ELIMINATING SPURIOUS STATES FROM THE DUAL RESONANCE MODEL

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A B S T R A C T

Virasoro pointed out that the dual resonance model with trajectory intercept one ($\alpha_0 = 1$) has a larger number of linear dependences. Here we construct a linearly independent basis for the spurious states at $\alpha(\pi^2) = N$, and demonstrate that the remaining real states have the same mass (degeneracy), spin and parity as those states created by the spatial oscillators $\tilde{a}_n^+$. The problem of eliminating ghosts (negative norm states resulting from $[a_{on}, a_{on}^+] = -1$) is reformulated, and no ghosts or unanticipated null ("zero norm") states are found on the leading nine trajectories. The projection operator onto the real subspace is calculated and the corrections to the single loop diagrams are shown to be $(1-x) \prod_{n=1}^{\infty} (1-x^n)$ and $(1+x) \prod_{n=1}^{\infty} (1-(-x)^n)$ for orientable and non-orientable diagrams respectively. The sum of all single loop diagrams is expressed as a single integral.

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I. INTRODUCTION

Ever since its original proposal, the dual resonance model has been plagued with negative width resonances, commonly called "ghosts". The essential reason for this is that Lorentz scalars must be written as a sum of terms with indefinite signs, e.g.,

$$P \cdot Q = \underline{P} \cdot \underline{Q} - P_0 Q_0 .$$

Analogy with the field theory of quantum electrodynamics ^{1),2),3)}, where a similar problem arises, leads us to hope that the apparent ghost and particle spectrum obtained by naive factorization is much larger than required, i.e. there exist linear dependences, or "Ward-like identities" between ghost states and particle states. Indeed, when Bardakci and Mandelstam, and Fubini and Veneziano first discussed the factorization of the n point amplitude, they pointed out the existence of just such linear dependences. Unfortunately, although these linear dependences reduced the number of ghosts, they were insufficient to remove all ghosts from the model.

The apparent spectrum of the dual resonance model can be represented by four-vector creation and annihilation operators, $a_{\mu n}^+$ where μ is a four-vector index and n ranges from 1 to ∞ . The linear dependences referred to above related the "ghost" operator a_{01}^+ to particle operators. There was no analogous result for a_{0n}^+ , $n > 1$. But then Virasoro ^{4),5)} pointed out that when the Regge intercept α_0 is set equal to one, or when the lowest mass is imaginary ($\alpha' = 1$, $m^2 = -1$) there are Ward-like identities for each mode. Thus arose the hope that if we introduced tachyons into the theory, we might be able to remove all negative width particles. Of course, tachyons ($m^2 < 0$) are still unphysical objects, but they may be necessitated by the

linear trajectories required by the narrow resonance approximation. With a little optimism, we may hope that unitarization will either bend the trajectories away from the negative axis or lead to a zero residue where the trajectory crosses the real axis, perhaps only for a particular value of the coupling constant.

In this paper, we leave aside such speculations and attempt to analyze the Virasoro identities to see if they really do the job of removing ghosts. All these identities can be formulated in terms of an infinite set of operators,

$$L_n^+ = L_{-n} \quad , \quad -\infty < n < \infty .$$

For each $n > 0$ the operator,

$$W_n = L_n - L_0 - (n-1) ,$$

satisfies

$$W_n G(P, a^+) |0\rangle = 0 ,$$

where $G(P, a^+) |0\rangle$ is a vertex function in the operator formalism of Fubini, Gordon and Veneziano⁶⁾. Thus the states at $\alpha(\pi^2) = N$,

$$\langle \lambda, N-n | W_n = \langle \lambda, N-n | L_n \quad (1.1)$$

have zero coupling to n scalars. The L_n^+ 's do not commute. In fact they can all be expressed as multiple commutators of L_1^+ and L_2^+ , and even the states ⁷⁾ generated by L_1^+ and L_2^+ cease to be linearly independent above $\alpha(\pi^2) = 4$. Thus it is not obvious that there are a sufficient number of independent Ward identities to remove all ghosts. In section II, we exhibit a 1 - 1 correspondence between linearly independent states of the form (1.1) and all states $\langle \lambda, N |$ which contain at least one "ghost" operator a_{n0}^+ . This discussion requires one assumption about null (zero norm) states. Di Vecchia and del Giudice ³⁾ have pointed out that states of the form $\langle \chi, N-1 | L_1$ where $\langle \chi, N-1 | L_n^+ = 0$ are null vectors within the spurious subspace and, as a consequence, cause two states instead of one to decouple. In section II, we assume that these are the only null states. While our attempts to prove this assumption have so far been frustrated, we have explicitly checked that it is true for the leading nine trajectories. We do take proper account of the null states which we know exist and show that in addition to the states (1), states $\langle \chi, N-1 | L_1^c$ must also decouple. Using these results, we define what we mean by "physical states".

In section III, we construct via a recursive scheme a projection operator which removes all spurious states described in section II. This operator exists if and only if our assumption about null states is valid. In this section, we also outline a general method for studying the ghost structure of the real states. In particular, we explicitly verify that there are no ghosts on the first nine trajectories. We regard this as an extremely suggestive result, particularly in view of the correspondence (in terms of mass, spin and parity) of our real states to those states created by a_{ni}^+ ($i = 1, 2, 3$).

In section IV, we apply the results of sections II and III to deduce the modification of single loop ⁸⁾ diagrams if only "physical states" are included. We conclude that the correction factor for planar and non-planar loop is

$$(1-x) \prod_{r=1}^{\infty} (1-x^r),$$

while that for non-orientable loops is

$$(1+x) \prod_{r=1}^{\infty} (1 - (-x)^r).$$

The extra factor of $(1 \pm x)$ arises because of the extra states removed when null states are properly taken into account. Notice the very satisfying result that planar and non-planar diagrams have the same correction factor. For the case $\alpha_0 \neq 1$, this was not true: the non-planar graph had a factor $(1-x)^2$ as compared to $(1-x)$ for the planar diagram. The reason for this is that for $\alpha_0 \neq 1$ factors raised to the power $(\alpha_0 - 1)$ gave different apparent spectra for twisted and untwisted trees. Thus, in the non-planar case linear dependences remove twice as many states as in the planar case, hence the extra factor $(1-x)$. But when $\alpha_0 = 1$, the apparent spectrum is the same for all trees and, when we properly take linear dependences into account, we obtain symmetric results.

Let us remember that the symmetrized ⁹⁾ Born term B for $\alpha_0 = 1$ can be written as a single integral in Koba-Nielson variables,

$$B = \int_{-\infty}^{\infty} \prod_{i \neq 1, N-1, N} d z_i \prod_{\substack{i \neq j \\ i, j \neq N}} |z_i - z_j|^{p_i \cdot p_j}, \quad (1.2)$$

where $z_1 = 0$, $z_{N-1} = 1$, $z_N = \infty$ and the other z_i 's are integrated over the whole real axis. The permutation symmetry of the integrand in Koba-Nielson variables is also present in second order. This permutation symmetry and the symmetry between orientable and non-orientable ($x \rightarrow -x$) allows us to write all second order diagrams as a single integral.

$$\int d^4 \pi \int_{-1}^1 dx \int \prod_{i \neq N} d z_i (1-x) \left[\prod_{r=1}^{\infty} (1-x^r) \right]^{-3} x^{\pi^2-2} \prod_i |z_i|^{2\pi \cdot p_i} \prod_{\substack{r=1, \infty \\ \text{for } r \neq 0}} \prod_{\substack{i \neq j \\ \text{for } r \neq 0}} |z_i - x^r z_j|^{p_i \cdot p_j} \quad (1.3)$$

where each variable z_i satisfies

$$-1 \leq z_i \leq -|x| \text{ or } |x| \leq z_i \leq 1 = z_N$$

independently of the other z_i 's. Moreover, the relative weight ¹⁰⁾ given to each diagram in (1.2) is exactly that required by perturbative unitarity.

On the basis of recent work on multiloops ¹¹⁾, one feels that the single integral structure will probably extend to the higher orders. Also a more elegant analysis of the symmetrized ¹²⁾ problem may reveal a more fundamental connection between the permutation symmetry and the additional Ward identities.

II. A LINEARLY INDEPENDENT BASIS FOR THE SPURIOUS SUBSPACE

Throughout our discussion, we shall analyze the states at the particular pole $\alpha = 1 - \pi^2 = N$. In the harmonic oscillator formalism, such states (in subspace V_N) are of the form :

$$|\lambda, N\rangle = \frac{(a_1^+)^{\lambda_1}}{\sqrt{\lambda_1!}} \cdot \frac{(a_2^+)^{\lambda_2}}{\sqrt{\lambda_2!}} \dots \frac{(a_N^+)^{\lambda_N}}{\sqrt{\lambda_N!}} |0\rangle$$

with $\sum_i \lambda_i = N$ and the a^+ 's obeying the harmonic oscillator commutation relations,

$$[a_{\mu m}, a_{\nu n}^+] = \delta_{mn} g_{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu} \delta_{\mu\nu}, \quad \eta_{\mu} = (-1, 1, 1, 1).$$

The property $\sum_i \lambda_i = N$ is ensured if $|\lambda, N\rangle$ satisfies

$$R |\lambda, N\rangle \equiv \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n |\lambda, N\rangle = N |\lambda, N\rangle$$

Virasoro ⁴⁾ has shown that states of the form,

$$L_n^\dagger |\lambda, N-n\rangle, \quad 0 < n \leq N \quad (2.1)$$

do not couple to any number of scalars. Here L_n is defined by the relations (π is the momentum of the resonance)

$$L_n = i\sqrt{2n} \pi \cdot a_n + \sum_{l=1}^{\infty} \sqrt{l(n+l)} a_l^\dagger \cdot a_{n+l} - \frac{1}{2} \sum_{l=1}^{n-1} \sqrt{l(n-l)} a_l \cdot a_{n-l},$$

$$L_{-n} = L_n^\dagger, \quad n < 0,$$

$$L_0 = R + \pi^2 = R - d(\pi^2) + 1.$$

It is straightforward to verify that the L_n obey the commutation relations ^{5), 13)},

$$[L_n^\dagger, L_m^\dagger] = (m-n) L_{m+n}^\dagger + \frac{(m-1)m(m+1)}{3} \delta_{n,-m}. \quad (2.2)$$

Because of (2.2), it is obvious that the states (2.1) are not linearly independent. In this section, we shall exhibit a linearly independent set of vectors of the form (2.1) which span the subspace generated by these vectors.

For purposes of our construction, we define sequences of subspaces V_n and $S_n \subseteq V_n$, $n = 0, 1, \dots, N$ as follows ¹⁴⁾.

$$\begin{aligned} V_n &= \{ |\psi\rangle \mid R|\psi\rangle = n|\psi\rangle \}, \\ S_n &= \{ |\psi\rangle \in V_n \mid |\psi\rangle = \sum_i' \alpha_i L_i^\dagger |\phi_i\rangle \}. \end{aligned} \quad (2.3)$$

Throughout this paper, we shall assume that S_n contains no null vectors for $n < N$. Furthermore, we shall assume that the only null vectors in S_N are those of the form $L_1^\dagger |\psi\rangle$, where $L_i |\psi\rangle = 0$ for all $i > 0$. (By null vector of S_n we mean a zero norm vector in S_n orthogonal to all vectors in S_n). We can now define a sequence of subspaces $R_n \subseteq V_n$, for $n < N$, as the orthogonal complement of S_n in V_n . Thus we have $V_n = R_n \oplus S_n$ for $n < N$. This definition is completely equivalent to the definition ³⁾,

$$R_n = \{ |\psi\rangle \in V_n \mid L_i |\psi\rangle = 0, \forall i > 0 \}. \quad (2.4)$$

Finally, we introduce an orthogonal basis $|\mu, n\rangle$ of R_n , normalized as follows :

$$\langle \mu', n' | \mu, n \rangle = \epsilon_\mu \delta_{\mu\mu'} \delta_{nn'}, \quad \epsilon_\mu = \pm 1.$$

In the Appendix, we show that the set of vectors,

$$\begin{aligned} |\{\lambda\}, \mu, m\rangle &= (L_1^\dagger)^{\lambda_1} (L_2^\dagger)^{\lambda_2} \dots (L_m^\dagger)^{\lambda_m} |\mu, n-m\rangle \\ \sum_i \lambda_i &= m \end{aligned} \quad (2.5)$$

form a basis (i.e., a linearly independent and complete span) of S_n . This basis is orthogonal in the μ and m indices, but not in $\{\lambda\}$, as we see immediately from the metric tensor,

$$\langle \{\lambda'\}, \mu', m' | \{\lambda\}, \mu, m \rangle = \epsilon_{\lambda} \delta_{\mu'\mu} \delta_{m'm} \mathcal{M}_{\{\lambda'\}\{\lambda\}}^m(\ell_0). \quad (2.6)$$

By repeated application of the commutation relations (2.2) to move all $L_i(L_i^+)$ to the right (left) and use of $L_i|\mu\rangle$, $n - m > 0$, one sees that the only contributions come from matrix elements proportional to $\langle \mu', n-m' | (L_0)^i | \mu, n-m \rangle = \epsilon_{\lambda} \delta_{\mu\mu'} \delta_{mm'} (\ell_0)^i$, where $\ell_0 = (n-m + \pi^2)$ is the eigenvalue of L_0 for $|\mu, n-m\rangle$. Hence the basis is diagonalized up to a matrix $\mathcal{M}^m(\ell_0)$, that is solely determined by the algebraic properties of the L 's.

Null States

To define the orthogonal complement of S_n in V_n (i.e., R_n), one needs the existence of an inverse for the metric tensor,

$$\epsilon_{\lambda} \delta_{\mu'\mu} \delta_{m'm} \mathcal{M}_{\{\lambda\}\{\lambda'\}}^{m-1}. \quad (2.7)$$

The inverse of $\mathcal{M}^m$ exists, if the determinant is non zero or equivalently all the eigenvalues η^m are non zero. In general, we make this assumption, and we have checked it directly for $m = 2, \dots, 8$.

For $m = 1$, and $n = N$, there is such a zero eigenvalue, and therefore, we must deal with this null state to construct R_N .

As noted in Ref. (3), the states ,

$$L_1^+ |M, N-1\rangle, \quad (2.8)$$

all have zero norm

$$\langle M, N-1 | L_1 L_1^+ | M, N-1 \rangle = 2\epsilon_M \delta_{M, N-1+\pi^2} = 0.$$

Furthermore, since all other states in S_N are orthogonal to it, it is a null vector in the subspace S_N . We know there are no null states in V_N , so we must conclude that there is another state $|\psi\rangle \in S_N$ such that $\langle \psi | L_1^+ | M, N-1 \rangle \neq 0$. Del Giudice and Di Vecchia have pointed out that if a zero norm state $|z\rangle$ decouples, then some other zero norm state $|\bar{z}\rangle$ with $\langle \bar{z} | z \rangle \neq 0$ must also decouple. This is because one can write the identity onto $|z\rangle, |\bar{z}\rangle$ as

$$I_{z, \bar{z}} = \frac{|z\rangle \langle \bar{z}|}{\langle \bar{z} | z \rangle} + \frac{|\bar{z}\rangle \langle z|}{\langle z | \bar{z} \rangle}$$

and both $|z\rangle$ and $\langle z|$ decouple. An explicit construction of the conjugate ¹⁵⁾ of the state (2.5) is (we assume $N \geq 2$)

$$L_1^{c\dagger} |M, N-1\rangle \quad (2.9)$$

where

$$L_1^{c+} = \left(1 - \frac{1}{8(N-1)} L_1^+ \mathcal{P}^{R_{N-1}} L_1^* \right) L_1^{*+},$$

$$L_1^* = -i\sqrt{2} \pi \cdot a_1 + L_1(0),$$

$$\mathcal{P}^{R_{N-1}} = \sum_{\mu} |\mu, N-1\rangle \epsilon_{\mu} \langle \mu, N-1|.$$

We verify that $L_1^{c+} |\mu, N-1\rangle$ is zero norm

$$\langle \mu', N-1 | L_1^c L_1^{c+} | \mu, N-1 \rangle$$

$$= \langle \mu', N-1 | L_1^* L_1^{*+} | \mu, N-1 \rangle$$

$$- \frac{1}{8(N-1)} \langle \mu', N-1 | L_1^* L_1^{*+} \mathcal{P}^{R_{N-1}} L_1 L_1^{*+} | \mu, N-1 \rangle$$

$$- \frac{1}{8(N-1)} \langle \mu', N-1 | L_1^* L_1^+ \mathcal{P}^{R_{N-1}} L_1^* L_1^{*+} | \mu, N-1 \rangle.$$

But

$$\langle \mu', N-1 | L_1^* L_1^+ | \mu, N-1 \rangle = \langle \mu', N-1 | [L_1^*, L_1^+] | \mu, N-1 \rangle$$

$$= \langle \mu', N-1 | (-2\pi^2 + 2R) | \mu, N-1 \rangle$$

$$= 4(N-1) \epsilon_{\mu} \delta_{\mu\mu'}.$$

Thus

$$\langle \mu', N-1 | L_i^c L_i^{c\dagger} | \mu, N-1 \rangle = 0.$$

Furthermore, we can easily verify that

$$\langle \mu', N-1 | L_i^c L_i^{c\dagger} | \mu, N-1 \rangle = 4(N-1) \epsilon_{\mu} \delta_{\mu\mu'}.$$

So we can construct a projection operator,

$$P^{Z_N} = \frac{L_i^{c\dagger} P^{R_{N-1}} L_i^c}{4(N-1)} + \frac{L_i^{c\dagger} P^{R_{N-1}} L_i^c}{4(N-1)} \quad (2.10)$$

which projects onto the subspace Z_N generated by the zero norm states

$$L_i^{c\dagger} | \mu, N-1 \rangle, \quad L_i^{c\dagger} | \mu, N-1 \rangle. \quad (2.11)$$

We have already observed that states of the form,

$$(L_i^{c\dagger})^{\lambda_1} \dots (L_m^{c\dagger})^{\lambda_m} | \mu, N-m \rangle, \quad m = \sum_i \lambda_i \geq 1$$

are linearly independent. We can conclude also that the states,

$$(1 - P^{Z_N}) (L_i^{c\dagger})^{\lambda_1} \dots (L_m^{c\dagger})^{\lambda_m} | \mu, N-m \rangle \quad (2.12)$$

are linearly independent since they have the same scalar products $[(2.11)]$. In fact, it is now clear that (2.11) and (2.12) form a

linearly independent span of the subspace $\bar{S}_N = \{L_1^+|\lambda, N-1\rangle, L_2^+|\lambda, N-2\rangle, L_1^{C+}|\mu, N-1\rangle\}$. By adjoining $L_1^{C+}|\mu, N-1\rangle$ to S_N , we have created a space $\bar{S}_N$ which has a finite norm basis. It is clear also that all states in $\bar{S}_N$ decouple from the n scalar vertex. Thus, we conclude that the residue at the pole $\alpha(\pi^2) = N$ can be expanded in terms of particle states in R_N ($V_N = R_N \oplus S_N$).

Since R_N is minimal (i.e., we have used all known linear dependences to reduce the degeneracy as much as possible), we take its states to define the physical states at the pole $\alpha(\pi)^2 = N$.

Clearly, our conjugate vector ($|\bar{z}\rangle$) for our null state ($|z\rangle, \langle \bar{z}|z\rangle = 1$) is somewhat arbitrary. Given a vector $|\psi\rangle$ orthogonal to these, the state,

$$|\bar{z}\rangle = \alpha|z\rangle + \left(\frac{\alpha + \alpha^*}{\langle\psi|\psi\rangle}\right)^{1/2} |\psi\rangle \quad (2.13)$$

is also a suitable conjugate. Consequently, the physical space is not uniquely defined. However, the dimension and ghost structure is unique, and that is the real question.

III. CONSTRUCTION OF PROJECTION OPERATOR ONTO R_N AND REMARKS ON GHOST STRUCTURE

Our construction of the projection operator onto physical states R_N is a recursive one. In general, the hermitian projection operator onto a space $S = \{|s_i\rangle\}$ spanned by a non-orthogonal basis $|s_i\rangle$ is

$$P^S = \sum_{i,j} |s_i\rangle M_{ij}^{-1} \langle s_j|, \quad (P^S)^2 = P^S$$

where M_{ii}^{-1} is the inverse of the metric tensor $M_{ii} = \langle s_i | s_i \rangle$. In our case, the metric tensor for S_N is a real symmetric matrix, which

has an inverse (2.7), only if there are no null vectors¹⁶⁾ (e.g., zero eigenvalues). Making this assumption, the projection into S_n is

$$P^{S_n} = \sum_{\{\lambda\} \{\lambda'\}}' (L_1^{+\lambda_1} \dots L_m^{+\lambda_m}) \sum_{\mu, m}' \epsilon_{\mu} |\mu, n-m\rangle \mathcal{M}_{\{\lambda\} \{\lambda'\}}^{m-1} \langle \mu, n-m | (L_m^{\lambda'_m} \dots L_1^{\lambda'_1}).$$

Because of the partial diagonalization (in μ) and the independence of $\mathcal{M}$ on μ , the sum over μ gives

$$P^{R_{n-m}} = \sum_{\mu}' |\mu, n-m\rangle \epsilon_{\mu} \langle \mu, n-m|.$$

Hence a recursive definition of $P^{R_n} = 1_n - P^{S_n}$ is obtained,

$$P^{R_n} = 1_n - \sum_{\{\lambda\} \{\lambda'\}}' \sum_{m=1}^n L_1^{+\lambda_1} \dots L_m^{+\lambda_m} P^{R_{n-m}} L_m^{\lambda'_m} \dots L_1^{\lambda'_1} \mathcal{M}_{\{\lambda\} \{\lambda'\}}^{m-1} \quad (3.1)$$

where $P^{R_0} = |0\rangle \langle 0|$.

For $n = N$, the null states require special treatment. The $m = 1$ term, $L_1^+ P^{R_{N-1}} L_1$ has a singular matrix, $\mathcal{M}_{11}^1 = 0$, so we must replace this with our operator P^{Z_N} . Hence, $P^{R_N} = 1 - P^{S_N} = 1 - P^{Z_N} - P^{F_N}$ where P^{F_N} projects onto the finite norm states $(1 - P^{Z_N})(L_1^+)^{\lambda_1} \dots (L_m^+)^{\lambda_m} |\mu, N-m\rangle$, for $m = 2, 3, \dots, N$. We have remarked that the presence of $(1 - P^{Z_N})$ does not affect the metric tensor and that linear independence is preserved. Therefore,

$$P^{F_N} = (1 - P^{Z_N}) \sum_{\{\lambda\} \{\lambda'\}}' \sum_{m=2}^N L_1^{+\lambda_1} \dots L_m^{+\lambda_m} P^{R_{N-m}} L_m^{\lambda'_m} \dots L_1^{\lambda'_1} \times \mathcal{M}_{\{\lambda\} \{\lambda'\}}^{m-1} (1 - P^{Z_N}). \quad (3.2)$$

Finally, it is often convenient to go to a basis $|\alpha, \mu, n\rangle$ for S_n that diagonalizes $\mathcal{M}^m$. Suppose $U_{\alpha\{\lambda\}}^m$ is that orthogonal transformation that diagonalizes the real symmetric matrix $\mathcal{M}^m$,

$$\mathcal{M}_{\{\lambda\}\{\lambda'\}}^m = U_{\{\lambda\}\alpha}^m \eta_{\alpha}^m \delta_{\alpha\alpha'} U_{\alpha'\{\lambda'\}}^m.$$

If we also define the operator,

$$O_{\alpha}^{m\dagger} = \sum_{\{\lambda\}} (L_1^{\dagger})^{\lambda_1} \dots (L_m^{\dagger})^{\lambda_m} U_{\{\lambda\}\alpha}^m,$$

we have the more compact expressions,

$$\begin{aligned} \mathcal{P}^{R_n} &= I_n - \sum_{m=1}^n (\eta_{\alpha}^m)^{-1} O_{\alpha}^{m\dagger} \mathcal{P}^{R_{n-m}} O_{\alpha}^m, \\ \mathcal{P}^{R_N} &= (1 - \mathcal{P}^{Z_N}) \left(1 - \sum_{m=2}^N (\eta_{\alpha}^m)^{-1} O_{\alpha}^{m\dagger} \mathcal{P}^{R_{N-m}} O_{\alpha}^m \right) \\ &\quad \times (1 - \mathcal{P}^{Z_N}). \end{aligned} \tag{3.3}$$

Since the inverse of the eigenvalue $(\eta^m)^{-1}$ enters, clearly our assumption on the absence of null vectors ($\eta^m \neq 0$) is required.

We now make some observations. Using the recursion relation for $\mathcal{P}^{R_n}$ in terms of $\mathcal{P}^{R_k}$, $k < n$, we can express $\mathcal{P}^{R_N}$ entirely in terms of $\mathcal{P}^{R_0} = |0\rangle\langle 0|$, the operators $L_n^{\dagger}(L_n)$ and the matrices $\mathcal{M}^m$. Consequently, $\mathcal{P}^{R_N}$ is related to the vacuum state entirely in terms of the operators L_n [see Eq. (3.5)]. All our ignorance lies in the matrices $\mathcal{M}_{\{\lambda\}\{\lambda'\}}^m$, but if it is non-singular, we can say a good deal without knowing it explicitly. For example, in section IV, we can deduce the correction to single closed loops. The eigenvalues of $\mathcal{M}$ tell us

about the ghost structure of the spurious subspace. If we can prove that the number of ghosts in the spurious subspace $\bar{S}_N$ is equal to the number of ghosts in the whole space $V_N = \bar{S}_N \oplus R_N$, then we have proved that there are no ghosts in R_N .

As we have indicated in the Introduction, our attempts at a general proof that m is non-singular (i.e., that our assumption about null states is true) have so far been frustrated. However, by studying (2.6) for low values of m , we have been able to support this assumption and to analyze the ghost structure of the first few trajectories. We conclude this section with an outline of our analysis.

Structure of Real Subspace

If one writes an arbitrary state (at rest) in V_n as

$$(a_{01}^+)^{\lambda_1} \dots (a_{0m}^+)^{\lambda_m} (\tilde{a}_1^+)^{\mu_1} \dots (\tilde{a}_{n-m}^+)^{\mu_n} |0\rangle,$$

one sees immediately an analogy between our subspaces S_n and the subspace T_n with one or more time-components ($\sum_i \lambda_i = m > 0$). Indeed, if one starts with the space $V_0 = \{ |0\rangle \}$, and iterates to obtain S_n and T_n (or R_n and the subspace X_n with only space components), one sees that both subspaces have exactly the same number of states of each spin and parity (since a_{on}^+ and L_n^+ are scalars). Finally, to ensure that all ghosts are removed, we can ask if the signature (number of positive and negative eigenvalues) is also unchanged by the replacement $L_n^+ \rightarrow a_{on}^+$, $L_n \rightarrow a_{on}$.

To be more precise, let us look at the first few trajectories in detail. The leading trajectory at $\alpha(\pi^2) = N$ has a unique state $|\lambda, R = N, J = N\rangle = |\lambda, N, N\rangle$ with spin N built totally from N space components $\tilde{a}_{(1)}^+$. This state has positive norm. The first spurious state of spin $J = N - 1$ is given uniquely by $L_1^+ |\lambda, N - 1, N - 1\rangle$ and the 1×1 matrix is

$$\begin{aligned} \mathcal{M}' &= \langle \lambda, N-1, N-1 | L, L_1^+ | \lambda, N-1, N-1 \rangle \\ &= 2(N-1 + \pi^2) = 0. \end{aligned}$$

Removing this null state and its conjugate of the same spin $J = N - 1$ (L_1^+ and L_1^{C+} are scalars), the two $J = N - 1$ states at $1 - \pi^2 = N$ are decoupled (see the Figure).

The important thing to note is that the matrix $\mathcal{M}^1(\ell_0)$ only depends on the eigenvalue ℓ_0 of L_0 ($L_0 |\mu, m - n\rangle = \ell_0 |\mu, m - n\rangle$), and this eigenvalue is unchanged as you move along a given trajectory. This is generally the case for all our matrices $\mathcal{M}_{\lambda\lambda\lambda\lambda}^m(\ell_0)$.

To go to the next lower trajectory (see the Figure), we can get two $J = N - 2$ spurious states with the operators L_2^+ and $(L_1^+)^2$ acting on the state $|\lambda, N - 2, N - 2\rangle$, on the leading trajectory, so the matrix $\mathcal{M}^2(\ell_0)$ for $\ell_0 = N - 2 + \pi^2 = -1$ is relevant. It is easy to show that this has one positive and one negative eigenvalue as it should (i.e., like $a_{02}^+|0\rangle$ and $(a_{01}^+)^2|0\rangle$). In addition, we can apply L_1^+ to the $J = N - 1$ state in R_{N-1} . This state $|\mu, N - 1, N - 2\rangle$ is not zero norm, since the "off-shell" spurious state $L_1^+|\lambda, N - 2, N - 2\rangle$ does not have zero norm. This time, we need the matrix $\mathcal{M}^1(\ell_0)$ for $\ell_0 = -N - 2 + \pi^2 = -1$.

The general situation is that for projecting at $\alpha(\pi^2) = N$, the matrix $\mathcal{M}^m(\ell_0)$ enters directly into $\mathcal{P}^R_N$ for $\ell_0 = N - m + \pi^2 = 1 - m$, but also indirectly into $\mathcal{P}^{R-N-k}$ for $\ell_0 = 1 - m - k$ for $k = 0, 1, \dots, N - m$. Hence, to check ghosts on the first N daughters, we need to check the eigenvalues of

$$\mathcal{M}^m(\ell_0) \quad \text{for } \ell_0 = 1 - m, -m, \dots, 1 - N, \quad (3.4)$$

for the matrices $m = 1, 2, \dots, N$. Since these matrices depend only on the eigenvalue of L_0 , they can just as well be written as vacuum expectation values, where we choose $\pi^2 = l_0$ for the vacuum state.

$$\mathcal{M}_{\xi\lambda\zeta\xi'\lambda'}^m(l_0) = \langle 0 | L_m^{\lambda'_m} \dots L_1^{\lambda'_1} L_1^{+\lambda_1} \dots L_m^{+\lambda_m} | 0 \rangle$$

for $\sum_i i \lambda_i = \sum_i i \lambda'_i = m, \quad l_0 = \pi^2.$ (3.5)

Consequently, the removal of ghosts can be formulated in terms of the following purely algebraic statement

The signature of $\mathcal{M}^m(l_0)$ for $m > 1$ (3.5) must be unaffected by the substitution

$$L_n^+ = -i\sqrt{2n} \pi \cdot \alpha_n^+ + L_n^+(0) \rightarrow -i\sqrt{2n} \pi \cdot \alpha_n^+$$

where L_n^+ obey the Lie algebra (2.2). (3.6)

We are optimistic that the essentially group theoretical character of this problem will allow it to be solved analytically. In the meantime, we have checked it for small m , and furthermore, we note in the Appendix its validity for $l_0 \rightarrow -\infty$ and all m .

We quote the results of our analysis ¹⁷⁾ for the first nine trajectories : 1) no further null states, 2) all ghosts are removed. As an added bit of information, we have determined the degeneracy $d(m)$ for the first six trajectories,

$$d(m) = 1, 0, 3, 3, 9, 10 \quad \text{for } m = 0, \dots, 5.$$

In particular, we note the existence of three trajectories on the level of the second odd-daughter ($m = 4$), even though no $J = 0$, $N = 3$ state is present ³⁾.

In general, if $n(N, J)$ is the number of states with spin J and $R = N$ that can be constructed using only the space components operators (a_n^+), and if there are no more null states, then

$$D(N, J) = n(N, J) - n(N-1, J)$$

is the number ¹⁸⁾ of states with spin J at $\alpha(\pi^2) = N$. Except for low spin states, this also gives the degeneracy of the trajectory $[d(m) = D(N, N-m) \text{ unless } J = N-m \approx 0]$.

IV. SUBTRACTION OF SPURIOUS STATE CONTRIBUTIONS FROM THE SINGLE LOOP DIAGRAMS

The calculation of the corrected planar loop integrand proceeds along standard lines. We first take note of a valuable relation. Defining,

$$\Gamma^n = V^{k_m} D(R+n, s_m) \dots D(R+n, s_2) V^{k_1}$$

where $s_2 = (\pi + k_1)^2, \dots, s_m = (\pi + k_1 + \dots + k_{m-1})^2$, $\sum_i k_i = 0$ and $\alpha(\pi^2) = N$, we have the identity,

$$\begin{aligned} & \langle \mu, n-m | L_m^{\lambda_m} \dots L_1^{\lambda_1} \Gamma^{N-n} | n \rangle \\ &= \langle \mu, n-m | \Gamma^{N-n+m} | n-m \rangle L_m^{\lambda_m} \dots L_1^{\lambda_1} \\ &= \langle \mu, n-m | \Gamma^{N-n+m} \mathcal{P}^{R_{n-m}} L_m^{\lambda_m} \dots L_1^{\lambda_1} \end{aligned} \quad (4.1)$$

where $1 \leq m = \sum_i \lambda_i \leq n$, $0 \leq n \leq N-1$, and it is understood that $\langle n, n-m | \in R_{n-m}$. This identity (4.1) is easily demonstrated by noting that since each $L_n(\pi)$ is on shell $\alpha(\pi^2) = N$, it can be replaced by $W_n - j$ for a suitable integer j . Then, by repeated use of the commutators,

$$(W_n(p) - j) V^k = V^k (W_n(p+k) + n - j),$$

$$(W_n(p) + n - j) D(R, p^2) = D(R + j + n, p^2) (W(p) - j),$$

we push each factor through Γ^{N-n} and change it back to L_n when it gets to 1_n , since it is again on shell ($\sum_i k_i = 0$).

The residue of the planar loop integrand (without spurious states) is

$$\begin{aligned} \text{Res}_{\alpha(\pi^2)=N} M(\pi) &= \text{Tr} \{ \mathcal{P}^{R_N} \Gamma^0 \} \\ &= \text{Tr} \{ \Gamma^0 \} - \text{Tr} \{ \mathcal{P}^{\tilde{N}} \Gamma^0 \} \\ &= \text{Tr} \{ \Gamma^0 \} - \text{Tr} \{ \mathcal{P}^{Z_N} \Gamma^0 \} - \text{Tr} \{ \mathcal{P}^{F_N} \Gamma^0 \}. \end{aligned}$$

Now

$$\begin{aligned} \text{Tr} \{ \mathcal{P}^{F_N} \Gamma^0 \} &= \text{Tr} \{ (1_N - \mathcal{P}^{Z_N}) \sum_{\alpha, m}^1 (\gamma_\alpha^m)^{-1} O_\alpha^{m\dagger} \mathcal{P}^{R_{N-m}} O_\alpha^m (1 - \mathcal{P}^{Z_N}) \Gamma^0 \} \\ &= \text{Tr} \left\{ \sum_{\alpha, m} (\gamma_\alpha^m)^{-1} O_\alpha^{m\dagger} \mathcal{P}^{R_{N-m}} O_\alpha^m \Gamma^0 \right\} \\ &= \sum_{\alpha, m}^1 (\gamma_\alpha^m)^{-1} \text{Tr} \{ \mathcal{P}^{R_{N-m}} O_\alpha^m O_\alpha^{m\dagger} \mathcal{P}^{R_{N-m}} \Gamma^m \} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{m=2}^N \sum_{\alpha} \text{Tr} \{ \mathcal{P}^{R_{N-m}} \Gamma^m \} \\
 &= \sum_{m=2}^N \delta_m \text{Tr} \{ \mathcal{P}^{R_{N-m}} \Gamma^m \},
 \end{aligned}$$

where δ_m = number of partitions of the integer m (i.e., δ_m is the dimension of $\mathcal{M}^m$). Also,

$$\begin{aligned}
 \text{Tr} \{ \mathcal{P}^{2N} \Gamma^0 \} &= \frac{1}{4(N-1)} \left[\text{Tr} \{ L_1^{c\dagger} \mathcal{P}^{R_{N-1}} L_1 \Gamma^0 \} + \text{Tr} \{ L_1^\dagger \mathcal{P}^{R_{N-1}} L_1^c \Gamma^0 \} \right] \\
 &= \frac{1}{4(N-1)} \left[\text{Tr} \{ \mathcal{P}^{R_{N-1}} L_1 L_1^{c\dagger} \mathcal{P}^{R_{N-1}} \Gamma^0 \} + \text{Tr} \{ \mathcal{P}^{R_{N-1}} L_1^c L_1^\dagger \mathcal{P}^{R_{N-1}} \Gamma^0 \} \right] \\
 &= 2 \text{Tr} \{ \mathcal{P}^{R_{N-1}} \Gamma^0 \}.
 \end{aligned}$$

Thus

$$\text{Res}_{\alpha(\pi^0)=N} M(\pi) = \text{Tr} \{ \Gamma^0 \} - \sum_{m=1}^N \delta_m \text{Tr} \{ \mathcal{P}^{R_{N-m}} \Gamma^m \} - \text{Tr} \{ \mathcal{P}^{R_{N-1}} \Gamma^0 \} \quad (4.2)$$

But now we can write

$$\begin{aligned}
 \text{Tr} \{ \mathcal{P}^{R_i} \Gamma^m \} &= \text{Tr} \{ \Gamma^m \} - \text{Tr} \{ \mathcal{P}^{S_i} \Gamma^m \} \\
 &= \text{Tr} \{ \Gamma^m \} - \sum_{k=1}^i \delta_k \text{Tr} \{ \mathcal{P}^{R_{i-k}} \Gamma^{m+k} \}
 \end{aligned} \quad (4.3)$$

Recalling that $\text{Tr} \{ \Gamma^m \}$ modifies the loop by an extra factor x^m where $x = \prod_i u_i$ and u_i is the integration variable of the i^{th} propagator, we see clearly by (4.2) and (4.3) that the loop integrand is merely modified by the extra factor,

$$1 - x f^{N-1}(x) - \delta_2 x^2 f^{N-2}(x) - \dots - \delta_N x^N f^0(x) - x f^{N-1}(x)$$

where $f^k(x)$ is defined recursively by

$$f^0(x) = 1$$

$$f^k(x) = 1 - x f^{k-1}(x) - \delta_2 x^2 f^{k-2}(x) - \dots - \delta_k x^k f^0(x),$$

so that the correction factor to the loop integrand is just

$$f^N(x) = x f^{N-1}(x).$$

We now give a proof by induction that

$$f^k(x) = \prod_{r=1}^{\infty} (1 - x^r) + O(x^{k+1}).$$

For this purpose define

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{1 - x^n} = \sum_{n=0}^{\infty} \delta_n x^n$$

and we prove that $F(x) f^k(x) = 1 + O(x^{k+1})$ which will establish

our result. For $k = 0$ it is trivial ; so let us assume that $F(x) f^m(x) = 1 + O(x^{m+1})$ for all $m < k$. Then,

$$\begin{aligned} F(x) f^k(x) &= \sum_{r=0}^{\infty} \delta_r x^r - F(x) \sum_{m=1}^k x^m f^{k-m}(x) \delta_m \\ &= \sum_{r=0}^k \delta_r x^r - \sum_{m=1}^k x^m [1 + O(x^{k-m+1})] \delta_m + O(x^{k+1}) \\ &= 1 + O(x^{k+1}), \end{aligned}$$

as desired. So finally, our correction factor to the loop is

$$f^N(x) - x f^{N-1}(x) = (1-x) \prod_{r=1}^{\infty} (1-x^r) + O(x^{N+1}).$$

But since terms of order x^{N+1} do not contribute to the pole at $\alpha(\pi^2) = N$, the correction factor $(1-x) \prod_{r=1}^{\infty} (1-x^r)$ ensures that no spurious particles are included at any of the poles.

Diagrams with any number of twists are equally as easy to calculate since at the poles the twist operator is just $(-1)^R$. For example for two twists we can calculate the double residue,

$$\text{Res}_{\substack{\alpha_i = N \\ \alpha_j = M}} M(\pi) = \text{Tr} \left\{ V^{k_m} D_m \dots V^{k_i} \mathcal{P}^{R_m} \Omega \dots V^{k_j} \mathcal{P}^{R_N} \Omega \dots \right\}.$$

Now

$$\mathcal{P}^{R_N} \Omega = \mathcal{P}^{R_N} \sum_{n=0}^{\infty} \frac{(L_1^+)^n}{n!} (-)^R = \sum_{n=0}^{N'} \frac{(L_1^+)^n}{n!} \mathcal{P}^{R_N-n} (-)^R.$$

But all of these terms decouple except $n = 0$. Thus

$$\text{Res}_{\substack{d_i=N \\ d_j=M}} M(\pi) = (-1)^{N+M} \text{Tr} \{ V^{k_m} D_m \dots V^{k_i} \rho^{R_m} \dots V^{k_j} \rho^{R_N} \dots \}.$$

The trace is calculated exactly as before. With an even number of twists, the residues come out right with the correction,

$$(1-x) \prod_{r=1}^{\infty} (1-x^r). \quad (4.4)$$

But for an odd number of twists the result is the factor

$$(1+x) \prod_{r=1}^{\infty} (1-(-x)^r). \quad (4.5)$$

IV. CONCLUDING REMARKS

We have accomplished two main objectives. First, we have constructed a basis for the spurious subspace, which is partially diagonalized - the remaining diagonalization being dependent only on the conformal group generated by the Lie algebra for L_n^+ . For example, Lorentz properties (spin, parity, etc...) are quite literally orthogonal to this problem. While we hope this will lead to an analytic ¹⁹⁾ proof of Virasoro's no ghost conjecture, it already greatly facilitates its empirical verification. By checking the eigenvalues of at most a 22×22 dimensional matrix ($m = 8$), we are able to eliminate all (several hundred) ghosts below $\alpha = 9$. This demonstration and the identification of an independent Ward identity for each state with one or more a_{on}^+ make Virasoro's bold conjecture quite plausible.

Second, we have constructed projection operators and demonstrated their utility by correcting the single loop diagrams. Here we note the basic simplicity of the symmetrized Born term for $\alpha_0 = 1$ extends to higher order. Apparently, the increased symmetry for $\alpha_0 = 1$ allows an even more elegant multiloop formulation.

Finally, we must emphasize that the whole analysis rests on the assumption of no further null states beyond those of Ref. 3. Beyond the first nine trajectories where this assumption has been explicitly verified, new null states would cause additional states to be removed. A well-defined new set of null states could probably be incorporated in the present scheme in close analogy to those already found. However, we feel the evidence against such additional null vectors is about as strong as our evidence against ghost states.

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A P P E N D I X

A BASIS FOR THE SUBSPACE S_n

We wish to show that the states (2.5),

$$|\{\lambda\}, \mu, m\rangle = (L_1^+)^{\lambda_1} \dots (L_m^+)^{\lambda_m} |\mu, n-m\rangle \quad (A1)$$

for all μ ($|\mu, n-m\rangle \in R_{n-m}$), $m = 1, \dots, n$ and $\sum_i \lambda_i = m$ form a basis for S_n . The idea behind the construction is quite simple. Any product of L^+ 's (e.g., $L_{\alpha_1}^+ L_{\alpha_2}^+ \dots L_{\alpha_K}^+$, $\sum \alpha_i = m$) can be expressed as a sum over canonically ordered products,

$$L_{\alpha_1}^+ L_{\alpha_2}^+ \dots L_{\alpha_K}^+ = \sum_{\{\lambda\}} C(\lambda, \alpha) (L_1^+)^{\lambda_1} \dots (L_m^+)^{\lambda_m} \quad (A2)$$

by repeated use of the commutators. There is one term with K L 's ($\sum \lambda_i = K$, $C = 1$), and the rest have fewer L 's, so the proof of Eq. (A2) goes by induction. As a consequence, the commutation relations ensure completeness, but since (A2) is unique, they give no dependencies. Note that all the L^+ 's are on an equal footing, as we could have chosen any fixed ordering (e.g., $2, 3, \dots, m, 1$ or $m, m-1, \dots, 1$ etc.). The results will be unchanged; indeed, it is easy to show that the signature of the metric tensor would be exactly the same.

(1) Completeness. We assume completeness for our basis for $S_{n'}$, $n' < n$, so that any state $|\psi, n-j\rangle$ in V_{n-j} can be written

$$|\psi, n-j\rangle = \sum_{\mu} C_{\mu} |\mu, n-j\rangle + \sum_{\{\lambda\}} d_{\{\lambda\}}^{\mu, m} L_1^{+\lambda_1} \dots L_m^{+\lambda_m} |\mu, n-j-m\rangle \quad (A3)$$

Hence any state $\alpha_j L_j^+ |\Psi, n-j\rangle$ in S_n is expressed as a sum of states $L_j^+ |\mu, n-j\rangle$, and states $L_j^+ (L_1^+)^{\lambda_1} \dots (L_m^+)^{\lambda_m} |\mu', n-j-m\rangle$. The latter can be put in canonical form using (A2).

(2) Independence. We assume there are no null states in S_n for $n < N$. Consequently, the orthogonality of states (A1) with different μ or m ensures that the only dependences are of the form

$$\sum_{\{\lambda\}} C_{\{\lambda\}} (L_1^+)^{\lambda_1} \dots (L_m^+)^{\lambda_m} |\mu, n-m\rangle = 0.$$

Select the term $a_1^+ \dots a_{n-1}^+$ in L_n^+ that creates the first type of excitation.

$$\sum_{\{\lambda\}} C'_{\{\lambda\}} (\pi \cdot a_1^+)^{\lambda_1} (a_1^+ \cdot a_1^+)^{\lambda_2} \dots (a_1^+ \cdot a_{m-1}^+)^{\lambda_m} |\mu, n-m\rangle$$

Clearly those terms that maximize $\lambda_1 + 2\lambda_2 + \lambda_3 + \dots + \lambda_m$ must cancel among themselves. But they are evidently independent, so inductively we show $C_{\{\lambda\}} \equiv 0$.

(3) Basis for $\bar{S}_N$. At $\alpha(\pi^2) = N$, we have the basis with null states (in subspace Z_N),

$$L_1^+ |\mu, N-1\rangle, \quad L_1^{c+} |\mu, N-1\rangle \quad (A4)$$

and the orthogonal part of the remaining states (A1),

$$(1 - \sigma^{\pi^2}) |\{\lambda\}, \mu, m\rangle, \quad m = 2, \dots, N. \quad (A5)$$

Since $\bar{S}_N$ is the union of S_N and Z_N , clearly completeness is unaffected.

Next we note that from the definition of $\mathcal{P}^{Z_N}$,

$$\mathcal{P}^{Z_N} = \frac{L_1^+ \mathcal{P}^{R_{N-1}} L_1^c}{4(N-1)} + \frac{L_1^c \mathcal{P}^{R_{N-1}} L_1}{4(N-1)}$$

and the orthogonality of $L_1^+ |\mu, N-1\rangle$ with $|\xi\lambda\zeta, \mu, m\rangle$ ($m \neq 1$), we can show that the metric tensor is unaffected by the $1 - \mathcal{P}^{Z_N}$ factor.

$$\begin{aligned} & \langle \xi\lambda'\zeta, \mu', m' | (1 - \mathcal{P}^{Z_N}) (1 - \mathcal{P}^{Z_N}) | \xi\lambda\zeta, \mu, m \rangle \\ &= \langle \xi\lambda'\zeta, \mu', m' | \xi\lambda\zeta, \mu, m \rangle = \epsilon_{\lambda} \delta_{\mu'\mu} \delta_{m'm} \mathcal{M}_{\xi\lambda'\zeta\xi\lambda\zeta}^m(l_0) \end{aligned}$$

The statement of linear dependence is just Eq. (A4) projected with $(1 - \mathcal{P}^{Z_N})$. This gives one new term from $\mathcal{P}^{Z_N}$,

$$L_1^+ \mathcal{P}^{R_{N-1}} L_1^c \sum_{\xi\lambda\zeta}' C_{\xi\lambda\zeta} L_1^{+\lambda_1} \dots L_m^{+\lambda_m} |\mu, N-m\rangle / (4(N-1))$$

proportional to $L_1^+ |\psi, N-1\rangle$, $|\psi, N-1\rangle \in R_{N-1}$. Thus we see that assuming no null vectors in the space spanned by $|\xi\lambda\zeta, \mu, N-m\rangle$ for $m > 1$, we again have linear independence.

(4) Metric tensor. The only lack of rigour in our present construction is the "no null vector assumption", which is actually the statement that

$$\det(\mathcal{M}^m(l_0)) \neq 0, \text{ for } l_0 = 1-m, -m, \dots, -\infty.$$

We have checked this directly for $m = 2, \dots, 8$ and $\ell_0 \geq -7$. Also it is reassuring to note that it is valid as $\ell_0 \rightarrow -\infty$ for all m . Indeed, in this limit all the eigenvalues are given by

$$\zeta_{\{\lambda\}}^m = \prod_{i=1}^m (2i\ell_0)^{\lambda_i} \lambda_i! (1 + O(\frac{1}{\ell_0})) \quad (A6)$$

which also gives the correct signature for ghost killing. To see this, we renormalize the spurious states,

$$|\{\lambda\}\rangle = |\{\lambda\}, m\rangle / \prod_i (2i|\ell_0|)^{\frac{\lambda_i}{2}} (\lambda_i!)^{\frac{1}{2}}$$

In this "new" basis the metric tensor $\tilde{m}^m$ is easily shown to be diagonal as $|\ell_0| \rightarrow -\infty$.

$$\tilde{m}_{\{\lambda\}\{\lambda'\}}^m = \prod_i \left(\frac{\ell_0}{i|\ell_0|}\right)^{\lambda_i} \delta_{\lambda_i \lambda'_i} + O(\frac{1}{|\ell_0|})$$

Hence, these states have eigenvalue ± 1 for $\ell_0 \rightarrow -\infty$.

Since all eigenvalues are positive for $\ell_0 > 0$ [Eq. (3.5), π , is spacelike], the sign of the negative eigenvalues must change somewhere on the negative real axis. If these zeros avoid the integers $\ell_0 = 1 - m, -m, -\dots, -\infty$, the no null state hypothesis ($m \neq 1$) is correct. In the few cases we have computed, the change in sign occurs in the interval $[0, 1 - m]$, so that the no ghost hypothesis is also supported. It could well be that there are no zeros for $\det \eta^m$ for $\ell_0 \leq 1 - m$ ($m \neq 1$), so the two hypotheses arise from essentially the same mechanism.

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- 13) We thank Dr. J.H. Weis for drawing our attention to this c number term.
- 14) The states in S_n for $n < N$ are not purely spurious, since they can couple to scalars for $1 - \pi^2 = N$, but the true off-shell spurious states $[L_n^+ - L_0 - (n-1)]|\psi\rangle$ are inconvenient to use since they mix eigenvalues of R .
- 15) For $N = 1$, S_1 contains the single null vector $L_1^+|0\rangle = -i\pi \cdot a_1^+|0\rangle$ and the conjugate can be taken to be $|c\rangle = -i(\pi \cdot a_1^+ + \pi_0 a_0^+)|0\rangle$ instead of $L_1^{+c}|0\rangle$. The projection operator is then

$$P^{\pm 1} = \frac{|c\rangle\langle 0|L_1}{2\pi_0} + \frac{L_1^+|0\rangle\langle c|}{2\pi_0}$$

and $\bar{S}_1 = \{L_1^+|0\rangle, |c\rangle\}$.

- 16) For $\alpha_0 \neq 1$, the projection operator with the inverse metric tensor [C.B. Thorn, Phys.Rev. (to be published)], has been expressed on shell $[\alpha(\pi^2) = N]$ as a finite series in $L_{\pm}, L_0$ [R.C. Brower and J.H. Weis, Nuovo Cimento Lett. 3, 285 (1970)], so the existence of the inverse and the absence of null states is guaranteed. No generalization to $\alpha_0 = 1$ has been found.

- 17) The matrices above m^4 were calculated directly from recursion relations by computer programmed in ALGO. For this ingenious bit of programming, I am indebted to the skill of Dr. Lyle Smith, Data Handling Division, CERN.
- 18) Since the twist operator $\mathcal{Q} = (-)^R e^{-L_1}$ has eigenvalue $(-)^N$ for the real state at $\alpha(\pi^2) = N$, signature will remove all $N = \text{odd}$ states [or in a model with isospin factor $\text{Tr}(\tau_1 \dots \tau_N)$, $N + 1 = \text{odd}$ states]. The definite signature of each trajectory for $\alpha_0 = 1$ was first demonstrated by Z. Koba and H.B. Nielsen, University of Copenhagen Preprint (1969).
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Matrices, $m^m(\lambda_0)$ giving spurious states at $1 - \pi^2 = N$ on leading four trajectories. Only $m^1(0)$ has a zero determinant, and therefore yields a null state.