RELATIVISTIC SPIN OPERATOR FOR DIRAC PARTICLES

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In honour of the sixtieth birthday of Friedrich Hehl

Abstract

It is shown that a relativistic spin operator, obeying the required $SU(2)$ commutation relations, may be defined in terms of the Pauli-Lubanski vector W^μ . In the case of Dirac particles, this operator reduces to the Foldy-Wouthuysen "mean-spin" operator for states of positive energy.

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1. Introduction and summary

It is an honour for me to be asked to contribute to the celebration of Friedrich Hehl's sixtieth birthday. His work on fermions and gravity over many years has contributed substantially to our understanding of this unduly neglected area of physics. I take pleasure in wishing him well on this occasion and expressing the hope that he has many more active years of research ahead of him.

Some years ago Hehl and Ni published a very interesting paper [1] which analysed the behaviour of a Dirac particle in a non-inertial frame of reference. An effect they were looking for - and found - was a coupling of intrinsic spin to rotations. The existence of such a coupling had been predicted by Mashhoon [2], and of course will manifest itself in a term proportional to $\sigma \cdot \omega$ in the Hamiltonian. Hehl and Ni found that this term indeed appeared in the nonrelativistic limit of the Dirac equation in a non-inertial frame. Their work was generalised by the present author [3] to the relativistic regime. From a physical perspective this generalisation becomes the more desirable as it becomes more realistic to look for relativistic spin effects in astrophysics and cosmology.

It was shown in [3] that a relativistic spin operator could be defined by demanding covariance in the usual Wigner basis $|p, s\rangle$, in which the spin s is defined by the "little group" of the inhomogeneous Lorentz group. The transformation relating this Wigner basis to the basis defined by the right- and left-handed spinors $\phi_R(p)$ and $\phi_L(p)$ transforming covariantly under the

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homogeneous Lorentz group was derived and the covariant spin operator in this basis was found to be the Foldy-Wouthuysen "mean spin" operator. It was then a simple matter to show that the Hehl-Ni prediction of the existence of a

spin-rotation coupling could be cast in a manifestly relativistic form. The present work is a continuation of the work of ref [3]. We use a different reasoning, starting from the observation that, in searching for a relativistic spin operator, the most logical place to start looking is at the Pauli-Lubanski vector

$$W^\mu = -\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \quad (1.1)$$

It is clear that the only contribution to W^μ from $J_{\mu\nu}$ is from the intrinsic spin contribution; the orbital part will not contribute. W^μ is therefore a promising operator to consider. It is well known that $W^\mu W_\mu$ and $P^\mu P_\mu$ are the two Casimir operators of the Poincaré group, giving, in essence, the mass and spin of the states. But what are the spin operators? For timelike states, there are three spin operators which generate the group SU(2), whereas W^μ has 4 components and these operators by themselves certainly do not generate SU(2). It will be shown in the present paper that the correct way to proceed is first to define the commutator

$$W^{\mu\nu} = [W^\mu, W^\nu], \quad (1.2)$$

where M stands for rest-mass, and its dual

$$W^{D\kappa\lambda} = -\varepsilon^{\kappa\lambda\mu\nu} W_{\mu\nu}. \quad (1.3)$$

The combinations

$$X^{\mu\nu} = -i(W^{\mu\nu} + iW^{D\mu\nu}), \quad Y^{\mu\nu} = -i(W^{\mu\nu} - iW^{D\mu\nu}) = iX^{D\mu\nu}, \quad (1.4)$$

may then be shown to obey the commutation relations

$$[X_{\mu\nu}, X_{\kappa\lambda}] = -i(g_{\mu\kappa}X_{\nu\lambda} - g_{\mu\lambda}X_{\nu\kappa} + g_{\nu\lambda}X_{\mu\kappa} - g_{\nu\kappa}X_{\mu\lambda}) \quad (1.5)$$

and similarly for Y . These are the same in form as the commutation relations for the angular momentum (orbital plus spin) operators $J_{\mu\nu}$, and it therefore follows, by familiar arguments, that the two sets of three operators

$$X_i = -\varepsilon_{ijk}X_{jk}, \quad Y_i = -\varepsilon_{ijk}Y_{jk} \quad (1.6)$$

obey the SU(2) commutation relations

$$[X_i, X_j] = i\varepsilon_{ijk}X_k, \quad [Y_i, Y_j] = i\varepsilon_{ijk}Y_k. \quad (1.7)$$

These operators therefore both generate an SU(2) group and are both intrinsic spin operators. In fact they are the operators appropriate for left-handed and right-handed states. The appropriate spin operator for Dirac particles is then

$$Z_i = \frac{1}{2} (1 - \gamma_5) X_i + \frac{1}{2} (1 + \gamma_5) Y_i. \quad (1.8)$$

It will be shown that this is indeed equivalent to the Foldy - Wouthuysen mean spin operator, when acting on positive energy states.

2. Spin operators and the Pauli-Lubanski vector

From the commutation relations obeyed by the generators of the Poincaré group

$J_{\mu\nu}$ and P_λ , it may be shown that (see equation (1.2))

$$W_{\rho\sigma} = \epsilon_{\rho\sigma\kappa\lambda} W^\kappa P^\lambda \quad (2.1)$$

and

$$[W_{\mu\nu}, W_{\kappa\lambda}] = (\epsilon_{\mu\nu\kappa\sigma} W_\lambda - \epsilon_{\mu\nu\lambda\sigma} W_\kappa) P^\sigma \quad (2.2)$$

In addition it is straightforward to show that

$$[W_{\mu\nu}, W^D_{\kappa\lambda}] = [(g_{\mu\kappa} W_\nu - g_{\nu\kappa} W_\mu) P_\lambda + (g_{\nu\lambda} W_\mu - g_{\mu\lambda} W_\nu) P_\kappa] \quad (2.3)$$

and

$$[W^D_{\mu\nu}, W^D_{\kappa\lambda}] = (\epsilon_{\mu\nu\kappa\sigma} W^\sigma P_\lambda - \epsilon_{\mu\nu\lambda\sigma} W^\sigma P_\kappa) \quad (2.4)$$

It follows from (1.3) that

$$W_\mu P_\nu - W_\nu P_\mu = -i W^D_{\mu\nu}, \quad (2.5)$$

and it then follows immediately that equations (2.2) to (2.5) imply equation (1.5), which holds both for $X_{\mu\nu}$ and for $Y_{\mu\nu}$. Equations (1.7) therefore follow.

We now have to find expressions for X_i and Y_i in terms of the familiar variables used to characterise Dirac fields, viz. p_i and σ_i . We proceed by calculating the components of W^μ in the particles rest frame, denoted by a prime. It is clear from equation (1.1) that in this frame

$$P'_i = 0, \quad P'_0 = M, \quad W'_0 = 0 \quad [\text{rest-frame}] \quad (2.6)$$

and

$$[W'_i, W'_j] = i M \varepsilon_{ijk} W'_k \quad (2.7)$$

and hence that *in this frame*

$$S_i = W'_i \quad (2.8)$$

is a suitable spin operator, obeying, as it does, the commutation relations

$$[S_i, S_j] = i \varepsilon_{ijk} S_k. \quad (2.9)$$

It remains to find an expression for S_i in an arbitrary frame. The relevant Lorentz transformation matrix $L^\mu{}_\nu$ has components (we work in a metric $+ - - -$)

$$L^0{}_0 = 1, \quad L^0{}_i = -v_i, \quad L^i{}_j = \delta^i_j - \frac{v^i v_j}{1 - v^2}, \quad (2.10)$$

so W_i in an arbitrary frame has components given by

$$W'_i = L_i{}^\mu W_\mu = W_i - \frac{v_i}{1 - v^2} (W_0 - \mathbf{v} \cdot \mathbf{W}) \quad (2.11)$$

where the equation $W^\mu P_\mu = 0$ has been used. It then follows from (2.8) and (2.11) that

$$S_i = [W_i - \frac{v_i}{1 - v^2} (W_0 - \mathbf{v} \cdot \mathbf{W})]. \quad (2.12)$$

This expression for the spin operator has been found many years ago; see for example [4].

On the other hand the expressions for the operators X_i and Y_i are found, from equations (1.2) to (1.6), to be

$$\begin{aligned} X_i &= [W_i P_0 - W_0 P^i + i \varepsilon_{ijk} W^j P^k] \\ Y_i &= [W_i P_0 - W_0 P^i - i \varepsilon_{ijk} W^j P^k] \end{aligned} \quad (2.13)$$

and the spin operator Z_i for Dirac particles, defined by (1.8), turns out to be

$$Z_i = [W_i P_0 - W_0 P_i] - \gamma_5 \varepsilon_{ijk} W^j P^k. \quad (2.14)$$

Finally, for Dirac particles we put $S_i = \sigma_i/2$, so equations (2.1) and (2.12) give (with $M \rightarrow m$)

$$\begin{aligned} W_0 &= -\boldsymbol{\sigma} \cdot \mathbf{p} \\ W_i &= +, \end{aligned} \quad (2.15)$$

which result in $\mathbf{Z}$ taking on the form

$$\mathbf{Z} = \boldsymbol{\sigma} - \mathbf{p} (\boldsymbol{\sigma} \cdot \mathbf{p}) - i\gamma_5. \quad (2.16)$$

On positive energy states, i.e. those for which

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \gamma^5 E - \gamma^5 \gamma^0 m \quad (2.17)$$

$\mathbf{Z}$ becomes

$$\mathbf{Z} = \gamma^0 - \gamma^5 (1 + \gamma^0). \quad (2.18)$$

This is also the form taken by the Foldy-Wouthuysen mean spin operator [5,3] when acting on positive energy states. Our contention is then vindicated, that the relativistic spin operator defined by (1.8) is equivalent to the Foldy-Wouthuysen mean spin operator.

It may be useful to remark that the formula (2.14) for Z_i may also be derived by a generalisation of the method used in ref [3] to derive the relativistic spin operator. In that work the relativistic spin operator was obtained from $\sigma^i/2$ by transforming to another basis. The generalisation of equation (3.12) of ref [3] to the case of arbitrary spin is

$$X_i = \exp(\mathbf{S} \cdot \boldsymbol{\varphi}) S_i \exp(-\mathbf{S} \cdot \boldsymbol{\varphi}) \quad (2.19)$$

where $\boldsymbol{\varphi}$ is the usual Lorentz boost parameter

$$\boldsymbol{\varphi} = \tanh^{-1} (\mathbf{p}/E) \quad (2.20)$$

On observing that (see equation (2.15)) $\mathbf{S} \cdot \boldsymbol{\varphi} = \varphi$, on expanding the exponential factors and evaluating the commutators, X_i may be shown to yield the expression (2.13) above. The expression for Y_i follows by substituting $\boldsymbol{\varphi} \rightarrow -\boldsymbol{\varphi}$ in (2.19).

3. Conclusion

It has been shown that two relativistic spin operators may be defined in terms of the operators $[W^\mu, W^\nu]$ and its dual, where W^μ is the Pauli-Lubanski vector. These are operators for right- and left-handed spin states and the complete spin operator for Dirac particles then turns out to be the same as the Foldy-Wouthuysen mean spin operator, for particles of positive energy.

References

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