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CONSTRAINTS ON THE DERIVATIVES OF THE $\pi\pi$ SCATTERING AMPLITUDE
FROM POSITIVITY

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A B S T R A C T

Conditions derived from
positivity are given both above and
below threshold for the derivatives
of the $\pi\pi$ scattering amplitude.

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1. - INTRODUCTION

From the general assumptions of unitarity, crossing and analyticity in a domain derivable from axiomatic field theory, it can be shown that the $\pi\pi$ scattering amplitudes satisfy twice subtracted dispersion relations ¹⁾. Hence D and higher partial waves have the Froissart-Gribov representation ²⁾ so that if $f_\ell(s)$ are the partial waves in the s channel for even isotopic spin states where s is the centre-of-mass energy squared ^{*}),

$$f_\ell(s) = \frac{4}{\pi(4-s)} \int_4^\infty A_t(s,t) Q_\ell\left(\frac{2t}{4-s}-1\right) dt; \ell=2,4,\dots \quad (1.1)$$

when $0 < s < 4$. $A_t(s,t)$ is the absorptive part of the scattering amplitude ^{**)} and for certain isotopic spin combinations ^{***)} in the s channel it has the expansion

$$A_t(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) \alpha_\ell(t) P_\ell\left(1 + \frac{2s}{t-4}\right) \quad (1.2)$$

where the $\alpha_\ell(t) \geq 0$ from unitarity.

It follows from (1.2) that for $t \geq 4$ and $0 < s < 4$,

$$A_t(s,t) \geq 0 \quad (1.3)$$

^{*}) We take units such that the mass of the pion is unity.

^{**)} We will use the suffices s and t to indicate physical quantities in the s channel and t channel respectively.

^{***)} See Eq. (2.1).

In previous papers ^{3),4)} it was shown that the condition (1.3) puts strong constraints on the $f_\ell(s)$ when $0 < s < 4$ and in fact they are sufficient to prove that the f_ℓ 's are solutions of a Stieltjes moment problem.

However, these results did not take into account the full content of positivity which requires $\alpha_\ell(s) \geq 0$ for all ℓ or equivalently

$$\int_{-1}^{+1} \cos \theta_t P_\ell(\cos \theta_t) A_t(s, t) d \cos \theta_t \geq 0, \ell = 0, 1, \dots; 0 \leq s \leq 4 \quad (1.4)$$

In this article we consider the problem of finding the constraints that this positivity places on the derivatives $(\partial^m T(s, t)) / ((\partial \cos \theta_s)^m)$ of the $\pi\pi$ scattering amplitude. Three types of constraints are considered.

- i) - Firstly conditions on the derivatives of $T(s, t)$ at any angle when s is restricted to $0 \leq s \leq 4$, which are derived from $A_t(s, t) \geq 0$.
- ii) - Secondly constraints on the quantities

$$\left. \frac{\partial^{n+m} T(s, t)}{\partial s^n (\partial \cos \theta_s)^m} \right|_{\cos \theta_s = 0} \quad n, m = 0, 1, 2, \dots$$

which ensure that

$$\frac{\partial^n A_t(s, t)}{(\partial \cos \theta_t)^n} \geq 0 \quad n = 0, 1, 2, \dots \quad (1.5)$$

Although the conditions (1.5) are weaker than positivity, they are already sufficient to obtain many of the consequences of the full positivity ⁵⁾.

iii) - Finally we derive constraints on the quantities

$$\frac{\partial^n A_t(s,t)}{(\partial \cos \theta_t)^n} \Big|_{\cos \theta_t = 1}$$

with t in the physical region which are necessary for the full positivity (1.4). This set of constraints can be extended to give a set of sufficient conditions for (1.4) to hold.

These three different types of constraints may prove useful for the parametrization of the $\pi\pi$ amplitudes, as they can be used both above and below threshold and thus may be directly inserted into dispersion relations. Also it is easier to handle "crossing" when dealing with derivatives of $T(s,t)$ than when working with the partial wave amplitudes $f_\ell(s)$ and it is therefore useful to have constraints on the former quantities.

2. - CONSTRAINTS ON THE DERIVATIVES OF THE SCATTERING AMPLITUDE $T(s,t)$ AT ANY ANGLE FOR FIXED s BELOW THRESHOLD

We work below threshold, i.e., $0 < s < 4$, for $\pi\pi$ scattering in any of the isospin combinations,

$$T(s,t) = (1 + \lambda/3) T^{(0)}(s,t) + (2\lambda/3) T^{(2)}(s,t), \quad \lambda \gg -1 \quad (2.1)$$

where $T^{(I)}$ is the amplitude for scattering in the s channel state with total isospin I . These combinations are such that the absorptive part in the t channel $A_t(s,t)$ satisfies the conditions (1.4)³⁾.

For $0 < s < 4$ there exists the fixed s dispersion relation,

$$T(s,t) = (\text{polynomial of first degree in } \cos \theta_s) + \frac{1}{\pi} \int_{t_0}^{\infty} \frac{1}{t'^2} \left[\frac{t^2}{t'-t} + \frac{(4-s-t)^2}{t'-4+s+t} \right] A_t(s,t') dt' \quad (2.2)$$

Let

$$\mu_m(s, \cos \theta_s) = \frac{\pi}{m!} \frac{\partial^m T(s, t)}{(\partial \cos \theta_s)^m} \quad (2.3)$$

Then from (2.2) for $m \geq 2$,

$$\mu_m(s, \cos \theta_s) = \int_{z_0}^{\infty} \left\{ \frac{1}{(z - \cos \theta_s)^{m+1}} - \frac{1}{(-z - \cos \theta_s)^{m+1}} \right\} A_4[s, (z+)(z-\frac{1}{2})] dz \quad (2.4)$$

where

$$z_0 = \left[\frac{4+s}{4-s} \right]$$

We consider the case when $\cos \theta_s \geq 0$. Then defining $\chi_{\pm} \equiv 1/(\pm z - \cos \theta_s)$, we have that as z varies between ∞ and z_0 , χ_+ varies between 0 and $\chi_+^0 \equiv 1/(z_0 - \cos \theta_s)$. Furthermore for any $z \geq z_0$ and when $\cos \theta_s \geq 0$, $\chi_+ \geq |\chi_-|$, we are now in a position to prove the following result.

Theorem 1

When $4 > s > 0$ and $\cos \theta_s \geq 0$, then for any odd integer $m \geq 2$,

$$\sum_{j=0}^n \binom{n}{j} (-1)^j (\chi_+^0)^{2(n-j)} \mu_{m+2j}(s, \cos \theta_s) / (m+2j+1) \geq 0, \quad n=0,1,2,\dots \quad (2.5)$$

and for any even integer $m \geq 2$,

$$\sum_{j=0}^n \binom{n}{j} (\chi_+^0)^{2(n-j)} \mu_{m+2j}(s, \cos \theta_s) \geq 0, \quad n=0,1,2,\dots \quad (2.6)$$

Proof

Consider first the case when m is odd. The left-hand side of (2.5) is equal to

$$\begin{aligned} & \int_{z_0}^{\infty} \sum_{j=0}^n \binom{n}{j} (-1)^j (\chi_+^0)^{2(n-j)} \frac{\{\chi_+^{m+2j+1} - \chi_-^{m+2j+1}\}}{(m+2j+1)} A_+ [s, (z+1)(2-s_2)] dz \\ &= \int_{z_0}^{\infty} \sum_{j=0}^n \binom{n}{j} (-1)^j (\chi_+^0)^{2(n-j)} \left\{ \int_0^{\chi_+} \chi^{m+2j} d\chi + (-1)^{m+2j} \int_0^{|\chi_-|} \chi^{m+2j} d\chi \right\} A_+ [s, (z+1)(2-s_2)] dz \\ &= \int_{z_0}^{\infty} \left\{ \int_0^{\chi_+} \chi^m [(\chi_+^0)^2 - \chi^2]^n d\chi + (-1)^{m+2j} \int_0^{|\chi_-|} \chi^m [(\chi_+^0)^2 - \chi^2]^n d\chi \right\} A_+ [s, (z+1)(2-s_2)] dz \end{aligned} \quad (2.7)$$

Since $\chi_+^0 \geq \chi_+ \geq |\chi_-|$ when $\cos \theta_s \geq 0$, the above quantity is positive for odd m so that (2.5) is proved. Similarly it can be proved that (2.6) is true for all even m .

Corollary

When $\cos \theta_s \leq 0$, the inequality (2.5) is reversed and (2.6) holds as it is. The second result is an immediate consequence of the fact that $\mu_m(s, \cos \theta_s) \geq 0$ for even m when $\cos \theta_s \leq 0$, while the first follows from the relation

$$\mu_m(s, \cos \theta_s) = -\mu_m(s, -\cos \theta_s) \quad (2.8)$$

when m is odd.

Although conditions (2.5) and (2.6) on the $\mu_n(s, \cos\theta_s)$ are very stringent, they are only necessary but not sufficient conditions for $A_t[s, (z+1)(2-s/2)] \geq 0$. However, when $\cos\theta_s = 0$ they coincide with the conditions of Ref. 4) on the quantities $\mu_m \equiv \mu_m(s, 0)$ and are then also sufficient to ensure the positivity of A_t .

3. - CONSTRAINTS ON THE DERIVATIVES OF $T(s, t)$ WITH RESPECT TO s

We will in this Section obtain sets of conditions on the quantities $\partial^{m+j} T(s, t) / \partial s^m (\partial \cos\theta_s)^j$ which ensure that Martin's positivity

$$\frac{\partial^n A_t(s, t)}{\partial s^n} \geq 0, \quad n=0, 1, 2, \dots \quad (3.1)$$

holds for $0 < s < 4$.

In fact we will consider only the case when $\cos\theta_s = 0$, as it is only then that simple conditions can be obtained even for the first condition of (3.1), viz., $A_t \geq 0$ to hold. Taking (2.4) for $\cos\theta_s = 0$ and defining for even m

$$R_m(s) = \frac{1}{4} (4-s)^{-m} \mu_m(s, 0) \quad (3.2)$$

we obtain the representation

$$R_m(s) = \int_4^{\infty} dt A_t(s, t) (2t-4+s)^{-m-1} \quad (3.3)$$

where the change of variables $z \rightarrow t$ has been made and we take this to be also the definition of $k_m(s)$ for m odd.

Differentiating with respect s ,

$$k_m^{(1)}(s) = \int_4^\infty dt A_t^{(1)}(s,t) (2t-4+s)^{-m-1} - (m+1) \int_4^\infty dt A_t(s,t) (2t-4+s)^{-m-2} \quad (3.4)$$

where $k_m^{(1)} \equiv \frac{\partial k_m}{\partial s}$ and $A_t^{(1)} \equiv \frac{\partial}{\partial s} A_t(s,t)$. Hence we see that the quantity $h_m^{(1)} \equiv k_m^{(1)} + (m+1) k_{m+1}$ has the representation

$$h_m^{(1)}(s) = \int_4^\infty dt A_t^{(1)}(s,t) (2t-4+s)^{-m-1} \quad (3.5)$$

Quite generally we will have

$$\begin{aligned} h_m^{(n)}(s) &\equiv k_m^{(n)}(s) - \sum_{l=1}^n \frac{\Gamma(n+1) \Gamma(m+l+1) (-1)^l}{\Gamma(n-l+1) \Gamma(l+1) \Gamma(m+1)} h_{l+m}^{(n-l)}(s) \\ &= \int_4^\infty A_t^{(n)}(s,t) (2t-4+s)^{-m-1} dt \end{aligned} \quad (3.6)$$

By a straightforward application of the methods of Refs. 2), 6), the following theorem may be proved.

Theorem 2

A set of necessary and sufficient conditions for (3.1) to be true is that the quantities $h_m^{(n)}(s)$ defined in (3.6) satisfy

$$\sum_{j=0}^l \binom{l}{j} (-1)^j h_{m+2j}^{(n)}(4+s)^{2(l-j)} \geq 0; \quad n, m = 0, 1, \dots; \quad 0 < s < 4 \quad (3.7)$$

We want to write the conditions (3.7) as constraints on the quantities

$$\frac{\partial^n \mu_m(s,0)}{\partial s^n} = \frac{\partial^{m+n} T(s,t)}{\partial s^n (\partial \omega \theta)^m} \Big|_{\omega \theta = 0} \quad (3.8)$$

However, they have the unpleasant feature of depending on $k_m^{(n)}(s)$ for both odd and even values of m . But it is only for even values of m that we can use (3.2) to get $k_m^{(n)}(s)$ in terms of the $\partial^n \mu_m(s,0)/\partial s^n$.

To obtain $k_m(s)$ as defined by (3.3) for odd m from its value for even m , we can use a very accurate method of interpolation which has been discussed by one of us in Section 5 of the second paper of Ref. 3). Similar results are true for the quantities $k_m^{(n)}(s)$. Using these values the conditions (3.7) can be tested to a high degree of accuracy.

4. - CONSTRAINTS FOR PHYSICAL t

So far we have not derived constraints which give the full positivity conditions (1.4) and this is what we will do now. These conditions are equivalent to having the expansion

$$A_t(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) \alpha_{\ell}(t) P_{\ell}(\omega \theta_t) \quad (4.1)$$

for $t \geq 4$, where $\alpha_{\ell}(t) = \text{Im} f_{\ell}(t)$ are positive. This expansion can be compared with the Taylor's series expansion,

$$A_t(s,t) = \sum_{n=0}^{\infty} \frac{[\cos \theta_t - 1]^n}{n!} \frac{\partial^n A_t(s,t)}{(\partial \omega \theta_t)^n} \Big|_{\omega \theta_t = 1} \quad (4.2)$$

Since 7)

$$\left. \frac{d^n R(x)}{dx^n} \right|_{x=1} = \frac{(\ell+n)!}{2^n n! (\ell-n)!} \quad (4.3)$$

we get

$$\left. \frac{\partial^n A_+(s,t)}{(\partial \cos \theta_t)^n} \right|_{\cos \theta_t=1} = \frac{1}{2^n n!} \sum_{\ell=0}^{\infty} \frac{(2\ell+1)(\ell+n)!}{(\ell-n)!} \alpha_{\ell}(s) \quad (4.4)$$

Then since

$$(\ell+n)!/(\ell-n)! = (\ell+n)(\ell+n-1) \dots (\ell-n+2)(\ell-n+1)$$

and

$$(\ell+n)(\ell-n+1) = \ell(\ell+1) - n(n-1)$$

we see that each coefficient of α_{ℓ} in (4.4) is proportional to the same polynomial in $\ell(\ell+1)$. Defining the quantities $\beta_n(s)$ to be the left-hand side of (4.4), then

$$\beta_n(s) \equiv \left. \frac{\partial^n A_+(s,t)}{(\partial \cos \theta_t)^n} \right|_{\cos \theta_t=1} = \frac{1}{2^n n!} \sum_{\ell=0}^{\infty} (2\ell+1) \alpha_{\ell}(s) \left\{ \prod_{m=0}^n [\ell(\ell+1) - m(m-1)] \right\} \quad (4.5)$$

We now define $g_n(s)$ to be

$$g_n(s) \equiv \sum_{\ell=0}^{\infty} (2\ell+1) [\ell(\ell+1)]^n \alpha_{\ell}(s) \quad (4.6)$$

The $\beta_n(s)$ may be obtained from the $g_n(s)$ and conversely by a non-singular linear transformation of the form

$$\begin{aligned} \beta_n(s) &= a_{n,n} g_n(s) + a_{n,n-1} g_{n-1}(s) + \dots + a_{n,0} g_0(s) \\ \beta_{n-1}(s) &= a_{n-1,n-1} g_{n-1}(s) + \dots + a_{n-1,0} g_0(s) \\ &\vdots \\ \beta_0(s) &= a_{0,0} g_0(s) \end{aligned} \quad (4.7)$$

where the coefficients $a_{n,m}$ can be determined from (4.5).

From (4.6) we notice that the $g_n(s)$ may be written in the form

$$g_n(s) = \int_0^\infty u^n d\phi(u) \quad , \quad n = 0, 1, 2, \dots \quad (4.8)$$

where $\phi(u)$ is a bounded non-decreasing function of u with points of increase at $u = l(l+1)$ where $l = 0, 1, 2, \dots$. The following theorem is then an immediate consequence of Theorem (1.3) given in Ref. 6) for sequences with such representations.

Theorem 3

A necessary set of conditions for $A_t(s, t)$ to have the expansion (4.1) with $\alpha_l \geq 0$ for all l is that the $g_n(s)$ defined by (4.5) and (4.7) satisfy

$$\begin{vmatrix} g_m(s) & g_{m+1}(s) & \dots & g_{m+n}(s) \\ g_{m+1}(s) & & & \\ \vdots & & & \\ g_{m+n}(s) & \dots & \dots & g_{m+2n}(s) \end{vmatrix} \geq 0 \quad ; \quad m, n = 0, 1, 2, \dots \quad (4.9)$$

By using the inverse of the transformation (4.7) the conditions (4.9) can be written in terms of the derivatives of $A_t(s, t)$ in the forward direction. For example we get taking $m = 0$ and $n = 1$ that

$$2\beta_0(s)\beta_2(s) + \beta_0(s)\beta_1(s) - [\beta_1(s)]^2 \geq 0 \quad (4.10)$$

The conditions (4.9) are also sufficient for the $g_n(s)$ to have the representation (4.8) with $\varnothing(u)$ bounded and non-decreasing but other conditions have to be added to ensure that $\varnothing(u)$ has only points of increase at $u = \ell(\ell+1)$ with $\ell = 0, 1, 2, \dots$ so that $g_n(s)$ has the representation (4.6) and $\beta_n(s)$ the corresponding representation (4.5). These extra conditions are given by the following theorem⁸⁾.

Theorem 4

Let

$$C_n = \frac{4^n}{n!} \sum_{m=n}^{\infty} (-\pi^2)^m \frac{m!}{(2m)!(m-n)!} \quad (4.11)$$

and suppose that the $g_n(s)$ satisfy the conditions (4.9) and are such that the series $\sum_{n=0}^{\infty} C_n g_n(s)$ converges absolutely. Then $g_n(s)$ has the representation (4.6) if and only if the sum of this series is $-g_0$.

Proof

We have the representation (4.8) for $g_n(s)$ since conditions (4.9) are satisfied and we have to prove that $\varnothing(u)$ increases only at the points $u = \ell(\ell+1)$. Now

$$\begin{aligned}
 \sum_{n=0}^{\infty} c_n g_n(s) &= \sum_{n=0}^{\infty} c_n \int_0^{\infty} d\phi(u) u^n \\
 &= \sum_{n=0}^{\infty} \frac{4^n}{n!} \left\{ \sum_{m=n}^{\infty} \frac{(-\pi^2)^m}{(2m)!} \frac{m!}{(m-n)!} \int_0^{\infty} d\phi(u) u^n \right\} \\
 &= \int_0^{\infty} d\phi(u) \left\{ \sum_{m=0}^{\infty} \frac{(-\pi^2)^m}{(2m)!} (1+4u)^m \right\} \\
 &= 2 \int_0^{\infty} d\phi(u) \sin^2 \frac{\pi}{2} [\sqrt{1+4u}-1] - g_0 \quad (4.12)
 \end{aligned}$$

where we have used the absolute convergence of the series to invert the order of summation and integration. The last integral vanishes if and only if $d\phi(u)$ is different from zero only when $u = l(l+1)$ with $l = 0, 1, \dots$, so the theorem is proved.

We have thus obtained sufficient conditions for $g_n(s)$ to have the representation (4.6). That they are not necessary is easily seen from the fact that the existence of the $g_n(s)$ for all $n = 0, 1, 2, \dots$, does not imply the absolute convergence of the series $\sum_{n=0}^{\infty} c_n g_n(s)$. Finally from unitarity we have the requirement that $\alpha_l(s)$ is smaller than unity. We have been unable to obtain either necessary or sufficient conditions on the derivatives of $A_t(s, t)$ for this to be so.

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