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PSEUDOSCALAR MESON DOMINANCE IN NEUTRINO-NUCLEUS REACTIONS

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1. - INTRODUCTION

You have heard much, at this school, of phenomena involving the coherent propagation in nuclei of electromagnetically generated vector mesons. This talk will be about similar phenomena for weak interactions. As well as the vector mesons, the axial vector and pseudoscalar can enter here. In particular the low mass of the pion implies that coherent propagation effects can develop at lower energies. Unfortunately it is not only in energy that available neutrino beams are inferior to photon and electron beams. Thus the experimental information available and the associated theoretical development are in no way comparable with what you have been hearing from Professors Leith ¹⁾ and Yennie ²⁾. Nevertheless preliminary attempts have been made ^{3),4)} to see a shadow effect in the nuclear size dependence of total cross-sections, and we will develop here the related theoretical notions.

Most of what will be said is based, in so far as it is novel, on a paper ⁵⁾ written with C.H. Llewellyn Smith, where more details of the working can be found. The introductory development has been set out more fully elsewhere ^{6),7)}.

2. - COHERENT MESON PROPAGATION

First let me remind you of some of the basic notions of coherent propagation that you have already learned for vector mesons. An optical model description will be used. The propagation of a meson in free space is described by the Klein-Gordon equation. To take account of refraction and absorption in nuclear matter, an optical potential V is added :

$$(m^2 + V - \partial_\mu \partial_\mu) \phi = 0$$

For the case of mesons incident on a nucleus from outside ϕ can be written as an incident plane wave plus a scattered wave

$$\phi = \phi_0 e^{iq_\mu x_\mu} + \chi$$

$$(m^2 + V - \partial_\mu \partial_\mu) \chi = -V \phi_0 e^{iq_\mu x_\mu}$$

Let us look for a plane wave solution

$$\chi \propto e^{iq_\mu x_\mu}$$

as might be appropriate deep inside a hypothetically very large nucleus, when surface effects have damped out. Then

$$\chi = \phi_0 e^{iq_\mu x_\mu} \left(\frac{-V}{m^2 + q^2 + V} \right)$$

For real mesons of course

$$m^2 + q^2 = 0$$

and the scattered wave χ is equal and opposite to, and extinguishes completely, the incident wave. This is simply a slightly complicated way of saying that the incident wave cannot penetrate indefinitely deeply into an absorbing medium.

Suppose next that instead of a meson incident we have some weakly interacting particle, say a photon, real or virtual, with wave-function

$$a_0 e^{iq_\mu x_\mu}$$

Then if coherent meson generation is possible, an equation

$$(m^2 + V - \partial_\mu \partial_\mu) \chi = -U a_0 e^{iq_\mu x_\mu}$$

will be appropriate, where U is a "transition potential". So now

$$\chi = a_0 e^{iq_\mu x_\mu} \left(\frac{-U}{m^2 + q^2 + V} \right)$$

For calculating the rate of reactions in any part of the nucleus we then have to allow for the coherent incidence of χ as well as of the original photon wave.

Suppose next that photon interactions occur only through virtual mesons. Then the photon wave

$$a_0 e^{iq_\mu x_\mu}$$

is equivalent to a meson wave

$$\frac{f}{m^2 + q^2} a_0 e^{iq_\mu x_\mu}$$

where f is the photon-meson coupling. So

$$U = V \frac{f}{m^2 + q^2}$$

Conversely the meson wave χ is equivalent to a photon wave

$$\frac{m^2 + q^2}{f} \chi$$

and the effective photon field in the nucleus is

$$a_c e^{iq_\mu x_\mu} + \left(\frac{m^2 + q^2}{f} \right) \chi$$

$$= a_c e^{iq_\mu x_\mu} \left(1 - \frac{V}{m^2 + q^2 + V} \right)$$

Thus although the weakly interacting photon is not much absorbed by the nuclear medium, the effective field can be damped to almost the same extent as for incident real mesons when the off-mass shell distance $m^2 + q^2$ is small compared with V . The damping factor in the amplitude is

$$\frac{m^2 + q^2}{m^2 + q^2 + V} \quad (1)$$

and in reaction rates

$$\left| \frac{m^2 + q^2}{m^2 + q^2 + V} \right|^2 \quad (2)$$

Thus even for weakly interacting particles, the bulk of the nucleus might be ineffective, and the reaction rate a surface rather than a volume effect.

To get some ideal of the magnitudes suppose that the optical potential V is purely absorptive, and that the absorption length, determined by considering exponentially damped plane wave solutions of the source-free equation, corresponds to a naïvely calculated mean free path

$$\lambda = (\rho \sigma)^{-1}$$

where ρ is nucleon density and σ the meson-nucleon cross-section averaged over nucleon spin and charge. Then

$$V = -i \sqrt{q_0^2 - m^2} \rho \sigma \quad (3)$$

We will suppose V does not vary much with q^2 . Then with

$$\sigma \approx 25 \text{ mb} \quad \rho \approx (1.8 \cdot 10^{-13})^{-3} \text{ cm}^{-3}$$

we have

$$\frac{V}{m^2} \approx \frac{\sqrt{q_0^2 - m^2}}{M_N} \frac{M_N}{m} \frac{1}{10} \quad (4)$$

where M_N is nucleon mass. With $m \approx 750 \text{ MeV}$ and $q^2 \approx 0$ the shadow effect is already well developed, and decreasing rapidly with energy, at a few GeV.

The serious application of such notions in the real photon case you have heard already from Leith and Yennie, and I will say nothing more on it. For virtual photons, i.e., electroproduction

some obvious extensions are necessary, e.g., the inclusion of longitudinal photon and vector meson waves.

3. - NEUTRINO REACTIONS

We are now concerned with the process

imagined for pictorial reasons to proceed through an intermediate boson W . At high enough energy transfers it would again be important to consider the propagation of vector mesons, and now also of axial vector mesons. However, neutrino beams available so far are effectively of energy a few GeV, and energy transfers are typically much smaller. Thus we will be mainly concerned with the propagation of the pion, and in fact will restrict our attention to this. Both because of the very low mass m [see Eq. (4)] and because of the very high cross-section in the resonance region we can expect the shadow effect to be highly developed even at these low energies - for reactions, if any, dominated by the pion. The question of pion dominance presents some peculiarities which we will now explain. But first let us mention what is probably a minor difference from the photon case : the W meson is charged, and it is with the coherent propagation of a charged pion in the nucleus that we are concerned. Both this pion and the outgoing charged lepton will be influenced by the nuclear Coulomb field, and one could worry that this might be a large effect in heavy nuclei. The question has been considered by Nachtmann⁸⁾, and it seems that the effect is likely to be quite small. We will not consider it here.

4. - THE PION POLE

Opening up the box in (5) one of the contributions will certainly be a pion pole diagram

and this will contribute an amplitude

$$M \frac{1}{m_\pi^2 + q^2} i q_\lambda f_\pi l_\lambda \quad (6)$$

where M is the amplitude for the same reaction with incident pions. l_λ is the factor arising from leptons and intermediate W meson, and f_π is the pion decay constant, whose magnitude can be obtained empirically from the rate of pion decay into leptons :

The factor $i q_\lambda$ is needed because this is the only way the pseudoscalar pion can couple to the vector meson W , (or directly to the lepton current if W does not exist). Now in detail

$$l_\lambda = \bar{\mu} i \gamma_\mu (1 + \gamma_5) \nu \quad g_W^2 \left(\delta_{\mu\lambda} - \frac{q_\mu q_\lambda}{M_W^2} \right) / (M_W^2 + q^2) \quad (7)$$

where g_W is the weak coupling constant, and so

$$q_\lambda l_\lambda = \bar{\mu} i \gamma_\mu q_\mu (1 + \gamma_5) \nu \quad g_W^2 / M_W^2$$

But

$$i\gamma_\mu q_\mu = i\gamma_\mu k_\mu - i\gamma_\mu k'_\mu$$

where k and k' are initial and final lepton four-momenta. Then using the Dirac equation

$$q_\lambda \ell_\lambda = m_\mu \bar{\mu} (1 + \gamma_5) \nu$$

where m_μ is the muon mass. However, the muon mass is a rather small quantity compared say with the lepton energies in which we will be interested. This, together with the rapid fall-off with q^2 of the pion pole term in (6), means that this contribution is relatively rather small. It could perhaps be important in a small region near $q^2 = 0$, but more detailed estimation shows that even there it is far from dominant^{9),7)}. Thus the pion certainly does not dominate the process in which we are interested in the same direct way as vector mesons are envisaged to dominate electromagnetic processes. Nevertheless for small q^2 the neutrino amplitude may be determined indirectly by the pion amplitude, as we will now explain.

5. - GOLDBERGER - TREIMAN CONSPIRACY

Let the full amplitude be written $\ell_\lambda M_\lambda$ and let M_λ be the sum of the pion pole term already considered and a remainder :

$$M_\lambda = i q_\lambda f_\pi \frac{1}{m_\pi^2 + q^2} M + M'_\lambda \quad (8)$$

Then it can be conjectured that the four amplitudes M'_λ , which by definition seem to have nothing to do with the pion, conspire together so that

$$iq_\lambda M'_\lambda = f_\pi M \quad (9)$$

- the Goldberger-Treiman conspiracy.

The main ground for this conjecture is that it works for "elastic" neutrino reactions ¹⁰⁾

$$\nu + n \rightarrow \mu + p$$

The amplitudes here can be expressed in terms of form factors

$$M'_\lambda = \bar{p} \{ i r_\mu f_1 + i r_\mu r_5 f_A - f_2 \sigma_{\mu\nu} i q_\nu \} n$$

$$M = \sqrt{2} g_{\pi N} \bar{p} i r_5 n$$

where $g_{\pi N}$ is the pion nucleon coupling constant, and we have supposed the pion pole to dominate the particular form factor (the induced pseudoscalar) in which it occurs. From the Dirac equation

$$iq_\lambda M'_\lambda = -2M_N f_A \bar{p} i r_5 n$$

where M_N is nucleon mass. So the GT conspiracy requires

$$f_A = f_\pi g_{\pi N} / (\sqrt{2} M_N)$$

This is the famous Goldberger-Treiman relation. It works quite well for low q^2 (and only for low q^2 is simple pion pole dominance of the induced pseudoscalar plausible). Note that it relates a quantity f_A , which does not itself contain a pion pole, to the residue of a pion pole in another function.

When the conspiracy occurs then from (8)

$$\begin{aligned}
 i q_\lambda M_\lambda &= i q_\lambda M'_\lambda - f_\pi \frac{q^2}{m_\pi^2 + q^2} M \\
 &= f_\pi M \left(1 - \frac{q^2}{m_\pi^2 + q^2} \right) \\
 &= \frac{f_\pi m_\pi}{m_\pi^2 + q^2} M
 \end{aligned}$$

- i.e., with M slowly varying $i q_\lambda M_\lambda$ (but not M_λ) is dominated by the pion pole. In terms of the hadronic weak vector and axial currents

$$M_\lambda = \langle F | J_\lambda^V + J_\lambda^A | I \rangle$$

where I and F are initial and final hadron states. Then

$$i q_\lambda M_\lambda = \langle F | \partial_\lambda J_\lambda^A | I \rangle$$

- since the divergence of the vector current is supposed to be zero. Thus the Goldberger-Treiman conspiracy implies, and is implied by, pion dominance of the divergence of the axial current - PDDAC (or PCAC).

Note again that the amplitudes M'_λ , of which a particular combination is related to the pion amplitude M , do not themselves contain a pion pole - nor does M . We insist on this because it is sometimes said that the phenomena with which we will be concerned must a priori be restricted to a very small q^2 region "because they are dominated by the pion pole". This is incorrect.

6. - ADLER'S FORMULA

For simplicity now neglect lepton mass m entirely, and with it the direct pion pole contribution to the matrix element :

$$\ell_\lambda M_\lambda = \ell_\lambda M'_\lambda$$

Consider the particular case of forward muon production - the muon continuing in the direction of the incident neutrino. With neglect of lepton mass the $\gamma_\mu(1+\gamma_5)$ coupling requires that lepton spin does not flip. It is then clear that the four-vector ℓ_λ must be some combination of the two four-vectors k_λ and k'_λ . But with neglect of lepton mass we have here two parallel null vectors :

$$k^2 = k'^2 = 0$$

So one is just a multiple of the other. Then we may take the particular combination

$$q_\mu = k_\mu - k'_\mu$$

(provided it does not vanish, as it would for elastic scattering) and say

$$\ell_\mu \propto q_\mu \tag{10}$$

with a proportionality factor that can be calculated. Then given the GT conspiracy the neutrino amplitude is a multiple of the pion amplitude. This observation is due to Adler ⁹⁾.

We have in particular, summing over all final hadron states, a relation between the double differential cross-section for secondary lepton production (with no change of strangeness)

$$\frac{d^2\sigma}{dq^2 dW^2}$$

[where $q^2 = (k_\mu - k'_\mu)^2 = 4k_0 k'_0 \sin^2 \frac{1}{2}\theta$ is the invariant momentum transfer, and $W^2 = M^2 + q^2 + 2Mq_0$ is the square of the hadron centre of mass energy] and the total cross-section $\sigma_\pi(W)$ for pions on the same target. Working out the kinematics and substituting numerical values for coupling constants

$$M_N^4 \frac{d^2\sigma}{dq^2 dW^2} \Big|_{\theta=0} = (3.99 \cdot 10^{-14}) \frac{k'_0}{k_0} \frac{M_N^2}{p_\pi W} \sigma_\pi(W) \quad (11)$$

where p_π is the pion centre of mass momentum for centre of mass energy W in pion nucleon collision.

The formal derivation of Adler's formula (11) does not mention the nature of the target. So why not apply it to nuclei? This seemed problematic at first because hadron cross-sections are expected to show a shadow effect - an " $A^{2/3}$ " dependence, but no such effect had been expected in reactions with energetic weakly interacting projectiles, which were expected to go essentially like A . It seemed probable therefore that for large nuclei the argument would be violated by some large q^2 dependences increasing with nuclear size - we have tacitly assumed that amplitudes not containing a pion pole vary slowly between $q^2 = -m_\pi^2$ (for real mesons) and $q^2 \approx 0$ (for forward neutrino reactions). This was only partially confirmed by the optical model analysis⁶⁾ already outlined. There is indeed a volume term at $q^2 \approx 0$ which is not present at $q^2 = -m_\pi^2$. But it is reduced by a factor

$$\left| \frac{m_\pi^2 + q^2}{m_\pi^2 + q^2 + V} \right|^2$$

relative to naïve expectation. Thus in appropriate conditions the weak interaction can also show a large shadow effect.

To get some idea of the importance of various regions we have made some simple calculations for a target which is the average of one proton and one neutron. The average of empirical $\bar{\pi}^+n$ and π^+p cross-sections was roughly fitted by a constant 35 millibarns for $W > 2.13 M_N$, and for $W < 2.13 M_N$ by a p wave resonance formula. Figure 1 shows the result of integrating (11) over W^2 , as a function of neutrino energy k_0 , and also the result of restricting the W^2 integration to the resonance region $W < 2.13 M_N$. It is clear that with neutrino energies used so far, effectively of a few GeV, the $N^*(1238)$ resonance region is rather important. Figure 2 shows the double differential cross-section (11), for the forward direction $q^2 \approx 0$, as a function of W^2 ; the curve shown is actually for a specific incident neutrino energy of 4 GeV, but it is clear from (11) that the shape does not depend sensitively on neutrino energy, which enters only in the factor (k'_0/k_0) . Also indicated in Fig. 2 is the effect of multiplying by a shadow factor (2), at $q^2 \approx 0$, with V determined in a simple forward scattering amplitude approximation.

On the basis of these observations, it might be hoped that the shadow effect could be demonstrated with available neutrino energies - if it were possible to select inelastic events of sufficiently forward character - whatever "sufficiently" may mean here - and study the A dependence of the rate.

Neutrino Energy k_0 (GeV)	Target	Calculated σ (mb)	Experimental σ (mb)
0.5	Fe	10.0 ± 1.0	0.15
0.5	Fe	10.0 ± 1.0	0.15
0.5	Fe	10.0 ± 1.0	0.15

7. - EXPERIMENT

A preliminary attempt in this direction was made during the 1967 CERN neutrino run ⁴⁾. A spark chamber assembly with plates of various materials was exposed to the beam. In this set-up one could see the material in which a given event originated, the angle of the muon direction, and could make a rough measurement of muon momentum by means of a magnetic field over part of the assembly. Not much could be seen of the character of each event, other than that the secondary particles included one of sufficient penetrating power to be probably a muon (in association with this a lower limit of about 1.2 GeV was imposed on the secondary energy), and of the incident neutrino energy it could be said only that it came from a spectrum of which one version is illustrated in Fig. 3.

So no control on degree of inelasticity was possible, and in particular "quasi-elastic" events could not be excluded - those involving rather low energy transfer, the recoil of a nucleon without meson production. No shadow effect is expected for such events. It was hoped, however, that at sufficiently small angles the inelastic events would contribute sufficient of the total for a shadow effect to appear.

The results were the following. What is quoted is the ratio of event rates per nucleon in various pairs of different materials. These ratios would all be unity for simple A dependence and the value indicated for simple $A^{2/3}$ dependence.

$\theta < 29^\circ$	Experiment	A ratio	$A^{2/3}$ ratio
Pb/C	0.97 ± 0.04	1	0.39
Fe/C	0.89 ± 0.08	1	0.60
Pb/Al	0.93 ± 0.07	1	0.51

Results for a smaller angle sample were presented for the Pb/C comparison :

$\theta < 5^\circ$	Experiment	A ratio	$A^{2/3}$ ratio
Pb/C	0.92 ± 0.15	1	0.39

These results certainly favour simple A dependence against $A^{2/3}$ dependence. However, we have to discuss the possible degree of dilution of the shadow effect by quasi-elastic scattering and by unshadowed inelastic scattering which must enter at angles other than zero. We will take up the latter first.

8. - GENERALIZED OPTICAL MODEL

We have now to deal with the effect of coherent pion propagation without the simplifications appropriate to the strictly forward reaction. The appropriate optical model equation is

$$(m_\pi^2 + V - \partial_\mu \partial_\mu) \chi = - U_\mu l_\mu e^{i q_\lambda x_\lambda} \quad (12)$$

where the source term takes account of the generation of the pion wave by the four different components of the lepton pair wave function. To link up with what has gone before let us suppose that U_μ is made up of a pion pole term and a remainder, in analogy with (8),

$$U_\mu = U_\mu^1 + i q_\mu f_\pi \frac{1}{m_\pi^2 + q^2} V \quad (13)$$

As before the pion pole term is eliminated by contraction with l_μ (in the approximation $m_\mu = q_\mu l_\mu = 0$) and does not contribute in our problem. Assuming the Goldberger-Treiman conspiracy in the form

$$i q_\mu U'_\mu = f_\pi V$$

we then reduce the problem, in the forward direction when $l_\mu \propto q_\mu$, to that of the incidence of off-shell pions, with the familiar results.

Returning to the general case (12) the desirable procedure would be to substitute realistic potentials V and U'_μ for finite distributions of nuclear matter, and solve the equations. This we will not do. A formal construction of operator U_μ and V from fundamental interactions can indeed be given. Apart from the question of what the fundamental interactions may be, we could not evaluate these expressions for strong interactions. So in practice progress depends on simple phenomenological approximations, and on experimental information. Even with real pions this study, for energies in the resonance region, is only beginning¹¹⁾. Realistic ideas of V will emerge - and then the question of U_μ can be tackled. Because of the uncertainties it does not seem reasonable to do very elaborate calculations at this time. We will give only some simple indications, for what they may be worth, based on the case of a plane wave in a homogeneous medium, and making a naïve forward scattering amplitude first approximation to the potentials.

For the homogeneous medium plane wave case, taking the direction of propagation as the three-axis, the terms in (12) invoking the transverse components l_1 and l_2 must vanish - by rotational symmetry about the three-axis:

$$(m_\pi^2 + V + q^2) \chi = -(U'_3 l_3 - U'_0 l_0) e^{i q_\mu x_\mu} \quad (14)$$

In these circumstances the transverse components of l_μ do not generate a coherent pion wave and so are not shadowed. As for the longitudinal components, we have, from the GT conspiracy, only that an incident lepton wave is equivalent to an incident pion wave when $(l_3/l_0) = (q_3/q_0)$. But in a moment we will adopt a model in which such an equivalence holds generally. Then for these contributions we have the shadow factor (2). Thus what we have to find is the relative importance, in reactions on a single nucleon, of the transverse components of l_μ on the one hand and of the longitudinal and scalar components on the other. This we will estimate in a simple model in which it is supposed that inelastic reactions, induced by either pions or intermediate weak bosons, go entirely through the 33 isobar. We have seen (Figs. 1 and 2) that although this resonance is not the whole story, it is certainly an important part. We will assume, although it might be more complicated, that the formation and decay of the isobar proceeds in nuclear matter very much as for an isolated nucleon.

9. - 33 ISOBAR MODEL

Working in the hadron centre of mass system, the spin state of the spin $\frac{3}{2}$ isobar can be specified by a Rarita-Schwinger three-vector two-spinor $\vec{R}_\alpha$ (in general the spinor index α will be suppressed) subject to the subsidiary condition

$$\vec{\sigma} \cdot \vec{R} = 0$$

[The vector indices having been saturated, $\vec{\sigma} \cdot \vec{R}$ would transform simply as a two-spinor, and we wish to have no such spin $\frac{1}{2}$ part in $\vec{R}$.] Consider first the process

and let P denote also the two-spinor giving (always in the hadron centre of mass system) the spin state of the proton. Then the unique coupling, by parity and rotational invariance, is

$$g_{\pi R} (\vec{R}^* P) \cdot i\vec{q} \quad (15)$$

where $\vec{q}$ is pion momentum and $g_{\pi R}$ a coupling constant, and the brackets () signify summation over suppressed spinor indices. The magnitude of $g_{\pi R}$ is determined by the observed width of the resonance.

Consider next the process

$$\nu + P \rightarrow \mu^- + R^{++}$$

The leptons must enter in the combination l_μ already defined, (7). There are eight different coupling schemes ¹²⁾. For the axial and vector currents respectively those of lowest order in the momentum $\vec{q}$ are

$$f_{AR} (\vec{R}^* P) \cdot \vec{l} + f_{VR} (\vec{R}^* P) \cdot \vec{l} \times i\vec{q} \quad (16)$$

These are the only terms which occur in a simple no-recoil model ¹³⁾. Dispersion theorists ¹⁴⁾ seem agreed that they are the most important terms, and that the relative sign is that given by the no-recoil theory. The magnitude of f_{AR} in terms of $g_{\pi R}$ can be fixed to give the Goldberger-Treiman conspiracy : insert in (9)

$$M'_\lambda = (f_{AR} (\vec{R}^* P), 0)$$

(with zero time component)

$$M = g_{\pi R} (\vec{R}^* P) \cdot i\vec{q}$$

and we have

$$f_{AR} = f_{\pi} g_{\pi R} \quad (17)$$

The magnitude of the vector coupling f_{VR} can be fixed by photoproduction data, via the isovector current hypothesis, and comes out [like (17)] close to the static theory value :

$$f_{VR} / f_{AR} = 2.0 M_N^{-1} \quad (18)$$

We spoke always of a proton here, but (17) and (18) apply equally to a neutron target. The isovector current, and the isovector pion, can couple in only one way with the isospinor nucleon to give isospin $\frac{3}{2}$, so that neutron and proton amplitudes are related by Clebsch-Gordan coefficients to a single amplitude.

Let us make a digression here. We are mainly interested in small q^2 . To consider large q^2 we would have to guess something about form factors. We have in fact made some calculations of total resonance production cross-sections, simply including in the amplitude (16) a factor

$$\left(1 + \frac{q^2}{81 M_N^2} \right)^{-2}$$

derived from nucleon electromagnetic form factors. Here, as elsewhere in this work, we allowed for resonance width by using the isobar model to generate relations between lepton and pion cross-sections, and using empirical fits for the latter. To compare with the work of Adler, based on a serious dispersion theory account, we integrate, as in one of his calculations, over $W < 1.39$ GeV. For the average over neutron and proton as target, the resulting production cross-section as a function of neutrino energy is plotted in Fig. 4. The curve is close to that of

Adler. Note, however, that the agreement would be less close for protons and neutrons separately ; we would have the pure $I = \frac{3}{2}$ ratio of three to one for protons as compared with neutrons. Adler, retaining some $I = \frac{1}{2}$ amplitudes, does not have such a large difference.

Other authors ¹⁵⁾ have obtained, with the isobar model, substantially larger N^* production cross-sections. Berman and Veltman ¹²⁾ used excessively strong couplings, as a result of not having detected a hidden convention with respect to Clebsch-Gordan coefficients in the photoproduction paper of Gourdin and Salin ¹⁶⁾ ; this point was cleared up by Bijtebier ¹⁷⁾. Salin ¹⁸⁾ on the other hand used a different axial current coupling which did not, as observed by Adler ¹⁹⁾, respect PDDAC.

Unfortunately experiment seems to favour the larger values ^{15),20)}. The discrepancy, with our model and with the theory of Adler, may arise from larger q^2 values. Then it does not concern us greatly here. If it is found to apply also for low q^2 values, then it will be a rather significant result, for Adler's forward-lepton pion equivalence theorem is built in to the models, and this theorem would then be in conflict with experiment.

10. - LONGITUDINAL AND TRANSVERSE INELASTIC

The isobar model amplitudes have been written in terms of quantities $\vec{l}$ and $\vec{q}$ defined in the hadron centre of mass system. To introduce the corresponding quantities of the laboratory system requires, by Lorentz transformation, the substitutions

$$\begin{aligned}\vec{l}_t &\rightarrow \vec{l}_t \\ \vec{l}_3 &\rightarrow (\vec{l}_3 - \vec{v} l_0) / \sqrt{1-v^2} \\ \vec{q} &\rightarrow (\vec{q} - \vec{v} q_0) / \sqrt{1-v^2}\end{aligned}\tag{19}$$

where $\vec{v}$ is the velocity (it is along $\vec{q}$) of the hadron centre of mass in the laboratory,

$$\vec{v} = \frac{\vec{q}}{M + q_0} \quad \frac{1}{1 - v^2} = \frac{M + q_0}{W}$$

and $\vec{l}$ has been written as a sum of parts $\vec{l}_3$ and $\vec{l}_t$ along and perpendicular to $\vec{q}$. Using

$$q_\mu l_\mu = 0$$

we can eliminate l_0 , and then find that (19) can be written

$$\vec{l}_t \rightarrow \vec{l}_t \quad \vec{l}_3 \rightarrow \vec{l}_3 \left(\frac{M}{W} - \frac{q^2}{W q_0} \right) \quad \vec{q} \rightarrow \vec{q} \frac{M}{W} \quad (20)$$

So we have the amplitudes for resonance formation

$$g_{\pi R} (\vec{R}^* P) \cdot i \vec{q} (M/W) \quad (21)$$

for incident pions, and

$$\left(f_{AR} \vec{l}_3 \frac{M q_0 - q^2}{W q_3} + f_{AR} \vec{l}_t + f_{VR} \vec{l}_t \times i \vec{q} \frac{M}{W} \right) (\vec{R}^* P) \quad (22)$$

for leptons.

From (22) and (21) it follows that the longitudinal and scalar components of l_μ , non-combined in the $\vec{l}_3$ term, are together equivalent to a certain pion wave, and so presumably it is this $\vec{l}_3$ term which is subject to shadowing.

If we consider now the total reaction rate with leptons it is clear, summing over spins of R and P, that this is proportional to the square of the coefficient of $(\vec{R}^* P)$ in (20),

$$|\vec{\kappa}|^2 = \left| f_{AR} \frac{M q_0 - q^2}{W q_3} \vec{l}_3 \right|^2 + \left| f_{VR} \vec{l}_t \times i \vec{q} \frac{M}{W} + f_{AR} \vec{l}_t \right|^2 \quad (23)$$

and that the coefficient of proportionality can be expressed in terms of the cross-section for incident pions. The result is, for a single nucleon,

$$\frac{d^2 \sigma}{dq^2 dW^2} = \frac{W^2}{8\pi^2} \frac{P(W)}{M^2 k_0^2} |\vec{\kappa}|^2 \quad (24)$$

with

$$P(W) = \sigma_{\pi}(W) / (4 p_{\pi} W) \quad (25)$$

The lepton factors have then to be evaluated from

$$l_{\mu} = \bar{\mu} i \gamma_{\mu} (1 + \gamma_5) V$$

and this has been done accurately for the numerical values to be quoted later. However, there is a simple approximation which gives some insight into what happens. It is valid for given energy and momentum transfer when the primary neutrino energy k_0 is very large, and so the angle θ between neutrino and muon directions is very small. Then in the explicit spinors

$$V = \sqrt{k_0} \begin{pmatrix} \phi \\ \sigma \cdot \hat{k} \phi \end{pmatrix} \quad \mu = \sqrt{k'_0} \begin{pmatrix} \phi' \\ \sigma \cdot \hat{k}' \phi' \end{pmatrix}$$

we can take $k_0 \approx k'_0$, $\hat{k} \approx \hat{k}'$, $\theta \approx \theta'$, and l_μ reduces to its forward scattering value (obtained by direct calculation or by covariance arguments)

$$l_\mu = 4 k_\mu$$

This can be regarded as a multiple of q_μ in the forward direction $\theta = 0$, but away from the forward direction $\vec{l}$ does not follow $\vec{q}$, but remains approximately fixed, and so the component transverse to $\vec{q}$ increases (it is $\vec{q}$ which moves away from $\vec{l}$ rather than vice versa) :

$$|\vec{l}_\perp| / |\vec{l}_3| \approx \theta' \quad (26)$$

where

$$\cos \theta' = \hat{q} \cdot \hat{k}$$

Under the conditions described θ' is much bigger than θ :

From this point of view the 5° sample of Borer et al., if we say that the typical energy transfer is of order 300 MeV and the typical primary energy of order 2 or 3 GeV, is really a $35-50^\circ$ sample ! For this reason, and because the longitudinal term in (23) is reduced especially away from the forward direction by the Lorentz transformation factor, the unshadowed transverse background can be expected to be substantial.

11. - QUASI-ELASTIC

It remains to consider the background of quasi-elastic events, events initiated by an elastic collision

$$\nu + n \rightarrow \mu + p$$

with a single neutron. Of course even to use this language is already to adopt the picture of a nucleus as a collection of almost - free almost - isolated nucleons.

For a single neutron the covariant matrix element is

$$M = \ell_\lambda L_\lambda$$

with

$$\ell_\lambda = \bar{\mu} i \gamma_\lambda (1 + \gamma_5) \nu (G \cos \theta_c / \sqrt{2})$$

where G is again the universal weak coupling and θ_c the Cabibbo angle (we neglect strangeness changing processes), and

$$L_\lambda = \bar{p} (f_1 i \gamma_\lambda - f_2 \sigma_{\lambda\nu} i q_\nu + f_A i \gamma_\lambda \gamma_5 - f_P q_\lambda \gamma_5) n$$

in terms of the usual form factors $f(q^2)$. The differential cross-section for a neutron at rest is,

$$\begin{aligned} \frac{d\sigma}{dq^2} = & \frac{G^2 \cos^2 \theta_c}{8\pi k_0^2} \left\{ \left[\frac{(k_0 + k'_0)^2}{1 + \frac{1}{4} q^2} - q^2 \right] g_1^2 + \left[2q^2 (k_0 + k'_0) \right] g_2 f_A \right. \\ & \left. + \left[\frac{(k_0 + k'_0)^2}{1 + \frac{1}{4} q^2} + q^2 \right] \left[g_2^2 q^2 + f_A^2 (1 + \frac{1}{4} q^2) \right] \right\} \quad (27) \end{aligned}$$

with units $\hbar = c = M_N = 1$ and

$$g_1 = f_1 - \frac{1}{2} q^2 f_2 \quad g_2 = f_2 + \frac{1}{2} f_1$$

$$q^2 = 4 k_0 k_0' \sin^2 \frac{1}{2} \theta$$

$$k_0' = k_0 - \frac{q^2}{2M_N} = k_0 / (1 + 2 k_0 \sin^2 \frac{1}{2} \theta)$$

In the numerical results, quoted already in Fig. 1 and to be quoted later, we have taken

$$(g_1, g_2, f_A) = (1, \frac{1}{2} \xi, \lambda) (1 + g^2 M_F^{-2})^{-2}$$

$$\xi = \mu_p - \mu_n = 4.7 \quad \lambda = f_A(0) = 1.2$$

$$M_F = 840 \text{ MeV}$$

The vector form factors are derived from the electromagnetic. To give the same q^2 dependence to the axial vector is not inconsistent with the rough experimental information available, but has no more basis than that. Since we are mainly interested here in small q^2 , the point is not vital.

Going to the nuclear case, two modifications immediately come up for consideration, namely Fermi motion and the exclusion principle. As for the first, nucleon velocities in nuclei are not very small ($v/c \lesssim 0.3$), but there is a good deal of cancellation between nucleons moving in different directions and we can omit this effect in a first orientation; Løvseth in particular has made calculations taking it into account. The exclusion principle, in a simple Fermi gas picture, then requires that only that fraction of the neutron Fermi sphere is effective for which the momentum transfer $\vec{q}$,

$$|\vec{q}|^2 = q^2 + q_0^2 = q^2 (1 + \frac{1}{4} q^2)$$

is sufficient to leave the final proton outside the occupied proton Fermi sphere. This fraction ²¹⁾ is, where N is neutron number, Z proton number, and $A = N + Z$,

$$(N - D) / N \quad (28)$$

with

$$\begin{aligned} D &= Z & \text{if } x < \frac{1}{2}(u-v) \\ D &= 0 & \text{if } x > \frac{1}{2}(u+v) \end{aligned} \quad (29)$$

$$D = \frac{A}{2} \left\{ 1 - \frac{3x}{4}(u^2+v^2) + \frac{x^3}{2} - \frac{3}{32x}(u^2-v^2)^2 \right\} \quad \text{otherwise}$$

where

$$x = |\vec{q}/2k_f| \quad u = |(2N/A)^{1/3}| \quad v = |(2Z/A)^{1/3}|$$

and in numerical calculation we take for the mean Fermi momentum

$$k_f = 268 \text{ MeV}/c$$

corresponding to a radius $R = (1.12)A^{1/3}$ Fermi for a sphere of nuclear matter. The factor (28) is plotted in Fig. 5 for $N/Z = 1$ and $N/Z = 126/82$.

Using these formulae and averaging over the CERN neutrino spectrum we have calculated the quasi-elastic contribution to the 5^0 sample of Borer et al. For $N = Z$ (carbon) it contributes a cross-section per nucleon of $0.84 \times 10^{-40} \text{ cm}^2$, without the Berman factor (28) it would have been $1.82 \times 10^{-40} \text{ cm}^2$. For $N/Z = (126/82)$ (lead) it contributes a cross-section per nucleon of $1.28 \times 10^{-40} \text{ cm}^2$, without the Berman factor this would have been $2.20 \times 10^{-40} \text{ cm}^2$.

The nucleons in lead are more efficient in this reaction than those of carbon ; to begin with a bigger fraction of them are neutrons, and then these neutrons are less inhibited by exclusion, the excess neutrons even at $|\vec{q}| = 0$ giving rise to protons outside the smaller proton Fermi sphere. The phenomenon goes in the opposite direction to the shadow effect, in comparing large and small nuclei.

The simple Fermi gas model is not at all immediately convincing in application to a nucleus as small as carbon. We have therefore made some additional calculations with shell model wave functions. The usual "pseudopotential" approach is followed ; the relativistic single-nucleon matrix element is expressed as the matrix element between Pauli two-spinors of an operator involving the Pauli spin operator $\vec{\sigma}$ and the momentum transfer $\vec{q}$; in general the initial nucleon momentum $\vec{K}$ would also appear, but is neglected in a first approximation. For the many-body problem a sum of such contributions from each nucleon is taken, and matrix elements evaluated between nuclear-model states. Finally the "closure" approximation is made in connection with summing over nuclear final states, and the result appears as a certain ground state expectation value. The details of our particular version of this will be presented elsewhere ²²⁾. Suffice to say here that instead of the single Berman factor D in (28) now appear three different factors appropriate for terms with different spin dependence in the basic operator :

$$D_{S,T,L} = -(\Psi, [\sum_{n \neq m} \sum e^{-i\vec{q} \cdot (\vec{r}_n - \vec{r}_m)} \vec{r}_n \vec{r}_m^+ (1, \sigma_{xn} \sigma_{xm}, \sigma_{zn} \sigma_{zm})] \Psi)$$

where Ψ is the ground state wave function, n and m enumerate nucleons, and σ_x and σ_z are components of $\vec{\sigma}$ transverse to and along $\vec{q}$. The cross-section per nucleon is obtained from (27) by the substitutions

$$g_1^2 \rightarrow g_1^2 (N - D_S) / A$$

$$g_2^2 \rightarrow g_2^2 (N - D_T) / A$$

$$f_A^2 \rightarrow f_A^2 (N - D_T) / A$$

$$g_2 f_A \rightarrow g_2 f_A (N - D_T) / A$$

(Actually D_L does not appear here, but does appear in the related problem of muon capture.)

The scalar exclusion factor can be written for $\vec{q} = 0$ as

$$\begin{aligned} (N - D_S) / A &= A^{-1} (\Psi, [\sum_n \frac{1}{2} (1 - \tau_3)_n + \sum_{n \neq m} \tau_n^- \tau_m^+] \Psi) \\ &= A^{-1} (\Psi, [\sum \tau_n^- \sum \tau_m^+] \Psi) \end{aligned}$$

Since $\sum \tau_m$ is the operator of total isospin, this vanishes for a nucleus of isospin zero. In the simplest shell model of a nucleus like O^{16} , where for a given orbital wave-function all spin states are occupied, it is easily seen that the three functions $D_{S,T,L}$ are in fact equal, and then the reaction is fully suppressed in the forward direction, $\vec{q} = 0$, as in the simple Fermi gas model with $N = Z$. This is not so for carbon. The simplest shell model description is then a closed $1s$ shell, a closed $1p_{3/2}$ shell, and an unoccupied $1p_{1/2}$ shell. The results of the calculations, with oscillator wave functions and a size parameter determined by the electromagnetic elastic form factor, are shown in Fig. 6. Also shown for comparison is the Berman exclusion factors for $(N/Z) = 1$. The difference in efficiency between neutrons in carbon and lead is diminished. Averaged over the 5° sample of Borer et al., the cross-section per nucleon in carbon comes out at $1.05 \times 10^{-40} \text{ cm}^2$ instead of $0.84 \times 10^{-40} \text{ cm}^2$, to be compared with $1.28 \times 10^{-40} \text{ cm}^2$ for lead (still with the Berman Fermi gas factor).

Actually it seems likely that these calculations exaggerate the difference for small $|\vec{q}|$ between D_T and D_L on the one hand and D_S on the other. The simplest shell model of carbon is not regarded as very realistic ²³⁾, and it is probable that configurations with some depopulation of the $p_{3/2}$ shell, and some population of the $p_{1/2}$ shell, are equally important - i.e., the spin-orbit splitting is not such a dominant force. As an experiment we therefore repeated the calculations substituting for D_T and D_L the value previously obtained for D_S . The cross-section for the five degree sample then fell to $0.94 \times 10^{-40} \text{ cm}^2$.

Before leaving this topic, it may be useful to make some general comments on the validity of the pseudopotential approach, and on the significance of using it with more realistic wave functions allowing for internucleon interactions. Although this procedure must contain part of the truth, and although most nuclear theory is based on such ideas, its limitations should not be overlooked. The correct operator must surely contain additional terms arising from the proximity and interaction of two or more nucleons. Moreover there is no unique division of the hypothetical correct operator into single-body and many-body terms. Given a picture involving wave functions Ψ and operators O , an equally valid picture is obtained by unitary transformation

$$\Psi' = e^{iS} \Psi \quad O' = e^{iS} O e^{-iS}$$

In general single and many-body terms in operators will be changed individually and mixed up by the transformation. Thus the same effect can appear, very roughly speaking, in one picture as due to a correlation in the wave function and in another as due to a many-body term in the operator concerned. Thus interactions and correlation functions are not separately well defined concepts. Yet it is not uncommon to find in the literature estimates of many-body terms in interactions without any definition whatever of the wave function for which they are appropriate. This is only acceptable so

long as the effects in question are small (as they usually are on the scale of precision of nuclear theory) and the calculations are regarded as only exploratory. To add up the results of different calculations of this kind, of different small effects of correlations in wave functions or many-body terms in operators, is very unreliable.

These remarks are made because of the possibility of renewed theoretical activity in this area, stimulated by a strange experimental result. The most relevant experiment ²³⁾ does not seem to show at all any inhibiting influence of the exclusion effect on small angle quasi-elastic neutrino reactions - although such an inhibition seems to show up strongly in the inverse reaction, muon capture. In this connection it is a pity that there is not more, and more comparable, electron or muon scattering data available ²⁴⁾.

CONCLUSION

The results of the various calculations, for the 5^0 sample of Borer et al., are set out in the Table. Because the use of (24) above the resonance region is quite arbitrary, we quote separately the inelastic contributions from low and high mass regions. The quasi-elastic cross-sections take account of nuclear effects only by the Berman exclusion factor (28). In the column labelled "unshadowed Pb", the inelastic cross-sections are less than those for "carbon" because with our pure isospin $-\frac{3}{2}$ amplitude neutrons are only one third as effective as protons. Of course the actual process is not pure $\frac{3}{2}$ and so we have probably exaggerated the effect of the neutron proton asymmetry. In the third column of the Table, a very crude allowance for shadow has been made in the longitudinal inelastic by putting a geometric ceiling

$$2\pi R^2 / A$$

(with $R = 1.2 A^{1/3}$ Fermi, $A = 208$) on σ_{π} in the formula. Again this is very likely an exaggeration of the Pb-C difference ; we would not expect the shadowing to be completely absent in the small nucleus carbon (see the experimental results quoted, for example, by Ericson and Locher ²⁵) nor to be completely developed even in the large nucleus Pb (see for example the calculations of Froyland and Løvseth ²⁶). Despite these exaggerations of the reduction in the inelastic, going from carbon to Pb, the increase in the elastic pretty well compensates for it.

With the figures given in the Table, the ratios of total cross-sections per nucleon in Pb and C are given by

$$\begin{aligned} \frac{\sigma(\text{Pb})}{\sigma(\text{C})} &= \frac{2.41}{2.10} = 1.15 && \text{without shadow} \\ &= \frac{2.02}{2.10} = .96 && \text{with shadow} \end{aligned}$$

If we use instead of the Berman exclusion factor D the shell model functions D_T, D_L

$$\begin{aligned} \frac{\sigma(\text{Pb})}{\sigma(\text{C})} &= \frac{2.41}{2.31} = 1.04 && \text{without shadow} \\ &= \frac{2.02}{2.31} = .88 && \text{with shadow} \end{aligned}$$

If we use the shell model D_S

$$\begin{aligned} \frac{\sigma(\text{Pb})}{\sigma(\text{C})} &= \frac{2.41}{2.20} = 1.09 && \text{without shadow} \\ &= \frac{2.02}{2.20} = .92 && \text{with shadow} \end{aligned}$$

The experimental result

... 0.92 ± 0.15 ... favours slightly the various shadowed values, but, as remarked by the experimenters ⁴⁾, the experimental uncertainty is too large for a serious conclusion. Moreover, we would not like to have to give a figure for the theoretical uncertainty.

The theory outlined here is naïve in conception, crude in execution, and lamentably lacking in experimental confirmation in its various parts. Nevertheless it seems to offer some understanding of how a substantial shadow effect in the appropriate component can be diluted away and lost in unshadowed background. The magnitude of this background is associated with the dominance of low energy transfer processes - quasi-elastic scattering and N^* production - in the experiment under study. Moreover these processes will remain rather important (see Figs. 1 and 2) even with much more energetic neutrinos, so long as the energy transfer cannot actually be controlled. The unshadowed inelastic background can be reduced by improved angular definition, but the quasi-elastic remains, and the importance of doubts about the nuclear effects here increases. It is quite frustrating to be obliged to theorize about everything and add up the results. The neutrino physicist can well envy the electrician's ability to isolate the various phenomena and come to terms with them one at a time.

	(N/Z) = 1 "carbon"	N/Z = 126/82 "unshadowed Pb"	N/Z = 126/82 "shadowed Pb"
-----	-----	-----	-----
Longitudinal inelastic			
$(W/M)^2 < 2.13$	0 . 452	0 . 405	0 . 115
$(W/M)^2 > 2.13$	0 . 218	0 . 195	0 . 096
-----	-----	-----	-----
Transverse inelastic			
$(W/M)^2 < 2.13$	0 . 436	0 . 390	0 . 390
$(W/M)^2 > 2.13$	0 . 165	0 . 148	0 . 148
-----	-----	-----	-----
Quasi-elastic	0 . 833	1 . 275	1 . 275
=====	=====	=====	=====
Total	2 . 104	2 . 413	2 . 024

TABLE 1 : Different contributions to the cross-section per nucleon for events with
 $\theta < 5^\circ$ and muon energy > 1.2 GeV.

REFERENCES

- 1) D.W.G.S. Leith - "Photoproduction and Vector Dominance", this volume.
- 2) D.R. Yennie - "Interaction of High-Energy Photons with Nuclei as a Test of Vector Meson Dominance", this volume.
- 3) M. Holder - Thesis work, Internal Report No 19 of the III. Physikalisches Institut, Technische Hochschule, Aachen.
- 4) K. Borer, B. Hahn, H. Hofer, F. Krienen and P.G. Seiler - Phys.Letters 30B, 572 (1969).
- 5) J.S. Bell and C.H. Llewellyn Smith - CERN Preprint TH.1198 (1970), to be published.
- 6) J.S. Bell - Phys.Rev.Letters 13, 57 (1964).
- 7) J.S. Bell - CERN Preprint TH. 887 (1968).
- 8) O. Nachtmann - CERN Preprint TH. 1162 (1970).
- 9) S.L. Adler - Phys.Rev. 135, B963 (1964).
- 10) Actually we can only infer that this will be so, from the case of beta decay, assuming electron-muon universality.
- 11) M. Krell and S. Barmo - "An Analysis of the Scattering $\pi^- 12\text{C}$ Around the N^* Resonance", to be published in Nuclear Phys.
- 12) S.M. Berman and M.J.G. Veltman - Nuovo Cimento 38, 993 (1965).
- 13) J.S. Bell and S.M. Berman - Nuovo Cimento 25, 404 (1962).
- 14) S.L. Adler - Ann.Phys. 50, 189 (1968), and references therein.
- 15) For a survey, and references, see : "Single Pion Production by High-Energy Neutrinos", Proceedings of the CERN Topical Conference on Weak Interactions, CERN Report No 69-7 (1969).
- 16) M. Gourdin and P. Salin - Nuovo Cimento 27, 193 and 309 (1963).
- 17) J. Bijtebier - Nuclear Phys., to be published.
- 18) P. Salin - Nuovo Cimento 48A, 506 (1967).
- 19) S.L. Adler - Private communication.

- 20) I. Budagov et al. - Phys.Letters 29B, 524 (1969).
- 21) S.M. Berman - Proceedings of the CERN High-Energy Theoretical Conference, CERN Report No 61-22 (1961).
- 22) J.S. Bell and C.H. Llewellyn Smith - in preparation.
Our approach is close to that of B. Goulard and
H. Primakoff - Phys.Rev. 135, B1139 (1964).
- 23) M. Hirooka et al. - Progr.Theor.Phys. 40, 808 (1968) ;
A. Fujii, M. Morita and H. Ohtsubo - Progr.Theor.Phys.,
Suppl.Extra No. 303 (1968).
- 24) G.R. Bishop, D.B. Isabelle and C. Betourne - Nuclear Phys.
54, 97 (1964).
- 25) T.E.O. Ericson and M.P. Locher - CERN Report No 69-30 (1969).
- 26) J. Frøyland and J. Løvseth - Nuovo Cimento 53A, 1 (1968).

FIGURE CAPTIONS

- Figure 1 The forward neutrino cross-section averaged over free neutrons and protons.
- a) The inelastic cross-section according to Adler's theorem using $\sigma_{\pi} = 35 \text{ mb}$ for missing mass $W > 2.13 M_p$ and a resonance formula, for $W < 2.13 M_p$.
- b) The contribution to the inelastic cross-section from $W < 2.13 M_p$.
- c) The elastic cross-section.
- Figure 2 The differential cross-section, Eq. (26), for the forward process in the resonance region for a neutrino energy $4 M_p$ (upper curve) and the result (lower curve) of multiplying by the square of the modulus of the shadow factor, Eq. (5), the potential V being computed in a modified forward scattering approximation. W is the invariant mass of the final hadron system.
- Figure 3 Normalized model of the CERN neutrino spectrum plotted against the neutrino energy in proton mass units.
- Figure 4 The inelastic neutrino cross-section, in our model, for invariant masses of the final hadron system $W < 1.39 \text{ GeV}$.
- Figure 5 Berman exclusion factors for symmetric ("carbon") and unsymmetric ("Pb") Fermi gases.
- Figure 6 The three shell model exclusion factors for carbon, and for comparison the symmetric Fermi gas factor. The calculations were done with oscillator wave functions and the simplest configuration, closed $1s$ and $1p_{\frac{3}{2}}$ shells, assumed.

FIG.1

FIG.2

FIG. 3

FIG.4

FIG.5

FIG.6