Exactly solvable models of two-dimensional dilaton gravity and quantum eternal black holes

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New approach to exact solvability of dilaton gravity theories is suggested which appeals directly to structure of field equations. It is shown that black holes regular at the horizon are static and their metric is found explicitly. If a metric possesses singularities the whole spacetime can be divided into different sheets with one horizon on each sheet between neighboring singularities with a finite value of dilaton field (addition horizons may arise at infinite value of it), neighboring sheets being glued along the singularity. The position of singularities coincide with the values of dilaton in solutions with a constant dilaton field. Quantum corrections to the Hawking temperature vanish. For a wide subset of these models the relationship between the total energy and the total entropy of the quantum finite size system is the same as in the classical limit. For another subset the metric itself does not acquire quantum corrections. The present paper generalizes Solodukhin’s results on the RST model in that instead of a particular model we deal with whole classes of them. Apart from this, the found models exhibit some qualitatively new properties which are absent in the RST model. The most important one is that there exist quantum black holes with geometry regular everywhere including infinity.

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I. INTRODUCTION

Two-dimensional theories of gravity count on better understanding the role of quantum effects in black hole physics in a more realistic four-dimensional case. In particular, new insight was gained due to reducing the problem of black hole evaporation to solving differential equations of the semiclassical theory [1]. However, these equations remain too complicated in the sense that exact solutions cannot be found. This obstacle was overcome in the approach based on modifying the form of the semiclassical action in such a way that solvability is restored. The well-known example is the Russo-Susskind-Thorlacius (RST) model [2]. In particular, this enabled one to give exhausting analysis of either geometry or thermodynamics of eternal black holes in the framework of RST dilaton gravity [3].

Examples of exactly solvable theories of dilaton gravity were discussed in [4], [5]. As was shown by Kazama, Satoh and Tsuichiya (KST), these models as well as the RST one can be found as particular cases of the unified approach [6] based on the symmetries of the nonlinear sigma model. A number of other exactly solvable models were suggested later [7], [8], [9], [10].

The aim of the present paper is two-fold. First, we suggest new approach to finding criteria for exact solvability and establish its equivalence to that of [6]. We also show that our approach encompasses all particular models mentioned above. Second, we solve the field equations in a closed form and study general properties of the found spacetimes. In this point we generalize the Solodukhin’s results for RST eternal black holes. The essential feature of models considered below consists in that we specify not the explicit dependence of the action coefficients on dilaton but, rather, relationship between these coefficients. This means that instead of one or several particular exactly solvable models we deal at once with the whole class of them.

The paper is organized as follows. In Sec. II we describe our approach and derive the basic equation which selects exactly-solvable models of two-dimensional dilaton gravity among all possible forms. We find the relation between action coefficients and demonstrate
how it enables one to cast the theory into the Liouville form. We also briefly show that the
this relation is equivalent to that obtained in [6] in a quite different approach. Further, we
prove that all black holes with a regular geometry are static and find their metric explicitly.
In Sec. III we discuss properties of found solutions - spacetime structure, thermodynamics,
the existence of a special class of solutions with a constant dilaton field value, etc. In Sec.
IV we show that all exactly solvable models mentioned above fall in the found class. In Sec.
V we summarize the main features of our approach and properties of obtained solutions and
outline some perspectives for future researches.

II. BASIC EQUATIONS

A. Conditions of exact solvability

Let us consider the system described by the action

$$ I = I_0 + I_{PL} $$

(1)

where

$$ I_0 = \frac{1}{2\pi} \int_M d^2x \sqrt{-g} \left[ F(\phi)R + V(\phi)(\nabla \phi)^2 + U(\phi) \right] + \frac{1}{\pi} \int_{\partial M} ds k F(\phi) $$

(2)

Here the boundary term with the second fundamental form $k$ makes the variational problem
self-consistent, $ds$ is the line element along the boundary $\partial M$ of the manifold $M$.

$I_{PL}$ is the Polyakov-Liouville action [12] incorporating effects of Hawking radiation and
its back reaction on the black hole metric for a multiplet of N scalar fields. It is convenient
to write it down in the form [3], [13]

$$ I_{PL} = -\frac{\kappa}{2\pi} \int_M d^2x \sqrt{-g} \left[ \frac{(\nabla \psi)^2}{2} + \psi R \right] - \frac{\kappa}{\pi} \int_{\partial M} \psi k ds $$

(3)

The function $\psi$ obeys the equation

$$ \Box \psi = R $$

(4)
where $\Box = \nabla_\mu \nabla^\mu$, $\kappa = N/24$ is the quantum coupling parameter.

We imply that all coefficients $F, V, U$ in (2) may, in general, contain terms with $\kappa$. For instance, in the RST model $F = e^{-2\phi} - \kappa \phi$.

Varying the action with respect to metric gives us ($T_{\mu\nu} = 2 \frac{\delta I}{\delta g_{\mu\nu}}$):

$$T_{\mu\nu} \equiv T^{(0)}_{\mu\nu} + T^{(PL)}_{\mu\nu} = 0$$

(5)

where

$$T^{(0)}_{\mu\nu} = \frac{1}{2\pi} \left\{ 2(g_{\mu\nu} \Box F - \nabla_\mu \nabla_\nu F) - Ug_{\mu\nu} + 2V\nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} V(\nabla \phi)^2 \right\},$$

(6)

$$T^{(PL)}_{\mu\nu} = -\frac{\kappa}{2\pi} \left\{ \partial_\mu \psi \partial_\nu \psi - 2\nabla_\mu \nabla_\nu \psi + g_{\mu\nu} \left[ 2R - \frac{1}{2}(\nabla \psi)^2 \right] \right\}$$

(7)

Variation of the action with respect to $\phi$ gives rise to the equation

$$RF' + U' = 2V\Box \phi + V'(\nabla \phi)^2,$$

(8)

where prime denotes derivative with respect to $\phi$.

In general, eqs. (4) - (8) cannot be solved exactly. In particular, the auxiliary function $\psi$ is a rather complicated nonlocal functional of dilaton field $\phi$ and its derivatives, the explicit form of which cannot be found. The key idea for finding exactly solvable models consists in two assumptions. First, we select among all variety of models such a subset for which the above equations admit the solution $\psi = \psi(\phi)$, i.e. a local connection between $\psi$ and $\phi$.

As was shown by Solodukhin [3], for eternal black holes in the RST model $\psi = -2\phi$. Now the dependence in question can be, generally speaking, nonlinear. Second, we impose such a constraint on the coefficients of the action that enables us to single out the terms with derivatives of $\phi$ with respect to coordinates. As a result, instead of solving the original (rather complicated equations) it is sufficient to ensure the cancellation of the coefficients at $\Box \phi$ and $(\nabla \phi)^2$ which themselves already do not contain derivatives with respect to coordinates. It turns out that such a constraint is not very tight and includes a large variety of models which may be of physical interest.
Inasmuch as the auxiliary function $\psi$ can be expressed in terms of $\phi$ directly, the action $I_0$ and the Polyakov-Liouville action are combined in such a way that field equations (5) - (7) can be formally obtained from the action $I_0$ only but with the ”renormalized” coefficients which receive some shifts: $F \to \tilde{F}, V \to \tilde{V}$ where

$$\tilde{F} = F - \kappa \psi,$$  \hspace{1cm} (9)

$$\tilde{V} = V - \frac{\kappa}{2} \psi'^2,$$  \hspace{1cm} (10)

Taking the trace of eq. (5) we get the relation

$$U = \Box \tilde{F},$$  \hspace{1cm} (11)

Then eq. (8), with eqs. (4), (9) - (11) taken into account, reads

$$A_1 \Box \phi + A_2 (\nabla \phi)^2 = 0,$$

$$A_1 = (u - \kappa \omega) \psi' + \omega u - 2V,$$  \hspace{1cm} (12)

$$A_2 = (u - \kappa \omega) \psi'' + \omega u' - V',$$

where $\omega \equiv U'/U, \ u \equiv F'$. For arbitrary coefficients $A_1(\phi), A_2(\phi)$ eq.(12) cannot be solved exactly. This can be done, however, if we demand that both coefficients in eq.(12) turn into zero. Such a demand represents the sufficient condition for eq.(12) to be satisfied. Then it follows from eq. $A_1 = 0$ that

$$\psi' = \frac{2V - \omega u}{u - \kappa \omega},$$  \hspace{1cm} (13)

which enables us to find at once $\psi$ in terms of known functions $u, V, \omega$ by direct integration. Demanding that both equations $A_1 = 0$ and $A_2 = 0$ be consistent with each other, we differentiate eq. $A_1 = 0$ and compare the result with $A_2 = 0$. Then we have

$$u'(2V - \omega u) + u(\omega u' - V') + \kappa(\omega V' - 2V \omega') = 0$$  \hspace{1cm} (14)

Thus, we made some selections among all possible models. Eq.(14) is the only constraint on the relationship between these three coefficients that leaves enough freedom in choosing
a model. Both eqs. (13), (14) represent direct consequences of our assumptions $\psi = \psi(\phi)$, $A_1 = A_2 = 0$ and do not hold true in an arbitrary model. In particular, for the original form of the action in the string-inspired gravity [14] $F = e^{-2\phi}, V = 4e^{-2\phi}, \omega = -2$ these equations are satisfied in the zero order in $\kappa$ only. However, for the RST action

$$F = e^{-2\phi} - \kappa \phi, V = 4e^{-2\phi}, \omega = -2$$

and this model does obey eq. (14). Then the integration of (13) leads to the relationship $\psi = -2\phi + \text{const}$ that agrees with [3].

It follows from eq. (14) that the function $\omega$ expressed in terms of $V$ and $u$ reads

$$\omega = \frac{u - D\sqrt{u^2 - 2V\kappa}}{\kappa}$$

Reverting this formula, we have

$$V = \omega(u - \frac{\kappa \omega}{2}) + C(u - \kappa \omega)^2$$

where $C = (2\kappa)^{-1}(1 - D^{-2})$. We must assume that the constant $D \rightarrow 1$ when $\kappa \rightarrow 0$ in order to have the well-defined classical limit. In what follows we mainly restrict ourselves to the simplest case $C \equiv 0, D \equiv 1$, when

$$\omega = \frac{u - \sqrt{u^2 - 2V\kappa}}{\kappa}, V = \omega(u - \frac{\kappa \omega}{2})$$

**B. Comparison with KST action**

In eq. (3.18) of Ref. [6] the following relation between different action coefficients was obtained:

$$a_q(1 - \frac{\kappa}{2}W_q) + \kappa W_{qq}(a + \frac{1}{2}W_q) = 0$$

where (in our notations) $W = \int d\phi \omega$ and by definition $V = aF^2 + F'\omega \equiv au^2 + u\omega$. Substituting these expressions into (19) we obtain after simple but rather lengthy calculation the equation which coincides with our eq. (14) exactly. It was pointed out in [6] that results
of [4], [5] are contained in the general formula following from (19). In our scheme the models of Ref. [5] (which include those of [4] as particular cases) follow from (18) if one writes down
\[ V = 4e^{-2\phi}[1 + h(\phi)], \quad u = -2e^{-2\phi}[1 + \tilde{h}]. \]

KST approach has the advantage of elucidating the hidden symmetry of the action in terms of a nonlinear \( \sigma \) model. On the other hand, the present approach, in our view, is much simpler in that it operates directly with the original action coefficients and demonstrates explicitly the origin of solvability as the cancellation of coefficients at \((\nabla \phi)^2\) and \(\Box \phi\) in (12). It also enables us to establish the staticity of black hole solutions (in the absence of external origins and matter fields) and find their explicit form. It is this issue that we now turn to.

C. Form of the metric

Let us return to general formulae without specifying the gauge. With eq. (11) taken into account, the field equations (5) - (7) can be rewritten in the form

\[ [\xi_1 \Box \phi + \xi_2 (\nabla \phi)^2] g_{\mu \nu} = 2(\xi_1 \nabla_\mu \nabla_\nu \phi + \xi_2 \nabla_\mu \phi \nabla_\nu \phi) \]

where \( \xi_1 = \tilde{F}', \quad \xi_2 = \tilde{F}'' - \tilde{V}. \) Let us multiply this equation by the factor \( \eta \) chosen in such a way that \((\xi_1 \eta)' = \xi_2 \eta\) whence \( \eta = \exp(-\int d\phi \tilde{V}/\tilde{F}'). \) Then eq. (20) turns into

\[ g_{\mu \nu} \Box \mu = 2\nabla_\mu \nabla_{\nu} \mu \]

where by definition \( \mu' = \xi_1 \eta. \) This equation takes the same form as eq.(2.24) from [3] and entails the same conclusions about properties of the geometry. It is convenient to choose the space-like coordinate as \( x = \mu(\phi)/B \) where \( B \) is some constant. Then it follows from eq.(21) that the metric takes the Schwarzschild-like form and is static:

\[ ds^2 = -gdt^2 + g^{-1}dx^2 \]

In this gauge \( R = -\frac{d^2g}{dx^2}. \) Substituting it into eq.(4) we get after integration:

\[ g = A \int_{\phi_1}^{\phi_2} d\phi \frac{d\mu(\phi)}{d\phi} e^{\psi(\phi)} - \psi(\phi) \]

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where \( A \) is a constant. Here it is supposed that a spacetime has an event horizon at \( x = x_h, \phi = \phi_h \).

Now we will see that the obtained expressions for the metric can be simplified further. Substituting eq. (17) into (23), (24) we find after some rearrangement that

\[
\alpha = -\frac{\tilde{V}}{\tilde{F}^2},
\]

and

\[
g = aC^{-1}(e^{CH} - e^{CH_h})e^{-\psi},
\]

where

\[
U = 4\lambda^2 \exp(\int d\phi \omega)
\]

and the limit of integration in the integral \( \int d\phi \omega \) is chosen equally in all formulae above.

In the case \( C = 0 \) this gives us

\[
g = (\tilde{F} - \tilde{F}_h) \exp[-\int d\phi \omega],
\]

where we put \( a = 1 \) for definiteness. For the RST model (15) we obtain in accordance with

\[
\mu = \phi - \frac{\varphi}{2}e^{2\phi}, \quad B = -2\lambda.
\]

In the above formulae it was tacitly assumed that the function \( \psi \) as well as the metric itself is regular at the horizon (when the temperature is equal to its Hawking value). As explained in detail in Ref. [3] (Sec. 2B), such a choice of boundary conditions corresponds to a black hole in equilibrium with Hawking radiation but does not describe formation of a hole from a flat space due to incoming matter.
D. Liouville theory

The suggested approach enables one to cast the original action into the Liouville form in a very simple way. In the conformal gauge

\[ ds^2 = -e^{2\rho} dx^+ dx^- \]  

(28)

Bearing in mind that in this gauge the Polyakov action [12] reads

\[ I_{PL} = -\frac{2\kappa}{\pi} \int d^2x \sqrt{g} (\nabla \rho)^2 \]  

and integrating in (2) the term with curvature by parts we have (we omit now all boundary terms)

\[ I = \frac{1}{2\pi} \int d^2x \sqrt{g} [V(\nabla \phi)^2 + 2\nabla \rho \nabla F + 2\kappa(\nabla \rho)^2 + 4\lambda^2 e^\eta], \]  

(29)

where by definition \( \eta = \int d\phi \omega \). Let now the potential \( V \) obey the condition (17). Then the action can be rewritten in the form

\[ I = \frac{1}{\pi} \int d^2x \sqrt{g} \{ \kappa ((\nabla \chi)^2 - (\nabla \Omega)^2(1 - 2C\kappa)) + 2\lambda^2 e^{2(\chi - \Omega - \rho)} \} \]  

(30)

where \( H = F - \kappa \eta \). Now introduce new fields \( \Omega \) and \( \chi \) instead of \( \phi \) and \( \rho \) according to \( H = 2\kappa \Omega, \eta(\phi) = 2(\chi - \Omega - \rho) \). Then after simple rearrangement we obtain

\[ I = \frac{1}{\pi} \int d^2x \sqrt{g} \{ 2\kappa [\partial_+ \Omega \partial_- \Omega (1 - 2C\kappa) - \partial_+ \chi \partial_- \chi] + \lambda^2 e^{2(\chi - \Omega - \rho)} \} \]  

(31)

Using the explicit formula for the conformal gauge (28) we obtain the action

\[ I = \frac{1}{\pi} \int d^2x \{ 2\kappa[(\partial_+ \Omega \partial_- \Omega (1 - 2C\kappa) - \partial_+ \chi \partial_- \chi] + \lambda^2 e^{2(\chi - \Omega - \rho)} \} \]  

(32)

which in the case \( C = 0 \) takes the familiar form. Meanwhile, it is instructive to rederive in the conformal gauge the obtained exact solutions with an arbitrary \( C \neq 0 \). In this gauge the equations of motion which follow from (32) read

\[ 2\kappa(1 - 2C\kappa)\partial_+ \partial_- \Omega = -\lambda^2 e^{2(\chi - \Omega)}, \]  

(33)

\[ 2\kappa \partial_+ \partial_- \chi = -\lambda^2 e^{2(\chi - \Omega)} \]
As usual, they should be supplemented by the constraint equations $T_{++} = T_{--} = 0$. The expressions for the classical part of $T_{++}$ and $T_{--}$ follow directly from (6). The formula for the quantum contribution can be obtained from the conservation law and conformal anomaly and has the form

$$T_{\pm\pm}^{(PL)} = \frac{2\kappa}{\pi} [(\partial_{\pm\rho})^2 - \partial_{\pm}^2 \rho + t_{\pm}]$$  \hspace{1cm} (34)

The functions $t_{\pm}$ depend on the choice of boundary conditions. For the Hartle-Hawking state we deal with $t_{\pm} = 0$. With the potential $V$ from (17) the constraint equations take the form

$$-\partial_\pm^2 F + 2\partial_\pm \rho \partial_\pm \rho + (\partial_\pm \phi)^2 V + 2\kappa [(\partial_{\pm\rho})^2 - \partial_{\pm}^2 \rho] = 0$$ \hspace{1cm} (35)

As seen from eq.(33), it is convenient to use the gauge in which $\chi = \Omega (1 - 2C\kappa)$, so $\chi - \Omega = -CH$. Then after some algebra we find that the factor $(1 - 2C\kappa)$ in a remarkable way is singled out in constraints and we arrive at equations obtained from (35) and (33)

$$\partial_\pm^2 H + C(\partial_{\pm} H)^2 = 0, \hspace{1cm} (36)$$

$$(1 - 2\kappa C) \partial_\pm \partial_- H = -\lambda^2 e^{-2CH}$$

It is convenient to make the substitution $z = e^{C(H - H_h)}$ where $H_h$ is some constant. Then the first equation in (36) turns into $\partial_\pm^2 z = 0$ whence $z = bx^+x^- + d$ (linear terms can always be removed by shifts in coordinates). By substitution in the second eq.(36) we obtain the restriction $bd = -\tilde{\lambda}^2$ where $\tilde{\lambda}^2 = \lambda^2 C(1 - 2\kappa C)^{-1}e^{-2CH_h}$. Let us choose $d = 1, \ b = -\tilde{\lambda}^2$ and make transformation to new coordinates according to $\tilde{\lambda}x^+ = e^{\lambda^+}, \ -\tilde{\lambda}x^- = -e^{-\lambda^-}$. Then the metric (28) turns into

$$ds^2 = -g d\sigma^+ d\sigma^-$$ \hspace{1cm} (37)

where

$$g = e^{-\psi} \frac{(e^{CH} - e^{CH_h})}{e^{CH_h}}, \ \psi = (1 - 2\kappa C) \int d\phi \omega + 2CF$$ \hspace{1cm} (38)
To rewrite the metric in the Schwarzschild gauge (22), let us introduce coordinates \( \sigma^\pm = t \pm \sigma \) and \( d\sigma = dxg^{-1} \). Then

\[
x'_{i}(\phi) = \frac{CH'}{2\lambda} e^{-CH_h-\lambda C-(1-\kappa C)} \int d\phi \omega \tag{39}
\]

We see that (38), (39) coincide with (25) if the constants are identified according to 
\[a = Ce^{-CH_h}, \quad B = 2\lambda C^{-1}(1-2\kappa C)e^{CH_h} \].

### III. PROPERTIES OF SOLUTIONS

#### A. Structure of spacetime

Now we can make general conclusions about the structure of spacetime. Let us restrict ourselves by the case \( C = 0 \). It follows from (27), (26) that the curvature \( R = -\frac{d^2 \tilde{F}}{d\phi^2} \) is equal to

\[
R = \frac{4\lambda^2}{\tilde{F}'} \left[ \frac{\omega(\tilde{F} - \tilde{F}_h)}{\tilde{F}'} \right] \exp(\int^\phi d\tilde{\phi} \omega) \tag{40}
\]

Let, by definition, \( \tilde{F}'(\phi_c) = 0, x_c = x(\phi_c) \). Near \( \phi_c \) the function \( \tilde{F}' \propto \phi - \phi_c, x - x_c \propto (\phi - \phi_c)^2 \) and, in general, \( R \propto (\phi - \phi_c)^{-3} \propto (x - x_c)^{-3/2} \). The exceptional case arises when \( \phi_c = \phi_h \). Then the above expression in square brackets is finite and \( R \propto (\phi - \phi_c)^{-1} \propto (x - x_c)^{-1/2} \). Thus, singularity becomes weaker but does not disappear. Such behavior, found earlier for the RST model [3], is inherent to any model under consideration.

Thus, the metric possesses singularities in the points \( \phi = \phi_c \) where \( \tilde{F}' = 0 \). The spacetime splits into intervals between zeros of \( \tilde{F}' \) which can be viewed as different sheets that generalizes the corresponding feature of the RST model [3]. Within each of them the function \( \tilde{F}' \) does not alter its sign, so the function \( \tilde{F}(\phi) \) is monotonic and the equation \( \tilde{F} = \tilde{F}_h \) has only one root. Then, according to (27), there is only one horizon at \( \phi = \phi_h \) on every sheet between any two singularities with a finite value of \( \phi \). In principle, it may happen that on a sheet between infinity and a singularity nearest to it there exists an additional horizon due to the factor \( e^{-\psi} \) in which case the coordinate \( x \) calculated according to (27) takes,
generally speaking, a finite value in this limit. To obtain the maximally extended analytical
continuation of spacetime, one is led to accept the possibility of complex dilaton field values
[15]. We will not, however, discuss such possibilities further (a more detailed description of
spacetime structure will be done elsewhere [16]). For the RST model there exist only two
sheets but, depending on properties of the function $\tilde{F}(\phi)$, the number of sheets in a general
case can be made arbitrary. Any two neighboring sheets are separated by the singularity
located at $\phi = \phi_c$.

As the singular points of the curvature represent zeros of the function $\tilde{F}'$, the case under
consideration admits solutions regular everywhere, if the function $\tilde{F}'$ does not turn into
zero. Let, for instance, $F = e^{\omega \phi} + \delta \kappa \omega \phi$ where $\delta$ is a pure number and $\omega$ is constant. Then

$$\tilde{F}' = \omega [e^{\omega \phi} + \kappa (\delta - 1)].$$

If $\delta > 1$, the expression in square brackets changes its sign nowhere and does not tend to zero at infinity. It follows from eq. (27) that the coefficient $V$ for such solutions is everywhere positive, so the action is well-defined. As far as relationship between
dilaton and coordinates is concerned, it follows from (27) that $\mu' = \omega + \omega \kappa (\delta - 1)e^{-\omega \phi}$, so $x = (2\lambda)^{-1}[\omega \phi - \kappa (\delta - 1)e^{-\omega \phi}]$ and $x \to \pm \infty$ when $\omega \phi \to \pm \infty$. The metric function
g = 1 + e^{-\omega \phi}[(\delta - 1)\kappa \omega \phi - b], b = [e^{\omega \phi} + \delta - 1)\kappa \omega \phi]_{\phi = \phi_h}$ has one zero at $\phi = \phi_h$, $g \to 1$
at $\omega \phi \to \infty$ and $g \to -\infty$ at $\omega \phi \to -\infty$. The curvature $R \simeq 4\lambda^2[\kappa (\delta - 1)]^{-1}e^{-|\omega \phi|} \to 0$
when $\omega \phi \to -\infty$ and $R \simeq -4\lambda^2(\delta - 1)\omega \kappa \phi e^{-\omega \phi} \to 0$ when $\omega \phi \to \infty$. Thus, not only the
curvature is finite at both infinities but, moreover, the spacetime is flat there. If $\kappa \to 0$ we
return to the string inspired dilaton gravity for which the singularity lies at $\omega \phi \to -\infty$. In
this sense, it is quantum effects which are responsible for removing the singularity. (On the
other hand, one can obtain the geometry regular everywhere already on a classical level due
to the choice, for example, $F = e^{\omega \phi} + K \omega \phi$ with $K > 0$.)

Following general rules [17], [18] we can draw the Penrose diagram of this nonsingular
spacetime starting from fundamental building blocks. In so doing, the form of such a block
depends crucially on whether or not $f(x) = f^x dyg(y)^{-1}$remains finite at the boundary: it is
triangle if $f$ is finite and square if $f$ diverges. In our case the function $f$ can be calculated
exactly from (27): $f = (2\lambda)^{-1} \ln(\tilde{F} - \tilde{F}_h)$. Collecting all this information we obtain the
Penrose diagram which is depicted at Fig. 1. The geodesic distance \( \tau = \int dx (c - g)^{-1/2} \) where \( c \) is a constant, so timelike geodesics can reach either on plus or minus infinity only for an infinite interval of time. It is seen from Fig. 1 that a horizon is rather acceleration horizon than a true black hole one. The structure of spacetime is similar to that of Rindler (with extension to the complete Minkowski spacetime). In the limit \( \omega \to 0, \delta \to \infty \) with \( \omega \delta = \text{const} \) this spacetime turns into the Rindler one directly as it immediately follows from the above formulae for the metric and coordinate.

FIG. 1. Penrose diagram for a nonsingular spacetime with \( \tilde{F}' \neq 0 \) everywhere

It is worth noting that, as we shall see now, there exists also quite another type of solutions regular everywhere and having a black hole horizon: those for which \( \tilde{F}' \to 0 \) at infinity in such a way that cancellation of \( \tilde{F}' \) is compensated by \( \exp(\int \omega d\phi) \), so that the whole expression (40) remains finite. Consider the following example. Let \( \tilde{F} \) be monotonic function of \( \phi \) such that \( \tilde{F} \simeq e^{\gamma_{\pm} \phi} \) when \( \phi \to \pm \infty \) with \( \gamma_{\pm} < 0 \). Let also the function \( \psi(\phi) = \int_{-\infty}^{\phi} d\phi \omega \) have the asymptotics \( \psi \simeq \omega_{\pm} \phi + \psi_{\pm} \) when \( \phi \to \pm \infty \). Then at both infinities the curvature (40) behaves like \( R \simeq A_{\pm} \exp [(\omega_\pm - 2\gamma_{\pm})\phi] \) where \( A_{\pm} \) are constants. We choose \( \omega_+ - 2\gamma_+ < 0 \) and \( \omega_- - 2\gamma_- > 0 \), so \( R \to 0 \) when \( \phi \to \pm \infty \). Let us also choose, for definiteness, \( \gamma_- = \omega_- \). Then \( g \to 1 \) when \( \phi \to -\infty \) and \( g \sim -e^{-\omega_+ \phi} \to -\infty \) when \( \phi \to \infty \). The coordinate \( x \simeq -2\lambda |\gamma_-| e^{-\psi_-} \phi \to \infty \) at \( \phi \to -\infty \) and \( x \simeq -|\gamma_+| (\gamma_+ - \omega_+)^{-1} e^{(\gamma_+ - \omega_+)} \phi \to -\infty \) at \( \phi \to \infty \) if \( \gamma_+ > \omega_+ \). Thus, the coordinate \( -\infty < x < \infty \). All restrictions imposed on parameters of the solutions read \( \omega_+ - 2\gamma_+ < 0 \) and \( \omega_- = \omega_- < 0 \) and are self-consistent. With all these properties, the Penrose diagrams looks like Fig. 2 where the horizontal lines represent not a singularity (as it would be for the Schwarzschild metric) but regular spatial infinities which can be reached along time-like geodesics only for a infinite proper time. The boundaries of the spacetime under discussion are null- and time-like complete [18], as \( x \to \pm \infty \) and \( \tau \) diverges.

FIG. 2. Penrose diagram for a nonsingular black hole with \( \tilde{F}' \to 0 \) at \( \phi = \infty \).
As a matter of fact, constructing both types of diagrams relies only on the asymptotic behavior of the functions $\tilde{F}$ and $\omega$, so they describe the whole classes of spacetimes. Now we will show that our metrics (27) contain also the third type of regular spacetimes - those with a constant curvature. Indeed, let $\tilde{F} = \exp(\frac{1}{2} \int_0^\phi \omega d\phi)$. Substituting this into (40) we find after simple manipulations that $R = 8\lambda^2 \tilde{F}_h = const > 0$, so spacetime in question is of de Sitter type. This generalizes the observation made in [15] for the particular choice $\omega = -2a$ and $\kappa = 0$.

The metrics (27) corresponding to the choice $C = 0$ possess the following interesting property. If the functions $\omega(\phi)$ and $\tilde{F}(\phi)$, whatever their form would be, do not contain $\kappa$, the metric also does not contain $\kappa$. Thus, within one-loop accuracy, we obtained the whole classes of models for which a classical geometry is the exact solution of field equations derived from quantum Lagrangians.

B. Solutions with a constant dilaton value

Apart from solutions discussed above, there is one more class of them. It is seen from eq. (12) that this equation turns into identity when $\phi = const \equiv \phi_0$. For such solutions field eq.(11) gives us $U = -\kappa R$. Substituting it into eq.(8) we have $R\tilde{F}'' = 0$ where we have taken into account that $\tilde{F}'' = u - \kappa \omega$. This means that nontrivial solutions ($R \neq 0$) exist only for values of the dilaton field $\phi_0 = \phi_c$. Let me recall that this is just the point where the curvature for solutions described by eq.(27) diverges. This gives nontrivial interplay between two branches of solutions, also found for the particular case of the RST model [3]: the values of the dilaton field for constant dilaton solutions coincides with the singularity of non-constant ones. In particular, it follows from the contents of the present paragraph that the class of models under discussion does not contain constant dilaton solutions with two horizons found in [19]. It is not surprising since the latter solutions exist only under the presence of an electromagnetic field which is now absent.
C. Thermodynamics

Let us now return to the solutions with \( \phi \neq \text{const} \) and discuss their thermodynamic properties. From eqs. (25), (26) we obtain the formula for the Hawking temperature \( T_H \) which turns out surprisingly simple:

\[
T_H = (4\pi)^{-1} \left( \frac{dg}{dx} \right)_{x=x_h} = \frac{aB}{4\pi(1 - 2\kappa C)}
\]

(41)

If the spacetime is not asymptotically flat, the choice of the Euclidean time is ambiguous and this is reflected in the appearance of the constants \( a, B \) in the metric (25) and temperature. More interesting is, however, the case when a spacetime is flat at infinity where there is a clear definition of time and a distant observer measures a temperature in an unambiguous way. Let us assume that \( C = 0 \) and, for definiteness, the asymptotically flat region corresponds to \( \phi \to -\infty \) and \( \tilde{F} \to \infty \), \( \exp(-f^0 d\phi \omega) \to 0 \) in such a way that their product is constant. Adjusting the limit of integration, one can always achieve \( \lim_{\phi \to -\infty} \tilde{F} \exp(-f^0 d\phi \omega) = a = 1 \) to have \( g = 1 \) at infinity. Then it follows from (27), (41) that the temperature in this case

\[
T_H = (2\pi)^{-1} \lambda
\]

(42)

and does not acquire quantum corrections. Moreover, as this temperature is a constant, it turns out that all horizons present in the solution have the same temperature. Both properties generalize the similar feature of the RST model [3]. In the latter case \( \omega = \text{const}, \tilde{F} \propto e^{\omega \phi} \).

It turns out that the general structure of the theories under consideration enables one to relate entropy and energy of a system in a rather simple manner. If the Hamiltonian constraint \( T_0^0 = 0 \) is taken into account one can infer that the action takes the thermodynamic form \( I = \beta E - S \). This expression for the zero loop action can be obtained by direct generalization of the procedure elaborated in [20] for the string-inspired dilaton gravity. Here \( \beta = T_H^{-1} \sqrt{g} \) is the inverse Tolman temperature on the boundary, the energy \( E = E_o - (2\pi)^{-1} (\frac{\partial \tilde{F}}{\partial l})_B \) where \( E_o \) is the subtraction constant irrelevant for our purposes, \( dl \) is
the line element, the index "B" indicates that the corresponding quantities are calculated at the boundary. If quantum effects are taken into account the entropy entering this formula is to be understood as 
\[ S = S_0 + S_q \] where zero loop contribution \( S_0 = 2F(\phi_h) \) and that of Hawking radiation \( S_q = 2\kappa[\psi(\phi_B) - \psi(\phi_h)] \) [21] (we choose the constant in the definition of entropy in such a way that \( S_q = 0 \) in the limit \( \phi_B = \phi_h \) when there is no room for radiation).

For the total entropy we have 
\[ S = S_0 + S_q = 2[\tilde{F}(\phi_h) + \kappa\psi(\phi_B)]. \]

It follows directly from (27) with \( a = 1, B = 2\lambda \) that \( E - E_0 = -\frac{\lambda}{\pi}\exp\left(\frac{\kappa}{2}\right)\sqrt{F_B - S/2}. \)

The general first law \( \beta\delta E = \delta S \) then tells us that the Hawking temperature \( T_H = \lambda/2\pi \) does not acquire quantum correction in agreement with the above formula (42). It was pointed out in [3] that not only the temperature but the energy and entropy as well do not acquire quantum corrections in the RST model. Strictly speaking, however, the physical meaning of this statement is not quite clear since it refers to characteristics of a black hole itself and is based on subtracting contributions of a hot gas [3] whereas in the microcanonical ensemble approach the total energy (but not its constituents) should be fixed. Meanwhile, it is seen from the above formula directly that if the function \( F(\phi) \) is such that it does not contain \( \kappa \), the dependence of the energy on total entropy retains the same form either in the classical or quantum domain, so there are no quantum corrections for characteristic of the whole system in this sense. The difference between this situation and that discussed in [3] consists in that in the RST model \( F \) contains \( \kappa \) but \( V \) does not, whereas in our case the situation is reverse according to eqs. (18), (27). It is worth noting that the expression for \( V \) can be rewritten as \( V = -2\tilde{F}' + 2\kappa \). Therefore, although this coefficient may change its sign it happens only after \( \tilde{F}' \) so does, i.e. beyond the singularity. On the whole sheet with \( \tilde{F}' \leq 0 \) the quantity \( V \) is positive as it should be for the action principle to be well-defined.

It is also worth noting that if \( \omega = const \) the formula for the entropy of Hawking radiation can be written as \( S_q = 2\kappa\omega(\phi_B - \phi_h) \) that coincides with the entropy of hot gas in a flat space in the container of the length \( L = |\phi_B - \phi_h| = \lambda|x_B - x_h| \) with a temperature \( T = \lambda/2\pi \).

This generalizes the corresponding observation [3] for the RST model. One can say that, in some sense, the entropy of thermal radiation in the backgrounds under discussion does not
acquire curvature corrections.

D. Factorization

It is instructive to look at the approach in question from somewhat another view point. Eqs. (8), (11) lead to the relation

\[ R(u - \kappa \omega) = (2V - \omega u) \Box \phi + (V' - \omega u')(\nabla \phi)^2 \] (43)

If the potential \( V \) satisfies eq. (27), eq. (43) takes the form

\[ (u - \kappa \omega)[R - \Box(\int d\phi \omega)] = 0 \] (44)

Thus, our choice of relationship between action coefficients gives rise to the important property of factorization, generalizing observation made for the RST model in [3]. If \( u = \kappa \omega \) we return to the constant dilaton field solutions discussed above. Otherwise \( R = \Box \omega \) and, according to (4), \( \omega = \psi' \) in agreement with (27).

One can try to gain the factorization property without referring to eq. (13) and the form of \( V \) following from it. Let us choose the functions, entering the action of the model, in such a way that the coefficient at \((\nabla \phi)^2\) cancel and, besides, the coefficient at \(\Box \phi\) be proportional to that at curvature:

\[ V' = \omega u', \quad 2V - \omega u = -\omega_0(u - \kappa \omega) \] (45)

where \( \omega_0 \) is some constant. Then eq.(43) turns into

\[ (u - \kappa \omega)(R + \omega_0 \Box \phi) = 0 \] (46)

However, one can check directly that eq. (13) is satisfied in this case automatically, so we do not obtain new solutions. It is worth noting that now \( D \neq 1 \) where \( D \) is a constant entering (16). There is also the choice \( V' - \omega u' = p(u - \kappa \omega), \quad 2V - \omega u = q(u - \kappa \omega) \) with arbitrary functions \( p,q \) that ensures factorization but we will not discuss this possibility further.
IV. COMPARISON WITH KNOWN MODELS

We demonstrated above that the RST model enters our scheme as a particular case. Here we show that the same is true for other known exactly solvable models.

A. BPP model

This model [9] is characterized in our notations by

\[ F = e^{-2\phi} - 2\kappa\phi, \quad \tilde{F} = e^{-2\phi}, \quad \omega = -2, \quad \psi = 2\phi, \quad V = 4e^{-2\phi} + 2\kappa \]  \hspace{1cm} (47)

It is seen from (47) immediately that (18) is satisfied. Substituting (47) into (27) we obtain

\[ g = 1 - e^{2\phi - 2\phi_h}, \quad -\lambda x = \phi \]  \hspace{1cm} (48)

Thus, this model possesses rather unexpected feature: the metric has the same form in terms of \( \phi \) and \( \phi_h \) (or \( x \) and \( x_h \)) as its classical counterpart! In other words, not only quantum corrections to the Hawking temperature vanish but also so do quantum correction to the metric itself. It is worth stressing that although this property sharply contrasts with the explicit form of the metric listed in [9] there is no contradiction here: authors of [9] consider solutions which are radiationless at infinity and have a singular horizon (analog of the Boulware state) whereas we deal with the Hartle-Hawking state that implies that the stress-energy tensor of radiation does not vanish at infinity, an event horizon being regular. In fact, as was explained in Sec. IIIA, any model from our set for which \( F = \tilde{F}(\phi) + \kappa \int d\phi \omega \) where \( \tilde{F}(\phi) \) does not contain \( \kappa \) will give the metric function without quantum corrections as follows from (27).

B. Other models

The model discussed by Michaud and Myers [7] is characterized by

\[ F = e^{-2\phi} - \frac{\kappa}{2}(\alpha + \sum_{n=2}^{K} a_n \phi^n), \quad V = 4e^{-2\phi} - \frac{\kappa}{2}(\beta + \sum_{n=2}^{K} b_n \phi^{n-1}), \quad \omega = -2 \text{ where } b_n = -2na_n, \quad \beta = 4 - 2\alpha. \]
For the Fabbri and Russo (FR) model [8] \( F = \exp\left(-\frac{2\phi}{n}\right) + \kappa \frac{(1-2n)}{n} \phi, \ V = \frac{4}{n} \exp\left(-\frac{2\phi}{n}\right) + 2\kappa \frac{(n-1)}{n}, \ \omega = -2 \). When \( n = 1 \) the RST result is reproduced.

The model which interpolates between the RST and BPP ones is considered in [10] (CN model). In this case \( F = \exp(-2\phi) + 2\kappa (a - 1) \phi, \ V = 4 \exp(-2\phi) + 2(1 - 2a)\kappa, \ \omega = -2 \). It reduces to the RST model when \( a = \frac{1}{2} \) and to the BPP one when \( a = 0 \).

It is easily seen that eq. (18) is satisfied in all these cases. Moreover, one can suggest, for instance, a new model which incorporates at once features of the RST, FR, CN and BPP: \( F = \exp(-\frac{2\phi}{n}) + 2\kappa (a - 1), \ V = 4 \exp(-\frac{2\phi}{n}) + 2\kappa (1 - 2a), \ \omega = -2 \). When \( a = 2n^{-1} \) we return to the BPP, \( n = 1 \) corresponds to the CN model.

For the exponential models [11] \( F = \phi, \ u = 1, \ V = 0, \ \omega = const \). This case is described by eq. (16) with \( D \neq 1 \).

V. SUMMARY AND OUTLOOK

Thus, the models considered in the present paper have the following properties:

1) they are exactly solvable in the sense that either the metric or dilaton field are found in a closed form; 2) their geometry is static; 3) quantum corrections to the Hawking temperature vanish; 4) for wide subsets of these models one of the following properties holds: a) the relationship between the total energy and the total entropy of the quantum finite size system is the same as in the classical limit, b) the metric itself does not contain quantum corrections and has the same form either in the classical or the quantum domain; 5) there exists the special class of solutions with a constant dilaton field \( \phi = \phi_c \), the geometry of non-constant dilaton solutions become singular in the point \( \phi_c \); 6) all spacetime for \( \phi \neq const \) can be divided to separate sheets with one and only one horizon on every sheet between two neighboring singularities with finite \( \phi_c \) (plus, perhaps, additional horizons due to \( \phi = \infty \) or \( \phi = -\infty \)), different sheets are glued in the singular points; all horizons on different sheets share the same temperature; 7) there exists the solution with one horizon and without singularities.
The most part of these properties is inherent to the RST model [3]. It does not possess, however, the properties 4a) and 4b). Besides, in 6) the number of sheets for the RST model is equal to two, whereas in general it can be arbitrary. Thus, the RST model turns out to be only a representative of a more wide class of models sharing common features. Moreover, we gained also the qualitatively new property which was absent in the RST model: the existence of quantum black holes without singularities whose metric is found explicitly. In this respect the corresponding solutions resemble those in string theory obtained in exact (non-perturbative) approach [22], [23]. On the other hand, regular black holes solutions found in the present paper differ from similar ones in [24] in that solutions discussed in [24] represent the extreme black holes whereas in our case they are essentially nonextreme since their temperature is nonzero constant.

The crucial point in which generalization of the RST model is performed consists in that we do not specify the form of the action coefficients and only impose one restriction (14) on them, so instead of particular models we operate with whole classes of them. In this respect our approach is similar to that of [6] but, in contrast to it, we appeal directly to properties of field equations and do not rely on the general structure of the nonlinear sigma model from which the dilaton-gravity action can be obtained. In so doing, we only assumed (i) the local connection between two unknown functions $\psi$ and $\phi$; (ii) cancellations of coefficients at first and second derivatives of $\phi$ in eq.(12) that saved us from the trouble to solve a generic complicated differential equation for $\phi(x)$ and enabled us to introduce variables in terms of which the field equations are greatly simplified.

It is of interest to generalize the elaborated approach to theories with higher derivatives (in particular, to generalize exact solutions for black holes found at the classical level [25]), additional scalar and gauge fields, interactions between black holes spacetimes and shock waves, etc. The problem deserving separate attention is detailed description and classification of different types of spacetime structure of considered quantum black holes similarly to what has been done for classical dilaton black holes [15], [26]. Of special interest is the issue of regular black holes including their formation by gravitation collapse starting form the
vacuum. It would be tempting to derive general criteria for Lagrangians admitting nonsingular black hole solutions either in the nonextreme or extreme case and select among them those with exact solutions. Besides the issues connected with black hole physics, exactly solvable models can be useful for the analysis of conceptual problems in different schemes of quantization of dilaton gravity theories [27].

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