Currents and Superpotentials in classical gauge invariant theories I. Local results with applications to Perfect Fluids and General Relativity.

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E. Noether’s general analysis of conservation laws has to be completed in a Lagrangian theory with local gauge invariance. Bulk charges are replaced by fluxes at a suitable singularity (in general at infinity) of so-called superpotentials, namely local functions of the gauge fields (or more generally of the gauge forms). Some gauge invariant bulk charges and current densities may subsist when distinguished one-dimensional subgroups are present. We shall study mostly local consequences of gauge invariance. Quite generally there exist local superpotentials analogous to those of Freud or Bergmann for General Relativity. They are parametrized by infinitesimal gauge transformations but are afflicted by topological ambiguities which one must handle case by case. The choice of variational principle: variables, surface terms and boundary conditions is crucial.

As a first illustration we propose a new Affine action that reduces to General Relativity upon gauge fixing the dilatation (Weyl 1918 like) part of the connection and elimination of auxiliary fields. We can also reduce it by similar considerations either to the Palatini action or to the Cartan-Weyl moving frame action and compare the associated superpotentials. This illustrates the concept of Noether identities. We formulate a vanishing theorem for the superpotential and the current when there is a (Killing) global isometry or its generalisation. We distinguish between, asymptotic symmetries and symmetries defined in the bulk.

A second and independent application is a geometrical reinterpretation of the convection of vorticity in barotropic nonviscous fluids first established by Helmholtz-Kelvin, Eckart and Ertel. In the homentropic case it can be seen to follow by a general theorem from the vanishing of the superpotential corresponding to the time independent relabelling symmetry. The special diffeomorphism symmetry is, in the absence of dynamical gauge field and spin, associated to a vanishing internal transverse momentum flux density. We consider also the nonhomentropic case. We identify the one-dimensional subgroups responsible for the bulk charges and thus propose an impulsive forcing for creating or destroying selectively helicity resp. enstrophies in odd resp. even dimensions. This is an example of a new and general Forcing Rule.
1 Introduction

1.1 Generalities

Eighty years ago E. Noether [1] assembled together in a series of theorems some consequences of continuous symmetries of classical actions. Any rigid (Lie) symmetry gives rise to a current with the general formula

\[ J := \Sigma - \delta \phi \wedge \frac{\partial L}{\partial \delta \phi} \quad (1) \]

The current is a \((D - 1)\)-form, \(\delta \phi\) is the variation of the field under an infinitesimal rigid symmetry, \(d\Sigma\) represents the total divergence by which the Lagrangian density changes (so \(\Sigma\) itself is defined up to an exact form, in other words up to a topological current) and a sum over the independent fields \(\phi\) is implied. The form notation should not deter the reader as we shall return to components for the simplest applications. The first theorem of Noether says that the symmetry and some equations of motion are encoded as the closure of \(J\) modulo the equations of motion. Conversely such a conserved current or more precisely the equality of a linear combination of the equations of motion to a total divergence implies a rigid symmetry. The precise statement eliminates the topological ambiguity in the Noether currents, there is no classical symmetry associated to a topological current unless one dualizes the theory!

Secondly local (gauge) invariances imply relations between the equations of motion, these are now called Noether identities. They follow from the triviality of the gauge variations which reduce the effectiveness of the variational principle. Conversely the identities imply local invariances. In the Hamiltonian formalism time dependent gauge invariance leads to primary constraints whereas secondary constraints follow from space dependent gauge invariance. This is now well understood in relativistic theories such as electromagnetism...

A third theorem formulated with Hilbert is in modern language that in the case of gauge invariance, any current \(J_\xi\) associated to a one parameter subgroup of generator \(\xi\) of the gauge group is equal, modulo the equations of motion, to an identically conserved (i.e. topological) local current. The idea then was that local invariance destroyed the physical relevance of the charges of any rigid subgroup. This claim can be made stronger as was shown in the case of General Relativity by Bergmann and his school around
1950 [2, 3, 4], namely there are local superpotentials $U$ ($(D - 2)$-forms) such that on shell, assuming only fields and their first derivatives contribute to the action:

$$J_\xi := \xi \cdot J + d\xi \wedge U$$  \hspace{1cm} (2)

is conserved for all $\xi$'s and hence $J = dU$. Bergmann, see [3] introduced the term strong conservation laws. In fact local invariance is still widely and wrongly believed to actually prevent the existence of any invariant conserved charge, see however the recent [5]. Note that the locality of $U$ does not follow from that of $J$ even when spacetime is contractible. If one restricts attention to infinitesimal gauge transformations along a fixed generator, one may still multiply the latter by a scalar coefficient depending arbitrarily on spacetime coordinates and apply the Hilbert-Noether-Bergmann construction to that subalgebra, one obtains then

$$J_\xi \approx dU_\xi := d(\xi \cdot U)$$  \hspace{1cm} (3)

Independently of these results it was shown in 1981 [6] that a $p$-form gauge invariance corresponding to a $(p + 1)$-form potential leads to a $(D - p - 2)$-form $J$ that is closed on shell. In other words $dJ \approx 0$ modulo the equations of motion, generalizing the $p = 0$ and $p = -1$ cases. If one views Yang-Mills invariance as a mixture of $p = 0$ and $p = -1$ invariances one recovers the analog of Bergmann’s analysis. We recognize one half of Maxwell’s equations in the strong conservation equation

$$J \approx dU = d*F$$  \hspace{1cm} (4)

In the nonabelian case we may still pick a direction of gauge transformations with arbitrary (scalar and $x$ dependent) magnitude $\xi(x)$ then for this particular abelian subgroup of gauge transformations we have the same formula (3). This is the origin of the ’t Hooft abelian charges of the dyons, see for instance [7].

The discussion of higher conservation laws has been recently carefully extended to a generalised Noether theorem relating symmetries of various types with generalised charges [8] in a cohomological framework.

Now in the case of rigid symmetry, $J$ is already afflicted by ambiguities, it is well known that they permit the constructions of the symmetrized or improved energy-momentum tensors. This arbitrariness becomes much more serious in the case of gauge invariance as the ambiguity of the superpotential $U$ seems to be total. In fact the litterature on General Relativity
is littered with a host of superpotentials without clear status of respectability. We shall concentrate in this paper on the local aspects of the theory, in particular on the formulas that generalize (1), their dependence on the order of differentiation of the fields and on possible surface terms.

But let us recall that the physical measurement of the force leads to the value of a gravitational mass far from its source by actually assuming asymptotic flatness all around it. One could also expand around an arbitrary background near infinity and define a mass parameter there, this has been carried out in particular for anti de Sitter asymptotics. In this paper we shall focus on the asymptotically flat case. One puts the laboratory at infinite distance from the source(s) in some direction in the sense that the metric becomes flat up to order $1/r$ corrections then the limit of $\frac{r}{2}(g_{00}+1)$ or $\frac{r}{2}(g_{ii}-1)$ in asymptotic rest frame coordinates is the physical mass deviating test particles. The use of arbitrary coordinates requires a geometrical definition of asymptotics, in other words of the boundary at infinity (we shall consider spatial infinity in this first paper). We must choose a model manifold for the neighbourhood of infinity but not its coordinates. Note that this manifold does not have to be close to ours except there. A side remark is that local but not global asymptotic flatness would force us to distinguish between a local definition of mass from formulas involving total fluxes. Let us take the example of electromagnetism and consider an orbifold ALM space obtained by quotienting $\mathbb{R}^4$ by $\mathbb{Z}_2$ (the sphere at infinity is replaced by $\mathbb{R}P^2$). Clearly if the electric field is $\frac{2}{r}$ in Gaussian units its flux is equal to $2\pi e$ and not $4\pi e$, the conical singularity at the origin affects the relation between the total flux and the local (asymptotic) field. Similarly total angular momentum perpendicular to the direction of the source is measured by the limit of $\frac{\mathcal{E}}{2} g^{ijk} g_{j0} r_k$ again if one assumes global trivial topology at infinity. We leave this global issue for subsequent work.

To the above physical and local definition of charge one can compare mathematical formulas, for instance the charge may take the form of a flux at infinity, this is the case for the celebrated ADM expression [9] for the total mass of a curved spacetime or the generalisation by Regge and Teitelboim [10] in a Hamiltonian description. We shall follow here a Lagrangian approach and invert the conventional order of the constructions: we shall look for a bulk density such that its integral is equal to the physical mass given by such a flux at infinity. It has not been widely recognized that when there is a singularity, even if it is hidden behind an horizon and contrary to the abelian case, bulk integrals may not make physical sense.

The Nester-Witten form [11, 12] can be used outside horizons and will be
discussed in the next paper of this series hereafter called paper II. The proof
of the positivity of total ADM mass for a general solution with black holes
uses the existence of a supersymmetric extension of the theory it localises all
the energy outside the horizons, and the “energy density” is positive there.

The special case of a global Killing vector is essentially bringing us into
an abelian framework as Kaluza-Klein inspired ideas may suggest. In a gen-
eral gauge theory we shall call Killing symmetry a Lie algebra generator
preserving the value of the gauge field, for instance isometries of a metric,
isotropy gauge transformations in Yang-Mills theory, Killing spinors for su-
persymmetry etc... The existence of a global (bulk) Killing symmetry leads
in general to the vanishing of the gauge part of the current density as a
generalisation of the vanishing charge of the photons and of all the Fourier
zero modes of the Kaluza-Klein dimensional reduction. In the case of diffeo-
morphisms the gauge current may fail to vanish because of a surface term
but it does vanish for spatial Killing directions and in the vacuum as we
shall see. We shall discuss the general formalism in section 2 but mostly
focus on the rich case of diffeomorphisms.

1.2 Perfect fluids

The reader interested in the conservation laws of fluids will at this stage be
able to skip the middle sections (3-5) on our new formulation of General Rel-
ativity and should go directly to section 6, if he so wishes. Relabelling sym-
metries allow a physically suggestive interpretation of the conserved quan-
tities of perfect fluids. These fluids obey a variational principle involving
independent Lagrangian coordinates (the labels), the fields are simply the
Eulerian, or laboratory, coordinates, they admit time independent space
relabeling gauge invariance without any gauge field. In the homentropic
(possibly compressible) case the relabelings are arbitrary volume preserving
diffeomorphisms. The corresponding spatial Noether current is purely lon-
gitudinal because there is no propagating gauge field in this gauge invariant
theory, this is the local vorticity conservation in comoving cordinates [13].
Noether’s theorem has been invoked before but without superpotentials (see
the nice review [14]) and when it was precisely formulated it was the global
theorem that was used as in Taub’s description of flows (see the review
[15]) where the roles of Lagrangian and Eulerian variables are exchanged.
It turns out that global (bulk) conservation laws do exist even in the ab-
sence of boundaries. This may seem surprising to a field theorist; we shall
explain this phenomenon and identify the rigid symmetries responsible for these charges. We hope to return to the effect of boundaries in the future.

We shall also identify a simple mechanism of creation of these charges by forcing with an optimum scheme that could be implemented numerically and almost experimentally. The problem with experimental implementation is not serious, in most (=slow speed) situations the incompressible approximation is valid and thus one may identify at any chosen instant Lagrangian and Eulerian coordinates so the forcing mechanism can be formulated either theoretically in Lagrangian coordinates or practically of course in Eulerian ones. We shall return to the incompressible case in the next paper.

This forcing, although impulsive, is reminiscent of the generation of the electric charge of electromagnetic dyons by uniform rotation in internal space [16] and of geodesic motion of quasistatic solutions of the variational problem of magnetic monopole theory [17]. We shall also explain the relation between homentropic and non homentropic situations, in fact a partial breaking from \((D+1)\) to \(D\)-dimensional relabeling symmetry by some marker like the value of the entropy changes dramatically the number of local invariants and exchanges the properties of even and odd numbers of space-dimensions.

1.3 General Relativity

The organisation of the rest of the paper is as follows. First the notions of cascade and abelian cascade of currents and superpotentials: \(J\) or \(T, U, V,...\) are introduced in subsection 2.1 and illustrated on simple examples including electromagnetism, Yang-Mills theory, p-forms gauge fields, in the rest of section 2. The identification of Noether currents for selected generators reduces the problem of finding the invariant charges to the selection of abelian subgroups of gauge symmetries.

In section 3, Hilbert's action in second order form is analysed and the need for longer cascades appears. The spin term of Belinfante’s symmetrized energy-momentum tensor [18] for matter is derived from the matter contribution to the superpotential. The mechanism is that tensor fields with spin do transform under diffeomorphisms with derivative terms as gauge fields do.

In the fourth section a new first order affine gauge theory of gravitation is defined. Its symmetries include diffeomorphisms, local linear frame transformations and a new gauge symmetry without gauge field, let us call it the
Einstein-Weyl symmetry, we shall see why momentarily. The latter gauge symmetry does not have any propagating gauge fields, as a consequence the associated superpotential and currents vanish. This is a special case of the so-called Noether identities which is formulated as a general vanishing theorem in subsection 4.2. The various superpotentials are easily analysed. In subsection 4.3 other Noether identities are discussed and the Sparling-Dubois-Violette-Madore rewriting of Einstein’s equations as a closure, or conservation, condition [19, 20] is adapted to our affine theory. Supergravity practitioners should not be surprised by such a result, see for instance [21]. What happens here is that the conservation laws encode all the equations of motion and not only some combinations of them. Gauge invariance far from being a nuisance has the power to determine the dynamics.

The affine theory leads, see subsection 4.4, either to first order Poincaré (Cartan-Weyl) theory by going to orthonormal frames and using the metricity of the connection or to the Palatini formalism by going to a coordinate frame and eliminating the torsion. In both cases one eliminates part of the linear connection by its equation of motion and by fixing a residual 1-form gauge invariance (without gauge 2-form): the arbitrariness of the scaling part of the linear connection. In other words the invariance of the action under the shift of $\Gamma^\rho_{\mu\nu}$ by a scaling (Weyl) component, $A_\mu(x) \delta^\rho_\nu$, is a gauge symmetry that generalizes the so-called Einstein symmetry [22]. In summary, modulo this “Einstein-Weyl” arbitrariness which is due to the form of the scalar curvature, the vanishing of torsion and nonmetricity follow from the variations of suitable components of the connection field. The name of H. Weyl is associated with the invention of (scaling) gauge invariance and is appropriate despite differences in the implementation. For this new gauge invariance one explains again the vanishing of the superpotential and hence of the current by the absence of propagating gauge field. Recall the examples of kappa-symmetry or (string theory) Weyl currents...

The local invariance of our affine action with respect to those $gl(D, \mathbb{R})$ generators that are not in the Lorentz subalgebra defined by the metric (and consequently do not propagate) leads also to the vanishing of the associated $U^{(ab)}$ superpotentials. Frauendiener [23] also considered the full frame bundle to investigate energy-momentum pseudotensors. Finally the Hilbert action follows from our action by going to second order formalism via the Cartan-Weyl action for instance.

In Subsection 4.5 we compare the superpotentials associated to these four actions, they may be called respectively affine, Cartan-Weyl, Palatini and Møller. In order to recover the right mass for the Schwarzschild solu-
tion Møller did actually rescale arbitrarily the potential derived canonically from the Hilbert action and multiplied by a factor of two [24] the honest one. A linear combination of the energy-momentum tensor and its associated superpotential involving an infinitesimal gauge parameter gives (3) the ordinary Noether current for the one dimensional subgroup along a given gauge direction. Ignoring extra terms due to higher derivatives it would have the form

$$J_\xi = \xi^\rho J_\rho + d\xi^\rho \wedge U_\rho$$  \hspace{1cm} (5)$$

This current has its own superpotential $U_\xi := \xi^\rho U_\rho$. In the Palatini case the latter becomes the Komar superpotential [25] after suitable modification by the frame change, it has the property to be a tensor. The Palatini superpotential differs from the affine superpotential by a contribution induced by the choice of coordinate frame: one must compensate the change of coordinates by a local linear transformation and this mixes the energy-momentum tensor and the $gl(D)$ current. Finally as explained in the previous section the antisymmetry in the two indices $\mu$ and $\nu$ of $U_\mu \nu$ is spoiled by the presence of higher derivatives present in the second order formalism. The reconciliation of first and second order formalisms requires also some mixing with another symmetry in the case of the orthonormal frame choice, that is in the Cartan-Weyl formalism: one can check that a compensating Lorentz transformation allows us to relate the superpotentials of the two.

The whole picture can be studied for the three theories above Hilbert’s scalar action as was just presented or for the corresponding theories above the Einstein metric action which is noncovariant but has only first order derivatives of the metric. The Einstein action differs from Hilbert’s by a surface term and leads to the Einstein energy-momentum complex sometimes called pseudotensor. It was a big surprise when Freud [26] discovered the relevant local superpotential, its origin was clarified by Bergmann but it could have been conjectured by Noether and Hilbert! Surface terms have been considered also in the Hamiltonian formalism [10] and for the path integral quantization they are reviewed in [28]. We identify on the Einstein side both the Freud superpotential and the Sparling one in section 5.1. In the rest of section 5 we consider the issue of boundary conditions and surface terms building on the previous examples.

Let us recall that the gauge field part of the superpotential and hence the corresponding part of the current do vanish either when there is no propagating gauge field but only a compensator or when there is a global (bulk) spacelike Killing vector or its analog in a general gauge theory. The
asymptotic symmetry of the set of allowed configurations (and solutions), at the “end” of spacetime where one does the experiment, is needed to define global charges because we need distinguished subgroups at infinity. It turns out that the contribution to these charges from the gauge fields vanishes asymptotically despite their infinite range in the case of spatial Killing vectors defined also in the bulk, at least near infinity.

We list along the way some projects for part II.

2 The general formalism and first examples

2.1 The general formalism

A local action that depends for simplicity on the fields and their first derivatives \( S = \int_M L(\phi, \partial \phi) \) may be invariant under a continuous (Lie) transformation. In this case one has:

\[
\delta S = 0 \iff \delta L = \partial_\mu S^\mu = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \partial_\mu \phi} \delta \partial_\mu \phi
\]  

(6)

Using the fact that \( \partial_\mu \) and \( \delta \) commute, we obtain

\[
\partial_\mu S^\mu = \left[ \frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} \right] \delta \phi + \partial_\mu \left[ \frac{\partial L}{\partial \partial_\mu \phi} \delta \phi \right]
\]  

(7)

This implies the existence of a conserved Noether current \( J^\mu \) for each generator of the Lie group:

\[
J^\mu := S^\mu - \frac{\partial L}{\partial \partial_\mu \phi} \delta \phi
\]  

(8)

\[
\partial_\mu J^\mu \approx 0
\]  

(9)

where \( \approx \) means on shell. Note that \( S^\mu \) is not uniquely defined without more choices.

The classical theorem expresses the conservation of this current as a consequence of the Euler-Lagrange field equations. In differential form notations \( J^\mu \) has a Hodge-dual \( (D-1) \) form noted by \( J \) (where \( D \) is the spacetime dimension). \( J \) is a local function of the fields but we can only deduce from its closedness \( (dJ \approx 0) \) that it is exact \( (J \approx dU) \) if spacetime is contractible and for a given solution of the equations of motion, in particular the \( (D-2) \)
form $U$ is not guaranteed to be “local”, i.e., can be written locally in terms of the fundamental fields of the theory. The total charge $Q = \int_{V_{D-1}} J$ is conserved given sufficient decay at spatial infinity, more covariantly $\int_{V_D} J \approx 0$ ($V_{D-1}$ is a space like hypersurface). The addition of a topological term to the Noether current is allowed if its topological charge vanishes so that the Noether charge is unaffected.

Gauge theory: the cascade equations
Let us look at the case of a general gauge symmetry. That means that the transformation of the fields can be parametrized by a local parameter $\xi^A(x)$ (here $A$ will denote an internal or spacetime index) and its derivatives, for example:

$$\delta \varphi = \xi^A \Delta_A(\varphi) + \partial_\nu \xi^A \Delta^\nu_A(\varphi)$$ (10)

Note that this is just a special case. In fact there could be more terms with an arbitrary number of derivatives of $\xi^A(x)$. The surface term can also be expanded in a similar way:

$$S^\mu = \xi^A \Sigma^\mu_A(\varphi) + \partial_\nu \xi^A \Sigma^{\mu\nu}_A(\varphi)$$ (11)

If we insert this decomposition in (8) and (9) we simply obtain after a trivial rearrangement,

$$\partial_\mu (\xi^A J^\mu_A + \partial_\nu \xi^A U^\mu_A) \approx 0$$ (12)

where

$$J^\mu_A := \Sigma^\mu_A - \frac{\partial L}{\partial \partial_\mu \varphi} \Delta_A(\varphi)$$ (13)

$$U^{\mu\nu}_A := \Sigma^{\mu\nu}_A - \frac{\partial L}{\partial \partial_\mu \varphi} \Delta^{\nu}_A(\varphi)$$ (14)

Note that $J^\mu_A$ which is the coefficient of the undifferentiated $\xi^A(x)$ term in the total current $J^\mu$ (see equations (8) and (12) ) is nothing more but the usual Noether current. In fact, we recover the well known result by just putting $\xi^A(x) = C^A$ in (12). The extra information due to the locality of the symmetry is encoded in the cascade equations which follow from (12) by using the arbitrariness and independence of $\xi^A(x)$ and their derivatives:

$$\partial_\mu \partial_\nu \xi^A(x) [U^{\mu\nu}_A] = 0 \Rightarrow U^{\mu\nu}_A = U^{[\mu\nu]}_A$$ (15)
\[
\partial_\mu \xi^A(x) [J^\mu_A + \partial_\nu U^{\nu\mu}_A \approx 0] \tag{16}
\]

\[
\xi^A(x) [\partial_\mu J^\mu_A \approx 0] \tag{17}
\]

As usual, (…) means symmetrization of the indices and [...] antisymmetrization. Note that the first equation is an identity whereas the other two are just on-shell equations (as emphasized by the \( \approx \) symbol).

The main result of this computation is that the so-called Noether current \( J^\mu_A \) is locally exact modulo the equations of motion when the symmetry is local. The corresponding superpotential \( U^{\nu\mu}_A \) has to be antisymmetric and can be computed directly from the Lagrangian of the theory by the use of equation (14). This antisymmetric property is particular to the case with at most first order derivatives of the fields.

The Noether identities

We would like here to recall the famous second Noether theorem which gives some relations between equations of motion when some gauge symmetry is present. We will show how they can be deduced in our formalism. If we use the decompositions (10) and (11), the definitions (13) and (14) in the equation (7) we obtain that

\[
\frac{\delta L}{\delta \varphi} \left( \xi^A \Delta_A + \partial_\nu \xi^A \Delta^\nu_A \right) = \partial_\mu \left( \xi^A J^\mu_A + \partial_\nu \xi^A U^{\nu\mu}_A \right) \tag{18}
\]

where \( \frac{\delta L}{\delta \varphi} \) are just the Euler-Lagrange equations derived from \( \varphi \) and an abstract summation over all the fields of the theory is understood. These equations are now exact and we can look for the cascade equations corresponding to this equality:

\[
\xi^A \left[ \frac{\delta L}{\delta \varphi} \Delta_A = \partial_\mu J^\mu_A \right] \tag{19}
\]

\[
\partial_\mu \xi^A \left[ \frac{\delta L}{\delta \varphi} \Delta^\mu_A = J^\mu_A + \partial_\nu U^{\nu\mu}_A \right] \tag{20}
\]

\[
\partial_\mu \partial_\nu \xi^A \left[ U^{\mu\nu}_A = 0 \right] \tag{21}
\]

These last equations replace the cascade equations when no use of the equations of motion is permitted. We now see why equation (15) was exact.
If we now replace $J^\mu_A$ as given by equation (20) into (19) and make use of the antisymmetry of $U^{\mu\nu}_A$ we easily obtain the Noether identities:

$$\frac{\delta L}{\delta \varphi} \Delta_A(\varphi) = \partial_\mu \left( \frac{\delta L}{\delta \varphi} \Delta^\mu_A(\varphi) \right)$$  \hspace{1cm} (22)

Note that the Noether identities do not depend on any surface term because all of them are hidden in $J^\mu_A$. This old result will be used in section 4 for a better understanding of the affine gauge theory and its reduction to Einstein theory.

Remember that we have just treated the simple case where the decompositions (10) and (11) go only up to first derivative in $\xi^A(x)$. In a more general case, the above conclusions have to be modified as we will see in section 3 in the specific example of General Relativity.

The abelian cascade: the $U^{\mu\nu}_\xi$ superpotential

In the next subsection we will give some examples but before that, let us just show that $J^\mu(\xi^A(x))$ of (8) (noted now simply $J^\mu_\xi$) where $\delta \varphi$ is given by (10) and $\Sigma^\mu$ by (11), can be expressed as a divergence. In fact the corresponding (local) parameter dependent superpotentials $U^{\mu\nu}(\xi^A(x))$ (noted $U^{\mu\nu}_\xi$) will be the most important object in gauge theories to compute conserved charges as we shall see in specific examples. Let us use the decomposition

$$\xi^A(x) := \epsilon(x)\xi^A_0(x)$$

in equation (12). $\epsilon(x)$ is just the local parameter for an abelian \(^1\) subgroup with $\xi^A_0(x)$ fixed. The cascade in terms of $\epsilon(x)$ and its derivatives gives after some trivial algebra the main result:

$$J^\mu_\xi \approx -\partial_\nu U^{\nu\mu}_\xi$$ \hspace{1cm} (23)

where in this case $U^{\nu\mu}_\xi$ is simply $U^{\nu\mu}_A \xi^A_0$. As we shall see the case of General Relativity in its 2nd order formulation (section 3) is only slightly more difficult but the idea is similar. We will introduce its affine formulation in section 4 where the computations will be easier and the comprehension more profound maybe. Let us insist that $U^{\nu\mu}_\xi$ is the fundamental object we shall use to compute physical charges in gauge theories. The main difficulty is to select the appropriate $\xi_0$’s to get gauge invariant results.

\(^1\)Actually the subgroup is not really abelian in the case of diffeomorphisms but there all changes of coordinates are linearly related or unidimensional.
The conserved charges

The usual Noether conservation law \( dJ \approx 0 \) can be used to define a conserved charge. For that purpose we may integrate this equation on a D-dimensional spacetime bounded by two spatial hypersurfaces (say \( \Sigma_1 \) and \( \Sigma_2 \)) and by spatial infinity (say \( \Sigma_\infty \)). If we impose the physical condition that \( J \) has a vanishing flux through \( \Sigma_\infty \) (in other words that the charge does not leave spacetime between \( \Sigma_1 \) and \( \Sigma_2 \)), Stokes’ theorem implies that \( \int_{\Sigma_1} J = \int_{\Sigma_2} J \) and so the charge defined as the integral of \( J \) on a spatial hypersurface is conserved and is independent on the choice of spacelike hypersurface.

The case of a local symmetry is more subtle because the usual Noether conservation may be replaced by

\[
J \approx dU \tag{24}
\]

This has radical consequences for the meaning of what is a conserved quantity and how to define it. In fact now equation (24) can be integrated on a \((D-1)\) dimensional manifold, in two different ways:

- On \( \Sigma_\infty \): This will be the right choice to define a conserved charge in General Relativity and Yang-Mills theories. Let \( B_{1\\infty} \) and \( B_{2\infty} \) the boundaries of \( \Sigma_\infty \) at time \( t_1 \) and \( t_2 \) respectively (these are actually \( D-2 \) dimensional closed manifolds). If we again assume that the flux of \( J \) vanishes on \( \Sigma_\infty \) then Stokes law applied to equation (24) will imply that \( \int_{B_{1\infty}} U = \int_{B_{2\infty}} U \). Then the conserved charge may be defined as the integral of \( U \) on the infinite spatial boundary of a time-fixed hypersurface:

\[
Q = \int_{B_\infty} U \tag{25}
\]

This definition is completely independent of the fact that there exist or not an interior black hole horizon or singularity inside space time. The key point is that this construction never leaves the asymptotic region and is both robust and physical as that is precisely where charges are measured. As we will discuss in more detail in the next examples (Yang-Mills and Gravitation), there exist relations between the boundary conditions we have to impose on our fields to define the variational principle, the form of asymptotic Killing vectors and the associated gauge invariant conserved charges. In some very special cases (for instance in presence of a global spatial Killing vector), we should be able to use other timelike hypersurface than \( \Sigma_\infty \), for instance at finite distance (say \( \Sigma_{r_0} \)), leading to the notion of quasi-local
charges. We postpone this discussion to section 5.2 for the gravitational case.

- On $\Sigma_1$ : In that case, we will obtain relations between quantities computed at a fixed time. Take for example the next simplest case where space-time has one interior boundary $B_{1H}$ (for instance a black hole horizon). Then we will obtain the general relation:

\[
\int_{B_{1\infty}} U = \int_{B_{1H}} U + \int_{\Sigma_1} J \tag{26}
\]

This kind of equation has been used successfully in General Relativity, for instance to prove the positivity of the ADM mass and the first law of thermodynamics. Our purpose here is not to repeat nor give other demonstrations of these crucial results. We will just show in section 5.2 that their starting point is nothing but a well understood version of equation (26) which is a direct consequence of the locality of diffeomorphism invariance.

Let us anticipate and remark here that the left hand side is the charge defined by the assumption of asymptotic symmetry of the fields, which allows us to avoid the nogo theorem of Hilbert and Noether. However there is no general prescription yet to define separately either of the terms on the right hand side, this will be studied in paper II of this series. Clearly the choice of inner boundary has to obey the zero flux condition and must involve some dynamical knowledge. This could be useful for the present studies of anti de Sitter spaces.

2.2 Yang-Mills case

General formalism:

Let us start with the usual Yang-Mills Lagrangian, eventually coupled to a matter term:

\[
L_{YM} = -\frac{1}{4} F_{\mu\nu}^A F_{\mu\nu}^A + L_{mat}(\varphi, \partial_\mu \varphi, A_\mu^A) \tag{27}
\]

where $A_\mu^A$ is the gauge potential, $F_{\mu\nu}^A$ its associated curvature and $\varphi$ a matter field lying in some representation $R$ of the gauge group. We also assume that $L_{mat}$ depends on $A_\mu^A$ only through the covariant derivative of $\varphi$. 

15
This Lagrangian is invariant under the local gauge transformation:
\[
\delta \xi A_\mu = \partial_\mu \xi^A + f_{BC}^A A_\mu \xi^C = D_\mu \xi^A
\]
\[
\delta \xi \varphi = \xi^A R_A \varphi
\]
\[
\Rightarrow \delta \xi L_{YM} = 0
\]
Of course \( f_{BC}^A \) are the structure constants and \( R_A \) the specific infinitesimal generators in the representation \( R \) of the group.

We can now use this symmetry in equation (8) and then rewrite it as in equation (12). The useful quantities (13) and (14) can now be computed for the Yang-Mills Lagrangian (27):
\[
J_\mu^A = -f_{BC}^B A_\nu^B F_{\mu \nu}^C + \partial L_{\text{mat}} / \partial \partial_\mu \varphi R_A \varphi
\]
\[
U_{\mu \nu}^A = F_{\mu \nu}^A
\]
\[
J_\xi^A = \xi^A R_A + \partial_\nu \xi^A U_{\mu \nu}^A
\]
We see that \( F_{\mu \nu}^A \) is just the superpotential of the naive Noether current \( J_\mu^A \). Now the cascade equations are nothing more but the Yang-Mills equations:
\[
\partial_\nu (F_{\mu \nu}^A) \approx -f_{BC}^B A_\nu^B F_{\mu \nu}^C + \partial L_{\text{mat}} / \partial \partial_\nu \varphi R_A \varphi
\]
Our purpose is to study the fate of conserved quantities in the presence of local invariance so it is important to recognize these equations as conservation equations of the type \( J \approx dU \). Here we have a superpotential which is not anymore a gauge scalar. In fact the integral at spatial infinity of \( F_{\mu \nu}^A \) does not make any sense as a conserved quantity because it is not gauge invariant. The good gauge independent superpotentials are thus parameter dependent ones and can be obtained by the abelian cascade method. The result is obvious:
\[
U_{\mu \nu}^A = F_{\mu \nu}^A \xi^A
\]
\[
J_\xi^A \approx \partial_\nu U_{\mu \nu}^A
\]
If we recall the discussion of conserved charges of the previous subsection, we obtain that the gauge invariant conserved charge is (equation (25)):
\[
Q(\xi^A(x)) = \int_{B^\infty} U_\xi
\]
where as usual \( U_\xi \) is the \( D - 2 \) form associated to the Hodge dual of \( U_{\mu \nu}^A \).

The point is now that in order to obtain physical charges we have to specify and select what \( \xi^A(x) \) can be. This is treated in the following.

The Yang-Mills Charges
First we would like to recall an important point which has to be taken into account in a variational principle. A variational principle is defined only when boundary conditions are specified. In addition, if a boundary condition is chosen, we cannot add anymore an arbitrary total derivative to the Lagrangian because in general when the fundamental fields of the theory do not vanish on the boundary (say at infinity) the variational principle (i.e. \( \delta S = 0 \Rightarrow \text{Equations of motion} \)) will not be satisfied.

For example, the variational principle for the Yang-Mills Lagrangian (27) implies that

\[
\int \frac{\partial L}{\partial A^A_{\mu}} \delta A^A_{\mu} + \frac{\partial L}{\partial \phi} \delta \phi
\]  

has to vanish for an arbitrary variation. We do not want to analyse here the behaviour of the solutions of this equation say at spatial infinity in terms of power series in \( \frac{1}{r} \). We will consider the simplified Dirichlet case as if infinity was at a finite distance like in a compactification of spacelike infinity:

\[
\lim_{r \to \infty} \delta A^A_{\mu} = 0
\]

\[
\lim_{r \to \infty} \delta \phi = 0
\]

The full mathematical analysis is deferred to our second paper.

If we use this for the special case of a gauge variation, we obtain the boundary “Killing” equations:

\[
\lim_{r \to \infty} D_\mu \xi^A = 0
\]  

\[
\lim_{r \to \infty} \xi^A R_A \phi = 0
\]

The last two equations tell us which asymptotic \( \xi \)'s are allowed in equation (28). These can form an infinite asymptotic group (see for instance the gravitational case) but only their asymptotic form is used and these can be a finite number of those charge by means of equation (28).

Note also that equation (29) used for the simple case \( \delta = \delta_\xi \) will imply the vanishing of \( J^\mu_\xi \) at spatial infinity and so the existence of a conserved charge showing the consistency of our framework. We can go even further in the analysis when there exists some global Killing parameter (i.e. \( D_\mu \xi^A_K = 0 \)).
In that case $J_{\xi_K}^\mu = 0$ everywhere and so the corresponding charge $Q(\xi_K) = \int_B U_{\xi_K}^{\mu\nu}$ can be computed on any (D-2) dimensional surface outside matter sources.

We want to insist here on the following points

- The case of Dirichlet conditions (30) and (31) is just the simplest solution for the vanishing of equation (29). The general solution to this condition has to be treated in the asymptotic regime with the appropriate decrease.

- Physical conditions will specify the boundary condition (as in the case of free or fixed-ends strings) which will not only fix part of the surface term of the Lagrangian but also give some conditions on the asymptotically allowed gauge parameters.

- Boundary conditions should be gauge invariant. In the case of General Relativity this is made possible by introducing a reference space at infinity (where it is needed).

Some well known examples are:

- The Maxwell case with matter fields which vanish at infinity. In that case the asymptotic Killing equation just becomes $\lim_{r \to \infty} \partial_\mu \xi = 0$. The subalgebra which will give a non vanishing finite charge will be $\mathbb{R}$. Thus the number of charges is just 1 (the dimension of $\mathbb{R}$, which is also the number of independent Casimir operators of the subgroup). In addition, $\xi = C^t$ is a global Killing parameter and so we recover the well known result that the electric charge can be computed on any closed surface which surrounds the charged matter distribution.

- The SU(2) Yang-Mills-Higgs system where a particular solution to the asymptotic Killing equations is just $\xi^A = \Phi^A_0$ (the direction of the Higgs field at infinity), see for instance [7].

2.3 The p-form theory

We can consider the abelian p-form Lagrangian given essentially by

$$L = G \wedge \ast G$$

(32)

Where $G = dB$, $B$ being the p-form abelian gauge field (see [6]).

The local gauge invariance is just $\delta_\xi B = d\xi$; where $\xi$ is an arbitrary (p-1)-form gauge parameter. We will not repeat all the computations but just give the final result which is that the parameter dependent conserved charge is given by:
\[ Q(\xi) = \int_{B_\infty} \xi \wedge \ast G \]

If we again impose Dirichlet type boundary conditions, the analogue of (30) and (31) for \( \xi \) is thus:

\[ \lim_{r \to \infty} d\xi = 0 \]

It is obvious from the definition of \( Q(\xi) \) and the equations of motion of \( B (d \ast G \approx 0) \) that when \( \xi = d\beta \), the charge will vanish on shell (remember that \( B_\infty \) is already a boundary hence is closed and that partial integration can be done without any boundary term). Thus in the case of the p-form, the subgroup which could potentially give some non trivial conserved charge is just the set of (p-1)-forms which are closed but not exact or in other words the \((p - 1)\)th De Rham cohomology group \( H^{p-1} \) of \( B_\infty \). The number of conserved charges will then be given by the \((p - 1)\)th-Betti number \( b^{p-1}(B_\infty) = \dim (H^{p-1}(B_\infty)) \). For example for a spacetime with 2 infinite boundaries components (wormhole) we recover 2 ordinary charges.

The reader interested only in fluid dynamics can now skip to section 6.

3 The classical case of General Relativity

3.1 Second order form of gravitation: the cascade Equations for diffeomorphisms

Let \( L(g, \partial g, \partial^2 g) = \frac{1}{2k} \sqrt{-g} R \) be the scalar Hilbert Lagrangian density of our theory. It is equal to the so-called Einstein Lagrangian up to the surface term that eliminates second derivatives of the metric, see section 5. A variation of \( L \) is given by

\[ \delta L = \frac{\partial L}{\partial g} \delta g + \frac{\partial L}{\partial \partial_{\mu} g} \delta \partial_{\mu} g + \frac{\partial L}{\partial \partial_{\mu} \partial_{\nu} g} \delta \partial_{\mu} \partial_{\nu} g \] (33)

Where we omitted the spin indices of \( g_{\alpha\beta} \) for notational simplicity. Using the second order equations of motion \( \frac{\delta L}{\delta g} = \frac{\partial L}{\partial g} - \partial_{\mu} \frac{\partial L}{\partial \partial_{\mu} g} + \partial_{\nu} \partial_{\mu} \frac{\partial L}{\partial \partial_{\nu} \partial_{\mu} g} \approx 0 \)

we can write equation (33) as a total derivative,

\[ \partial_{\mu} J_{\xi}^{\mu} := \delta L - \frac{\partial L}{\partial \partial_{\mu} g} \delta g - \partial_{\nu} \left( \frac{\partial L}{\partial \partial_{\mu} \partial_{\nu} g} \right) \delta g + \frac{\partial L}{\partial \partial_{\mu} \partial_{\nu} g} \partial_{\nu} \delta g \approx 0 \] (34)
Our Lagrangian density is again such that the action is invariant under a reparameterization \( x^\rho \rightarrow x^\rho + \xi^\rho (x) \). Putting the well known expressions for the variation \( \delta L = \mathcal{L}_\xi \delta\mathcal{L} = \xi^\rho \partial_\rho g_{\alpha\beta} + \partial_\alpha \xi^\rho g_{\rho\beta} + \partial_\beta \xi^\rho g_{\alpha\rho} \) in (34) and sorting out the factors of \( \xi^\rho, \partial_\alpha \xi^\rho \) and \( \partial_\beta \partial_\alpha \xi^\rho \), we obtain

\[
\Leftrightarrow \partial_\mu (\xi^\rho T_\rho^\mu + \partial_\nu \xi^\rho U_\rho^{\mu\nu} + \partial_5 \partial_\rho \xi^\rho V_\rho^{\mu(\nu\delta)}) = 0
\]  

(35)

Where \( T_\rho^\mu \) and \( U_\rho^{\mu\nu} \) are the canonical energy-momentum complex (called sometimes canonical energy-momentum pseudotensor and noted \( t_\rho^\mu \)) and the canonical spin complex respectively,

\[
T_\rho^\mu := \delta^\mu_\rho L - \left( \frac{\partial L}{\partial g_{\alpha\beta}} - \partial_\nu \frac{\partial L}{\partial g_{\nu\alpha} g_{\rho\beta}} \right) \partial_\rho g_{\alpha\beta} - \frac{\partial L}{\partial g_{\rho\alpha} g_{\rho\beta}} \partial_\rho g_{\alpha\beta}
= \sqrt{-g} \left( R \delta^\mu_\rho + \Gamma^\alpha_{\alpha\beta,\rho} g^{\beta\mu} - \Gamma^\mu_{\alpha\beta,\rho} g^{\alpha\beta} \right)
\]

\[
U_\rho^{\mu\nu} := \left( \partial_\eta \frac{\partial L}{\partial g_{\nu\alpha} g_{\rho\beta}} - \frac{\partial L}{\partial g_{\rho\alpha} g_{\rho\beta}} \right) \Lambda^{\nu\gamma}_{\alpha\beta} g_{\rho\gamma} - \frac{\partial L}{\partial g_{\rho\alpha} g_{\rho\beta}} \partial_\rho g_{\alpha\beta} - \frac{\partial L}{\partial g_{\rho\alpha} g_{\rho\beta}} \Lambda^{\nu\gamma}_{\alpha\beta} \partial_\rho g_{\rho\gamma}
= \sqrt{-g} \left[ \delta^\mu_\rho \Gamma^\nu_{\nu\beta} g^{\beta\alpha} + \Gamma^\alpha_{\alpha\beta} g^{\mu\nu} - 2 \Gamma^\mu_{\alpha\beta} g^{\nu\rho} \right]
\]

\[
V_\rho^{\mu(\nu\delta)} := - \frac{\partial L}{\partial g_{\nu\alpha} g_{\rho\beta}} \Lambda^{\nu\gamma}_{\alpha\beta} g_{\rho\gamma} \text{ (symmetrized in } \nu\delta) \]

\[
= \sqrt{-g} \left[ \frac{1}{2} g^{\mu\delta} \delta^\nu_\rho + \frac{1}{2} g^{\mu\nu} \delta^\delta_\rho - g^{\nu\delta} \delta^\mu_\rho \right]
\]

\[\Lambda^{\nu\gamma}_{\alpha\beta} := \delta^\nu_\alpha \delta^\gamma_\beta + \delta^\gamma_\alpha \delta^\nu_\beta \text{ and } \Gamma = \Gamma^{(g)} \text{ is the Levi-Civita connection.}
\]

Then, as in the previous examples, we derive the cascade Equations:

\[
\partial_\mu \partial_\nu \partial_\delta \xi^\rho \left[ V_\rho^{\mu(\nu\delta)} \right] = 0 \Leftrightarrow V_\rho^{\mu(\nu\delta)} = 0
\]  

(36)

\[
\partial_\mu \partial_\nu \xi^\rho \left[ U_\rho^{\mu\nu} + \partial_3 V_\rho^{\delta\nu\mu} \right] = 0 \Leftrightarrow U_\rho^{\mu\nu} + \partial_3 V_\rho^{\delta\nu\mu} = F_\rho^{\mu(\nu\|})
\]  

(37)

\[
\partial_\mu \xi^\rho \left[ T_\rho^\mu + \partial_5 U_\rho^{\mu\nu} \right] \approx 0
\]  

(38)

\[
\xi^\rho \left[ \partial_\mu T_\rho^\mu \right] \approx 0
\]  

(39)

And

\[
F_\rho^{\mu\nu} := \frac{\sqrt{-g}}{2k} \left[ 2 \delta^\mu_\rho \Gamma^\nu_{\alpha\beta} g^{\alpha\beta} - 2 \delta^\nu_\rho \Gamma^\mu_{\alpha\beta} g^{\alpha\beta} + g^{\mu\alpha} \Gamma^{\nu\alpha}_{\rho\alpha} - g^{\nu\alpha} \Gamma^{\mu\alpha}_{\rho\alpha} \right]
\]  

(40)

Note that \( U_\rho^{\mu\nu} \) is not antisymmetric and that the first two equations are exact but that the last two are on-shell.
We again see an important fact: equation (38) shows that the current (in this case the canonical energy-momentum complex) can be written down as a divergence. The gauge diffeomorphism invariance of General Relativity implies then that the charge associated with this symmetry may be expressed as a surface integral.

We already saw in the case of Yang-Mills theory that the most important quantity to define conserved charges is the (local) parameter-dependent superpotential. To derive it, we will use the abelian cascade trick in this non trivial example. We shall just give the formal result without trying to obtain physical consequences for the moment. We shall postpone this question to section 5.2. The motivation to construct such an object is just that it will provide a single formula for conserved quantities like total mass or angular momentum. The point is that none of the tricks Landau-Lifshitz or Weinberg used to construct the gravitational angular momentum starting from a symmetrized canonical energy-momentum complex is needed. The connection with the Komar or Katz superpotentials will be established in the following sections using the affine gauge formalism, where it is much simpler.

Let us start with equation (35):

\[ \partial_\mu J^\mu_\xi \approx 0 \tag{41} \]

where

\[ J^\mu_\xi = \xi^\rho T^\mu_\rho + \partial_\nu \xi^\rho U^\mu_\rho + \partial_\delta \partial_\nu \xi^\rho V^\mu_\rho \delta \]

Now, let us define \( \xi^\rho_0(x) \epsilon(x) = \xi^\rho(x) \). Again, \( \epsilon(x) \) is a local parameter for an abelian subgroup. \( \xi^\rho_0(x) \) is kept fixed and will be determined for each conserved charge to be computed. Using this decomposition in equation (41) we get that

\[ \partial_\mu (\epsilon J^\mu_\xi + \partial_\nu \epsilon U^\mu_\nu \xi_0 + \partial_\delta \partial_\nu \epsilon V^\mu_\nu \delta \xi_0) \approx 0 \tag{42} \]

The abelian cascade equations are computed in terms of derivatives of \( \epsilon \) and give:

\[ \partial_\mu \partial_\nu \partial_\delta \epsilon : V^{(\mu \nu \delta)}_{\xi_0} = 0 \tag{43} \]

\[ \partial_\mu \partial_\nu \epsilon : U^\mu_\nu \xi_0 + \partial_\delta V^\nu_\delta \xi_0 = F^{[\mu \nu]}_{\xi_0} \tag{44} \]

\[ \partial_\mu \epsilon : J^\mu_\xi + \partial_\nu U^\nu_\xi \approx 0 \tag{45} \]
\[ \epsilon : \partial_\mu J_{\xi_0}^\mu \approx 0 \]  

(46)

So equation (45) shows, as one expects, that the total Noether current \( J_{\xi_0}^\mu \) can be written in terms of a divergence modulo equations of motion due to the locality of the symmetry. For completeness let us write down the formula for \( U_{\xi_0}^{\nu\mu} \) in the gravitational 2nd order formalism:

\[ U_{\xi_0}^{\nu\mu} = \xi_0^\rho U_{\rho}^{\nu\mu} + 2\partial_\delta \xi_0^\rho V_{\rho}^{\nu\mu\delta} \]  

(47)

In what follows, we will omit the \( 0 \) subscript from \( \xi_0 \).

We may conclude this discussion by just giving the connection of the above formulas with some well known results:

- \( T_{\rho}^{\mu} \) is nothing but one half the originally rescaled Møller energy-momentum pseudotensor [24].

- This pseudotensor can be written as the divergence of the canonical spin complex which is not antisymmetric (equation (38)). However there exists an antisymmetric superpotential which does the same job:

\[ M U_{\rho}^{\mu\nu} := U_{\rho}^{\mu\nu} - \partial_\delta W_{\rho}^{\nu[\mu\delta]} \]  

(48)

\[ W_{\rho}^{\nu[\mu\delta]} := \frac{\sqrt{-g}}{2k} \left( g^{\nu\mu} \delta_\rho^\delta - g^{\nu\delta} \delta_\rho^\mu \right) \]  

(49)

hence equation (38) and the above definitions imply that

\[ T_{\rho}^{\mu} \approx \partial_\nu M U_{\rho}^{\mu\nu} \]  

(50)

Where \( M U_{\rho}^{\mu\nu} \) is one half the superpotential introduced (and rescaled) by Møller [24], and is equal to:

\[ M U_{\rho}^{\mu\nu} = \frac{1}{2k} \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} (\partial_\alpha g_{\beta\rho} - \partial_\beta g_{\alpha\rho}) \]  

(51)

\[ = - \frac{1}{2k} \sqrt{-g} \left( \Gamma_{\alpha\rho}^\mu g^{\alpha\nu} - \Gamma_{\alpha\rho}^\nu g^{\alpha\mu} \right) \]

The same is true for \( U_{\xi}^{\mu\nu} \), equation (47): there exists an antisymmetric version of it,

\[ K U_{\xi}^{\mu\nu} := U_{\xi}^{\mu\nu} - \partial_\delta \left( \xi_0^\rho V_{\rho}^{\nu[\mu\delta]} \right) \]  

(52)
\[ J_{\xi}^\mu \approx \partial_\nu K U_{\xi}^{\mu\nu} \quad (53) \]

Where now \( K U_{\xi}^{\mu\nu} \) is one half the Komar [25] superpotential,

\[ K U_{\xi}^{\mu\nu} := \frac{1}{2k} \sqrt{-g} (\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) \quad (54) \]

\[ = MU_{\rho}^{\mu\nu} \xi^\rho + W_{\delta}^{[\mu} \partial_{\nu]} \xi^\rho \quad (55) \]

Where we used in the last equation that \( V_{\rho}^{\mu\delta} = \frac{1}{2}(W_{\rho}^{\nu\mu\delta} + W_{\rho}^{\delta\nu\mu}) \).

One could think that we have been lucky that such superpotentials exist for equations (48) and (52). We shall show, in the Affine Gauge formalism (section 3) that their existence is due to some local symmetry, just as the existence of the canonical spin complex \( U_{\rho}^{\mu\nu} \). We will also understand in an elegant way the lack of antisymmetry of the latter. The point is that the Affine Gauge formalism is a first order formulation so the antisymmetry is guaranteed from the beginning as we saw in section 2.1. Equation (55) will then be derived in a natural way. We will also understand in an elegant way its non-antisymmetry. Before that let us show that the addition of a matter field will not essentially change the above formulas.

### 3.2 Matter’s contribution: The symmetric tensor

All the previous discussion was for vacuum gravitational theory. However we would like to add some matter, i.e. \( L_{\text{Matter}}(\Phi, \partial \Phi) \), and see how this can affect our equations.

The basic Noether theorem gives a formula which allows us to calculate a conserved current coming from a global symmetry of our Lagrangian. When this symmetry is the translation invariance in a flat background then the conserved current is just the so-called canonical energy-momentum matter tensor \( c_{t\rho}^{\mu} \) (lower case letters will be used for the contribution coming from the matter fields), which is given by the usual formula:

\[ c_{t\rho}^{\mu} = \delta_{\rho}^{\mu} L_M - \frac{\partial L_M}{\partial \partial_{\rho} \Phi} \partial_{\rho} \Phi \quad \text{and} \quad \partial_{\mu} c_{t\rho}^{\mu} \approx 0 \quad (56) \]

Then, the time conserved physical quantity is the 4-momentum vector \( P_{\rho} = \int_V c_{t\rho}^{\mu} dV \).

In general, this energy-momentum tensor is not symmetric. As Belinfante has shown [18], it is possible to add an antisymmetric surface term to symmetrize it without changing the physics, i.e.
\[ B^\mu_\rho = c^\mu_\rho + \partial_\nu B^{\nu\rho}_\Sigma^{[\mu\nu]} \]  

and \( \int_\infty \Sigma^\rho dS_i \longrightarrow 0 \), \( B^\sigma_\mu = \eta^\sigma_\rho B^\mu_\rho = B^{\mu\sigma} \).

When our background space-time becomes a dynamical variable \( g_{\alpha\beta} \), then the matter energy momentum tensor \( s^\sigma_\mu = 2 \partial L_M / \partial g^{\sigma\mu} \) appears as the source in Einstein’s equations. The symmetry of its upper indices is guaranteed by the symmetry of the metric and it has been verified that this quantity effectively coincides with the above Belinfante version. In what follows, we shall give another proof of this important fact and see how this can affect the total energy-momentum of the gravitational part. The antisymmetric surface term needed to symmetrize \( c^\mu_\rho \) appears naturally as the matter superpotential.

Let \( L_M(g, \Phi_m, \partial \Phi_m) \), the scalar Lagrangian density, be a functional of a set of fields \( \Phi_m \) (where \( m \) is a spin index) and their first derivatives and of the background metric. We now use the fact that we can write the matter variation of the Lagrangian in a total derivative form by making use of the equations of motion of \( \Phi_m \) only:

\[ \delta L_M \approx \partial_\mu \left( \frac{\partial L_M}{\partial \partial_\mu \Phi_m} \delta \Phi_m \right) + \frac{\partial L_M}{\partial g^{\sigma\mu}} \delta g^{\sigma\mu} \]  

The variations of all the fields are given by their Lie derivative. Let us define the matrix \( \Delta \) which acts on the spin index of \( \Phi_m \) by:

\[ \delta \Phi_m = \xi^\rho \partial_\rho \Phi_m + \partial_\nu \xi^\rho (\Delta^\nu_\rho m) \Phi_n \]  

Equation (58) becomes

\[ \partial_\nu \left( \xi^\rho c^\mu_\rho + \partial_\mu \xi^\rho u^{\nu\rho}_\mu \right) \approx \frac{s^\sigma_\mu}{2} (\xi^\rho \partial_\rho g^{\sigma\mu} + 2 \partial_\mu \xi^\rho g_{\sigma\rho}) \]  

Where \( u^{\nu\mu}_\rho := -\frac{\partial L_M}{\partial \partial_\rho \Phi_m} (\Delta^\nu_\rho m) \Phi_n \) is the matter superpotential and \( c^\mu_\rho \) and \( s^\sigma_\mu \) have been defined above. Now, the cascade equations are

\[ \partial_\nu \partial_\mu \xi^\rho \left[ u^{\nu\mu}_\rho \right] = 0 \Leftrightarrow u^{\nu\mu}_\rho = u^{[\nu\mu]}_\rho \]  

\[ \partial_\rho \xi^\rho \left[ c^\mu_\rho + \partial_\nu u^{\nu\mu}_\rho \right] \approx s^\sigma_\mu g_{\sigma\rho} \]  

24
\[ \xi^\rho \left[ \partial_\mu c t^\rho \mu \approx \frac{s \tau^\mu}{2} \partial_\rho g_{\sigma \mu} \right] \] (63)

Then equation (62) shows that if the Euler-Lagrange equations of \( \Phi_m \) hold then the matter superpotential is the antisymmetric quantity that we have to add to the canonical energy-momentum tensor to obtain the symmetric one. The last equation cannot be identified with a conservation law unless \( s \tau^\mu \approx 0 \) which is the case only when no dynamical term for the metric are present.

Let us summarize: The addition of a matter Lagrangian to the gravitational one affects the equations of section 3.1 by just adding a symmetric tensor term to the gravitational canonical energy-momentum complex. In fact, it is easy to see that with the simple change \( T^\mu_\rho \rightarrow T^\mu_\rho + t^\mu_\rho \), all the above equations remain unchanged, always keeping in mind that the previous vacuum equations of motion are modified by matter terms. Finally we have gained a deeper understanding of the relation between the canonical matter tensor and the symmetric one. In what follows, we will return to the vacuum case. We just need to keep in mind the fact that the addition of an integer spin matter field doesn’t change anything if we proceed with the above substitution. The case where the matter field is a finite spinorial representation of the Lorentz group is more subtle. In that case we will not be able to use our affine gauge theory (because the universal covering group of \( GL(D, \mathbb{R}) \) requires infinite spinorial representations, see for instance [22]), for the same reason that we cannot use Einstein formalism to deal with spinors. However, the results derived for the affine formalism can be reduced to the orthogonal case, allowing the addition of for example Dirac spinors. Let us now turn to our new (affine) first order formalism, which generalizes Cartan-Weyl or Palatini formalism. Note that if we deal with spinors or gravitinos, first order formalism can introduce torsion. So we will allow torsion and even nonmetricity.

4 Superpotentials of Affine gauge relativity

4.1 Definition of the theory

The simplest mathematical form of general relativity uses moving linear frames, it is the so-called first order formalism. The classical case is
so(1, D − 1; \mathbb{R}) formulation. Let L(M) be the linear frame bundle over a D-dimensional Riemannian manifold M with metric g (D > 2). Let us consider on L(M) a Lagrangian D-form L, function of a linear 1-form connection $\omega^a_b$ (Yang-Mills $\mathfrak{gl}(D, \mathbb{R})$), of the canonical 1-form $\theta^a$ ($\mathbb{R}^D$ valued) and of the metric $g^{ab}$ (which will be used to lift and lower the $\mathbb{R}^D$-valued indices), as well as their first derivatives:

$$L = \frac{1}{2k} R^a_c \wedge \sqrt{-g} g^{eb} \Sigma_{ab}$$

(64)

$k = 8\pi G$ is the usual normalisation factor. We defined

$$\Sigma_{a_1...a_r} := \frac{1}{(D-r)!} \epsilon_{a_1...a_r c_{r+1}...c_D} \theta^{c_{r+1}} \wedge ... \wedge \theta^{c_D}$$

where $\epsilon_{a_1...a_D}$ is the Levi-Civita tensor density symbol. The curvature 2-form $R^a_b$ is as usual defined by

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$$

(65)

We can also define the torsion $\Theta^a$ and the nonmetricity $\Xi^{ab}$ as the covariant derivatives (we will say curvatures) of the moving frame and the metric respectively:

$$\Theta^a := D\theta^a = d\theta^a + \omega^a_b \wedge \theta^b$$

(66)

$$\Xi^{ab} := Dg^{ab} = dg^{ab} + \omega^{ab} + \omega^{ba}$$

(67)

where $D$ is the $\mathfrak{gl}(D, \mathbb{R})$ covariant derivative. We remember that the torsion measures some covariant anholonomy and the nonmetricity the failure of the metric $g^{ab}$ to be compatible with the connection. Their vanishing defines uniquely the Levi-Civita connection as a function of the metric and its derivatives.

Due to their definitions these three “curvatures” obey the following Bianchi identities:

$$DR^a_b = 0$$

(68)

$$D\Theta^a = R^a_b \wedge \theta^b$$

(69)

$$D\Xi^{ab} = R^{ab} + R^{ba}$$

(70)
In the previous sections we learned that to obtain some cascade equations coming from a gauge symmetry of the Lagrangian we need a variational principle. We will then take the pullback along an arbitrary local section of the above quantities. We will keep the same letter to note the pullback quantity for notational simplification. The two usual sections are

- the coordinate or holonomic section where \( s^* (\theta^a) = dx^a \) and \( s^* (\omega^a_b) = \Gamma^a_{cb} dx^c \) (\( \Gamma^a_{cb} \) is an affine connection on \( M \)) and
- the rigid or orthogonal section where \( s^* (\theta^a) = e^a \) and \( s^* (\omega^a_b) = \gamma^a_{bc} e^c \) (\( e^a \) are the well known orthogonal Lorentz frames, sometimes called tetrads or vielbein, and \( \gamma^a_{bc} \) the Ricci “rotation” coefficients).

We will allow the fields \( \omega^a_b, \theta^a (s^* (\theta^a) \text{ in fact}) \) and \( g^{ab} \) to vary independently. The Euler-Lagrange equations corresponding to the Lagrangian (64) are

\[
\frac{2k}{\sqrt{-g}} \frac{\delta L}{\delta g^{ab}} = R^c_{\quad a} \Sigma_{cb} + R^c_{\quad b} \Sigma_{ca} - g_{ab} R^{cd} \Sigma_{cd} \approx 0 \tag{71}
\]

\[
\frac{2k}{\sqrt{-g}} \frac{\delta L}{\delta \theta^a} = R^{bc} \Sigma_{bca} \approx 0 \tag{72}
\]

\[
\frac{2k}{\sqrt{-g}} \frac{\delta L}{\delta \omega^a_b} = \Xi^{bc} \Sigma_{bac} + \Theta^c \Sigma_{ace} g^{cb} := \frac{1}{\sqrt{-g}} D \left( \sqrt{-g} \Xi^{bc} \Sigma_{ac} \right) \approx 0 \tag{73}
\]

where we used the definitions: \( \Xi^{ab} := \Xi^{ab} - g^{ab} \Xi \) and \( \Xi := \Xi^{ab} g_{ab} \).

We will analyse the meaning of these equations of motion and their relation with usual formulations of General Relativity in the next subsections.

4.2 Local symmetries and associated superpotentials

The gravitational Lagrangian (64) has three distinct gauge symmetries. Let us apply the cascade machinery for each one.

1) The local “frame choice” freedom which is of the Yang-Mills type.

\[ \delta \lambda \theta^a (x) = \lambda^a_b (x) \theta^b (x) \]

\[ \delta \lambda \omega^a_b (x) = \lambda^a_c (x) \omega^b_c (x) - \lambda^a_c (x) \omega^c_b (x) - d \lambda^a_b (x) \]

It is easy to check that the Lagrangian (64) remains invariant under this transformation, i.e. \( \delta \lambda L = 0 \). Since \( L \) depends on derivatives of the \( \omega^a_b \)
field only (there is no $d\theta^a$ or $dg^{ab}$ terms in (64)), equation (8) of the general discussion is in this case simply
\[ \delta_\lambda L - d \left( \delta_\lambda \omega^a_b \wedge \frac{\partial L}{\partial d\omega^a_b} \right) \approx 0 \quad (74) \]
Which becomes after substitution
\[ dJ_\lambda \approx 0 \quad (75) \]

Where
\[ J_\lambda := \lambda^a_b J^b_a + d\lambda^a_b \wedge U^b_a \]
\[ J^b_a := \frac{\sqrt{-g}}{2k} \left( -\omega^b_c \Sigma_{adg}^{dc} + \omega^c_a \Sigma_{cdg}^{db} \right) \]
\[ U^b_a := \frac{\sqrt{-g}}{2k} \Sigma_{adg}^{db} \]
The cascade equations are derived as before: we just impose the fact that $\lambda^a_b$ and its derivatives are arbitrary:
\[ d\lambda^a_b \wedge [J^b_a \approx dU^b_a] \quad (76) \]
\[ \lambda^a_b \left[ dJ^b_a \approx 0 \right] \quad (77) \]

Note that the 'zero' cascade equation which in component language was the antisymmetry of the superpotential here is automatically satisfied because we use differential forms.

These equations are nothing other than the equations of motion of $\omega^a_b$ (see equation (73)) as in the Yang-Mills case.

Now, if we use the abelian cascade trick, we can rewrite the total current $J_\lambda$ as a superpotential. The result is obvious:
\[ J_\lambda \approx dU_\lambda \quad (78) \]

Where $U_\lambda := \lambda^a_b U^b_a$.

Later we will analyse the connection of this theory with ordinary General Relativity and we will see that this superpotential appears in the conservation of angular momentum. But before that, let us pursue the discussion of local symmetries.

2) The diffeomorphism invariance. Under an infinitesimal local reparameterisation $x^\rho \rightarrow x^\rho + \xi^\rho(x)$, the variation of the Lagrangian is given by its Lie derivative $\mathcal{L}_\xi L$ along $\xi^\rho$. Now $dL = 0$ because $L$ is a top form and $\mathcal{L}_\xi = d \cdot i_\xi + i_\xi \cdot d$ so we see that the variation of the Lagrangian is a
total derivative, i.e. $L_{\xi}L = d \cdot i_{\xi}L$. In addition, $\delta \omega^a_b = L_{\xi} \omega^a_b$ and so the conservation equation becomes

$$d \left( i_{\xi}L - \frac{\sqrt{-g}}{2k} [(d \cdot i_{\xi} + i_{\xi} \cdot d) \omega^a_b] \wedge g^{bd} \Sigma_{ad} \right) \approx 0$$  \hspace{1cm} (79)

$$\Leftrightarrow d \left( \xi^\rho i_{\rho}L - \frac{\sqrt{-g}}{2k} [(d\xi^\rho : i_{\rho} + \xi^\rho L) \omega^a_b] \wedge g^{bd} \Sigma_{ad} \right) \approx 0$$  \hspace{1cm} (80)

Where for notational convenience we defined $i_{\rho} := i_{\frac{\partial}{\partial \rho}}$ and $L_{\rho} := L_{\frac{\partial}{\partial \rho}}$.

We can then separate out the factors of $\xi^\rho$ and $d\xi^\rho$ and, define the (D-1) and (D-2) forms respectively,

$$\tau^\rho := i_{\rho}L - \frac{\sqrt{-g}}{2k} (L_{\rho} \omega^a_b) \wedge g^{bd} \Sigma_{ad}$$

$$\sigma^\rho := -\frac{\sqrt{-g}}{2k} (i_{\rho} \omega^a_b) \Sigma_{ad}$$

to finally get

$$dJ_{\xi} := d(\xi^\rho \tau^\rho + d\xi^\rho \wedge \sigma^\rho) \approx 0$$  \hspace{1cm} (81)

Again, the factors in front of $\xi^\rho$ and $d\xi^\rho$ must vanish separately :

$$d\xi^\rho \wedge [\tau^\rho - d\sigma^\rho] \approx 0$$  \hspace{1cm} (82)

$$\xi^\rho[d\tau^\rho] \approx 0$$  \hspace{1cm} (83)

The abelian cascade gives us:

$$J_{\xi} \approx dU_{\xi}$$  \hspace{1cm} (84)

where $U_{\xi} := \xi^\rho \sigma^\rho$.

The cascade equations of $gl(D, \mathbb{R})$ gauge invariance was just a rewriting of the equations of motion for $\omega^a_b$. In the next subsection we will explain how to recover General Relativity, the cascade equations (82) and (83) will encode some Einstein equations derived by variation of $\theta^a$ and $g^{ab}$ (see equations (71) and (72)). We will see that in section 4.3.

Before that, let us analyse the third symmetry of the Lagrangian (64).

3) The Projective Symmetry is a generalisation of what is called the $\lambda$-Einstein symmetry [22]. It is very easy to check that (64) is invariant under $\delta_{\kappa} \omega^a_b = \kappa \delta^a_b$
\[ \delta_\kappa \theta^a = \delta_\kappa g^{ab} = 0 \]

where \( \kappa \) is an arbitrary 1-form. For completeness we note that under such a variation, the curvature varies as \( \delta_\kappa R^a_b = d\kappa \delta^a_b \) and then if \( \kappa = d\lambda \) (the \( \lambda \)-Einstein symmetry) it stays invariant. More generally the projective symmetry follows from the elimination of the diagonal \( \mathbb{R} \) subgroup of \( GL(D, \mathbb{R}) \) by our choice of dynamics.

Again, the conservation equation for this symmetry reads

\[ dJ_\kappa := d \left( \kappa \wedge \frac{\partial L}{\partial d\omega^a_a} \right) \approx 0 \quad (85) \]

If we follow the cascade we obtain as first equation that

\[ d\kappa \wedge \frac{\partial L}{\partial d\omega^a_a} \approx 0 \]

And the Noether current has to vanish. This is due to the vanishing of the local superpotential, or in other words to the fact that we have a local symmetry without any propagating field that transforms as a derivative of the parameter (in particular there is no gauge potential for this local symmetry). This is a theorem:

**Theorem 1:** If propagating fields have transformation rules that do not contain derivatives of the local *arbitrary* parameter and if the variation of the Lagrangian can be written as the divergence of a surface term with that same property, then the Noether current has to vanish.

There are many well known examples of this fact, for instance kappa symmetry or Weyl symmetry of string theory. The fact that the parameter of the symmetry is unconstrained is fundamental for the theorem. As we will see in the last section for the case of (non)homentropic fluids, a constraint on the parameter allows conserved charges, even an infinite number of them in even space dimension. Note that the theorem may break down for diffeomorphisms symmetry if fields have spin because of the derivative terms or because of surface terms as in Einstein Lagrangian.

Although this projective symmetry gives no contribution to the total Noether current or to the superpotentials, its existence is crucial if we want to identify somehow the affine gauge theory with General Relativity. In fact the Einstein theory will be recovered as a special gauge choice as we will see in the following subsection.
4.3 Noether identities and integrability condition for gravitation

The Noether Identities

First of all we would like to add some precisions about the equations of motion (71), (72) and (73). The Noether identities due to the gauge symmetry are relations between the equations of motion. For instance if we use equation (22) for the $gl(D, \mathbb{R})$ symmetry we obtain after some rearrangement that

$$D \left( \frac{\delta L}{\delta \omega_{ab}^c} \right) + \theta^b \wedge \frac{\delta L}{\delta \theta^a} + 2 g_{eb} \frac{\delta L}{\delta g^{ac}} = 0$$  \hspace{1cm} (86)

Using now (73) and the Bianchi identities (68), (69) and (70) we easily find that (86) implies (after lowering the $b$ index)

$$R^c_{a} \wedge \Sigma_{bc} - R^c_{b} \wedge \Sigma_{ac} + \theta_b \wedge \frac{\delta L}{\delta \theta^a} + 2 \frac{\delta L}{\delta g^{ab}} = (R_{bc} + R_{cb}) \wedge \Sigma_a^c$$  \hspace{1cm} (87)

The Einstein theory has zero nonmetricity. In that case, the Bianchi identity (70) implies that the symmetric part of the curvature has to vanish $R^{(ab)} = R_{(ab)} = 0$, and so the r.h.s of (87) can be set to zero. Using this vanishing condition the first pair of terms of this equation is completely antisymmetric in $ab$ but the last term is completely symmetric. The conclusion of the Noether identity is then that the equations of motion of the metric are identical to the symmetric part of those of the moving frame when the nonmetricity vanishes. Of course these equations are nothing more than the vacuum Einstein equations after we identify $R_{a}^c$ with the Riemann tensor $R_{a}^c(g)$. It is also interesting to discuss the antisymmetric part of (87). Again assuming metricity (and so the r.h.s. of (87) vanishes), the expression $\theta_b \wedge \frac{\delta L}{\delta \theta^a}$ is actually a combination of the torsion Bianchi identity (69). It is an easy exercise to use the invariance of $\epsilon^{a_1\cdots a_D}$ under $sl(D, \mathbb{R})$ (using again metricity) to prove this fact.

The Noether identity due to the projective symmetry is rather simple:

$$\frac{\delta L}{\delta \omega_{a}^a} = 0$$  \hspace{1cm} (88)

Its meaning will be explained in the following. Before that, we would like to analyse the Noether identities due to the diffeomorphism invariance.
The Einstein equations are conservation laws.
The purpose is to reanalyse with our technology the following theorem:

Theorem 2: The following three affirmations are equivalent:
1- Spacetime is Ricci-flat with null torsion and null nonmetricity in the Einstein gauge.
2- \( d\tau_\rho = 0 \forall \rho \).
3- \( \tau_\rho = d\sigma_\rho \forall \rho \).

A similar theorem was obtained some time ago by Dubois-Violette and Madore [20] but considering neither variational principle nor gauge symmetry.

Proof:

1. \( 2 \Rightarrow 1 \)

First note that we can rewrite the on-shell equation (83) as an off-shell equation by use of the formula (19) of the general discussion (section 2.1) applied to the diffeomorphism invariance. The result is:

\[
d\tau_\rho = L_\rho \left( g^{ab} \right) \frac{\delta L}{\delta g^{ab}} + L_\rho \left( \theta^a \right) \wedge \frac{\delta L}{\delta \theta^a} + L_\rho \left( \omega^a_b \right) \wedge \frac{\delta L}{\delta \omega^a_b} \tag{89}
\]

In addition to that, the Noether identity (equation (22)) due to the diffeomorphism invariance gives:

\[
L_\rho \left( g^{ab} \right) \frac{\delta L}{\delta g^{ab}} + L_\rho \left( \theta^a \right) \wedge \frac{\delta L}{\delta \theta^a} + L_\rho \left( \omega^a_b \right) \wedge \frac{\delta L}{\delta \omega^a_b} = d \left( i_\rho \left( \theta^a \right) \frac{\delta L}{\delta \theta^a} + i_\rho \left( \omega^a_b \right) \frac{\delta L}{\delta \omega^a_b} \right) \tag{90}
\]

Then equation (89) just becomes

\[
d\tau_\rho = d \left( i_\rho \left( \theta^a \right) \frac{\delta L}{\delta \theta^a} + i_\rho \left( \omega^a_b \right) \frac{\delta L}{\delta \omega^a_b} \right) \tag{91}
\]

Note that the equations of motion of the theory (see explicit computations, equations (71), (72) and (73)) transform as \( gl(D, \mathbb{R}) \)-tensors. So, the only part of the r.h.s. of (91) which is not a \( gl(D, \mathbb{R}) \) scalar is the last term which is proportional to \( i_\rho \left( \omega^a_b \right) \). Thus, if we suppose that the l.h.s. is zero, this last term has to vanish identically (otherwise a \( gl(D, \mathbb{R}) \) gauge transformation parametrized by \( \lambda^a_b \) would generate the non vanishing term \( i_\rho \left( d\lambda^a_b \right) \frac{\delta L}{\delta \omega^a_b} \)), so we must have that \( \frac{\delta L}{\delta \omega^a_b} = 0 \). We will admit now that this
last equation implies that the torsion $\Theta^a$ and the nonmetricity $\Xi^{ab}$ have to vanish in what will be called the Einstein gauge. This important statement will be proved in the following discussion: “fixing the projective symmetry” of section 4.4. If we use all this in equation (91) we obtain that affirmation 2 implies that

$$d \left( \theta^a_{\rho} \frac{\delta L}{\delta \theta^a} \right) = D \left( \theta^a_{\rho} \frac{\delta L}{\delta \theta^a} \right) = 0 \quad (92)$$

Now we can use equation (72) for $\frac{\delta L}{\delta \theta^a}$, the Bianchi identity (68) for $R^a_b$ and the vanishing of torsion and nonmetricity to rewrite equation (91) as:

$$- R^{bc} \wedge \Sigma_{bca} \wedge D \theta^a_{\rho} = 0 \quad (93)$$

This holds for arbitrary coordinate frames and so the Einstein equations are satisfied.

• $3 \Rightarrow 2$

This statement is obvious.

• $1 \Rightarrow 3$

We can use equation (20) of the general discussion (section 2.1) applied to the diffeomorphism invariance to obtain:

$$\tau^a_{\rho} - d \sigma^a_{\rho} = \theta^a_{\rho} \frac{\delta L}{\delta \theta^a} + \omega^a_{\rho b} \frac{\delta L}{\delta \omega^b} \quad (94)$$

If the torsion and the nonmetricity vanish and the Einstein equations are satisfied the r.h.s. is obvious zero, which implies affirmation 3.

What we have obtained is thus just a new way to rewrite the Einstein equations in D dimensions for vanishing torsion and nonmetricity. The theorem of Dubois-Violette and Madore [20] uses objects that are not our $\tau^a_{\rho}$ and $\sigma^a_{\rho}$ and were just postulated in D dimensions after the work of Sparling [19] and Nester-Witten [11] [12] in 4 dimensions. To be more precise, these objects were first defined directly on the bundle of orthonormal frames (i.e. with gauge group $so(D, \mathbb{R})$) and later on the linear frame bundle $L(M)$ by Frauendiener [23]. Our formulas are based on a variational principle and on symmetry arguments and so cannot be obtained on the bundle. We have to choose an arbitrary section first and then the computations can begin. We can then hope to pullback the quantities of Dubois-Violette and Madore along this arbitrary section. This will be done in section 5.1.
4.4 Gauge fixing and equivalence with Palatini and Cartan-Weyl actions

“Gauge fixing the Projective Symmetry”

At the end, we would like to recover the basic statements of Einstein theory which are the vanishing of torsion and nonmetricity. The first one imposes on our theory $\frac{D^2(D-1)}{2}$ constraints while the second imposes $\frac{D^2(D+1)}{2}$ more constraints, so we just need $D^3$ constraints. One could have hoped that these constraints would be given by the naively $D^3$ equations of motion of $\omega^a_b$ (73). However the projective symmetry with its Noether identity is telling us that the $\omega^a_b$ give only $D^3 - D$ independent equations ($D$ of them are identically zero, equation (88)). After some algebraic work we derive from equations (73) that their general solutions are given by:

$$\Theta^a \approx \Lambda \wedge \theta^a$$

$$\Xi^{ab} \approx (2 - D)\Lambda g^{ab}$$

Where $\Lambda$ is an arbitrary (undetermined by the equations of motion) one form. In fact it is just proportional to the trace of the nonmetricity, namely $\Lambda = \frac{\Xi}{2D}$. Thus we obtain only $(D^3 - D)$ independent equations which are not enough by themselves to get a null torsion and nonmetricity. In other words, the trace of the nonmetricity will be a free field, not fixed by the equation of motion of $\omega^a_b$. However, if we fix the projective symmetry in the Einstein gauge defined by:

$$\tilde{\omega}^a_b = \omega^a_b - \Lambda \delta^a_b$$

then we obtain the wanted result

$$\Theta^a = \Xi^{ab} = 0$$

We will henceforth that the Einstein gauge is used.

“Gauge fixing the $gl(D, \mathbb{R})$ Symmetry”

We will use the following three gauge choices:

- the holonomic or coordinate gauge: $\theta^a_\mu = \delta^a_\mu$ (or in differential form notation, $s^* (\theta^a) = dx^a$). Here we will be able to use indifferently greek (for the curved base manifold) or latin indices ($gl(D, \mathbb{R})$ ones).

A frame choice makes use of all the $D^2$ degrees of freedom of $gl(D, \mathbb{R})$ but the theory still has the diffeomorphism symmetry. The above gauge fixing has to be preserved under a Lie-derivative transformation, and if it is
not the case, has to be compensated by a $gl(D, \mathbb{R})$ transformation. In fact we see that
\[
\begin{align*}
\mathcal{L}_{\xi} & \left( \frac{\partial a}{\partial \xi} \right)_{\mu} \left|_{\theta=\delta} = \xi^\rho \partial_\rho \delta^a_\mu + \partial_\mu \xi^\rho \delta^a_\rho = \partial_\mu \xi^a, \\
\lambda^a_{b} \theta_{b}^{a} |_{\theta=\delta} = \lambda^a_{\mu}
\end{align*}
\]

Thus, the diffeomorphism transformation which preserves the coordinate gauge is a Lie derivation of parameter $\xi^a$ in addition to a $\text{gl}(D, \mathbb{R})$ rotation of parameter $\lambda^a_{\mu} = -\partial_\mu \xi^a$. Note that with this we recover for example the usual Lie derivative formula for the metric:
\[
\delta g^{ab} = \xi^\rho \partial_\rho g^{ab} + \lambda^a_{\mu} g^{cb} + \lambda^b_{\mu} g^{ae} = \xi^\rho \partial_\rho g^{ab} - \partial_\xi \xi^a g^{cb} - \partial_\xi \xi^b g^{ae}.
\]
The same is true for the connection field which now has to be identified with the Christoffel symbols (the Einstein gauge is used), $\omega_{a}^{b} = \Gamma^{a}_{\mu b} d^x^\mu$ (the non tensorial part of the connection transformation is given by the $- d\lambda^{a}_{b} = \partial_\mu \partial_\nu \xi^a d^x^\mu$ term).

This will be used in the next section to study the gravitational superpotentials.

We would like now to reduce our theory to the Palatini formalism: in the Einstein gauge, the equations of motion of $\Gamma^{a}_{[\mu b]}$ will imply the vanishing of the torsion. The point is that in a coordinate frame, the vanishing of torsion is the vanishing of $\Gamma^{a}_{\rho [\mu \nu]}$ (remember that $\Theta_{a} = d\delta_{a} + \omega_{a}^{e} \wedge \delta_{b} = \Gamma^{a}_{\mu b} d^x^\mu \wedge d^x^b = 0$).

If we eliminate the $\Gamma^{a}_{[\mu e]}$ fields from the Lagrangian (64) we recover the Palatini first order formulation of gravity depending on the fields $\Gamma^{a}_{\mu e}$ and $g^{\mu e}$. The equations of motion of the first give the metricity condition (i.e. the equation which gives $\Gamma$ in terms of $g$ and its derivatives) and those of the second the Einstein equations. More precisely the equations of motion of the $\Theta^{a}$ field which is eliminated by going to a coordinate frame must be recovered as equations of motion of $\Gamma^{a}_{[\mu e]}$ and $g^{\mu e}$. We have seen in the study of Noether identities of $gl(D, \mathbb{R})$ how the symmetric and antisymmetric parts of (87) are indeed deductible from them when torsion vanishes.

- the orthogonal gauge: $g^{ab} = \eta^{ab}$ ($\eta^{ab}$ is the usual flat metric).

This gauge condition fixes only $\frac{D(D+1)}{2}$ of the $D^2$ degrees of freedom allowed by $gl(D, \mathbb{R})$. The remaining symmetries are now the Lie derivative diffeomorphism invariance and the local $so(D, \mathbb{R})$ Lorentz invariance, parametrized now by an infinitesimal antisymmetric tensor $\epsilon^{e}_{a}, \epsilon^{ab} = \epsilon^{a}_{e} \eta^{eb} = \epsilon^{[ab]}$.

The easy part here is that we do not need to modify these symmetries by a compensating “symmetric $gl(D, \mathbb{R})$ rotation” because the gauge choice is automatically preserved:
\( \mathcal{L}_\xi g^{ab} \mid_{g=\eta} = \xi^\rho \partial_\rho \eta^{ab} = 0 \) \hspace{1cm} (95)

\( (\epsilon_a^e g^{eb} + \epsilon_b^e g^{ea}) \mid_{g=\eta} = \epsilon^{ab} + \epsilon^{ba} = 0 \) \hspace{1cm} (96)

The relation with the usual tetrad or vielbein (Cartan Weyl) theory is as follows: In the Einstein gauge the equations of motion of \( \omega^{(ab)} \) imply that the nonmetricity has to vanish. Now the vanishing of the nonmetricity is the vanishing of \( \omega^{(ab)} \) when the orthogonal gauge is used (remember that \( \Xi^{ab} = d\eta^{ab} + \omega^{ab} + \omega^{ba} = 2\omega^{(ab)} = 0 \)).

There is a nice way to understand this: let us decompose \( \omega^{ab} \) in its irreducible parts \( \omega^{ab}_s = \omega^{(ab)} - \omega^c \epsilon^{ab}_{\,\,cd} \) (symmetric traceless), \( \omega^{ab}_a = \omega^{[ab]} \) (antisymmetric) and \( \omega^{ab}_T = \omega^{c} \epsilon^{ab}_{\,\,cd} \) (trace). The curvature can also be decomposed in the same way \( R^{ab}_S = d\omega^{ab}_S + \omega^a c \omega^{cb}_b + \omega^b c \omega^{ab}_S \) \( R^{ab}_A = d\omega^{ab}_A + \omega^a c \omega^{ab}_A + \omega^b c \omega^{ab}_S \) \( R^{ab}_T = d\omega^{ab}_T \). Note that only the skew part of the curvature (\( R^{ab}_A \)) contributes to the Lagrangian (64) (due to the contraction of the curvature with the antisymmetric tensor \( \Sigma_{ab} \)) and so \( \omega^{ab}_T \) completely decouples, the Einstein gauge corresponds to set it to zero. The symmetric \( \omega^{ab}_s \) has no kinetic term and its equation of motion equals it to zero. We recover in that way the above conclusion.

So the field \( \omega^{ab}_S \) can be eliminated from the Lagrangian (64) to obtain the first order tetrad formulation of gravity, i.e. a theory which depends on the fields \( \omega^{[ab]} \) and \( \theta^a \) where the equations of motion of the first give the null torsion equation and of the second the Einstein equations. Note that the equations of motion of \( g^{ab} \) have not been lost by the gauge choice \( g^{ab} = \eta^{ab} \) as we discuss at the beginning of section 4.3. They are indeed in those of the form \( \theta^a \) (see equation (87)).

- the arbitrary fixed frame: \( \theta^{\mu}_a = \bar{\theta}^{\mu}_a \).

This case is slightly more complicated than the previous ones. Now the frame is chosen to be an arbitrary \( x \)-dependent frame \( \bar{\theta}^{\mu}_a(x) \). Again, all the \( gl(D, \mathbb{R}) \) gauge invariance has been used. The modified remaining diffeomorphism invariance can be computed as in the previous example:

\[ \mathcal{L}_\xi \theta^a_{\mu} \mid_{\theta=\bar{\theta}} = \xi^\rho \partial_\rho \bar{\theta}^a_{\mu} + \partial_\mu \xi^\rho \bar{\theta}^a_{\rho} \]
\[ \lambda^a_b \theta^b_{\mu} \mid_{\theta=\bar{\theta}} = \lambda^a_b \bar{\theta}^b_{\mu} \]

Then the remaining symmetry combines a Lie derivation of parameter \( \xi^\rho \) plus a \( gl(D, \mathbb{R}) \) rotation of parameter \( \lambda^a_b = -\bar{\theta}^b_{\rho} (\xi^\rho \partial_\rho \bar{\theta}^a_{\mu} + \partial_\mu \xi^\rho \bar{\theta}^a_{\rho}) \).
4.5 Comparison of superpotentials

The purpose of this subsection is to collect all the above results to recover some of the well known superpotentials. The connection with the second order formulation of section 3 will also be established.

We learned in previous sections that the local $gl(D, \mathbb{R})$ invariance implies the existence of a total Noether current $J_\lambda$ together with an associated superpotential $U_\lambda$ (equation (78)) both $\lambda^a_b(x)$ dependent. The same is true for the local diffeomorphism invariance (parametrized by $\xi(x)$), with corresponding quantities $J_\xi$ and $U_\xi$ (equation (84)). The $\kappa$ local symmetry had vanishing currents for the reason already explained at the end of section 4.2. With all this we can construct a big Noether current:

$$J_{\xi,\lambda} \approx dU_{\xi,\lambda}$$ (97)

where

$$J_{\xi,\lambda} := J_\xi + J_\lambda = \xi^\alpha \tau_\alpha + d\xi^\alpha \wedge \sigma_\alpha + \lambda^a_b J^b_a + d\lambda^a_b \wedge U^b_a$$

$$U_{\xi,\lambda} := U_\xi + U_\lambda = \xi^\alpha \sigma_\alpha + \lambda^a_b U^b_a$$

Now, if we use an holonomic section $s$ of $L(M)$ (in other words if we use a coordinate frame), i.e. $\theta^a = dx^a$, then:

- the dual of the $(D-2)$-form $\sigma_\rho$ is what we called the Møller superpotential in equation (51). In fact, Møller multiplied it by a factor of two in a desesperate attempt to gain weight [24]. From the definition of $\sigma_\rho$ (equation (81)) we have that:

$$\sigma_\rho = -\sqrt{-\det g} \omega^a_b \epsilon^c_d s^*(\Sigma_{ab})$$

and, using the fact that $s$ is a coordinate section, $\omega^a_b = \Gamma^a_{bc}$ with the definition (51) we obtain,

$$s^*(\sigma_\rho) = \frac{1}{2} M U^i_{[ab]} s^*(\Sigma_{ab})$$

In components, the dual 2-form of this is just $M U^i_{[ab]}$ (the $\frac{1}{2}$ factor is killed as usual by the 2 coming from the $\epsilon_{abc3...eD} d\omega^a_b \epsilon^c_d dx^c \wedge ... \wedge dx^e$ contraction).

We also find that

$$s^*(d\sigma_\rho) = ds^*(\sigma_\rho) = \frac{1}{2} \partial_\alpha M U^i_{[\mu\nu]} dx^\alpha \wedge \frac{1}{(D-2)!} \epsilon_{abc3...eD} dx^c \wedge ... \wedge dx^e$$
\[ \partial_{\nu} M U^{[\mu \nu]}_{\rho} \frac{1}{(D-1)!} \epsilon_{\mu \nu_{2} \ldots \nu_{D}} dx^{\nu_{2}} \wedge \ldots \wedge dx^{\nu_{D}} \]

- the dual of the \((D - 1)\)-form \(\tau_{\rho}\) is just the canonical energy momentum complex (of equations (38) and (39)) or one half of Møller’s original gravitational energy-momentum pseudotensor. Let us recall that \(L = \frac{\sqrt{-g}}{2k} R \Sigma\) where \(R\) is the scalar curvature and \(\Sigma = \frac{1}{2^k} \epsilon_{\epsilon_{1} \ldots \epsilon_{D}} dx^{\epsilon_{1}} \wedge \ldots \wedge dx^{\epsilon_{D}}\) the volume form, we obtain from the definition of \(\tau_{\rho}\):

\[
\tau_{\rho} = \frac{\sqrt{-g}}{2k} R s^{*}(\rho \Sigma) - \frac{\sqrt{-g}}{2k} \partial_{\rho} \omega_{[a} b_{d]} g^{[a] \gamma} s^{*}(\theta^\mu \Sigma_{ad})
= \frac{\sqrt{-g}}{2k} R s^{*}(\Sigma_{\rho}) - 2 \frac{\sqrt{-g}}{2k} \partial_{\rho} \omega_{[a} b_{d]} g^{[a] \gamma} s^{*}(\Sigma_{\gamma})
= M T^{\mu}_{\rho} \frac{1}{(D-1)!} \epsilon_{\mu \nu_{2} \ldots \nu_{D}} dx^{\nu_{2}} \wedge \ldots \wedge dx^{\nu_{D}}
\]

- the dual of the \((D - 2)\)-form \(U_{\xi, \lambda}\) with \(\lambda^{a}_{b} = -\partial_{b} \xi^{a}\) (see section 4.4) is just the Komar superpotential (54). To show that we first use (from equation (75)) that:

\[
U_{a}^{b} = \frac{\sqrt{-g}}{2k} g^{bd} s^{*}(\Sigma_{ad})
= \frac{1}{2} W_{a}^{b[dc]} s^{*}(\Sigma_{cd})
\]

Where \(W_{a}^{b[dc]}\) has been defined in (49). We thus see that

\[
U_{\xi, -\partial_{\xi}} = \xi^{\rho} \sigma_{\rho} - \partial_{a} \xi^{b} s^{*}(U_{a}^{b})
= \frac{1}{2} \left( M U^{[\mu | | \nu]}_{\rho} \xi^{\rho} + W^{a | \mu | | \nu} \partial_{a} \xi^{\rho} \right) s^{*}(\Sigma_{\mu \nu})
= \frac{1}{2} K U^{[\mu | | \nu]}_{\xi} s^{*}(\Sigma_{\mu \nu})
\]

- the dual of the \((D - 1)\)-form \(J_{\xi, \lambda}\) with \(\lambda^{a}_{b} = -\partial_{b} \xi^{a}\) is just the total Noether current, equation (41) or (35). To simplify the analysis and obtain a better understanding of what is going on, let us first make the following comment: in the second order formalism the definition of \(\Gamma^{\mu}_{\rho \sigma}\) in terms of the metric and its derivatives is assumed, in our language, this means that the equations of motion of \(\omega^{a}_{b}\) have been used (in the Einstein gauge). This means that equation (76) becomes an identity and then the big Noether current (definition (97)) specializes for \(\lambda = -\partial_{\xi}:\)

\[
J_{\xi, -\partial_{\xi}} := J_{\xi} + J_{\lambda = -\partial_{\xi}} = \xi^{\rho} \tau_{\rho} + d \xi^{\rho} \wedge \sigma_{\rho} - \partial_{b} \xi^{a} d U_{a}^{b} - d \partial_{b} \xi^{a} \wedge U_{a}^{b}
\]

We then obtain:

\[
J_{\xi, -\partial_{\xi}} = \xi^{\rho} M T^{\mu}_{\rho} \xi^{\nu} (\Sigma_{\mu}) + \left( \partial_{a} \xi^{\rho} M U^{[\mu | | \nu]}_{\rho} + \partial_{b} \xi^{a} \partial_{a} W_{a}^{b [\mu | | \nu]} \right.
\] 

\[
+ \partial_{a} \partial_{b} \xi^{a} W_{a}^{b [\mu | | \nu]} \right) \left( \frac{1}{2} dx^{a} \wedge s^{*}(\Sigma_{\mu}) \right) = \left[ \xi^{\rho} M T^{\mu}_{\rho} + \partial_{b} \xi^{a} \left( M U^{[\mu | | \nu]}_{\rho} + \partial_{b} W^{\nu | | \delta}_{b} \right) + \partial_{b} \partial_{c} \xi^{d} W^{b \nu | | \delta}_{b} \right] \wedge s^{*}(\Sigma_{\mu}) \quad (98)
\]

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Now we are in a position to understand the objects found in the second order formulation: the second term in (98) is just what was called the canonical spin complex $U_\mu^{\rho\nu}$ and the last term in (98) symmetrized on the $(\nu\delta)$ indices of course is just $V_\mu^{\rho\nu\delta}$ of section 3.1. The somewhat mysterious relation (48), and the consequent non antisymmetry of the spin complex becomes natural in this framework. The completely antisymmetric contribution comes from the diffeomorphism but another term comes from a $gl(D, \mathbb{R})$ compensating transformation. We can understand in a similar way equation (52). We point out now that the difference between the canonical spin complex and the Møller superpotential is not a physical observable since the conserved charge is always obtained by its integral over a closed boundary (for instance, see equation (25)).

We will not analyse in more details the results of the previous discussion for orthonormal sections. We just want to point that for an arbitrary fixed section, we obtain again the Komar superpotential if we use the modified $\lambda_{\xi}$ given at the end of section 4.4 also for that case.

Finally, we have obtained

- An explicit derivation of superpotentials from first principles (gauge invariance) and a general formula (14).

- The Affine formulation gives the superpotentials in a clearer geometrical way, if reduces to the usual formulations of gravity by using some equations of motion and fixing the Einstein gauge. This is why we will study it further in section 5.

- Remember that the abelian cascade defined in section 2.1 is used to obtain a parameter dependent superpotential. When we used it before compensating for the choice of a coordinate section we obtained an explicitly antisymmetric superpotential, $U_{\xi}$, (which gives the Komar superpotential when the compensation is taken into account). When we used it directly in the 2nd order formulation we got a non antisymmetric one (see equation (47)), which has been shown to be identical to the first one up to a divergence term (52). It seems that more geometrical objects are obtained when the abelian cascade is used in the $gl(D, \mathbb{R})$ formalism. This is the method we will use in the following.

- About the mysterious one half factor between the superpotentials we found from our general formula (14) and the ad-hoc definitions given in the early days by Møller [24] and Komar [25], the point is that, as we will show in the next section, the variational principle and the Lagrangian (64)
used in this section are not compatible with standard asymptotically flat solutions of general relativity as for example the Kerr solution. It then does not make sense to use the corresponding superpotentials to compute for example the conserved mass or angular momentum. If we in fact compute the conserved charge associated with the constant asymptotic Killing time like vector for the Schwarzschild solution, we obtain with our definitions $m^2$. We understand now why Møller and then Komar doubled the above expressions in the hope to obtain correct superpotentials. As we will see in next section, to correct the above anomaly we will need to add a surface term to our first Lagrangian (64). The associated superpotentials will be now those of Freud [26] and Katz [27] (actually, a non background version of it). They are comparable with the ADM formula (see for example [10]) of the Hamiltonian formalism.

5 Surface terms, well posed variational principles and physical charges

5.1 Physical Lagrangian, modified superpotentials

As we discussed in the Yang-Mills situation (section 2.2) the choice of boundary conditions will specify the surface term we have to add to the Lagrangian of the theory. For instance for the Lagrangian (64), the variational principle implies that (equation (29) in the Yang-Mills case)

$$\int_{\partial M} \delta \omega^a_b \wedge \frac{\partial L}{\partial d \omega^a_b}$$

has to vanish.

Later, we would like to impose the vanishing of metricity and torsion which, as we know, allows us to rewrite the $\omega^a_b$ (or the $\Gamma^a_{\mu b}$ in more familiar notation) in terms of the metric and its derivatives. But the condition $\delta \omega^a_b = 0$ on boundaries is a Neumann type boundary condition (for the metric). If we look for conserved charges, then we would like instead to impose Dirichlet boundary conditions. For that, we have to eliminate all the derivatives of $\omega^a_b$ from the Lagrangian (64) by adding a surface term. The most obvious way to do that (and the only one, see [28]) is to define:

$$\hat{L} = L - dS$$
\[
\begin{align*}
\frac{\sqrt{-g}}{2k} \omega^a \wedge \sqrt{-g} e^b \wedge \Sigma_{ab} + \frac{1}{2k} \omega^a \wedge D(\sqrt{-g} e^b \Sigma_{ab}) & \quad (100) \\
= \frac{\sqrt{-g}}{2k} \omega^a \wedge \omega^b \wedge \Sigma_{ab} + \frac{1}{2k} \omega^a \wedge D(\sqrt{-g} e^b \Sigma_{ab})
\end{align*}
\]

Where \( S := \frac{1}{2k} \omega^a \wedge \sqrt{-g} e^b \Sigma_{ab} \). Note that this Lagrangian computed with a coordinate section and using the null torsion and metricity conditions is the classical Einstein Lagrangian \( \hat{L} = \hat{L}(g, \partial g) \). Of course the equations of motion are the same as in the section 4.1.

The asymptotic conditions should be coordinate independent of course so we need some asymptotic reference manifold, ideally a boundary.

Our purpose is to derive the superpotentials associated to \( \hat{L} \). We will analyse the conserved charges in the next section.

As for the previous Lagrangian (64), only two of the three gauge symmetries will give non trivial Noether currents and associated superpotentials (as before, the \( \kappa \) local projective symmetry does not contribute and will be fixed in the Einstein gauge). The analogous equation to (74) for the Einstein Lagrangian \( \hat{L} \) (which now depends on the derivatives of \( \theta^a \) and \( g^{ab} \) only) is

\[
\delta \hat{L} - d \left( \delta \theta^a \wedge \frac{\partial \hat{L}}{\partial \theta^a} + \delta g^{ab} \frac{\partial \hat{L}}{\partial g^{ab}} \right) \approx 0 \quad (101)
\]

This formula can be used for both symmetries:

- The \( gl(D, \mathbb{R}) \) symmetry: Due to the surface term added, the variation of the Einstein Lagrangian does not vanish anymore but is equal to a surface term \( \delta \lambda \hat{L} = d \left( d \lambda^a \wedge \sqrt{-g} e^b \Sigma_{ae} \right) \). Note that this is in some sense a definition of \( \delta \lambda \hat{L} \) because the Noether method only gives quantities up to a exact form. If we use this in (101), the corresponding Noether current and superpotentials for \( \hat{L} \) are exactly the same as for \( \hat{L} \), equation (75) (and the definitions which follow it):

\[
\begin{align*}
\hat{J}^a &= J^a \\
\hat{U}^a &= U^a \\
\hat{J}_\lambda &= J_\lambda \\
\hat{U}_\lambda &= U_\lambda
\end{align*}
\]

and the corresponding cascade equations are obviously the same.

- The diffeomorphism invariance: we use that under a Lie derivative the Einstein Lagrangian transforms as usual, i.e., \( \delta \xi \hat{L} = d \cdot i_\xi \hat{L} \) and that \( \delta \xi \theta^a =
\[ \mathcal{L}_\xi \theta^a, \: \delta_\xi g^{ab} = \mathcal{L}_\xi g^{ab} = i_\xi \cdot dg^{ab}, \text{ then formula (101) becomes (compare equation (81))} \]
\[ d\hat{J}_\xi := d(\xi^\rho \hat{\tau}_\rho + d\xi^a \wedge \hat{\sigma}_\rho) \approx 0 \quad (102) \]

and direct computations show that
\[ \hat{\tau}_\rho = i_\rho \hat{L} - \mathcal{L}_\rho (\theta^a \wedge \frac{\partial L}{\partial \theta^a}) - \mathcal{L}_\rho g^{ab} \cdot \frac{\partial L}{\partial g^{ab}} \]
\[ = \tau_\rho + d \cdot i_\rho \mathcal{S} \]
\[ = - \frac{\sqrt{-g}}{2k} \left( \omega^c \wedge \omega^{ab} \wedge \Sigma_{cab} + \omega^{ac} \wedge \omega^b \wedge \Sigma_{pab} \right) + di_\rho \theta^a \wedge \left( - \frac{\sqrt{-g}}{2} \omega^{bc} \wedge \Sigma_{lca} \right) \]
\[ + \frac{1}{2k} i_\rho (\theta^a) \mathcal{D} \left( \sqrt{-g} g^{bc} \Sigma_{lca} \right) \wedge \omega^b_d \]

\[ \hat{\sigma}_\rho = -i_\rho \theta^a \frac{\partial L}{\partial g^{ab}} \]
\[ = \sigma_\rho + i_\rho \mathcal{S} \]
\[ = - \frac{\sqrt{-g}}{2k} \omega^{ab} \wedge \Sigma_{pab} \]

Note that \( \hat{J}_\xi = J_\xi + d(\xi^a \cdot \mathcal{S}) \) where \( J_\xi \) was given in (81).

So all the cascade equations derived in the past sections can be used here by just hatting them and using these new formulas for practical computations. Note also that the theorem 2 of section 4.3 and its proof can be repeated identically.

The relation between \( \tau_\rho \) and \( \sigma_\rho \) with the \( gl(D, \mathbb{R}) \) objects of Frauendiener [23] (see discussion which follow theorem 2 of section 4.3) is as follows:

Let us derive the cascade equations for the the \( gl(D, \mathbb{R}) \) vector \( \xi^a \) defined as:

\[ \xi^a = \xi^\rho \theta^a_\rho \]

where \( \theta^a_\rho \) will be fixed. Using this decomposition in (102) we obtain that:

\[ d(\xi^a \hat{\tau}_a + d\xi^a \wedge \hat{\sigma}_a) \approx 0 \quad (103) \]

where now
\[ \hat{\tau}_a = \hat{\tau}_\rho \theta^a_\rho + d\theta^a_\rho \wedge \sigma_\rho \]
\[ = - \frac{\sqrt{-g}}{2k} \left( \omega^b_a \wedge \omega^{cd} \wedge \Sigma_{bcd} + \omega^{bc} \wedge \omega^d \wedge \Sigma_{abd} \right) \]
\[ + \frac{1}{2k} \omega^b_a \mathcal{D} \left( \sqrt{-g} g^{cd} \Sigma_{bd} \right) + \frac{1}{2k} \mathcal{D} \left( \sqrt{-g} g^{de} \Sigma_{lca} \right) \wedge \omega^b_d \]

\[ \hat{\sigma}_a = \hat{\sigma}_\rho \theta^a_\rho \]
\[ = - \frac{1}{2k} \omega^{bc} \wedge \Sigma_{abc} \]
What we did in the last equations is just a trick to change the $\rho$ index (curved manifold index) into a $a$ index ($gl(D, \mathbb{R})$ index).

Similar objects have been encountered before in the works of Sparling, Nester-Witten, Dubois-Violette & Madore and Frauendiener. As we already said, it is Frauendiener who finally defined their most general version, namely on the linear frame bundle $L(M)$ in $D$ dimensions:

$$\tilde{\tau}_a = -\frac{1}{2k} \left( \omega^a_b \wedge \omega^{cd} \wedge \Sigma_{bcd} + \omega^b_c \wedge \omega^d_c \wedge \Sigma_{abd} \right)$$

$$\tilde{\sigma}_a = \hat{\sigma}_a$$

It is easy to see that the $\sigma_a$’s (hatted and tilded) coincide and also the $\tau_a$’s if we set the nonmetricity and the torsion to zero (more precisely the hatted quantities coincide with the pullback along an arbitrary section of the tilded ones). What we have just achieved is to derive from a variational principal and symmetry arguments some objects whose existence appeared before rather mysterious.

Finally, we can define as in section 4.5 the total hatted Noether $\xi$-dependent superpotential, exactly in the same way as equation (97).

Let us now make contact with the ordinary second order formalism. We are not going to repeat here the complete analysis of section 3 for the Einstein Lagrangian case:

$$\hat{L} = \frac{1}{2k} \sqrt{-g} R - \partial_\mu S^\mu = \frac{1}{2k} \sqrt{-g} g^{\alpha\beta} (\Gamma^\eta_{\alpha\delta} \Gamma^\delta_{\eta\beta} - \Gamma^\eta_{\eta\beta} \Gamma^\delta_{\alpha\delta})$$  \hspace{1cm} (104)

Where

$$S^\mu := \frac{1}{2k} \sqrt{-g} \left( \Gamma^\mu_{\alpha\beta} g^{\alpha\beta} - \Gamma^\beta_{\alpha\beta} g^{\mu\alpha} \right)$$  \hspace{1cm} (105)

Let us just give crucial differences and the principal results:

- The analogue of equation (35) is

$$\partial_\mu \hat{j}_\xi^\mu \approx 0$$  \hspace{1cm} (106)

$$\Leftrightarrow \partial_\mu (\xi^\rho \hat{T}_\rho^\mu + \partial_\nu \xi^\rho \hat{U}_\rho^{\mu\nu} + \partial_\nu \partial_\xi^\rho \hat{V}_\rho^{\mu\nu\delta}) \approx 0$$  \hspace{1cm} (107)

Where

$$\hat{T}_\rho^\mu := \delta_\rho^\mu \hat{L} - \frac{\partial \hat{L}}{\partial \nu g_{\alpha\beta}} \partial_\rho g_{\alpha\beta}$$
\[
\frac{\sqrt{-g}}{2k} \left[ \delta^\mu_\nu (\Gamma^\alpha_\beta_\gamma - \Gamma^\alpha_\beta_\gamma) g^{\gamma \delta} + \Gamma^\beta_\mu_\nu_\gamma g^{\mu \alpha} - \Gamma^\beta_\rho_\gamma_\alpha g^{\mu \alpha} + \Gamma^\alpha_\rho_\gamma_\beta g^{\beta \gamma} + \Gamma^\alpha_\rho_\gamma_\beta g^{\beta \gamma} - 2 \Gamma^\mu_\alpha_\beta_\rho_\gamma g^{\beta \gamma} \right]
\]

\[
\hat{U}^\mu_\nu := - \frac{\partial \hat{L}}{\partial \partial^\mu}(\delta^\nu_\rho g^\rho_\beta + \delta^\nu_\beta g^\rho_\rho) = \sqrt{-g} \frac{1}{2k} \left[ \delta^\nu_\rho (\Gamma^\mu_\alpha_\beta g^{\alpha \beta} - \Gamma^\beta_\alpha_\beta g^{\alpha \beta}) + \delta^\mu_\rho \Gamma^\beta_\alpha_\beta + \Gamma^\alpha_\rho_\gamma_\beta g^{\beta \gamma} - 2 \Gamma^\mu_\alpha_\beta_\rho_\gamma g^{\beta \gamma} \right]
\]

\[
\hat{V}^\mu_\nu_\delta = \hat{V}_\rho^\mu_\nu_\delta = \sqrt{-g} \frac{1}{2} \left[ \frac{1}{2} g^{\mu \delta} \delta_\rho^\nu + \frac{1}{2} g^{\nu \mu} \delta_\rho^\delta - g^{\delta \nu} \delta_\rho^\mu \right]
\]

-The cascade equations are the same equations as (36)-(39) but hatted.

- \( \hat{L} \) is now a Lagrangian which depends only on the metric and its first derivative. One then may ask where the \( \hat{V}^\mu_\nu_\delta \) term comes from? The answer is quite simple and is that now \( \hat{L} \) is not anymore a scalar. In fact, it is easy to show that its variation under a diffeomorphism induces an inhomogeneous surface term due to the non tensorial part of the \( \Gamma \)'s. In other words, \( \delta_\xi \hat{L} = \partial^\mu (\xi^\mu \hat{L}) - \frac{1}{2k} \partial_\mu (\sqrt{-g}(\partial_\delta \partial_\rho \xi^{\mu} g^{\rho \delta} - \partial_\delta \partial_\rho \xi^{\rho} g^{\mu \delta})) \). We note finally that \( \hat{V}^\mu_\nu_\delta = V^\mu_\nu_\delta \).

- \( \hat{T}^\mu_\rho \) has been found by Einstein himself and is ususaly called the “canonical energy momentum Einstein pseudotensor”.

- The Einstein canonical spin complex \( \hat{U}^\mu_\nu_\rho \) is not more antisymmetric than its Hilbert brother. Its associated antisymmetric quantity is what is called the Freud superpotential [26] (analogous to the Møller one, equation (48)):

\[
F \hat{U}^\mu_\nu_\rho := \hat{U}^\mu_\nu_\rho - \partial_\delta W^\nu_\rho_\delta
\]

\[
\hat{T}^\mu_\rho \approx \partial_\nu F \hat{U}^\mu_\nu_\rho
\]

Where,

\[
F \hat{U}^\mu_\nu_\rho := \frac{1}{2k} \frac{1}{\sqrt{-g}} \frac{1}{2k} g^{\rho_\alpha_\beta} \partial_\beta (-g(g^{\mu \alpha} g^{\nu \beta} - g^{\mu \beta} g^{\nu \alpha}))
\]

Note that \( F \hat{U}^\mu_\nu_\rho = M \hat{U}^\mu_\nu_\rho + S^\mu_\rho_\delta_\nu - S^\nu_\rho_\delta_\mu \).

The abelian cascade trick will again naturally give us a non antisymmetric \( \xi^\rho \) dependent superpotential \( \hat{U}_\xi^\mu_\nu \) defined exactly as equation (47) hatted. Its corresponding antisymmetric quantity is just a non background version of the Katz superpotential [27].

\[
K_\alpha \hat{U}_\xi^\mu_\nu := \hat{U}_\xi^\mu_\nu - \partial_\delta \left( \xi^\rho W^\nu_\rho_\delta \right)
\]
\[ \hat{J}_\xi^\mu \Rightarrow \partial_\nu K_a \hat{U}_\xi^{\mu\nu} \]  

where we have that:

\[
K_a \hat{U}_\xi^{\mu\nu} := F \hat{U}_\rho^{\mu\nu} \xi^\rho + g^{\mu\alpha} \partial_\alpha \xi^\nu - g^{\nu\alpha} \partial_\alpha \xi^\mu \\
= K U_\xi^{\mu\nu} + S^\mu \xi^\nu - S^\nu \xi_\mu
\]  

What we have just obtained is the Katz superpotential if we add the condition that only asymptotically cartesian coordinates can be used to compute the conserved charges. We will come back to this important point in the next subsection.

- The contact between the \( gl(D, \mathbb{R}) \) objects and the above formulas has been partially established by Frauendiener [23] and Szabados [29]. Remember that their definitions of \( \tilde{\tau}_a \) and \( \tilde{\sigma}_a \) coincide with our hatted quantities (with the null torsion and metricity conditions added for the \( \hat{\tau}_{\nu} \)). What they have shown is that:
  
  • The dual of the pullback of \( \hat{\sigma}_a \) (or \( \hat{\sigma}_\rho \)) along an holonomic section (in this case the \( a \) and \( \rho \) indices are indistinguishable) is the Freud superpotential.

  • The dual of the pullback of \( \hat{\tau}_a \) (or \( \hat{\tau}_\rho \) if \( \Theta^a = \Xi^{ab} = 0 \)) along an holonomic section is the Einstein pseudotensor.

  • The pullback of \( \tilde{\sigma}_a \) (or \( \hat{\sigma}_a \)) along an orthonormal section is the Nester-Witten form.

  • The pullback of \( \tilde{\tau}_a \) (or \( \hat{\tau}_a \) if \( \Theta^a = \Xi^{ab} = 0 \)) along an orthonormal section is the Sparling form.

- With no more work than the results found in section 4.5 and the above theorems we can easily complete the picture:
  
  • The dual of \( \hat{U}_\xi \) along an holonomic section is the Katz superpotential.

  • The dual of \( \hat{J}_\xi \) along an holonomic section is the total hatted Noether current \( \hat{J}_\xi^\mu \).

- Finally, the non antisymmetry of the Einstein canonical spin complex \( \hat{U}_\rho^{\mu\nu} \) is explained in exactly the same way as we did for \( U_\rho^{\mu\nu} \).

The conclusion of this subsection is that the superpotential associated to some theory depends strongly on the choice of boundary conditions. This
is expected because the cascade equations imply that in the case of a gauge symmetry all the charges have to be computed on a \((D-2)\) hypersurface. If we complicate the story and we use more general boundary conditions then a background superpotential has to be subtracted from the above defined superpotentials (see Rosen [30], Cornish [31], Katz [27], Katz, Bičák and Lynden-Bell [32], Chruściel [33]). This is part of the subject of the following subsection, we shall return to it in II.

5.2 Physical charges

We have just seen that specific boundary conditions correspond to specific surface terms in the action which naturally do not change the equations of motion. As we know, the result of adding this surface term is not so innocuous. In fact, the Lagrangian loses its explicit scalar form (as we just saw, \(\hat{L}\) transforms as a scalar plus an inhomogenous term). The major consequence of that is that the associated parameter dependent superpotentials are not anymore covariant. For instance although the Komar superpotential (equation (54)) is covariant the Katz one (equation (111)) is not. Unfortunately for the covariance it is the second one which is derivable with the asymptotic spatial Dirichlet boundary conditions for the metric (and so with the Kerr solution for example) as we will see in this section.

**asymptotic conditions**

The addition of a surface term \(S\) to the scalar Lagrangian \(L\) (see equation (100)) replaces the “Hilbert variational principle condition” (equation (99)) by the vanishing of

\[
\int_{\partial M} \delta \theta^a \wedge \frac{\partial \hat{L}}{\partial \theta^a} + \delta g^{ab} \frac{\partial \hat{L}}{\partial g^{ab}} \quad (114)
\]

This condition is more satisfactory in the light of the discussion which followed equation (99).

As in the Yang-Mills case (section 2.2), the purpose here is not to precisely solve this condition, say for example for an asymptotically flat boundary. We will just illustrate what is going on with a Dirichlet condition for the metric in the coordinate-Einstein gauge (i.e. \(\theta^a = dx^a\) and \(\Lambda\) chosen so that the torsion and the metricity vanish). The Dirichlet solution for the vanishing of equation (114) in that gauge is

\[
\lim_{r \to \infty} \delta g = 0 \quad (115)
\]
Where $g = g_{ab} \, dx^a \otimes dx^b$. The solution to that is just as usual $g|_{\partial M} = \bar{g}$, where $\bar{g}$ is a given asymptotic boundary metric.

Remember that in the coordinate-Einstein gauge the remaining gauge symmetry is a linear combination of a diffeomorphism $L_\xi$ and a $gl(D, \mathbb{R})$ gauge rotation with parameter $\lambda^a_b = - \partial_b \xi^a$ (section 4.4). The result of that is just the well known definition of the Lie derivative in a Riemannian manifold, which for simplicity will also be called $\mathcal{L}_\xi$. Now this symmetry has to preserve the above boundary condition, that means that $\mathcal{L}_\xi g|_{\partial M} = 0$.

We obtain just the asymptotic Killing condition:

$$\lim_{r \to \infty} \left( \xi^\rho \partial_\rho g^{ab} - \partial_\rho \xi^a g^{b\rho} - \partial_\rho \xi^b g^{a\rho} \right) = 0 \quad (116)$$

Again it is important to find all the $\xi^\rho(x)$ that satisfy this condition. After that, a recipe can be given to obtain all the conserved physical charges of the theory due to the gauge symmetry. Before that we will just comment on the non covariance of $K_a^\hat{U}$ and on the way Katz cured it.

The need of a background metric :

The problem of non covariance is important for practical computations. For example only Cartesian coordinates were allowed to compute say the mass of the Schwarzschild black hole in the Freud superpotential formula (which is equal to the Katz superpotential for constant $\xi^\rho$).

The way to remedy this is now well known and was introduced in the case where the asymptotic metric is flat by Rosen [30] and Cornish [31]. Technically it consists in introducing a background metric $\bar{g}_{\mu\nu}$ in the theory and replaces every non covariant piece by a covariant one with respect to this metric. We will just refer the interested reader to the works of Katz [27], Chrušciciel [33] and Katz, Bičák and Lynden-Bell [32] for flat and more general backgrounds (for example an anti de Sitter space). For completeness, let us note that the introduction of a non dynamical background connection was proposed in [34].

The point is that this background metric is nothing other than the boundary conditions we used to solve the Dirichlet problem (see above). Thus the conserved charge makes sense in general only on the boundary of the manifold $\partial M$, where its corresponding superpotential is well defined. The idea of using the background metric $g_{\mu\nu}$ everywhere would allow to define the non covariant superpotential on $M$; however the physical meaning of these objects in general is not clear (as for example a quasilocal mass).

In some special cases, when there exists a global space like Killing vector a
quasilocal charge can be defined. We postpone the discussion of the angular momentum to the last part of this subsection.

The conserved charges

Suppose as in our general discussion (section 2.1) that a portion of spacetime (between two times, say \( t_1 \) and \( t_2 \)) is bounded by a \((D - 1)\) timelike hypersurface at infinite distance \( \Sigma_\infty \) and by two spacelike \((D - 1)\) hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \). The possibility of having a black hole will not alter our discussion of the conserved charges, so we will ignore it. Without more computations, we give now the recipe to find all the conserved charges due to the local diffeomorphism invariance:

- Solve the Einstein equation with your chosen boundary conditions (we shall take the example of Dirichlet boundary conditions for the metric).
- Given these boundary conditions, derive the parameter dependent Noether current \( J_\xi \) with its associated superpotential \( U_\xi \) (for example \( \kappa a \hat{U}_\xi \) of equation (111) for the Dirichlet choice). If this superpotential is not covariant, use the appropriate background correction on the boundary of the manifold \( M \).
- Find all the \( \xi^\rho(x) \) that satisfy the asymptotic Killing equation (116) say for the Dirichlet condition, or its analogue for the Neumann case,

\[
\lim_{r \to \infty} \delta_\xi \omega^a_b = 0.
\]

They may form an infinite group.
- Integrate the equation \( J_\xi \approx dU_\xi \) on the \((D - 1)\) surface \( \Sigma_\infty \), bounded by \( B_1 \) at time \( t_1 \) and \( B_2 \) at time \( t_2 \) (as in our general discussion of section 2.1). The physical condition for charge conservation is that the charge does not “run away”, i.e. that \( J \) has to vanish on \( \Sigma_\infty \). This implies in particular that the integral of \( U_\xi \) on \( B_1 \) will be the integral of \( U_\xi \) on \( B_2 \). In other words, the integral of \( U_\xi \) on a \( D - 2 \) hypersurface (say now \( B_\infty \)) at infinite distance and fixed time is the conserved quantity, and nothing more. Warning: but for instance the integral of \( J_\xi \) over a fixed time spacelike hypersurface is not in general (for example when a black hole exists) conserved contrary to the usual Noether charge associated to a global symmetry.
- Compute for each \( \xi^\rho(x) \) the associated conserved charge given by:

\[
Q(\xi^\rho(x)) = \int_{B_\infty} U_\xi
\]

with the parameter dependent superpotential associated to your boundary choice.
- The number of conserved charges will be given by the number of finite Q’s which are “not always zero” (“not always zero” means that for example even if the linear momentum of Schwarzschild black hole is zero, a finite boost transformation can give it a non null contribution; in that case the linear momentum counts as a charge). The number of really independent charges (not connected by a asymptotic gauge transformations) will be given by the number of Casimirs of this subgroup.

Some comments on the total Noether current

We would like to conclude with some comments on the total Noether current \( J_\xi \), and its associated equation

\[
J_\xi = dU_\xi
\]

In our discussion on conserved charges we just integrated this equation at \( \Sigma_\infty \). The other obvious choice is to integrate it on a \((D-1)\) space like hypersurface \( \Sigma \), bounded say by \( B_1 \) and \( B_2 \), where \( 1 \) or \( 2 \) can stand for an asymptotic or black hole boundary, or any other finite distance. This is the problem of quasilocal charges.

However, we have to be very careful because to perform such an integration, the current and the superpotential must be defined everywhere and not only at infinity. Let us show some cases where this can be done:

- The quasilocal angular momentum associated to a global Killing vector, namely \( \nabla^{(\mu} \xi^{\nu)}|_\Sigma = 0 \). Remember that the Katz superpotential can be written as the Komar (one half of the usual definition) superpotential wich is covariant plus a non covariant part (see equation (113)):

\[
K_a U_{\xi}^{\mu\nu} = K U_{\xi}^{\mu\nu} + S^{\mu} \xi^{\nu} - S^{\nu} \xi^{\mu}
\]

One of the timelike normals to the boundary of the hypersurface \( \Sigma \) is orthogonal to the Killing vector \( \xi^\mu \) (which is spacelike). If \( \Sigma \) is chosen tangent to the Killing vector so that the second normal to its boundary is also orthogonal to \( \xi^\mu \), then the last two terms of (119) vanish after integration over \( \Sigma \).

In that case, the Katz superpotential becomes covariant and so can be defined at any point of the manifold, it becomes the covariant Komar superpotential.

On the other hand, the total Noether current \( J_\xi^\mu \) (see definition (34)) becomes in the Killing case just \( \xi^\mu L \) (remember that \( \delta_\xi g_{\mu\nu} = 0 \)). When
integrated over $\Sigma$ this term will also vanish (remember the orthogonality between $\xi^\mu$ and the normal of $\Sigma$). What we obtained is just that

$$\int_{B_1} \kappa U_\xi = \int_{B_2} \kappa U_\xi$$

which shows the quasilocal nature of the conserved quantity.

The angular momentum of an axi-symmetric spacetime obviously satisfies the above conditions. As was shown by Katz, the one half factor in front of the old Komar expression is welcome because it gives the right (absolute) value for the angular momentum.

Note finally that the existence of a quasilocal conserved charge could have been expected from Kaluza-Klein arguments. In fact, the presence of a global spacelike Killing vector allows us to dimensionally reduce the gravitational Lagrangian along this direction. It is well known that when such a reduction occurs, some abelian gauge field with its associated quasilocal (electric type) charge appears. Thus the quasilocal charge is just the one corresponding to the abelian internal symmetry left after a Kaluza-Klein reduction.

- The proof of positivity of the gravitational mass is also an example where the equation (118) has been used successfully [11] [12]. The integral of $J_\xi$ on $\Sigma$ was shown to be positive and the integral of $U_\xi$ on the black hole horizon to vanish [35]. This implied the positivity of the integral of $U_\xi$ at spatial infinity. We would like to point out here that the superpotential used by Witten [11] (and later in a covariant way by Nester [12]) was explicitly covariant, spinor dependent and reduces to what is called the Nester-Witten form ($\hat{\sigma}_a$ of section 5.1) in the constant spinor case. It has been shown to come from the asymptotic local supersymmetry invariances of N=1 D=4 supergravity [36].

- Finally another historical example where the equation (118) has been used is in the proof of the first law of black hole thermodynamics [37]. The Killing vector is timelike but in vacuum $L \approx 0$ and so $J_\xi$ (Komar current) vanishes in the bulk of a spacelike hypersurface. This implies a relation between the integral of $U_\xi$ at the horizon of the black hole and at spatial infinity which was the first step in the proof of the first law. Note that in that case, the covariant superpotential used was just the original Komar superpotential (multiplied by its historical factor of two). Some improved version of the first law was given by Wald [38].
6 Fluids: a case of constrained gauge parameter

This section is meant to be readable independently of the previous three.

6.1 The theory and the cascade equations associated to its \( sdiff(V_a) \) gauge symmetry

In this last section we will discuss another case where the cascade equations can be useful to find conserved quantities. We will treat the case of non relativistic fluids, (the relativistic case can be studied in a similar way, work in progress), in a Lagrangian formulation in contrast with the more usual Eulerian discussion.

In this context, the basic fields of our theory are the fluid-particle Eulerian coordinates \( x^i(a^a, \tau) \). The cells of fluid are labelled by the \( a^a \) at a given time \( \tau \). The \( i, j, k... \) indices will be used for the laboratory space (called \( x \)-space) whereas the \( a, b, c... \) will be for the internal label space (called \( a \)-space) both with same dimension \( D \). The labelling follows the fluid particles along the dynamics. The labels are the Lagrangian coordinates. The domain of \( a^a \) is a manifold \( V_a \) without boundary (see paper II for boundaries).

The action to extremize is the integral of the following Lagrangian \[14\] over the \( a \)-space:

\[
L = \frac{1}{2} \left( \frac{\partial x^i}{\partial \tau} \right)^2 - e \left( \det \frac{\partial x^i}{\partial a^a}, s(a^a) \right) - \Phi(x^i) \tag{121}
\]

\( \Phi(x^i) \) is the potential of some external force.

Here \( e \) is just the specific internal energy, a given thermodynamic function of \( \det \frac{\partial x^i}{\partial a^a} \) and of the specific entropy \( s \). The important hypothesis here is that the entropy \( s \) depends on the labels but not on the time \( \tau \), it is an adiabaticity or isentropy condition. We could have used the pressure or any other macroscopic thermodynamic variable. We will see in the following that the presence of such a conserved non uniform function breaks the maximal infinite dimensional \( \tau \)-independent relabelling gauge group allowed by the theory, namely \( SDiff(V_a) \), the group of all diffeomorphisms that preserve the \( D \)-volume to (proper)subgroups. The amazing thing is that the number of mesurable physical charges will be dramatically altered by such reductions.

We will not analyse here the equations of motion of this Lagrangian and its relation with the more usual formulation of Euler equation. We just refer
the interested reader to the excellent review [14].

Another way to look at this theory is to say that it is the dynamics of the mapping \( x^i(a^a, \tau) \) from the \( a \)-space onto the \( x \)-space \((a^a \to x^i)\) as a function of the time \( \tau \). We will suppose that this mapping is invertible. This will allow us to come back to the Eulerian description (i.e. velocities of the fluid as functions of the \( x^i \)'s) using the inverse formula \( a^a = a^a(x^i, t) \) (see section 6.3).

The \( a \)-space contains of course a volume form to allow integration, its density has been normalised for convenience to one. Its pullback with the inverse \( x \)-map will then induce a volume form on \( x \)-space with density \( \rho \) which is given by the inverse of the Jacobian of the transformation \( a^a \to x^i \):

\[
\rho = \text{Det} \left| \frac{\partial x^i}{\partial a^a} \right|^{-1}
\] (122)

The labels will be taken so that \( \rho \) gives correctly the mass density of the fluid.

The \( x \)-space admits the Euclidean metric \( \eta_{ij} \). To simplify our example this metric was assumed to be flat but in more general cases it could depend on the \( x^i \) fields. This is the case of general relativistic fluids which can be analysed similarly. Its pullback along the \( x \)-map will induce a metric on the \( a \)-space, which will not be invariant, i.e.,

\[
g_{ab} = \frac{\partial x^i}{\partial a^a} \frac{\partial x^j}{\partial a^b} \eta_{ij}, \quad \frac{\partial g_{ab}}{\partial \tau} \neq 0
\] (123)

Let us first analyse the homentropic case (barotropic if the pressure is taken as the given thermodynamic function) where \( S(a^a) = S_0 = C^t \).

If we make a completely general (internal) coordinate transformation

\[
\delta a^a = \xi^a(a^a)
\] (124)

the Lagrangian (121) will not in general vary by a total derivative; instead, \( \delta L = \xi^a \partial_a L \). Something special happens in general relativity where the volume form is metric compatible of density \( \sqrt{-g} \). The variation of this term provides the missing \( \partial_a \xi^a L \) part which allows to complete the total derivative, in that case, the symmetry group contains all the spatial diffeomorphisms of the internal coordinates denoted by \( \text{Diff}(V_a) \). The absence of such an invariant metric in the \( a \)-space does not allow us to use that trick. However imposing on the gauge parameter to be divergenceless, \( \partial_a \xi^a = 0 \) the
transformation (124) is now a gauge symmetry of the fluid Lagrangian (121),
\( \delta L = \partial_a (\xi^a L) \). The infinite dimensional gauge group is then \( SDiff(V_a) \), the
group of \( \tau \)-independent diffeomorphisms which preserve the volume form.
Let us insist that this has nothing to do with incompressibility which does
not hold here, in fact the incompressible case will be discussed in paper II.
Under such a local transformation, the variation of the fluid field is given
by \( \delta x^i (a^a, \tau) = \xi^a \partial_a x^i \). Let us now apply the cascade machinery to the
Lagrangian (121), equations (8) and (9) give
\[
\partial_\alpha (T_\alpha^a \xi^a) \approx 0
\]
Where the \( \alpha \) indices combine \((\tau, a^a)\) and \( T_\alpha^a := \frac{\partial L}{\partial \partial_x x^i} \partial_a x^i - \delta_\alpha^a L \) is the
canonical energy momentum tensor. The cascade equations associated to
this symmetry are given by:
\[
\partial_\alpha \xi^a T_\alpha^a \approx 0
\]
\[
\xi^a \partial_\alpha T_\alpha^a \approx 0
\]
Since \( \xi^a \) is not arbitrary but constrained by the volume preserving con-
dition (and of course also time independent), the theorem 1 of the end of
section 4.2 is not applicable any more. Thus we cannot say that the Noether
current \( T_\alpha^a \) identically vanishes due to the locality of the symmetry but in-
stead that (see equation (126))
\[
\partial_b \xi^a T_\alpha^b = 0 \quad \Rightarrow \quad T_\alpha^b = \delta_\alpha^b \Lambda
\]
Where \( \Lambda \) is a function (a Lagrange multiplier) which can be explicitly
computed, \( \Lambda = \frac{P}{\rho} - L \), where \( P \) is the pressure and is defined as usual by
\( P = -\frac{\partial e}{\partial \rho} \). Using the definition of \( T_\alpha^b \) we easily see that this implies that
the Lagrangian (121) can depend only on the determinant of \( \frac{\partial x^i}{\partial x^a} \) (i.e. on
the density of the fluid \( \rho \)). With that, equation (127) becomes:
\[
\xi^a (\partial_\tau A_a \approx -\partial_a \Lambda)
\]
where we defined \( A_a := T_a^\tau \), which is just \( \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial a} \eta_{ij} \). For further discussion
in the next subsection, note that this can be rewritten as
\[
\partial_\tau (A_a \xi^a) = -\partial_a (\Lambda \xi^a)
\]
We will use this result to obtain the conserved charges of the theory.

But before that let us consider the case where the entropy is not anymore uniform. Even $s_{\text{diff}}(V_a)$ does not preserve the Lagrangian (121). In fact, its variation is given by $\delta L = -\frac{\partial e}{\partial s} \frac{\partial s}{\partial a} \xi^a$. So the gauge symmetry of the theory is reduced. In fact, we have to look for some $\xi^a \frac{\partial}{\partial x^a}$ such that its Lie derivative on $s(a^a)$ vanishes, i.e.

$$\xi^a \frac{\partial s}{\partial a^a} = 0 \quad (130)$$

This shows that we can only use the $s_{\text{diff}}(V_a)$ vectors which are tangent to the constant entropy $(D-1)$-dimensional hypersurfaces $W_a$. The entropy plays the role of a label and allows us to reduce the dimension of the problem with the associated partial breakdown of diffeomorphisms. The resulting gauge group is thus $SDiff(W_a)$. If there were more than one thermodynamic quantity in the Lagrangian which depended explicitly on the labels, say $p$ of them, then the gauge group would be $SDiff$ on each $D-p$ dimensional invariant set. The extreme case is when all the directions are broken, no more local symmetry remains, the system is then like frozen.

This restriction of $\xi^a$ to be orthogonal to $\partial_a s$ has important consequences on the cascade equation (128). In fact we can now only deduce that

$$(\partial_t A_a + \partial_a A) \approx \lambda \partial_a s \quad (131)$$

Where $\lambda$ is an arbitrary function, the Lagrange multiplier for $s$.

All these observations have very important consequences on the number of conserved charges of our theory as we will see in the subsection 6.3.

### 6.2 Conserved charges and forcing

We saw in the gravitational (or Yang-Mills) discussion that the important charge is the parameter dependent (or gauge invariant) one. In simple words it just means that the physical charges that we will be able to measure in our laboratory are going to be scalars of the gauge group (so, quantities with no floating $^a$ index and no $a^a$ dependence). Let us first integrate equation (129) over all the $a$-space with the supposition that there is no boundary:

$$\partial_t \int_{V_a} A_a \xi^a d^D a = 0 \quad (132)$$
What we have to do now is just to find all the allowed $\xi^a$ as functions of the local fields that satisfy the volume preserving constraint and the $\tau$-independence. The key formula (132) identifies the one dimensional subgroup of the infinite dimensional gauge group whose charge is computed.

Let us say in words what remains to be done: the problem to find gauge invariant charges is now that of finding the diffeomorphism parameters $\xi^a$ that can be constructed in a tensorial way from the physical fields of the theory. This is done in detail and full generality in the next section. Considering the importance of the helicity invariant of Moreau [39] and Moffatt [40] in turbulence, especially in magnetohydrodynamics (dynamo problem, Kolmogorov cascade...) let us exhibit the diffeomorphism in label space that is responsible for helicity, in other words that is such that rigid diffeomorphisms along it have helicity as their canonical Noether charge. Then we will give a general algorithm for impulsively modifying the charge (here the helicity) in a controlled fashion. We shall use components rather than differential forms to be read more widely despite some inevitable heaviness.

It turns out that the vorticity (co)vector which can be expressed in Lagrangian coordinates $\omega^a := (\text{curl} u)^i(\partial a^a/\partial x^i)$ is the simplest possible $\xi^a$ one can think of. It could have been guessed long ago that in order to increase the pseudoscalar density of helicity $u.\omega$ it is necessary to push (or kick) the fluid along its vorticity the resulting change in vorticity however might destroy the effect. In fact it does not, we shall presently show that pushing along the vorticity and proportionnally to it, that is only along vortex bodies is optimum.

We would like now to propose as a general result the

RULE: If there are a global or rigid Noether symmetry and the associated charge given with $x$ representing the field variables by $\delta x = \xi(x)$ and $Q = \int J_\xi$ from eqs. (23) (129), then the change of $Q$ under the impulsive forcing at some time $t$

$$\delta x = 0, \delta u = \xi(x)$$

is precisely equal to

$$\delta Q = (\partial^2 L/\partial u \partial u).\xi.\xi$$

This is a positive quantity in view of the positivity of the acceptable kinetic terms.

It is important to realise that the time independent symmetry variation dictates the form of a time dependent kick (not a symmetry anymore) along itself that does increase the charge (with the proper sign). This is obvious
for linear momentum and translation invariance but quite general. Note that dropping the surface term is allowed for any spatial symmetry; for the energy or boost charges a separate analysis is required.

Let us now consider a simple example. To create helicity let us take a vortex ring with vorticity concentrated for definiteness on the core, helicity is zero by parity. Then let us force it to rotate along itself in effect creating a second vorticity distribution through it clearly this is a situation of knotted vortex lines see [40] at the origin of helicity.

This can be adapted to 2 dimensions and leads to control of enstrophies. Let us recall that the first enstrophy is quite important in Kolmogorov’s cascade. We shall develop the applications of this general technique in further papers.

6.3 Explicit “abelian” symmetries

The best way to do that is to rewrite everything in differential form notation and we shall follow the review [41]:

- First remember that there is no available invariant metric in the $a$-space and that the Hodge dual operation is not allowed.
- There exists a $(D-1)$-form associated to $\xi^a$, namely $\xi = i_\xi \mu = \frac{1}{(D-1)!} \epsilon_{ab_2...b_D} \xi^a \, db_2 \wedge ... \wedge db_D$, where $\mu$ is the volume $D$-form of the $a$-space. The volume preserving condition is then just the closure of $\xi$, $d\xi = 0$, or locally, $\xi = d\zeta$, where $\zeta$ is some $(D-2)$-form to be determined. If there exists some non uniform thermodynamic function, say for instance the entropy $s$, then $\xi$ must satisfy the additional constraint $\xi \wedge ds = 0$, see equation (130).
- $A_a$ will now be written as a 1-form $A := A_a dx^a$. So equation (128) becomes $\partial_\tau A \approx -d\Lambda$ which implies that $\partial_\tau dA \approx 0$ (this is nothing but the Eckart law [13] derived also with Noether arguments by Salmon [14]). In the nonhomentropic case, equation (131) becomes $\partial_\tau dA = d\Lambda \wedge ds$. Thus the supposition that the entropy is time independent implies that the conserved quantity will be now $\partial_\tau dA \wedge ds = 0$.
- The conserved charge (132) is in this notation $Q(\xi) = \int A \wedge \xi$, with the extra condition (the $\tau$ independence of $\xi$) that $\partial_\tau \xi = 0$.

In summary, the problem of finding a conserved charge has been reduced to the problem of finding all $(D-1)$-forms $\xi$ which satisfy

- Closure, $d\xi = 0$. 

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- Time independence, $\partial_\tau \xi = 0$.
- preservation of the thermodynamic function(s), if any, $\xi \wedge ds = 0$.

To solve first the time independence condition we can observe that if the only available field of our theory is $A$ then any $\tau$ independent local function (or form) has to depend only on $dA$ in the homentropic case and on $dA \wedge ds$ in the nonhomentropic one. $dA$ is just the Lagrangian version of vorticity. So, the local, time-conserved,

- functions are
  - the entropy or any other thermodynamical given function, if any.
  - only in the even dimensional homentropic case ($D = 2n$, $n \geq 1$),
    \[ f_0 = \frac{dA \wedge \ldots \wedge dA}{\mu} \] (n number of $dA$).
  - only in the odd dimensional nonhomentropic case ($D = 2n + 1$, $n \geq 1$),
    \[ g_0 = \frac{ds \wedge dA \wedge \ldots \wedge dA}{\mu} \] (n number of $dA$).

Of course, any function of these functions is again a $\tau$ conserved function. Note in particular that in the homentropic (uniform entropy) odd dimensional case, there is simply no $\tau$ conserved local function.

- 1-forms are $df$, where $f$ is any of the above time conserved functions. In fact, the only other one form we have at our disposal is $A$ but it does not satisfy $\partial_\tau A = 0$. Again, any conserved function times one of these one forms is also conserved.

- 2-forms are any wedge product of any two of the above 1-forms and $dA$ in the homentropic case.

- 3-forms are any wedge product of the above 1-forms and 2-forms plus $dA \wedge ds$ in the case of nonhomentropic fluid.

- $p$-forms, $p \geq 4$ any wedge product of the above forms.

Let us show how the analysis continues for specific cases:

- The (homentropic) odd dimensional case, $D = 2n + 1$, $n \geq 1$:
  We are in the case where no conserved function exist. From the above discussion, we can see that only even conserved forms exist. Fortunately $\xi$ should be an even closed form (of degree $2n$). Then the only local possibility is $\xi = dA \wedge \ldots \wedge dA$ ($n$ times). The conserved charge is thus, following equation (132):
\[ Q = \int_{V_a} A \wedge (dA \wedge \ldots \wedge dA) \] (133)

The Eulerian version of these results already exists in the literature [42] [43] and are just generalisations of the helicity. The relation will be given in the last subsection.

- The (homentropic) even dimensional case, \( D = 2n \), \( n \geq 1 \): The most important difference with the above case is that now we have an infinite number of \( \tau \)-conserved functions, namely \( f_0 \) and any function of it. However, the only non vanishing conserved 2-form is \( dA \) because if \( f \) and \( g \) are two arbitrary functions of \( f_0 \), then 
  \[ df \wedge dg = \frac{\partial f}{\partial f_0} \frac{\partial g}{\partial f_0} df_0 \wedge df_0 = 0. \]
  We then can look for all the odd \((2n - 1)\)-dimensional closed forms \( \xi \) which can be constructed from the above ingredients. It is not difficult to see that the most general possibility is 
  \[ \xi = df \wedge dA \wedge \ldots \wedge dA \quad (n \text{-times}), \]
  where again \( f \) is any function of \( f_0 \). Then the conserved charges are
  \[ Q_f = \int_{V_a} A \wedge (df \wedge dA \wedge \ldots \wedge dA) \] (134)

Note that if we integrate by parts, the above definition is nothing but 
  \[ Q_{\bar{f}} = \int_{V_a} f \cdot f_0 \mu = \int_{V_a} \bar{f} \mu, \]
  where \( \bar{f} \) is another function of \( f_0 \) given by \( f \cdot f_0 \). Since \( f \) is arbitrary, \( \bar{f} \) is arbitrary too. The number of charges is infinite.

There is an intuitive way to understand the meaning of this infinity. If we take a Dirac \( \delta \) function for \( \bar{f} \) (say \( \delta(f_0 - C) \) where \( C \) is an arbitrary constant) the conserved charges \( Q_f \) guarantee that each surface of constant \( f_0 \) is conserved. The fluid looks like an infinite number of \((D-1)\)-dimensional subsystems at each slice of constant \( f_0 \). Then the arbitrary function \( \bar{f} \) gives the relative weight to assign to each hypersurface of constant \( f_0 \) when we compute a conserved charge as an integral over the whole \( D \)-dimensional label space.

An Eulerian translation of this result (which is a generalisation of what is called enstrophy) will be given in section 6.3.

- The (nonhomentropic) odd dimensional case, \( D = 2n + 1 \), \( n \geq 1 \): Now we have an extra data which is a \( \tau \) conserved function, namely the specific entropy \( s \) (again, the same is true for any other macroscopic thermodynamic function). The most important consequence of that is that now we can construct the analogue of \( f_0 \), namely
  \[ g_0 = \frac{ds \wedge dA \wedge \ldots \wedge dA}{\mu}, \]
  which is time conserved. With this extra data, the most general \( 2n \) closed form \( \xi \) which in addition satisfies the symmetry constraint \( \xi \wedge ds = 0 \) is simply

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\[ \xi = r(g_0, s) \, dg_0 \wedge ds \wedge dA \ldots \wedge dA \ (n - 1 \text{ times}) \] where \( r \) is any function of \( g_0 \) and \( s \). The doubly infinite number of conserved charges are then

\[ Q_r = \int_{V_n} A \wedge (r \, dg_0 \wedge ds \wedge dA \ldots \wedge dA) \quad (135) \]

This can be rewritten more simply after an integration by part as \( Q_{\bar{r}} = \int_{V_n} \bar{r} \mu \), where \( \bar{r} := \int_{g_0} f(\tilde{g}, s) d\tilde{g} \). \( g_0 \) is any function of \( g_0 \) and \( s \). We will see in section 6.3 that the Eulerian version of \( g_0 \) is just what is usually called the potential vorticity of Ertel \([44]\) in the three dimensional case.

The amazing point here is that in an odd dimensional space, the presence of a non uniform thermodynamic variable breaks the maximal symmetry group (namely SDiff(\( V_n \))) with only one conserved invariant charge to a proper subgroup (sdiff(\( W_a \))) (where \( W_a \) is again a \((D - 1)\)-dimensional manifold) with an infinite number of conserved charges again invariant under relabeling.

Note that the last formula (135) depends on the choice of the two parameter function \( \bar{r} \). What really happens is that the presence of a non uniform entropy breaks our odd \( D \)-dimensional fluid theory into an infinite number of even \((D - 1)\)-dimensional ones. Take the example of a three dimensional fluid. When a gradient of entropy exists we can define a bidimensional theory on each slice of constant entropy. As we saw in the previous example, each of these bidimensional systems can again be interpreted as an infinite number of one dimensional systems, each of constant \( g_0 \). The \( \bar{r} \) function gives the relative weight we assign to each one dimensional system when we compute some conserved charge.

As we can see in the following the same happens from even to odd dimensions. Of course all the above equations and definitions can be translated without major work to the Eulerian formalism.

- The (nonhomentropic) even dimensional case, \( D = 2n, n \geq 1 \):

Now we cannot use the function \( f_0 \) to construct conserved quantities because it is not any more \( \tau \)-conserved (remember that now \( \partial_\tau dA = d\alpha \wedge ds \)).

The only time conserved function that we can use to construct \( \xi \) is thus \( s \). We again have to look for all gauge parameter \( \xi \) \((D - 1)\)-forms that satisfy the \( \tau \) independence constraint \((\partial_\tau \xi = 0)\), the divergenceless constraint \((d\xi = 0)\) and the constraint \( \xi \wedge ds = 0 \). The result is that the only solution is \( \xi = dF \wedge dA \ldots \wedge dA \) (where the number of \( dA \) factors is just \((n - 1)\) and \( F \) is an arbitrary function of the entropy \( s \)). The associated charge is
\[ Q_F = \int_{V_a} A \wedge (dF \wedge dA \ldots \wedge dA) \] (136)

Again we find a single infinity of charges. The arbitrariness of the function \( F \) has a similar meaning as the arbitrariness of the \( \vec{r} \) function of the previous example. If for example we look for a four dimensional nonhomentropic fluid, then what really happens is that the theory is just an infinite number of three dimensional theories living in some slice of constant entropy. We can then compute the helicity for each of them which will be independently conserved. The total charge (136) is just each of these helicities weighted by an arbitrary function (for example a Dirac \( \delta \) function if we just want to look for the helicity of a specific slice). By analogy with the preceding example we will call these quantities the potential helicities.

In the case where we have more than one macroscopic thermodynamic function in our theory, the analysis, is completely straightforward if we follow the above examples. In fact the rule seems to be that if we have \( p \) thermodynamic variables the gauge group is broken from \( \text{SDiff}(V_a) \) to \( \text{SDiff}(W_a) \) (\( W_a \) is a \((D-p)\)-dimensional manifold), with an infinite number of conserved charges if the effective dimension \( D - p \) is even and just one in the odd case for each homogeneous submanifold of fluid (with uniform thermodynamic variable).

It is remarkable that the same group of volume preserving diffeomorphisms is absolutely essential in membrane theory and its application to define the mysterious M-theory by discretizing the surface and the group is under intensive study. It is important to note that all the even dimensional flows share this feature of an infinite number of invariants, it is not a particularity of two dimensions. This may be encouraging for M-theory that requires us absolutely to consider less familiar objects than strings or even membranes.

6.4 The Eulerian framework

The purpose of this last section is just to give the translation of the above results in the more usual Eulerian language. For that we just need to do an ordinary change of variables in the integral formulas between the \( a^a \) variables and the \( x^i \) variables.

Before starting with some precise example, let us remember the general points:
- The change of variables will be $a^a \rightarrow a^a(x^i,t)$, and $\tau = t$. $t$ is just the Eulerian time, $\tau$ is the material or Lagrangian time. A conserved charge has to be invariant under the $\frac{\partial}{\partial \tau}$ derivative, which is in Euler time just the total derivative $\frac{D}{Dt} := \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i}$. Of course $u^i$ is the Eulerian velocity.

- In the beginning we fixed the volume form of the $a$-space to one. Now, the corresponding volume form in the Euler framework is just the inverse of the Jacobian of the transformation $a \rightarrow x$. We then must be careful to use Levi-Civita tensor densities and in the following the $\epsilon_{ij...}$ or $\epsilon_{abc...}$ symbols will be always constant. The density $\rho$ will then appear explicitly.

We are now ready to translate all the results found in section 6.2 and 6.3:

- The total charge, equation (132) becomes
  \[ \partial_\tau \left[ Q(\xi) = \int_{V_x} u_i \xi^i \rho \, d^D x \right] = 0 \]  
  (137)
  Where $u_i = A_a \frac{\partial a^a}{\partial x^i}$ is the Eulerian velocity and $\xi^i = \xi^a \frac{\partial a^a}{\partial x^i}$.

- The helicity, equation (133) becomes
  \[ Q = \int_{V_x} \epsilon_{ijk_1...jnk_n} u_i \partial_j u_{k_1} \cdots \partial_j u_{k_n} \, d^{2n+1} x \]  
  (138)
  This charge was found by other methods by Tartar & Serre [42] and by Khesin & Chekanov [43] as a $D$-dimensional generalisation of the usual helicity [39] [40].

- The enstrophies, equation (134), become
  \[ Q_f = \int_{V_x} \tilde{f}(f_0) \rho \, d^{2n} x \]  
  (139)
  Where $f_0 = \rho^{-1} \epsilon_{ijj_1...jn} \partial_{j_1} u_{j_1} \cdots \partial_{j_n} u_{j_n}$ and $\tilde{f}(f_0)$ is of course an arbitrary function of $f_0$.

  Again this result can be found in the work of Serre [42] and Khesin & Chekanov [43].

- The potential vorticities, equation (135) becomes
  \[ Q_r = \int_{V_x} \tilde{r}(s,g_0) \rho \, d^{2n+1} x \]  
  (140)
  Where $g_0 = \rho^{-1} \epsilon_{ijj_1...jn} \partial_{j_1} s \partial_{j_1} u_{k_1} \cdots \partial_{j_n} u_{k_n}$ was already known in three dimensions as the potential vorticity [44], see for example [14]. $\tilde{r}$ is
an arbitrary function of \( g_0 \) and \( s \). Remember that the double infinity of conserved charges was explained in section 6.3. It is just an infinite number of \((D-2)\)-dimensional subsystems of constant entropy \( s \) and constant potential vorticity \( g_0 \) weighted by \( \bar{r} \). 

- The potential helicity, equation (136) becomes 

\[
Q_F = \int_{V_0} \epsilon^{ijkl\ldots} \partial_i F_{jkl} \partial_{k_1} u_{l_1} \ldots \partial_{k_n} u_{l_n} d^{2n+1}x \quad (141)
\]

Where \( F \) is an arbitrary function of \( s \). Here, the number of charges is just infinite. The reason is that our fluid, as in the previous example can be viewed as an infinite number of \((D-1)\)-dimensional subsystems of constant entropy weighted by \( F(s) \), whose odd dimensional character implies the existence of only one conserved charge, the helicity.

In conclusion let us comment on the impressive generality of the Noether approach, it did in fact impress her contemporaries. A poll among colleagues has shown that very few did actually know the full contents of the paper, and even fewer read it.

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