70), the energy

density is found to reduce to

$$\begin{aligned}
\langle T_0^0 \rangle_{ren} = & \frac{1}{7680\pi^2 t^4} - \frac{m^2}{192\pi^2 t^2} - \frac{m^4}{16\pi^2} \left\{ \ln \sqrt{\frac{t}{t_0}} + 1 - \left(1 + \frac{t_0}{t}\right) \left(\frac{3}{2} \right. \right. \\
& \left. \left. - \sqrt{\frac{t_0}{t}}\right) + \left(1 - \frac{t_0}{t}\right)^2 \left[C - 1 + \ln \left(2mt_0 \left| 1 - \sqrt{\frac{t}{t_0}} \right| \right) \right] \right. \\
& \left. - \frac{t_0}{t} \left(2 - \frac{t_0}{t} \right) \ln \frac{Q_0}{ma_0} \right\} + \frac{3m^2}{16\pi^2 t^2} (\Delta\xi) \left\{ \ln \sqrt{\frac{t}{t_0}} + \sqrt{\frac{t_0}{t}} \right. \\
& \left. - \frac{t_0}{t} + \left(1 + \frac{t_0}{t}\right) \left[C + 1 + \ln \left(2mt_0 \left| 1 - \sqrt{\frac{t}{t_0}} \right| \right) \right] \right. \\
& \left. + \frac{t_0}{t} \ln \frac{Q_0}{ma_0} + \frac{\kappa t}{b^2} \left(\frac{t_0}{t} - 2 \ln \frac{t_0}{t} - 4 \ln \frac{Q_0}{ma_0} \right) \right\} \quad (78) \\
& - \frac{9}{16\pi^2 t^4} (\Delta\xi)^2 \frac{\kappa t}{b^2} \left[\frac{\kappa t}{b^2} \left(1 + 2 \frac{\kappa t}{b^2} \right) \left(1 - \ln \frac{t_0}{t} - 2 \ln \frac{Q_0}{ma_0} \right) \right],
\end{aligned}$$

where now the conformal contributions are dominating. Moreover, it is possible to take the limit $t_0 \rightarrow 0$ in (78). In this case, the energy density simplifies to

$$\begin{aligned}
\langle T_0^0 \rangle_{ren} = & \frac{1}{7680\pi^2 t^4} - \frac{m^2}{192\pi^2 t^2} + \frac{m^4}{16\pi^2} \left(\ln \frac{1}{mt} - \ln 2 - C + \frac{3}{2} \right) \quad (79) \\
& - \frac{3m^2}{16\pi^2 t^2} (\Delta\xi) \left(\ln \frac{1}{mt} - \ln 2 - C - 1 + \frac{2\kappa t}{b^2} \ln \frac{(6\xi - 1)\kappa}{m^2 b^2 t} \right) \\
& - \frac{9}{16\pi^2 t^4} (\Delta\xi)^2 \frac{\kappa t}{b^2} \left[\frac{\kappa t}{b^2} \left(1 + 2 \frac{\kappa t}{b^2} \right) \left(1 - \ln \frac{(6\xi - 1)\kappa}{m^2 b^2 t} \right) \right].
\end{aligned}$$

If we additionally put $\kappa = 0$ in (79) we obtain the energy density for the quasi-Euclidean, radiation dominated universe in agreement with the known result [32].

Another interesting example that nicely illustrates the use of the representation (46) for the renormalized vacuum SET is given by the scale factor

$$a(t) = bt, \quad b > 0. \quad (80)$$

Substituting this expansion law into (46) readily yields

$$\langle T_0^0 \rangle_{ren} = -\frac{1}{240\pi^2 t^4} - \frac{m^2}{48\pi^2 t^2} + \frac{m^4}{16\pi^2} \left(\ln \frac{1}{mt} + \ln 2 - \frac{1}{4} \right)$$

$$\begin{aligned}
& -\frac{1}{8\pi^2 t^4}(\Delta\xi) \left(3 + \frac{\kappa}{b^2}\right) + \frac{3m^2}{4\pi^2 t^2}(\Delta\xi) \left[1 - \left(1 + \frac{\kappa}{b^2}\right) \ln \frac{Q_0}{ma}\right] \\
& -\frac{9}{4\pi^2 t^4}(\Delta\xi)^2 \left[\frac{3}{4} \left(1 + \frac{\kappa}{b^2}\right) + \left(3 - \frac{\kappa}{b^2}\right) \ln \frac{Q_0}{ma}\right], \quad (81)
\end{aligned}$$

where we have already set $t_0 = 0$. Note that the result (81) could not be obtained from (70) by a smooth transition $q \rightarrow 1$.

The expansion law (80) also includes the Milne universe ($b = 1$, $\kappa = -1$) which is just a coordinatization of flat spacetime. In this case, the energy density (81) becomes

$$\begin{aligned}
\langle T_0^0 \rangle_{ren} = & -\frac{1}{240\pi^2 t^4} - \frac{m^2}{48\pi^2 t^2} + \frac{m^4}{16\pi^2} \left(\ln \frac{1}{mt} + \ln 2 - \frac{1}{4} \right) \\
& -\frac{1}{4\pi^2 t^4}(\Delta\xi) (1 - 3m^2 t^2). \quad (82)
\end{aligned}$$

Like in the radiation dominated case, the condition (37) holds for all times and the early time approximation produces an exact result for the massless field:

$$\langle T_0^0 \rangle_{ren} = -\frac{1}{240\pi^2 t^4} - \frac{1}{4\pi^2 t^4}(\Delta\xi). \quad (83)$$

If we additionally impose conformal coupling ($\xi = 1/6$) in (83), the energy density only consists of the contribution due to the conformal anomaly. Note that the quantity (82) is nonzero because the vacuum state (13) does not coincide with the Minkowski vacuum.

The density of massless created particles $n(t)$ vanishes if the expansion law of the Milne universe is substituted into (63) (with $t_0 = 0$) regardless of the value of the coupling coefficient ξ . As it has been shown in sec. 4 production of massless particles according to (63) (with $t_0 = 0$) does not occur in the radiation dominated universe either. This is to be expected since the scalar curvature R vanishes identically in both cases and hence, it does not enter the mode equation (7).

6 Conclusion and discussion

In the present paper we have studied a scalar field with arbitrary mass and curvature coupling in a gravitational background of Robertson–Walker type.

The quantity Ω defined in (8) has been used as an adiabatic frequency in the construction of the adiabatic mode solutions. We have found a representation of the renormalized SET as a momentum integral. With the help of the n -wave regularization we were able to show that the subtracted terms coincide with the known ones. Hence, the subtractions are to be interpreted as a renormalization of the respective gravitational coupling coefficients.

The nonconformal early time approximation has carefully been elaborated. It is found to be well adapted to the conditions in the early universe. With its help it proved possible to perform the above mentioned momentum integrations in the representation of the renormalized vacuum SET. As a result, we obtained the renormalized vacuum SET as an explicit functional of the scale factor which is predestinated for investigating cosmological problems in the framework of the semiclassical Einstein equation (1). The vacuum SET was shown to possess the correct scaling properties under a change of the renormalization scale. The conformal trace anomaly has been derived correctly in the nonconformal early time approximation. In analogy with the vacuum SET the density of created particles could explicitly be calculated as well.

The vacuum SET was probed for the degree-type scale factors which are important in Friedmann cosmology. Except for the square root expansion law, the energy density is dominated by the new nonconformal contributions as the time evolves to the region $t \gg t_0$. These contributions may significantly influence the gravitational field and may give rise to new selfconsistent solutions of the semiclassical Einstein equations.

Since the Milne universe and the radiation dominated case have vanishing scalar curvature, the nonconformal early time approximation reduces to the condition (36) involving the mass of the field. In the massless limit, the early time approximation is trivially satisfied so that the result for the vacuum SET becomes exact. In both cases, massless particles are not created during the expansion (with $t_0 = 0$).

Appendix

In the appendix we wish to illustrate how the momentum integration in (34) can be performed with the help of the nonconformal early time approxima-

tion. For this purpose, we look at the integral

$$\int_0^\infty d\lambda \lambda^2 \left[V(\eta) - \frac{Q^{2'}}{4\omega^3} \right], \quad (\text{A.1})$$

which is a part of (34). This integral serves us as an example to demonstrate the basic features of the calculations. All other terms of (34) are treated likewise.

As a first step we split the momentum integration of (A.1) according to

$$\int_0^\infty d\lambda \dots = \int_0^{\lambda_0} d\lambda \dots + \int_{\lambda_0}^\infty d\lambda \dots, \quad (\text{A.2})$$

where λ_0 is the transitional momentum from (44). Since we are working in the early time approximation we can use the asymptotic solutions (40) and (43) for the momentum intervals $(0, \lambda_0)$ and (λ_0, ∞) , respectively. In the region $(0, \lambda_0)$ we have $V = 0$ from (40) and the corresponding part of the integral (A.1) gives

$$\int_0^{\lambda_0} d\lambda \lambda^2 \left(V - \frac{Q^{2'}}{4\omega^3} \right) \approx -\frac{Q^{2'}}{4} \left(-\frac{\lambda_0}{\omega} + \ln \frac{\lambda_0 + \omega}{ma} \right). \quad (\text{A.3})$$

When expanding this expression to leading order in the small parameter Q/λ_0 we arrive at the contribution

$$\frac{Q^{2'}}{4} \left(1 - \ln \frac{2\lambda_0}{ma} \right) \quad (\text{A.4})$$

from the low momentum region $(0, \lambda_0)$.

In the large momentum region we find with the help of (43)

$$\int_{\lambda_0}^\infty d\lambda \lambda^2 \left(V - \frac{Q^{2'}}{4\omega^3} \right) \approx \frac{1}{4} \int_{\eta_0}^\eta d\eta_1 Q^{2''} \text{Ci}(2\lambda_0 |\eta - \eta_1|), \quad (\text{A.5})$$

where we have utilized an integral representation of the Cosine integral $\text{Ci}(z)$ [33]. Note that the divergent contribution $Q^{2'}/4\lambda^3$ from (43) has been cancelled by the leading term of the subtraction $-Q^{2'}/4\omega^3$ from (A.1). In leading order of the parameter $\lambda|\eta - \eta_1|$ expression (A.5) reduces to

$$\frac{1}{4} Q_0^{2'} \ln |\eta - \eta_0| + \frac{1}{4} Q^{2'} [C + \ln(2\lambda_0)] + \frac{1}{4} \int_{\eta_0}^\eta d\eta_1 Q^{2''} \ln |\eta - \eta_1|. \quad (\text{A.6})$$

The full result for the integral (A.1) is now obtained by adding the contributions (A.4) and (A.6) from both momentum regions. We get

$$\int_0^\infty d\lambda \lambda^2 \left(V - \frac{Q^{2'}}{4\omega^3} \right) \approx \frac{1}{4} \left\{ Q_0^{2'} \ln |\eta - \eta_0| + Q^{2'} [C + 1 + \ln(ma)] + \int_{\eta_0}^\eta d\eta_1 Q^{2''} \ln |\eta - \eta_1| \right\}. \quad (\text{A.7})$$

As expected the dependence on the transitional momentum has disappeared. The result (A.7) is valid in the early time approximation.

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