Exact isotropic cosmologies with local fractal number counts

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Abstract. We construct an exact relativistic cosmology in which an inhomogeneous but isotropic local region has fractal number counts and matches to a homogeneous background at a scale of the order of $10^2$ Mpc. We show that Einstein’s equations and the matching conditions imply either a nonlinear Hubble law or a very low large-scale density.

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1. Introduction

There is doubt and argument about whether the data on the galactic number count $N$ can support what is called a fractal structure, i.e.,

$$N \propto y^\nu,$$

(1)

where $0 < \nu < 3$ is the fractal index and $y$ is some distance measure (see [1, 2, 3] and references therein). However, there is evidence for such a pattern in the number counts out to a distance of the order of $10^2 h^{-1}$ Mpc [1, 2, 4].

Even this evidence is beset with problems because it is difficult to agree adequate statistics which are model independent [2]. For instance, Peebles ([1], page 212) specifically excludes inhomogeneous spherically symmetric models in which we are at the centre. This may be reasonable but it serves to illustrate how aware one needs to be of the underlying model. It would help if good predictive dynamical models for the local distribution were available. The data could then be tested against these. Attempts in this direction are in their infancy but there are some that could yield a fractal number count [5]. At the same time, a number of workers are using recent large-scale structure observations and indirect evidence to argue that fractal-like clustering on small and possibly intermediate scales does not continue indefinitely, and that there is a transition to a near-homogeneous distribution on large scales [6]. This
is supported by the implications for large-scale structure of the near-isotropy of the cosmic microwave background radiation, since it has been shown that if all fundamental observers measure small anisotropy in the temperature, then the universe on large scales after last scattering is necessarily close to homogeneity [7].

Our study will consider the number counts as a function of the observer area distance (angular diameter distance) \( D \) or redshift \( z \), down the past light cone of the observer. As a consequence, even in the spatially homogeneous Friedman-Lemaître-Robertson-Walker (FLRW) case the number counts will be non-homogeneous. However the underlying spatial homogeneity of the FLRW geometry is reflected in the fact that the number count/redshift formula is precisely determined by the model [8]. This formula specifically rules out a simple fractal distribution of the form (1). Note that our approach precludes direct comparison with work in which the fractal distribution is assumed to hold on spacelike surfaces (see for example [9]).

To analyse the problem theoretically we assume (a) that locally we are in a spherically symmetric region which can be modelled by a Lemaître-Tolman-Bondi (LTB) geometry with a fractal number count of sources down our past light cone, and (b) that at some value of the observer area distance (of order \( 10^2 h^{-1} \text{ Mpc} \)) the universe becomes homogeneous and can be modelled by an appropriate FLRW spacetime. We assume that the LTB region matches smoothly to the FLRW region, with no surface density layer or shell crossing [10, 11]. These assumptions are not unreasonable on the basis of current observations. Of course, we are only considering a very restricted form of fractal distribution, i.e. a single fractal in the radial direction which maintains the overall spherical symmetry.

A similar model was investigated by Ribeiro [12], who was the first to construct a relativistic fractal-count model. Ribeiro developed sophisticated numerical computations to analyze his models. He showed that parabolic and elliptic models gave a nonlinear Hubble law, while hyperbolic models were able to support a linear Hubble law. Our results are in agreement with these conclusions, but we believe that we have identified more clearly and simply the underlying reasons for these features. We have also pursued the implications of parabolic models that match to open FLRW models – i.e. that the large-scale density is extremely low. Ribeiro’s work concentrated on numerical integration. Our approach is primarily analytical, and relies crucially on a systematic use of the central regularity conditions and the transition matching conditions. Thus our work may be seen as complementing Ribeiro’s by bringing analytical insights to some of his results and extending aspects of his work.

In principle, LTB models can accommodate a great variety of number count relations [8, 13, 14]. However, the assumed number count relation may not be compatible with matching to the large-scale homogeneous universe, with other observations, or with regularity conditions on the observer’s wordline.
The main result here is that if the fractal number counts are taken at face value and the universe can be modelled locally by an LTB geometry which matches to an FLRW background at some scale, then the universe either cannot have a linear Hubble law at low redshift (parabolic case), or it has a very low large-scale density (non-parabolic case). Perturbation of the model is unlikely to change this conclusion and increasing the scale of the fractal-count region (and there is some evidence for this) will exacerbate the low density problem. The low density problem identified by this model can only be avoided if the dark matter is strongly biased.

It would be of interest to construct a linear perturbative model to confirm the above. The objective here, however, is to use an exactly isotropic non-perturbative model and to study the consequences flowing from the field equations and matching conditions.

2. The cosmological model

In this section equations are presented for the general LTB metric, which, together with a Kantowski-Sachs-type solution [10], encompasses all regular, spherically symmetric dust models. These equations are used in section 3 to analyse the fractal number count subclass of LTB models.

The metric must satisfy regularity conditions, including the central conditions as \( D \rightarrow 0 \). (See [8] for the central conditions in general; a comprehensive analysis for LTB is given in [10] for central observers, and in [15] for off-centre observers.) For this problem we use coordinates based on the observer’s past light cones as described in [8] and usually called observational coordinates. Details of the construction for the LTB model are given in [13]. (See also [16], where semi-isotropic observational generalizations of LTB models are investigated.)

Explicitly the coordinates are \( \{w, y, \theta, \phi\} \), where \( \{w = \text{constant}\} \) are the observer’s past light cones, \( y \) is a distance from the observer along a past light ray and \( (\theta, \phi) \) label the direction of observation. In these coordinates the metric of the LTB model is

\[
\text{d}s^2 = -A(w, y)^2\text{d}w^2 + 2A(w, y)B(w, y)\text{d}w\text{d}y + C(w, y)^2 \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right).
\]

The Einstein field equations cannot be integrated explicitly in these coordinates [13], but they can be in the more familiar (1+3) formalism [17]. However the (1+3) coordinates are not directly observable. An advantage of the observational coordinates is that the null geodesic equations are already integrated, whereas they are not integrable in the (1+3) formalism. This explicit relation between the null geodesics and the coordinates facilitates the interpretations. We will use both formalisms below. The metric in (1+3) comoving coordinates has the form

\[
\text{d}s^2 = -\text{d}t^2 + \left[ \frac{\partial R(r, t)}{\partial r} \right]^2 \frac{\text{d}r^2}{1 - kf(r)^2} + R(r, t)^2 \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right),
\]
where \( f \) is arbitrary and relates to the ‘total energy’ of the LTB model. The cases \( k = 0, +1, -1 \) correspond respectively to parabolic, elliptic or hyperbolic geometry of the \( \{ t = \text{constant} \} \) hypersurfaces. If \( \{ w = w_0 \} \) is the past light cone of observation, then

\[
D(y(z)) = C(w_0, y(z)) = R(r(z), T(r(z))) ,
\]

where \( t = T(r) \) is the equation of a past light ray.

In the observational coordinates the number count (total number of sources out to distance \( y \)) is given by [8, 13]

\[
N(y) = 4\pi \int_0^y n(w_0, x) B(w_0, x) D(x)^2 dx ,
\]

where \( n(w_0, y) \) is the number density of sources. The dust density is

\[
\rho = mn ,
\]

where \( m \) is the average galactic mass, which we assume to be constant. For simplicity, we omit evolution and selection effects. It would be possible to take some account of these effects when choosing \( n \) and \( m \), but we have not done so.

The total number count in the inhomogeneous region is given in terms of the Bondi mass function \( M(r) \) (which arises from integrating the field equations) by [13]

\[
N(y) = \frac{1}{m} \int_0^y M'(x) \left[ 1 - k f(x)^2 \right]^{-1/2} dx ,
\]

where \( M(y) \) is short-hand for \( M(r(y)) \).

It is well known that the LTB metric is fully determined by three arbitrary functions: one can be removed by a coordinate choice but the other two have physical significance. In [13] quadratures are given which determine the LTB functions directly from the observational data \( D(z) \) and \( N(z) \). This determination forms the corner-stone of the current analysis, since the LTB functions are needed to determine the dynamics and (crucially) the matching to the FLRW model. If the LTB functions are identified as the Bondi mass \( M \), the energy function \( f \) and the big-bang time \( \beta \) (see [18]), then we can use the results of [13] to give these functions in terms of the observables as follows:

\[
\sqrt{1 - k f(z)^2} = \frac{(1 + z)}{2D(z)} \int_0^z \frac{1}{Q} \left\{ D'(x) + \left[ \frac{D(x)Q(x)^2}{(1 + x)^2} \right]' \right\} dx ,
\]

where

\[
Q(z) \equiv 1 - m \int_0^z \frac{(1 + x)N'(x)}{D(x)} dx .
\]

By a rearrangement of (3)

\[
M(z) = m \int_0^z \left[ 1 - k f(x)^2 \right]^{1/2} N'(x) dx ,
\]
\[ \beta(z) = t_0 - \int_0^z D'(x) \left\{ \left[ 1 - kf(x)^2 \right]^{1/2} - \left[ \frac{2M(x)}{D(x)} - kf(x)^2 \right]^{1/2} \right\}^{-1} \, dx \]

\[
\begin{cases}
-\frac{1}{3} [2D(z)^3 M(z)^{-1}]^{1/2} & \text{(if } k = 0) \\
M(z) [\sin \Gamma(z) - \Gamma(z)] |f(z)^{-3}| & \text{(if } k = +1) \\
M(z) [\Gamma(z) - \sinh \Gamma(z)] |f(z)^{-3}| & \text{(if } k = -1),
\end{cases}
\]

where

\[
\Gamma(z) \equiv \begin{cases}
2 \arcsin \left[ \frac{1}{2} D(z) f(z)^2 M(z)^{-1} \right]^{1/2} & \text{(if } k = +1) \\
2 \arcsinh \left[ \frac{1}{2} D(z) f(z)^2 M(z)^{-1} \right]^{1/2} & \text{(if } k = -1).
\end{cases}
\]

An arbitrary constant of integration appearing in (6) has been identified as \( t_0 \), the time of observation (\( t_0 \) is arbitrary since only the time elapsed after the big bang, \( t - \beta \), has physical significance). In these integrations the freedom in \( y \) on the light cone of observation has been used up by setting \( A(w_0, y) = B(w_0, y) \); this choice simplifies the calculation \[13\].

Now it is evident that in the \( k = 0 \) case only one of the functions \( N(z) \) and \( D(z) \) is arbitrary; for example \( N(z) \) may be obtained from (4) by setting \( k = 0 \), once \( D(z) \) is known. This covariant constraint on the observational data may be found explicitly as follows. By (5)

\[ 1 + z = -\frac{D}{m} \frac{dQ}{dD} \left( \frac{dN}{dD} \right)^{-1} \left( \frac{dQ}{dD} \right)^{-1} \],

and then (4) with \( k = 0 \) gives

\[ Q \frac{d}{dD} \left[ -2m \frac{dN}{dD} \left( \frac{dQ}{dD} \right)^{-1} \right] = \frac{d}{dD} \left\{ D \left[ 1 + \frac{Q^2}{(1 + z)^2} \right] \right\}. \]

Integrating by parts, and using the central conditions, we find that

\[ -2mN + 2m \frac{dN}{dD} \left( \frac{dQ}{dD} \right)^{-1} + D + \frac{m^2 Q^2}{D} \left( \frac{dN}{dD} \right)^2 \left( \frac{dQ}{dD} \right)^{-2} = 0. \]

Solving this as a quadratic (and again imposing the central conditions to eliminate a spurious root), gives

\[ \frac{1}{Q} \frac{dQ}{dD} = -\frac{m}{D} \frac{dN}{dD} \left( 1 - \sqrt{\frac{2mN}{D}} \right)^{-1}. \]
Together with (8), this gives the solution
\[ 1 + z = \left( 1 - \sqrt{\frac{2mN}{D}} \right)^{-1} \exp \left[ - \int \frac{m}{D} \frac{dN}{dD} \left( 1 - \sqrt{\frac{2mN}{D}} \right)^{-1} dD \right], \quad (9) \]
which appears to be a new result, and which is central to showing that the Hubble law is nonlinear (see below). Thus when \( k = 0 \), if \( N \) is known in terms of \( D \), then (9) gives \( z \) in terms of \( D \).

3. Fractal number counts

The model we construct is isotropic about the observer and inhomogeneous out to a distance \( D = D_h \) of the order of \( 10^2 \) Mpc, corresponding to a redshift \( z \) below \( 10^{-1} \). The LTB spacetime metric in this local region satisfies the Einstein equations with a dust source which yields a power-law number count with fractal index, of the form (1). To ensure consistency with observational data, we also require a linear distance/redshift relation out to the homogeneity scale \( D_h \), i.e. a linear Hubble law for small redshift. The Hubble constant is \( H_0 = 100h \) km s\(^{-1}\) Mpc\(^{-1}\), with \( 0 < h < 1 \). For the large-scale universe (\( D > D_h \)), we require a homogeneous FLRW geometry. The Darmois matching conditions at \( D_h \) uniquely determine the parameters of this FLRW solution.

In all cases the central conditions [13] demand that for small \( D \)
\[ N(D) = \left( \frac{4\pi\rho_0}{3m} \right) D^3 + O(D^4), \quad (10) \]
so that \( N \sim D^3 \) as \( D \to 0 \). This means that the number count cannot be fractal (i.e. with \( \nu \neq 3 \)) for \( D \) near zero. There is some minimum distance \( D_f \) for fractal counts, below which \( N \sim D^3 \). (Note that \( D_f \) should be above the averaging scale which is implicit in a continuum dust model of galactic matter.) A simple model of fractal number counts that incorporates the limiting behaviour (10) is the continuous power-law ansatz
\[ N(D) = \begin{cases} 
(\frac{4\pi\rho_0}{3m}) D^3 & \text{for } D \leq D_f \\
(\frac{4\pi\rho_0}{3m}) D_f^3 \left( \frac{D}{D_f} \right)^\nu & \text{for } D_f \leq D < D_h.
\end{cases} \quad (11) \]
(Matching conditions require \( N \) to be at least continuous at \( D = D_f \), otherwise there would be a surface layer.) This model represents an ‘instantaneous’ transition from non-fractal to fractal counts at the minimum fractal distance \( D_f \), and \( N \) is not differentiable at \( D_f \). An alternative model, in which \( N \) is a smooth function of \( D \), is
\[ N = \left( \frac{4\pi\rho_0}{3m} \right) D^3 \left( 1 + \frac{D}{D_f} \right)^{\nu-3}, \quad (12) \]
which satisfies $N \sim D^3$ for $D \ll D_f$ and $N \sim D^\nu$ for $D \gg D_f$.

Since $D_f$ has to be small, the region where $D < D_f$ can be treated as homogeneous and so $z_f \equiv z(D_f) \approx H_0 D_f$. At the homogeneity scale $D_h$, the number counts must match continuously to the FLRW number count relation $\bar{N}(D)$:

$$N(D_h) = \bar{N}(D_h).$$  \hspace{1cm} (13)

The functions $\bar{N}(D)$ are known. For example, the $k = 0$ FLRW spacetime has [13]

$$H_0 D = 2 \left( \frac{m H_0}{32 \pi} \bar{N} \right)^{1/3} \left[ 1 - \left( \frac{m H_0}{32 \pi} \bar{N} \right)^{1/3} \right]^2. \hspace{1cm} (14)$$

3.1. Parabolic fractal-count models

Consider now the possibility of modelling the fractal-count region $D_f < D < D_h$ by a parabolic LTB solution ($k = 0$). Assuming that the core region $0 \leq D < D_f$ is also parabolic, Einstein’s equation determines the Hubble constant in terms of the central density (as in a FLRW model) [13]:

$$H_0 = \sqrt{\frac{8}{3} \pi \rho_0}.$$

We assume that the fractal number counts are modelled by the power-law ansatz (11). Then the matching of $N$ at the homogeneity scale $D_h$, given by (13) and (14), implies

$$\left[ \frac{H_0 D_f \left( \frac{D_h}{D_f} \right)^{\nu/3} - 2}{8 \left( \frac{D_h}{D_f} \right)^{3-\nu}} \right]^6 = 0. \hspace{1cm} (15)$$

Thus the four parameters $H_0$, $D_f$, $D_h$ and $\nu$ are subject to the constraint (15) by virtue of number count continuity. If observations are used to determine $H_0$, $D_h$ and $\nu$, then this constraint fixes the minimum fractal scale $D_f$.

The problem with the parabolic fractal-count models arises from the nonlinear behaviour of the redshift/area distance relation for small $D$. From equations (8) and (9), we find that (11) implies

$$\frac{dz}{dD} \approx \frac{1}{2} H_0 \left[ (\nu - 1) \left( \frac{D_f}{D} \right)^{(3-\nu)/2} - \nu H_0 D_f \left( \frac{D_f}{D} \right)^{2-\nu} \right], \hspace{1cm} (16)$$

for small $D$. It follows that

$$\frac{dz}{dD} \sim \left\{ \begin{array}{ll}
-D^{(\nu-3)/2} & \text{for } 0 < \nu < 1 \\
-D^{\nu-2} & \text{for } 1 \leq \nu < 3.
\end{array} \right.$$  

Thus after the initial linear behaviour up to $D_f$ (by construction), the redshift/distance graph begins immediately to curve downwards. This nonlinearity contradicts the well-established linear Hubble law on scales up to the order of $10^2$ Mpc, and means that
parabolic fractal-count models are ruled out. Although we have used the power-law fractal count ansatz (11) to deduce this nonlinearity, it is clear that the feature will persist for any model that incorporates (1), since the argument depends only on the behaviour for small $D$.

Finally, we note that exact expressions for $z(D)$ may be obtained for any rational fractal index $\nu$, since in this case the quadrature in (9) may be performed exactly for the power-law relation (11). For example, with $\nu = \frac{3}{2}$ we find

$$1 + z = (1 + z_f) \left[ \frac{1 - H_0 D_f (D/D_f)^{1/4}}{1 - H_0 D_f} \right]^2 \exp \left[ 3 H_0 D_f \left\{ \left( \frac{D}{D_f} \right)^{1/4} - 1 \right\} \right],$$

(17)

for $D_f < D < D_h$. Nonlinearity for small $D$ is readily confirmed by plotting the graph of (17).

### 3.2. Non-parabolic fractal-count models

The interpretation of observations cannot be realised if the fractal-count region is parabolic, since then the redshift/distance relation is nonlinear. For $k = \pm 1$ there is no such problem, since (as discussed in section 2) the observable functions $N(z)$ and $D(z)$ are independent. This allows us to circumvent the nonlinearity problem – but a new problem arises, i.e. the problem of very low density in the large-scale FLRW region arising from fractal number counts and the matching conditions. Intuitively, fractal number counts imply an under-density (since $\nu < 3$), so that $k = -1$. Then matching conditions rule out $k = +1$ on large scales, and imply that the large-scale $k = -1$ FLRW model must have even lower density than the local LTB region.

From (3) the relationship between $M$ and $N$ is more complicated in LTB models with $k \neq 0$. If $k = +1$, one obtains $M/m < N$, and conversely for $k = -1$. Note that regularity in the current context dictates that $k$ must not increase with distance from the observer [10]. Therefore we cannot match a $k = -1$ LTB model to a $k = +1$ LTB or FLRW exterior. Furthermore, the matching conditions require continuity of $M$ and $kf^2$.

Numerical integrations are simpler with differentiable functions. For this purpose we replace (11) with the alternative smooth-transition ansatz (12). We now focus on the dynamics and the implications of such a number count profile (with the other observations) for the large-scale density. Using the formulas (4)–(7) we have integrated the field equations with number count formula (12) out to $z = 0.07$, where we assume the metric matches to the FLRW background. We take $D_f = 10$ Mpc. We label the central (local) density parameter by $\Omega_c$ and the large-scale (background) density parameter by $\Omega_0$. We take $D(z) = H_0^{-1} z$, which is the well established Hubble law for these redshifts. In the following tables the remaining parameters are varied in turn and the consequences for
the background density are shown.

Solutions with $h = 0.65$, $\Omega_c = 0.2$

<table>
<thead>
<tr>
<th>Fractal index $\nu$</th>
<th>Large-scale density $\Omega_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.0002</td>
</tr>
<tr>
<td>1.5</td>
<td>0.001</td>
</tr>
<tr>
<td>2.0</td>
<td>0.008</td>
</tr>
<tr>
<td>2.5</td>
<td>0.04</td>
</tr>
<tr>
<td>3.0</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Solutions with $h = 0.65$, $\nu = 1.5$

<table>
<thead>
<tr>
<th>Local density $\Omega_c$</th>
<th>Large-scale density $\Omega_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.001</td>
</tr>
<tr>
<td>0.35</td>
<td>0.0025</td>
</tr>
<tr>
<td>0.50</td>
<td>0.004</td>
</tr>
<tr>
<td>0.75</td>
<td>0.006</td>
</tr>
<tr>
<td>1.00</td>
<td>0.008</td>
</tr>
</tbody>
</table>

Solutions with $\nu = 1.5$, $\Omega_c = 0.2$

<table>
<thead>
<tr>
<th>Hubble rate $h$</th>
<th>Large-scale density $\Omega_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.00025</td>
</tr>
<tr>
<td>0.40</td>
<td>0.0007</td>
</tr>
<tr>
<td>0.65</td>
<td>0.001</td>
</tr>
<tr>
<td>0.80</td>
<td>0.002</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0025</td>
</tr>
</tbody>
</table>
In all of these cases it was found that \( k \) had to be equal \(-1\), corresponding to hyperbolic space geometry as one would expect with low densities. The background density is that of the uniquely defined FLRW model that matches at \( z = 0.07 \) \((D = D_h)\) to the interior fractal-count LTB model (for more details of the matching calculation, see [10]). In all cases the significant fact is that the background density \( \Omega_0 \) is extremely low. Qualitatively, what happens is that the under-density implied by \( N \sim D^\nu \) with \( \nu < 3 \) in the LTB region, is worsened by the fact that in the corresponding FLRW case, \( N \) has a steep non-power-law gradient beyond \( z \approx 0.4 \) [12].

It is of interest that the above numerical integrations confirmed that the no-shell-crossing conditions [11] were satisfied. The effect of the big-bang function \( \beta \) on the value of \( \Omega_0 \) is negligible at these redshifts.

4. Conclusion

The nonlinear Hubble law and low density problems of fractal-count universes, identified in this paper, apply to all the regular spherically symmetric dust spacetimes [10]. These are the spacetimes that can be constructed by piecing together regions with LTB metrics (including the homogeneous FLRW case), in which the matching satisfies the Darmois conditions and there are no surface layers or shell crossings.

The nonlinear Hubble law at very low redshift rules out the parabolic models. However, ways to avoid the low density problem of non-parabolic models are still conceivable. There could be a surface layer or a cosmological constant, neither of which has been included in this analysis, and their effects are not fully known. The most compelling explanation, however, is the possibility of a significant bias in the data – the number counts of luminous matter may not trace the actual distribution of density. It could be the result of selection effects, evolution or the presence of dark matter. This provides an interesting slant on our result because it gives a different emphasis to the search for evidence.
References