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GENERALIZED VIRASORO MODELS

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A B S T R A C T

We present a series of dual resonance amplitudes for positive integer intercept ($\alpha_0 = 1, 2, 3, \dots$) based on the projective group in n dimensions. In each case the amplitudes factorize with a non-degenerate leading trajectory and have four-point functions identical to the Virasoro model. The special properties of the $\alpha_0 = 1$ (symmetrized Veneziano) model and $\alpha_0 = 2$ (Shapiro-Virasoro) model are discussed.

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The familiar N point generalization of the Veneziano beta function model is also a generalization of the Virasoro four-point function ¹⁾

$$\frac{\Gamma(-\frac{1}{2}\alpha(s))\Gamma(-\frac{1}{2}\alpha(t))\Gamma(-\frac{1}{2}\alpha(u))}{\Gamma(-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))\Gamma(-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(-\frac{1}{2}\alpha(u)-\frac{1}{2}\alpha(s))} \quad (1)$$

if the trajectory intercept, α_0 , is set equal to one and we symmetrize over all external lines. This special case, $\alpha_0 = 1$, is particularly interesting in view of the growing evidence in favour of Virasoro's no ghost hypothesis ²⁾. Recently, Shapiro ³⁾ has generalized the Virasoro amplitude for $\alpha_0 = 2$, by extending the electrostatic analogue to the surface of a sphere. Factorization and the operator formalism have subsequently been discussed by Yoshimura ⁴⁾ and by Del Guidice and Di Vecchia ⁵⁾.

Here we present an N point generalization of the Virasoro amplitude for any positive integer intercept $\alpha_0 = n = 1, 2, 3, \dots$. While these intercepts are increasingly unphysical, this series of models casts new light on several fundamental aspects of dual theory, e.g., the rôle of the projective group, the absence of ghosts and the special features of $\alpha_0 = 1, 2$. Eventually, of course, one needs to understand the nature of the symmetry breaking which will allow lower intercepts $[\alpha_p(0) \simeq \frac{1}{2}, \alpha_\pi(0) \simeq 0]$ and the absence of low mass exotic resonances.

We consider the N point function

$$A_N(k_i) = \int \prod_{i \neq j} |z_i - z_j|^{k_i k_j} \prod_{i=1}^N d^n z_i / dw \quad (2)$$

defined as an integral over the whole of n dimensional real space R^n , with the length of an n dimensional real vector $z \equiv (z^{(1)}, z^{(2)}, \dots, z^{(n)})$ being defined in the usual way

$$|z| = (z \cdot z)^{\frac{1}{2}} = \left(\sum_{k=1}^n z^{(k)2} \right)^{\frac{1}{2}}$$

Clearly, the integral has simple poles; the differential $d\omega$ ensures that there can be only $N-3$ simultaneous poles and that the integral is finite.

$$d\omega = \frac{d^nz_a \, d^nz_b \, d^nz_c}{|z_a - z_b|^n |z_b - z_c|^n |z_c - z_a|^n} \quad (3)$$

Both $d\omega$ and the whole integrand of Eq. (2) are invariant under the projective (conformal) group on R^n . This is the $\frac{1}{2}(n+1)(n+2)$ parameter group isomorphic to $SO(n+1,1)$ generated by rotations, translations, inversions and scale transformations. Invariance of the integrand of (2) under rotations and translations is immediate. For invariance under inversions

$$z_i \rightarrow z_i / |z_i|^{-2}$$

we have invariance if, for each particle, the mass squared, $-k_i^2 = n$; that is the trajectory intercept $\alpha_0 = n$. The same condition gives invariance under scale transformations

$$z_i \rightarrow \lambda z_i$$

The fact that, given three points, in general there exists a projective transformation taking one set into the other, guarantees the independence of (2) of the choice of z_a, z_b, z_c . Thus, the projective group plays the rôle played by the Gliozzi group, Ref. 6) $SO(2,1)$, or $SL(2, \mathbb{C})$ for $n=1,2$ respectively. To evaluate the four-point function we may take as fixed points $z_1=0$, $z_3=\hat{1}$, a fixed unit vector, and $z_4=\infty$. It appears that there is an n dimensional integration left but the axial symmetry about $\hat{1}$ reduces this to two dimensions and the integral may be evaluated using the procedure of Shapiro and Sommerfeld³⁾ to give (1) except for the multiplicative factor $\pi^{\frac{1}{2}(n+1)} \Gamma(\frac{n+1}{2}) / \Gamma(\frac{1}{2}n)$.

a) Operator Formalism

The basic algebraic structure can be expressed quite conveniently in the operator formalism of Fubini and Veneziano ⁷⁾. For a given ordering of points $|z_i| < |z_j|$ we can expand

$$\begin{aligned} |z_i - z_j| &= |z_j| (1 + |z_i|^2/|z_j|^2 - 2 \cos \theta_{ij} |z_i|/|z_j|)^{\frac{1}{2}} \\ &= |z_j| \exp \left\{ - \sum_{r=1}^{\infty} \left(\frac{|z_i|}{|z_j|} \right)^r \frac{\cos r \theta_{ij}}{r} \right\} \end{aligned} \quad (4)$$

θ_{ij} being the angle between z_i and z_j . So, as for $\alpha_0 = 1, 2$ we introduce a four-vector "field" $Q(z) = Q^+(z) + Q^0(z) + Q^-(z)$ where

$$Q^0(z) = q_0 + i p_0 \log |z| \quad (5)$$

and such that

$$[Q^+(z_i), Q^-(z_j)] = g_{\mu\nu} \sum_{r=1}^{\infty} \left(\frac{|z_i|}{|z_j|} \right)^r \frac{\cos r \theta_{ij}}{r}, \quad |z_i| < |z_j| \quad (6)$$

The commutation relations (6) together with the standard conditions $Q^+|0\rangle = \langle 0|Q^- = 0$, $p_0|0\rangle = 0$, $[q_0, p_0] = i$, are sufficient to prove that the integrand for the $N+2$ point function of the Figure factorizes as follows:

$$\prod_{i=0}^{N+1} |z_i|^{k_0} \prod_{i \neq j} |z_i - z_j|^{k_i k_j} = \langle 0 | T \{ U_0 U_1 \dots U_{N+1} \} | 0 \rangle \quad (7)$$

where

$$\begin{aligned} U_i(z_i) &= \exp(i\sqrt{2} k_i Q^-(z_i)) \exp(i\sqrt{2} k_i Q^0(z_i)) \exp(i\sqrt{2} k_i Q^+(z_i)) \\ &= : \exp(i\sqrt{2} k_i Q(z_i)) : \end{aligned} \quad (8)$$

The T product ($\tau = \log|z|$) indicates an ordering of the $U_i(z_i)$ left to right as $|z_i|$ increases. To prove Eq. (7), one merely commutes all Q^+ (Q^-) factors to right (left) and then pushes p_0 factors to the right or left leaving $\exp(i\sqrt{2}(\sum k_i)q_0)$ which is one by momentum conservation. Thus, the plane wave character of the vacuum does not present any difficulty.

It remains to show we can construct Q^+ , Q^- satisfying Eq. (6). For this purpose we introduce a rotation R_i [in $SO(n)$] which takes $\hat{z}_i$, the unit vector in the direction of z_i , into the fixed unit vector $\hat{i}$.

$$z_i = |z_i| R_i^{-1}(\hat{i})$$

The central problem is to factorize $\cos r\theta_{ij} = \hat{i} \cdot (R_i R_j^{-1})(\hat{i})$. To do this we introduce a representation $D^{(\eta)}_{\ell}$ of $SO(n)$ such that

$$\begin{aligned} D^{(\eta)}_{\ell\ell'}(R_i R_j^{-1}) &= \frac{\eta_r}{r} \cos r\theta_{ij} & \eta_r > 0 \\ &= \sum_i D^{(\eta)}_{\ell i}(R_i) D^{(\eta)}_{i\ell'}(R_j^{-1}) \epsilon_i \end{aligned}$$

where $\epsilon_i = \pm 1$ allows the representation to be pseudo-unitarity (indefinite metric). Introducing operators $a_{(r)\mu}^\ell$ such that

$$[a_{(r)\mu}^\ell, a_{(s)\nu}^{k\dagger}] = g_{rs} \delta_{\mu\nu} \epsilon_i \delta_{ki} \quad (9)$$

it is clear that the fields can be defined by

$$Q_r^+(z_i) = \sum_{\ell} \frac{|z_i|^\ell}{\sqrt{\eta_r}} a_{(r)\mu}^\ell D_{\ell\ell'}^{(\eta)}(R_i) \quad (10a)$$

$$Q_s^-(z_i) = \sum_{\ell} \frac{|z_i|^\ell}{\sqrt{\eta_s}} a_{(s)\mu}^{\ell\dagger} D_{\ell\ell'}^{(\eta)}(R_j^{-1}) \quad (10b)$$

Finally it is convenient to introduce the generators of the projective group: J_{ij} (rotations), $L_{1(i)}$ (translations), L_0 (dilations) and $L_{1(i)}^+$ (special conformal transformations), acting in the space spanned by the occupation number states

$$|\lambda_r^t\rangle = \prod_{r,t} [\lambda_r^t!]^{-\frac{1}{2}} (a_r^{\dagger t})^{\lambda_r^t} |0\rangle$$

Their action on the Q fields is simply given by

$$[J_{ij}, Q(z)] = \left\{ z^{(i)} \frac{\partial}{\partial z^{(j)}} - z^{(j)} \frac{\partial}{\partial z^{(i)}} \right\} Q(z) \quad (11a)$$

$$[L_{1(i)}, Q(z)] = - \frac{\partial}{\partial z^{(i)}} Q(z) \quad (11b)$$

$$[L_0, Q(z)] = - \sum_k z^{(k)} \frac{\partial}{\partial z^{(k)}} Q(z) \quad (11c)$$

$$[L_{1(i)}^+, Q(z)] = - \sum_k \left\{ -|z|^2 \delta_{ik} + 2z^{(i)} z^{(k)} \right\} \frac{\partial}{\partial z^{(k)}} Q(z) \quad (11d)$$

b) Factorization

We now go to the standard factorization configuration of Fubini, Gordon and Veneziano ⁸⁾. The essential step is the transformation $\prod (|z_i|, R_i)$ that takes expression (8) to the "frame" in which $z_i = \hat{1}$.

$$: \exp \{ i\sqrt{2} k_i Q(z_i) \} : = \prod^{-1} (|z_i|, R_i) : \exp \{ i\sqrt{2} k_i Q(\hat{1}) \} : \prod (|z_i|, R_i) \quad (12)$$

The operator Π can be written explicitly in the canonical form of Alessandrini et al.⁹⁾ as

$$\Pi(|z_i|, R_i) = : \exp \left\{ \sum_{k,r} a_{(r)}^{+k} (|z_i|^r D_{rk}(R_i) - e_i \delta_{rk}) a_{(r)}^k \right\} : |z_i|^{k^2} \quad (13)$$

or directly as a dilation multiplied by a rotation

$$\Pi(|z_i|, R_i) = |z_i|^{L_0} \mathcal{D}(R_i) \quad (14)$$

where $\mathcal{D}$ is the representation of $SO(n)$ generated by J_{ij} . Since dilations and rotations commute the operators Π satisfy the group property

$$\Pi(|z|, R) \Pi(|z'|, R') = \Pi(|z||z'|, RR') \quad (15)$$

Now, for definiteness, consider the $N+2$ amplitude of the Figure with the ordering $|z_{i+1}| > |z_i|$ and with $z_0 = 0$, $z_N = 1$, $z_{N+1} = \infty$. Everything can be expressed, using Eqs. (12) and (15), in terms of the multiperipheral variables

$$\alpha_i = |z_i| |z_{i+1}|^{-1} \quad \text{and} \quad R_i = R_i R_{i+1}^{-1}$$

Defining the standard vertex

$$V(k) = : \exp \{ i \sqrt{2} k Q(\tau) \} : \quad (16)$$

and the propagator

$$\Delta_i(\pi_i^2) = \kappa \int_0^1 dx_i x_i^{L_0 - \alpha_i - 1} \int dR_i \mathcal{D}(R_i) \quad (17)$$

we obtain the standard factorization formula

$$A_{N+2} = \sum \langle k_0, 0 | V(k_1) \Delta_1 V(k_2) \dots \Delta_{N-1} V(k_N) | 0, k_{N+1} \rangle \quad (18)$$

where $\pi_i = k_0 + k_1 + \dots + k_i$, the constant $K = 2 \pi^{n/2} [\Gamma(\frac{1}{2}n)]^{-1}$ has been introduced because the group volume element dR is normalized to integrate to one and the sum in Eq. (18) is over permutations of $k_1, \dots, k_N$. The vacuum states have a label to indicate they carry the momentum of the end lines.

c) The Spectrum

We must first consider the construction of $D^{(r)}$. The case $n=1$ is trivial because the group integrations do not come in. The case $n=2$ is simple because $SO(2)$ is commutative. There, we may take $d(r)$, the dimension of $D^{(r)}$, equal to two, for each r , and

$$D^{(r)}(\theta) = \begin{bmatrix} \cos r\theta & -\sin r\theta \\ \sin r\theta & \cos r\theta \end{bmatrix}$$

We may easily regain the formalism of Yoshimura⁴⁾ and Del Guidice and Di Vecchia⁵⁾ by changing to a basis in which $D^{(r)}$ is diagonal. It is with the case $n=3$ that the essential features of the problem manifest themselves. We can build up $D^{(r)}$ by taking direct sums of irreducible representations of spin J , dimension $2J+1$. Such an irreducible representation has $P_J(\cos \theta)$ as a matrix element. And we can write

$$\cos r\theta = \sum_{J=0}^r C_{rJ} P_J(\cos \theta) \quad (19)$$

where $C_{rJ} = 0$ unless $r-J$ is even. In fact

$$C_{r, r-2j} = -\frac{1}{2} r \left(\frac{1}{2} r - j + \frac{1}{4} \right) \frac{\Gamma(r-j) \Gamma(j-\frac{1}{2})}{\Gamma(j+1) \Gamma(r-j+\frac{3}{2})}$$

which is positive for $j=0$ and negative thereafter. This introduces negative norm states because we must take the direct sum of the spin J representations attaching the sign of C_{rJ} to the metric of each respective representation space. The corresponding

dimension $d(r) = \frac{1}{2}(r+1)(r+2)$ might seem large but in fact it is the smallest possible, for any $D^{(r)}$ can be reduced to a direct sum of irreducible representations. This gives an expansion of $\cos r\theta$ in terms of spherical harmonics which, by uniqueness, must be (19). Thus, $D^{(r)}$ must contain all the irreducible representations we have used. For $n > 3$ this whole discussion can be generalized using n dimensional spherical harmonics.

To look at the spectrum we study the poles of the propagator

$$\Delta(\pi^2) = \frac{\kappa}{L_0 - \alpha_0} P_s \quad (20)$$

where P_s comes from the integral of $\mathcal{D}$ and is the projection operator on the space of states which are scalars with respect to $SO(n)$

$$L_0 = p_0^2 + \sum_{r=1}^{\infty} \sum_i r e_i a_{(r)}^{i\dagger} a_{(r)}^i$$

From Eq. (20) we see that the states coupling at any pole are $SO(n)$ scalars, or, equivalently zero eigenstates of all the J_{ij} . This generalizes the known result for $n=2$ where J_{12} is essentially R [notation of Ref. 4)] or $L_0 - \bar{L}_0$ [Ref. 5)]. Further, states on the leading trajectory must be built with the spatial parts of $a_{(1)}^{+\ell}$ which transforms as an n vector with respect to $SO(n)$. Because we can only form one symmetric scalar with an even number of n vectors and none with an odd number, the leading trajectory is non-degenerate and of positive signature. Further for n even all poles where α is an odd integer are excluded ¹⁰⁾.

These angular conditions drastically reduce the spectrum and it is further reduced by Ward identities. For $n=1,2$ the first Ward identity is known to be connected with a one-parameter subgroup of the conformal group with a generator of the type $L_0 - L_1^+$. We will derive an identity of this type using the method developed by Fubini and Veneziano ⁷⁾. Writing the commutation relations (11c) or (11d) in the form

$$[L_f, Q(z)] = \sum_k f_{(k)}(z) \frac{\partial}{\partial z} Q(z) \quad (21)$$

we may, in either case, deduce

$$[L_f, |z|^{-\alpha_0} U(z)] = \sum_k \frac{\partial}{\partial z} \{ f_{(k)} |z|^{-\alpha_0} U(z) \} \quad (22)$$

Now, consider the state

$$\begin{aligned} \langle k_i | &= \sum \langle k_0, 0 | V(k_1) \Delta_1 V(k_2) \dots \Delta_{N-1} V(k_N) \\ &= \sum \int \prod_{i=1}^{N-1} d^n z_i \langle k_0, 0 | \prod_{i=1}^N |z_i|^{-\alpha_0} U(z_i) \Big|_{z_N = \hat{1}} \end{aligned} \quad (23)$$

where the sum is over permutations of $1, 2, \dots, N$. Applying operators L_f , satisfying (22) to this state we see that their action may be easily evaluated if $f_k(0) = f_k(\hat{1}) = 0$. Thus, taking $f_k = z_k^+ |z|^{-2} \delta_{1k} - 2z_1 z_k$, so that $L_f = L_0 - L_1^+(1)$ we deduce

$$\langle k_i | (L_0 - L_1^+(1)) = 0 \quad (24)$$

using the cancellation of the surface terms arising from the right-hand side of (22). The existence of an operator satisfying (11d) is guaranteed by projective invariance and may be explicitly constructed in terms of the a operators by considering the corresponding infinitesimal change of co-ordinates in (23). In contrast to the $\alpha_0 = 1$ case, we note that the divergence condition (22) strongly constrains possible functions $f_{(k)}$ and permutation symmetry is essential for the cancellation of surface terms.

Of course, further, dependent restrictions on the spectrum may be obtained using commutators $[\bar{J}_{ij}, L_0 - L_1^+(1)]$ and we note that, for $n=2$, all the Ward operators may be derived from a single set $W_i + \bar{W}_i$, $i=1, 2, \dots$, [notation of Ref. 5] in this way. For $n > 2$

we have been unable to derive a corresponding infinity of Ward operators, essentially because $SO(n)$ is non-commutative then. On the other hand, a simple search for ghosts in the four-point function failed to reveal them for $1 \leq n \leq 45$, searching empirically down to the 30th non-trivial level on a computer. Incidentally, relaxing the constraint that n be integral a similar search revealed a remarkable absence of ghosts in (1) for $\alpha(0) > 4/\sqrt{3} - 2 \approx 0.31$.

Thus, while the original model with $n=1$, and, as a direct consequence that with $n=2$, seems likely to have no ghosts, the generalization to $n>2$ which we have presented may have ghosts either because of a deficiency of Ward identities or because of the indefinite metric needed for $D^{(r)}$. However, they make it clear that $SU(1,1)$, or its complexification $SL(2, \mathcal{C})$ is only one of a number of groups that can give duality and further suggest that the analogue model ¹¹⁾, which is also a special feature of $n=1,2$, is perhaps only a mathematical convenience.

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FIG.1