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RIGOROUS BOUNDS ON AVERAGED CROSS-SECTIONS
FOR THE SCATTERING OF PIONS ON ANY TARGET

A.K. Common *)

CERN - Geneva

and

F.J. Yndurain

University of Zaragoza, Zaragoza

A B S T R A C T

The analysis of two previous papers by the authors is generalized, showing that bounds valid at all energies and without unknown constants, can be obtained for pion scattering on nucleons or almost any other target. Asymptotically these bounds correspond to the usual Froissart bounds.

The above bounds were previously derived in terms of the partial-wave amplitudes in the t channel but it is shown that they may also be obtained from direct channel quantities. Numerical results for the scattering of pions on nucleons and helium are presented and indicate that the bounds are reasonably tight at intermediate energies.

The bounds even allow one to put constraints on the slope of the Pomeranchuk trajectory and suggest that the binding energy of nucleons in nuclei has to be < 10 MeV. Finally the method can be reversed to put lower bounds on t channel partial-waves.

*) On leave of absence from the University of Kent at Canterbury until October 1st, 1970.

1. - INTRODUCTION *)

In previous papers ^{1),2)} the authors have removed the two weaknesses of the usual Froissart bound for $\pi\pi$ scattering,

$$\sigma_{TOT}(s) \lesssim \frac{\pi}{\mu^2} \log^2 s/s_0; \mu = \text{pion mass}, \quad (1.1)$$

namely the appearance of the unknown constant s_0 and secondly the fact that the bound is supposed to hold only asymptotically (with no indication of when the asymptotic region is reached).

To do this average cross-sections were defined by

$$\overline{\sigma}_{TOT}(s) \equiv \frac{(m+1)}{(s-4\mu^2)^{m+1}} \int_{4\mu^2}^s ds' (s'-4\mu^2)^m \sigma_{TOT}(s'); m \geq 1 \quad (1.2)$$

One could have defined more general averages by using different weight functions, but (1.2) is general enough for our purposes and furthermore it can be proved ³⁾ that if σ_{tot} does not oscillate too much at high energy, $\overline{\sigma}_{tot} \simeq \sigma_{tot}$, so our integrated estimates become local estimates.

Then it was shown by one of us (F.J.Y.) that ¹⁾, using assumptions provable from, for example, axiomatic field theory, one obtains the bounds ^{**)}

$$\begin{aligned} \overline{\sigma}_{TOT}(s) \leq & \left[\frac{n+1}{n} \right]^2 \left[\frac{m+1}{m} \right] \frac{\pi}{\mu^2} \log^2 (s/\mu^2) + \\ & + \left[\frac{n+1}{n} \right] \left[\frac{m+1}{m+\frac{1}{2}} \right] \frac{4\pi}{\mu s^{1/2}} \log (s/\mu^2) + (m+1) 30\pi^2 M_n^{-1} \alpha_2^2; \quad n=1,2,3,\dots \end{aligned} \quad (1.3)$$

*) A brief survey of this paper has been presented to the XV Conference in High Energy Physics at Kiev, CERN Preprint TH. 1185, 1970.

**) Only the case when $m = 1$ was discussed in Ref. 1), while the more general form with $m \geq 1$ was given by A.K.C. in Ref. 2).

where

$$M_n = \frac{(2n)!}{2^n (n!)^2 \left[(2n+1)/(n+1) \right]^n}$$

and α_2^t is the D wave scattering length in the t channel. The bound (1.3) is valid at all energies ^{*}) and as $s \rightarrow \infty$ and letting $n, m \rightarrow \infty$ one recovers the usual Froissart bound (1.1).

This result was then refined by the other author ²⁾ (A.K.C.) by employing variational techniques developed by Lukaszuk and Martin ⁵⁾. Although this method does not yield an explicit formula for the bound, which has to be calculated numerically, it is an improvement on (1.3) by a factor ~ 3 at low energies and also at high energies where it can be computed explicitly to read

$$\sigma_{\text{Tot}}(s) \leq \left(\frac{m+1}{m} \right) \frac{\pi}{\mu^2} \log^2(s/s_0) \quad (1.4)$$

where

$$s_0 = \frac{s_1^{3/2}}{8\mu} \log \left\{ s / \left[\frac{3}{8} \sqrt{\frac{s_1}{4\mu^2}} \log(s/s_1) \right] \right\} \quad (1.5)$$

and

$$s_1 = 4e / (15\pi \mu^2 \alpha_2^t)$$

Inequality (1.4) is actually an improvement on (1.1) since s_0 increases with s .

^{*}) In fact the bound (1.3) is not quite exact except at high energies. The exact form of the bound for $s > 20 \mu^2$ has been given by Atkinson ⁴⁾.

The actual values of the bounds ^{*)} vary between ~ 100 times the expected value of 16 mb for q in the range between 3 and 10 GeV/c, (q is the c.m. momentum) and ~ 6 times the expected value in the resonance region, $q < 1.0$ GeV/c. Actually the bound even requires the absence of resonances in D waves or higher for $\sqrt{s} \leq 5\mu$, thus putting an upper bound on the slope of the Pomanchuk trajectory.

The basis of the method is that the quantity α_2^t defined by

$$\alpha_2^t = \lim_{q_t \rightarrow 0} f_2(t) / q_t^5 \quad (1.6)$$

(q_t is the c.m. momentum in the t channel and s, t, u will always denote the usual Mandelstam variables) is also given by

$$\alpha_2^t = \frac{8}{15\pi\mu} \int_{4\mu^2}^{\infty} ds' \operatorname{Im} F(s', 4\mu^2) / s'^3 \quad (1.7)$$

where $F(s, t)$ is the s channel scattering amplitude. This follows from the Froissart-Gribov representation,

$$f_2(t) = \frac{4}{\pi(4\mu^2 - t)} \int_{4\mu^2}^{\infty} ds' Q_2\left(\frac{2s'}{4\mu^2 - t} - 1\right) \operatorname{Im} F(s', t) \quad (1.8)$$

the existence of which for $0 < t < 4\mu^2$ can be proved by using results derivable ⁷⁾ from axiomatic field theory and unitarity ^{**)}.

Since

$$\operatorname{Im} F(s', 4\mu^2) = \frac{\sqrt{s'}}{q'} \sum_{\ell=0}^{\infty} (2\ell+1) \operatorname{Im} f_{\ell}(s') P_{\ell}(x') \quad (1.9)$$

^{*)} We take $\alpha_2^t = 1.3 \times 10^{-3} \mu^{-5}$; this figure has been obtained by Morgan and Shaw and by Le Guillou, Morel and Navelet ⁶⁾, and it is also predicted by the Veneziano model. This value disagrees with that quoted in Ref. 2) which is wrong. However the values of the bounds to $\pi\pi$ cross-sections given in the above work were evaluated using the value of α_2^t given here.

^{**)} For $t = 4\mu^2$ it is also necessary to assume that Martin's pathology ⁸⁾ does not occur. A discussion of pathological cases is given later.

where $x' = 1 + t/2q^2$, and $\text{Im } f_\ell(s')$, $P_\ell(x')$ are positive (the latter for $t \geq 0$), it is clear that (1.2) cannot be too large without $\text{Im } F$ violating the constraint (1.7) or the unitarity bounds $1 \geq \text{Im } f_\ell \geq |f_\ell|^2 \geq 0$.

The actual computations in Refs. 1), 2) were performed for $\pi^0 \pi^0$ scattering. In the present paper we generalize these results to include isospin (an almost trivial problem) and spin. As is clear from the above discussion the main problem in the generalization is to find a partial wave in the t channel possessing a Froissart-Gribov representation similar to (1.8), with $\text{Im } F(s', t)$ having the positivity properties of the expansion (1.9).

For πN scattering such partial waves have been found by Roy ⁹⁾ for $t = 0$ and by Cheung ¹⁰⁾ when $0 < t \leq 4\mu^2$ ^{*}). For the general spin case one follows the method of Mahoux and Martin ¹¹⁾. Actually the πN case is just a particular example of the general situation but we shall treat it separately because of its importance and to exemplify the method.

For πN scattering the analogue of (1.8) reads ^{**)}

$$\begin{aligned} p_2^{(+)}(t) - \frac{(2\mu^2 - t)G^2}{4\mu\sqrt{(M^2 - \frac{t}{4})(\mu^2 - \frac{t}{4})}} Q_2 \left[\frac{(\mu^2 - \frac{t}{4})}{2\sqrt{(M^2 - \frac{t}{4})(\mu^2 - \frac{t}{4})}} \right] &\gg \\ &\gg \frac{1}{2\pi\sqrt{(M^2 - \frac{t}{4})(\mu^2 - \frac{t}{4})}} \int_{(M+\mu)^2}^{\infty} ds' \text{Im } F^{(+)}(s', t) Q_2 \left[\frac{s' + \frac{t}{4} - M^2 - \mu^2}{2\sqrt{(M^2 - \frac{t}{4})(\mu^2 - \frac{t}{4})}} \right] \end{aligned} \quad (1.10)$$

where $F^{(+)}$ is the sum over helicities of the πN amplitudes, and $p_2^{(+)}(t)$ is the D wave for the process $\pi\pi \rightarrow \tilde{N}\tilde{N}$; (1.10) is valid throughout the unphysical region $0 < t \leq 4\mu^2$. The inequality sign

^{*}) The reader should note that Ref. 10) contains a number of mistakes.

^{**)} We use the notation of Hamilton ¹²⁾.

in (1.10) comes from the fact that we have dropped the (positive) contribution of the u channel to the r.h.s. Actually this contribution can be taken into account as will be shown later, improving (1.10) by a factor $\simeq 2$ when $q \gg \mu$. Finally to get bounds to πN cross-sections an estimate for $h_2^{(+)}(t)$ is needed and we will show how this may be obtained from knowledge of the coupling of the f^0 meson to nucleons.

In the most general case, we get formulae identical to (1.10) where now $F^{(+)}$ is the symmetrized regularized amplitude of Mahoux and Martin ¹¹⁾. The problem in the general case of scattering of anything on anything is that neither the necessary analyticity or positivity have been proved (in the case of the latter, the best one can do are the results of MacDowell and Mahoux ¹³⁾, which are clearly insufficient), so it is not clear that the desired results will hold.

An important exception, however, is the case of πA scattering where A is any nucleus. Here an inequality equivalent to (1.10) can still be proved from axiomatic field theory ¹⁴⁾ in a range $0 < t \leq t_0$ where t_0 is the lowest threshold in the channel $\pi\pi \rightarrow A\bar{A}$. Therefore t_0 equals $4\mu^2$ if no anomalous threshold occurs as in the case for, e.g., $A = {}^4\text{He}, {}^{16}\text{O}, \dots$

The method yields bounds for the "elastic" amplitudes. To obtain bounds for the isospin amplitudes is, as stated before, straightforward. Thus for example is the case of $\pi\pi$ scattering, it is sufficient to use

$$f_2^{(0)}(t) = \frac{4}{\pi(4\mu^2 - t)} \int_{4\mu^2}^{\infty} ds' Q_2 \left[\frac{2s'}{4\mu^2 - t} - 1 \right] \text{Im} \left[\frac{1}{3} F^{(0)}(s'; t) + F^{(1)}(s'; t) + \frac{5}{3} F^{(2)}(s'; t) \right] \quad (1.11)$$

in lieu of (1.8); here $f^{(0)}$ is the wave with isospin 0 in the t channel and $F^{(I)}$ the amplitude with isospin I in the s channel.

As remarked earlier, the bounds depend on the assumption that Martin's pathology ⁸⁾ does not occur. In any case rigorous (albeit weaker) bounds can be obtained for any process $\pi A \rightarrow \pi A$ using (1.10) with $t = t_1$ where $0 < t_2 < t_0$. It is also possible to give bounds for arbitrary processes if one is willing to accept that they contain an arbitrary constant embodying $h_2^{(+)}(t_1)$ as well as the low energy contribution of the imaginary part of the scattering amplitude to the analogue of Eq. (1.10). These bounds will thus involve, just like the usual Froissart bound, an arbitrary constant but will have the advantage over it of being valid at all energies. So far we have discussed how bounds on the scattering amplitudes in the s channel can be derived from t channel partial wave amplitudes. However, one can also derive bounds from a knowledge of the total amplitude in the direct channel at and below threshold. For example in the case of πN scattering it may be proved from axiomatic field theory ^{11), 15)} that $F^{(+)}(s, t)$ satisfies twice subtracted dispersion relations for $0 < t < 4\mu^2$ so that

$$\begin{aligned} & \frac{F^{(+)}(s, t)}{[s - (M + \mu)^2][s + t - (M - \mu)^2]} - \frac{F^{(+)}[(M + \mu)^2, t]}{[s - (M + \mu)^2][4M\mu + t]} + \frac{F^{(+)}[-t + (M - \mu)^2, t]}{[s + t - (M - \mu)^2][4M\mu + t]} \\ & - \frac{G^2}{2M} \left\{ \frac{\mu^2}{(s - M^2)[M^2 - (M + \mu)^2][M^2 + t - (M - \mu)^2]} + \frac{t - \mu^2}{(s + t - M^2 - 2\mu^2)[M^2 + 2\mu^2 - t - (M + \mu)^2](2M\mu - 3\mu^2)} \right\} \\ & = \frac{1}{2\pi i} \int_{(M + \mu)^2}^{\infty} \frac{F_s^{(+)}(s', t) ds'}{(s' - s)[s' - (M + \mu)^2][s' + t + (M - \mu)^2]} + \frac{1}{2\pi i} \int_{(M + \mu)^2}^{\infty} \frac{F_u^{(+)}(2M^2 + 2\mu^2 - s', t) ds'}{(s' - u)[s' + t - (M - \mu)^2][s' - (M + \mu)^2]} \end{aligned} \quad (1.12)$$

where $F_s^{(+)}$ and $F_u^{(+)}$ are the discontinuities of $F^{(+)}(s, t)$ across the s and u cuts respectively and are positive. So long as $(M - \mu)^2 - t \leq s \leq (M + \mu)^2$ and $0 \leq t < 4\mu^2$ the denominators in the integrals are positive and then bounds can be obtained on πN cross-sections once the l.h.s. of (1.12) is known. We will discuss later how the required values of $F^{(+)}(s, t)$ can be found to a good approximation by using experimental knowledge of low energy π nucleon scattering.

So far we have discussed explicitly only bounds on total cross-sections and will only give numerical results for this case, but bounds can similarly be derived for differential cross-sections [cf. Ref. 2].

Two important consequences of this work are that firstly from bounds for $\pi\pi$ scattering cross-sections we deduce bounds to the slope of the Pommeranchuk trajectory and secondly, under certain assumptions, we can show that the binding energy of each nucleon in large nuclei < 10 MeV.

Before continuing we now give the order in which the various topics mentioned above will be discussed in the following Sections. A brief review of the method for obtaining bounds in the $\pi\pi$ case is presented in Section 2 while Section 3 is devoted to treating in detail the case of πN scattering. In Section 4 the case of π nucleus scattering is considered and also the most general case. The estimate of the quantity $h_2^{(+)}(t)$ for $\pi\pi \rightarrow N\bar{N}$ or more generally for $\pi\pi \rightarrow A\bar{A}$ is then described in Section 5. Then Section 6 deals with direct channel bounds, while the implications of our results as well as their comparison with experimental quantities are presented in Section 7. The article is concluded in Section 8 with some comments and discussions.

2. - $\pi\pi$ SCATTERING

From (1.7), (1.9), one has the chain of inequalities ^{*)}

$$\begin{aligned} \alpha_2^t &> \frac{8}{15\pi\mu} \int_{4\mu^2}^s ds' \operatorname{Im} F(s', 4\mu^2)/s'^3 > \frac{8}{15\pi\mu s^3} \int_{4\mu^2}^s ds' \frac{\sqrt{s'}}{q'} \sum_{L+2}^{\infty} (2L+1) \operatorname{Im} f_L(s') P_L(\alpha) \\ &> \frac{8 P_{L+2}(\alpha) \sqrt{s}}{15\pi\mu s^3 q} \int_{4\mu^2}^s ds' \sum_{L+2}^{\infty} (2L+1) \operatorname{Im} f_L(s') \end{aligned} \quad (2.1)$$

*) The summations are over even values of L since we consider throughout this Section $\pi^0\pi^0$ scattering.

where $x = 1 + (8\mu^2/q^2)$. This together with the unitarity condition $0 \leq \text{Im } f_\ell(s') \leq 1$ and the optical theorem

$$\sigma_{\text{TOT}}(s) = \frac{32\pi}{s-4\mu^2} \left[\sum_{\ell=0}^L + \sum_{\ell=L+2}^{\infty} \right] (2\ell+1) \text{Im } f_\ell(s) \quad (2.2)$$

yields at once (1.3) by taking $L \sim \sqrt{s/\mu^2} \log(s/\mu^2)$.

It is clear that this method is not the best possible one because we are leaving out the sum $\sum_0^L (2\ell+1) \text{Im } f_\ell(s') P_\ell(x')$ in the expression for α_2^t . To obtain the best possible bounds a variational method is used.

We define $\bar{F}(s')$ and $G(s')$ by

$$\begin{aligned} \text{Im } F(s', 4\mu^2) &= \frac{\sqrt{s'}}{q'} \sum_{\ell=0}^{\infty} (2\ell+1) \text{Im } f_\ell(s') P_\ell(x') \\ &\geq \frac{\sqrt{s'}}{q'} \sum_{\ell=0}^{\infty} (2\ell+1) |f_\ell(s')|^2 P_\ell(x') \equiv G(s'), \end{aligned} \quad (2.3)$$

$$\text{Im } F(s', 0) \leq |F(s', 0)| \leq \frac{\sqrt{s'}}{q'} \sum_{\ell=0}^{\infty} (2\ell+1) |f_\ell(s')| \equiv \bar{F}(s') \quad (2.4)$$

where $x' = 1 + 8\mu^2/(s'-4\mu^2)$. For every fixed value of $\bar{F}(s')$ it can be proved ⁵⁾ that the unitarity condition $1 \geq |f_\ell|$ requires $G(s')$ to have a well-defined minimum different from zero. This is still true ¹⁶⁾ when the P_ℓ in (2.3) are replaced by any sequence of functions which are monotonically increasing with ℓ . This fact will be of importance later when we consider for example πN scattering.

The minimum value $\Phi(\bar{F}, s')$ of $G(s')$ is given by the following distribution of partial waves :

$$|f_\ell(s')| = 1, \ell \leq 2L; \quad |f_\ell(s')| = \frac{c}{P_\ell(x')}, \ell > 2L \quad (2.5)$$

where c is independent of L ; L is defined by requiring

$$P_{2L}(x') < c \leq P_{2L+2}(x')$$

and both c and L are uniquely determined once $\bar{F}(s')$ is given. From all this and the inequalities

$$\alpha_2^t \geq \frac{8}{15\pi\mu} \int_{4\mu^2}^s ds' \Phi(\bar{F}, s')/s'^3 \quad (2.7)$$

$$\bar{\sigma}_{\text{TOT}}(s) \leq \frac{8\pi(m+1)}{(s-4\mu^2)^{m+1}} \int_{4\mu^2}^s ds' (s'-4\mu^2)^m \bar{F}(s') \quad (2.8)$$

it follows, taking into account the fact that Φ increases with $\bar{F}$, that $\bar{\sigma}$ cannot be too large or otherwise Φ would have to be large and (2.7) would be violated.

By the calculus of variations the maximum of (2.8) when the r.h.s. of (2.7) stays fixed occurs when

$$\frac{\lambda}{s'^3} \frac{\partial \Phi}{\partial \bar{F}} = (s'-4\mu^2)^m \quad (2.9)$$

where λ is independent of s' but may depend on s . Since $\partial \Phi / \partial \bar{F} = 2c$, it is a simple matter to calculate the r.h.s. of (2.7) and (2.8) for given λ . Since furthermore $\partial^2 \Phi / \partial \bar{F}^2 > 0$, it is the smallest value of λ consistent with (2.7) that gives the maximum of the r.h.s. of (2.8) and hence the required upper bound for $\bar{\sigma}_{\text{tot}}$. As already mentioned this bound cannot be written explicitly [except for large s where it is given by (1.4)], but its numerical calculation offers no difficulty.

Thus we find bounds which depend on the quantity α_2^t . In principle this quantity should be measurable ; until this is done, however, we have to content ourselves with estimates of it which give ⁶⁾, $\alpha_2^t \simeq 1.3 \times 10^{-3} \mu^{-5}$.

It is now clear what we have to do in the general case is to find a representation like (1.8) [or (1.7)] with

$$\text{Im } F = \sum \text{Im } f_\ell x_\ell \quad (2.10)$$

where $\text{Im } f_\ell > 0$, $x_\ell > 0$ and x_ℓ increasing with ℓ . It is to this problem that we now turn.

3. - πN SCATTERING

Let us define [we follow the standard notation ; Ref. 12]

$$\begin{aligned} F^{(+)}(s,t) &= \frac{8\pi\sqrt{s}}{M} [f_1^{(+)}(s,t) + f_2^{(+)}(s,t)] \\ &= A^{(+)}(s,t) + \frac{(s-\mu^2-M^2)}{2M} B^{(+)}(s,t) \end{aligned} \quad (3.1)$$

M is the nucleon mass $f^{(+)}$ are the helicity amplitudes and $A^{(+)}$, $B^{(+)}$ are the analytic invariant amplitudes with the crossing properties

$$A^{(+)}(s,t) = A^{(+)}(u,t) ; B^{(+)}(s,t) = -B^{(+)}(u,t) \quad (3.2)$$

In terms of isospin amplitudes $A^{(+)} = \frac{1}{3}A^{(\frac{1}{2})} + \frac{2}{3}A^{(\frac{3}{2})}$ and the same for $B^{(+)}$. The partial wave decompositions of the helicity amplitudes are ^{*})

$$\begin{aligned} f_1^{(+)}(s,t) &= \frac{1}{q} \sum_{\ell=0}^{\infty} [f_{\ell+}^{(+)}(s) P'_{\ell+}(\cos\theta_s) - f_{\ell-}^{(+)}(s) P'_{\ell-}(\cos\theta_s)] \\ f_2^{(+)}(s,t) &= \frac{1}{q} \sum_{\ell=0}^{\infty} [f_{\ell+}^{(+)}(s) - f_{\ell-}^{(+)}(s)] P_{\ell}(\cos\theta_s) \end{aligned} \quad (3.3)$$

^{*}) ℓ is no longer restricted to even values.

where $\cos\theta_s = 1 + t/2q^2$.

The unitarity condition now reads

$$1 \geq \text{Im} f_{\ell \pm}^{(+)}(s) \geq |f_{\ell \pm}^{(+)}(s)|^2, \quad s \geq (M+\mu)^2 \quad (3.4)$$

Denoting by the suffices s and u the discontinuities of quantities across the s and u channel cuts respectively, we get for $s \geq (M+\mu)^2$ and $0 \leq t \leq 4\mu^2$,

$$\begin{aligned} \frac{1}{2i} F_s^{(+)}(s, t) &= \frac{1}{2i} \left\{ A_s^{(+)}(s, t) + \frac{(s - M^2 - \mu^2)}{2M} B_s^{(+)}(s, t) \right\} = \\ &= \frac{4\pi\sqrt{s}}{Mq} \left\{ \sum_{\ell=0}^{\infty} [P'_{\ell+1}(\cos\theta_s) - P'_{\ell}(\cos\theta_s)] \text{Im} f_{\ell+}^{(+)}(s) + \right. \\ &\quad \left. + \sum_{\ell=1}^{\infty} [P'_{\ell}(\cos\theta_s) - P'_{\ell-1}(\cos\theta_s)] \text{Im} f_{\ell-}^{(+)}(s) \right\} \geq 0. \end{aligned} \quad (3.5)$$

Similarly in the u channel where $u = 2(M^2 + \mu^2) - t - s \geq (M+\mu)^2$

$$\frac{1}{2i} F_u^{(+)}(s, t) = \frac{1}{2i} \left\{ A_s^{(+)}(u, t) + \frac{1}{2M}(u - M^2 - \mu^2) B_s^{(+)}(u, t) \right\} \quad (3.6)$$

Again we can prove that for $0 \leq t \leq 4\mu^2$, (3.6) is positive.

Actually we can prove more since if

$$S = \{[s - (M+\mu)^2][s - (M-\mu)^2]\}^{\frac{1}{2}} \equiv \lambda_s^{\frac{1}{2}}; \quad U = \sqrt{\lambda_u} \quad (3.7)$$

and

$$\sin \chi_{su} = -2M \sqrt{t[su - (M^2 - \mu^2)^2]} / Su \quad (3.8)$$

then

$$\frac{1}{2i} F_u^{(+)}(s, t) \geq \frac{1}{2i} F_s^{(+)}(u, t) \frac{\lambda_s}{\lambda_u} e^{-|\chi_{su}|} \quad (3.9)$$

We shall not present the proof of (3.9) here since this is a particular case of the situation treated in Section 4. For $0 < t \leq 4\mu^2$, $F^{(+)}$ satisfies the dispersion relation ^{*})

$$F^{(+)}(s, t) = \frac{1}{2\pi i} \int_{(M+\mu)^2}^{\infty} ds' \frac{F_s^{(+)}(s', t)}{(s' - s)} + \frac{1}{2\pi i} \int_{(M+\mu)^2}^{\infty} du' \frac{F_u^{(+)}(s, t)}{(u' - u)} \quad (3.10)$$

$$+ \frac{G^2}{2M} \left[\frac{\mu^2}{s - M^2} + \frac{\mu^2 - t}{u - M^2} \right]$$

where G^2 is the πNN coupling constant.

The t channel partial wave referred to in the Introduction is defined by

$$h_2^{(+)}(t) = \frac{1}{2} \int_{-1}^{+1} d\cos\theta_t P_2(\cos\theta_t) F^{(+)}(s, t) \quad (3.11)$$

where $\cos\theta_t$ is the scattering angle in the t channel ; i.e.,

$$\cos\theta_t = \frac{s + q_t^2 + Q_t^2}{2q_t Q_t} \quad \text{where} \quad q_t^2 = \frac{t}{4} - \mu^2, Q_t^2 = \frac{t}{4} - M^2 \quad (3.12)$$

^{*}) The dispersion relation (3.10) should be written with two subtractions ^{11), 15)}. Since they give no contribution to the t channel D wave we choose to ignore them.

Substituting (3.10) into (3.11) and using inequality (3.9) we get

$$\begin{aligned}
 h_2^{(+)}(t) + \frac{G^2(2\mu^2-t)}{4\pi q_t Q_t} Q_2 \left[\frac{M^2 + q_t^2 + Q_t^2}{2 q_t Q_t} \right] = \\
 = \frac{1}{4\pi i Q_t q_t} \left\{ \int_{(M+\mu)^2}^{\infty} ds' F_s^{(+)}(s'; t) Q_2 \left[\frac{s' + q_t^2 + Q_t^2}{2 q_t Q_t} \right] + \int_{(M+\mu)^2}^{\infty} du' F_u^{(+)}(s'; t) Q_2 \left[\frac{u' + q_t^2 + Q_t^2}{2 q_t Q_t} \right] \right\} \gg \\
 \gg \frac{1}{4\pi i Q_t q_t} \int_{(M+\mu)^2}^{\infty} ds' \left[1 + \frac{\lambda_{u'}}{\lambda_{s'}} e^{-|\chi_{su}|} \right] F_s^{(+)}(s'; t) Q_2 \left[\frac{s' + q_t^2 + Q_t^2}{2 q_t Q_t} \right]
 \end{aligned} \quad (3.13)$$

The inequality (1.10) is obtained by dropping the term proportional to $\exp \{-|\chi|\}$ on the r.h.s. of the above.

Taking the limit $t \rightarrow 4\mu^2$ and using (3.5)

$$\begin{aligned}
 \frac{15\pi M}{2(M^2-\mu^2)} \alpha_2^{(+)} - \frac{G^2\pi}{\mu^4} \gg M \int_{(M+\mu)^2}^s ds' \frac{[1+e^{-|\chi|}]}{(s'-M^2+\mu^2)^3} \left\{ A_s^{(+)}(s'; 4\mu^2) + \frac{(s'-\mu^2-M^2)}{2M} B_s^{(+)}(s'; 4\mu^2) \right\} \\
 = 4\pi \int_{(M+\mu)^2}^s ds' \frac{\sqrt{s'}}{q'} \left\{ \sum_{\ell=0}^{\infty} [P'_{\ell+1}(x') - P'_{\ell}(x')] \text{Im} f_{\ell+}^{(+)}(s') \right\} [1+e^{-|\chi|}] / (s'-M^2+\mu^2)^3 \quad (3.14) \\
 + 4\pi \int_{(M+\mu)^2}^s ds' \frac{\sqrt{s'}}{q'} \left\{ \sum_{\ell=1}^{\infty} [P'_{\ell}(x') - P'_{\ell-1}(x')] \text{Im} f_{\ell-}^{(+)}(s') \right\} [1+e^{-|\chi|}] / (s'-M^2+\mu^2)^3
 \end{aligned}$$

where

$$\alpha_2^{(+)} = \lim_{t \rightarrow 4\mu^2} h_2^{(+)}(t) / (4\mu^2 - t), \quad x' = 1 + 2\mu^2/q'^2$$

and we have used the well-known asymptotic expansion of the Legendre functions ¹⁷⁾. Since P'_{ℓ} increases with ℓ when $x \geq 1$, each integral in the r.h.s. of (3.14) is positive and hence bounded by the l.h.s.

Now the πN total cross-section is given through the optical theorem by

$$\begin{aligned}\sigma_{\text{TOT}}^{(+)}(s) &= \frac{4\pi}{q^2} \left\{ \sum_{\ell=0}^{\infty} [(\ell+1) \text{Im} f_{\ell+}^{(+)}(s) + \ell \text{Im} f_{\ell-}^{(+)}(s)] \right\} \\ &= \frac{1}{3} \sigma_{\text{TOT}}^{(\frac{1}{2})}(s) + \frac{2}{3} \sigma_{\text{TOT}}^{(\frac{3}{2})}(s)\end{aligned}\quad (3.16)$$

where the superscripts in $\sigma_{\text{tot}}^{(I)}$ refer to isospin states. Therefore if we define as for $\pi\pi$ scattering the averaged cross-sections,

$$\begin{aligned}\overline{\sigma}_{\text{TOT}}^{(+)}(s) &\equiv \frac{(m+1)}{[s-(M+\mu)^2]^{m+1}} \int_{(M+\mu)^2}^s ds' [s'-(M+\mu)^2]^m \sigma_{\text{TOT}}^{(+)}(s') \\ &\leq \frac{4\pi(m+1)}{[s-(M+\mu)^2]^{m+1}} \left\{ \int_{(M+\mu)^2}^s ds' \frac{[s'-(M+\mu)^2]^m}{q'^2} \sum_{\ell=0}^{\infty} (\ell+1) \text{Im} f_{\ell+}^{(+)}(s') + \right. \\ &\quad \left. + \int_{(M+\mu)^2}^s \frac{ds' [s'-(M+\mu)^2]^m}{q'^2} \sum_{\ell=1}^{\infty} \ell \text{Im} f_{\ell-}^{(+)}(s') \right\}\end{aligned}\quad (3.17)$$

we can proceed just as in the case of $\pi\pi$ scattering if we notice that $[\overline{P}_{\ell+1}'(x) - P_{\ell}'(x)]/(\ell+1)$ is an increasing function of ℓ . In order to get a meaningful bound, we have to use of course some value for the quantity $\alpha_2^{(+)}$. A method for obtaining such a value will be discussed in Section 5.

Finally we can easily get the asymptotic form of the bound (as remarked before, the actual bound can only be found by numerical computation ^{*}); the results will be presented in Section 7). The result is

^{*}) Of course, explicit although weaker bounds similar to (1.3) can also be written down, but we shall not do it here.

$$\overline{\sigma}_{\text{TOT}}(s) \lesssim \left(\frac{m+1}{m}\right) \frac{\pi}{\mu^2} \log^2(s/s_0) \quad (3.18)$$

where now

$$s_0 = (s_1^{3/2}/4\mu) \log \left\{ s / \left[s_1^{3/2} \frac{3}{8\mu} \log(s/s_1) \right] \right\}, \quad (3.19)$$

$$s_1 = \frac{4e}{8\mu} \left\{ \frac{15\pi}{2(M^2-\mu^2)} \alpha_2^{(*)} - \frac{G^2\pi}{\mu^4 M} \right\}^{-1}.$$

4. - THE GENERAL SPIN CASE

The results of the previous Sections will now be extended to the case of scattering of particles with arbitrary spins. To the problems due entirely to spin (which can be mastered quite easily with the help of techniques developed by Mahoux and Martin ¹¹⁾ and by Cohen-Tannoudji, Morel and Navelet ¹⁸⁾) are added two new ones. Firstly for processes with unphysical thresholds in the s or u channels, the partial waves lack positivity below the physical threshold. The only way we could proceed would be to assume this positivity and it is not clear whether this assumption could be justified ^{*}). Furthermore the existence of dispersion relations cannot be proved from axiomatic field theory (which only gives the results of Epstein, Glaser and Bros ¹⁴⁾), so the necessary analyticity also has to be postulated.

Secondly there are processes for which anomalous thresholds occur in the t channel. This would force us to use the analogue of (3.13), [rather than (3.14)] with $t < t_0$, where t_0 is the anomalous threshold. This will be discussed in detail later on.

^{*}) See Ref. 13) in this context.

The first trouble never occurs for pion scattering on any target ; so we shall begin by discussing πA scattering, A anything. Of course from the point of view of pure spinology, the π meson could be substituted by any other spin-zero object. We shall use the same notation as in Section 3, except that now M will denote the mass of the target A .

In the helicity formalism ¹⁹⁾ the πA scattering matrix can be written as

$$M_{\lambda\lambda'}(s,t) = \frac{\sqrt{s}}{2q} \sum_J (2J+1) f_{\lambda\lambda'}^J(s) d_{\lambda\lambda'}^J(\theta_s); \lambda\lambda' = -j, j+1, \dots, j; \quad (4.1)$$

here j is the spin of the target and λ, λ' are respectively its initial and final helicities. Also θ_s is the scattering angle in the s channel and the unitarity conditions for the partial waves $f_{\lambda\lambda'}^J$ read

$$1 \geq \text{Im} f_{\lambda\lambda'}^J = \sum_{\alpha} \alpha_{\lambda}^* \alpha_{\lambda'} \gg \sum_{\nu} f_{\nu\lambda}^{J*} f_{\nu\lambda'}^J \geq 0 \quad (4.2)$$

we are assuming invariance under time reversal although this is not necessary ; cf. footnote 13 in Ref. 11).

Following Mahoux and Martin ¹¹⁾ we define the regularized amplitude

$$\hat{F}(s,t) = \lambda_s^j \sum_{\lambda} M_{\lambda\lambda}(s,t) \times \begin{cases} 1 & \text{if } j = \text{integer} \\ \sqrt{\psi_{st}} & \text{if } 2j = \text{odd integer} \end{cases} \quad (4.3)$$

here λ_s is defined as in (3.7) and

$$\psi_{st} = [s - (m+\mu)^2]^2 + st \quad (4.4)$$

$\hat{F}$ has positivity properties in both s and u channels ¹¹⁾ and we can bound its discontinuity across the u channel cut in terms of its discontinuity across the s channel cut. We shall consider explicitly the case $j = \text{integer}$ only but the other case may be treated in exactly the same way.

Continuing $\hat{F}$ to the u channel we get the expression ¹⁸⁾

$$\text{Im } \hat{F}(s, t) = \frac{\sqrt{u}}{2q_u} \lambda_s^j \sum_J (2J+1) \sum_{\mu\nu} d_{\mu\nu}^j(2\chi) \text{Im} \bar{f}_{\nu\mu}^J(u) d_{\mu\nu}^J(\theta_u); \quad u \gg (M+\mu)^2 \quad (4.5)$$

where θ_u is the scattering angle in the u channel, $\bar{f}_{\nu\mu}$ are the u channel partial-wave amplitudes, and χ is the crossing angle defined in (3.8). We now prove that for $t \geq 0$, $u \geq (M+\mu)^2$,

$$\sum_{\mu\nu} d_{\mu\nu}^j(2\chi) \text{Im} \bar{f}_{\nu\mu}^J(u) d_{\mu\nu}^J(\theta_u) \geq e^{-2j|\chi|} \sum_{\lambda} \text{Im} \bar{f}_{\lambda\lambda}^J(u) d_{\lambda\lambda}^J(\theta_u) \quad (4.6)$$

Proof

Let A, B be any two positive-definite matrices and let $D^A = UAU^{-1}$, $B' = UBU^{-1}$ with U unitary and D^A diagonal. One has

$$\begin{aligned} \text{Tr } AB &= \text{Tr } D^A B' = \sum_n D_{nn}^A B'_{nn} \geq (\min_n B'_{nn}) \text{Tr } D^A \\ &= (\min_n B'_{nn}) \text{Tr } A \end{aligned} \quad (4.7)$$

where we have used the fact that B positive definite $\rightarrow B'$ positive definite $\rightarrow B'_{nn} \geq 0$.

Take $A_{\mu\nu} = d_{\mu\nu}^J(\theta_u) \text{Im} \bar{f}_{\nu\mu}^J(u)$, $B_{\mu\nu} = d_{\mu\nu}^j(2\chi)$. ¹¹⁾ With the help of the explicit expression

$$d_{\lambda\lambda'}^j(\beta) = \sum_{\nu} R_{\lambda\nu} R_{\nu\lambda'}^\dagger e^{-i\nu\beta} \quad (4.8)$$

we see that for any ϕ ,

$$\begin{aligned} \sum_{\mu\nu} \phi_{\mu}^* d_{\mu\nu}^J(\theta_u) \text{Im} \bar{f}_{\nu\mu}^J(\omega) \phi_{\nu} &= \sum_{\alpha} \sum_{\mu\nu} (\phi_{\mu} \alpha_{\mu}^*)^* d_{\mu\nu}^J(\theta_u) (\phi_{\nu} \alpha_{\nu}^*) = \\ &= \sum_{\rho} e^{-i\rho\theta_u} \left[\sum_{\alpha, \mu, \nu} (\phi_{\mu} \alpha_{\mu}^* R_{\mu\rho}^*)^* (\phi_{\nu} \alpha_{\nu} R_{\nu\rho}) \right] \end{aligned} \quad (4.9)$$

which is positive as θ_u is pure imaginary. Therefore A is positive definite and one also prove the same is true for B.

Therefore, using (4.7),

$$\begin{aligned} \sum_{\mu\nu} d_{\mu\nu}^j(2\chi) \text{Im} \bar{f}_{\nu\mu}^J(\omega) d_{\mu\nu}^J(\theta_u) &\equiv \text{Tr} AB \geq \\ &\geq (\min_n B'_{nn}) \text{Tr} A = (\min_n B'_{nn}) \sum_{\lambda} \text{Im} \bar{f}_{\lambda\lambda}^J(\omega) d_{\lambda\lambda}^J(\theta_u). \end{aligned} \quad (4.10)$$

Finally from (4.8) and with $U' = UR$

$$\begin{aligned} \min_n B'_{nn} &= \min_n \sum_{\lambda, \lambda', \rho} U_{n\lambda} R_{\lambda\rho} R_{\rho\lambda'}^{\dagger} U_{\lambda'n}^{\dagger} e^{-2i\rho\chi} = \\ &= \min_n \sum_{\rho} |U'_{n\rho}|^2 e^{-2i\rho\chi} \geq \min_n \{ [\min_{\rho} e^{-2i\rho\chi}] \sum_{\rho} |U'_{n\rho}|^2 \} \\ &= e^{-2j|\chi|} \end{aligned} \quad (4.11)$$

since U' unitary and χ is pure imaginary. Combining (4.11) with (4.10) we get (4.7).

Since furthermore $\lambda_s^j \geq \lambda_u^j$,

$$\text{Im} \hat{F}(s, t) \geq e^{-2j|\chi|} \frac{\sqrt{u}}{2q_u} \lambda_u^j \sum_J (2J+1) \sum_{\lambda} \text{Im} \bar{f}_{\lambda\lambda}^J(\omega) d_{\lambda\lambda}^J(\theta_u); u \geq (M+\mu)^2 \quad (4.12)$$

This is to be compared with the s channel expansion,

$$\text{Im } \hat{F}(s,t) = \frac{\sqrt{s}}{2q} \lambda_s^j \sum_J (2J+1) \sum_{\lambda} \text{Im } f_{\lambda\lambda}^J(s) d_{\lambda\lambda}^J(\theta_s) \quad ; \quad s \geq (m+\mu)^2 \quad (4.13)$$

It is proved in Ref. 18) that $\lambda_s^j M_{\lambda\lambda} / \psi_{st} |\lambda|$ is free from kinematical singularities so the same is true for $\hat{F}$. This together with the bound ¹¹⁾

$$|\hat{F}| < C s^{2+2j-\epsilon} \quad , \quad \epsilon > 0 \quad ; \quad t < 4\mu^2, \quad (4.14)$$

leads to a $(2j+2)$ subtracted dispersion relation,

$$\hat{F}(s,t) = (\text{pole terms}) + (\text{subtraction polynomial of degree } 2j+1) +$$

$$+ \frac{(s-s_0)^{2j+2}}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{ds' \hat{F}_s(s',t)}{(s'-s_0)^{2j+2} (s'-s)} + \quad (4.15)$$

$$+ \frac{(u-u_0)^{2j+2}}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{du' \hat{F}_u[2(m^2+\mu^2)-t-u',t]}{(u'-u_0)^{2j+2} (u'-u)}$$

Taking $s_0 = (2M^2 + 2\mu^2 - t)/2$ and u_0 symmetric with respect to this, we can rewrite (4.15) for the symmetric amplitude,

$$\begin{aligned} \hat{F}^{(+)}(s,t) &\equiv \frac{1}{2} [\hat{F}(s,t) + \hat{F}(u,t)] \\ &= \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \frac{\hat{A}^{(+)}(s',t)}{(s'-s_0)^{2j+2}} \left\{ \frac{(s-s_0)^{2j+2}}{s'-s} + \frac{(u-u_0)^{2j+2}}{s'+s-t-2(M^2+\mu^2)} \right\} \quad (4.16) \\ &\quad + (\text{pole terms}) + (\text{subtraction polynomial}) \end{aligned}$$

where from (4.12), (4.13),

$$\hat{A}^{(+)}(s,t) \geq \frac{\sqrt{s}}{2q} \lambda_s^j \mathcal{E}_{st} \sum_J (2J+1) \sum_{\lambda} \bar{I}_m f_{\lambda}^{(+)}(s) d_{\lambda}^J(\theta_s), \quad 0 \leq t \leq 4\mu^2 \quad (4.17)$$

with $f^{(+)} = \frac{1}{2}(f + \bar{f})$ and $\mathcal{E}_{st} = \frac{1}{2} [1 + e^{-2j|\chi|}]$.

Let us now define the t channel wave

$$h_2^{(+)}(t) = \frac{1}{2} \int_{-1}^1 d\cos\theta_t P_{2j+2}(\cos\theta_t) \hat{F}^{(+)}(s,t) \quad (4.18)$$

This quantity exists in the range $0 < t < 4\mu^2$ as follows from the analyticity domain of $\hat{F}^{(+)}(s,t)$ (see Fig. 1), given by the results of Refs. 14), 20). Proceeding as before we get the representation

$$\begin{aligned} & h_2^{(+)}(t) + (\text{pole contribution}) \\ &= \frac{1}{2\pi q_t \mathcal{Q}_t} \int_{(M+\mu)^2}^{\infty} ds' \hat{A}^{(+)}(s',t) \mathcal{Q}_{2j+2} \left[\frac{s' + q_t^2 + \mathcal{Q}_t^2}{2q_t \mathcal{Q}_t} \right] \end{aligned} \quad (4.19)$$

The pole terms can be treated explicitly while the subtraction polynomial gives no contribution.

To complete the parallelism of (4.19) with (3.13) all we have to show is that $h_2^{(+)}(t)$ here corresponds to the t channel D wave. At least for $t \simeq 4\mu^2$ this is easy to do : $\hat{F} = \lambda_s^j$ (we omit helicity indices) so that $h_2 = \int d\cos\theta_t P_{2j+2}(\cos\theta_t) \lambda_s^j$. Since λ_s^j is a polynomial of degree $2j$ in s and hence in $\cos\theta_t$, we have using the recursion formulae ¹⁷⁾ for the P_ℓ 's that

$$h_2 = \int d\cos\theta_t P_2(\cos\theta_t) M + \sum_{n \geq 3} \int d\cos\theta_t P_n(\cos\theta_t) M \quad (4.20)$$

and using threshold behaviour, it follows that near $t = 4\mu^2$ only the first term above survives.

Hence in particular if no anomalous thresholds occur we can take $t \rightarrow 4\mu^2$ and obtain the exact analogue of (3.14).

Finally we turn to the spinology of the general case. Then from Refs. 11), 18), we can construct the amplitude

$$\hat{F} = \lambda_s^{j_1+j_2} \sum_{\lambda_1, \lambda_2} M_{\lambda_1, \lambda_2, \lambda_1, \lambda_2} \times \begin{cases} 1 & \text{if } j_1+j_2 = \text{integer} \\ \sqrt{s_t} & \text{if } 2(j_1+j_2) = \text{odd integer} \end{cases} \quad (4.21)$$

which has the correct positivity properties in s and u channels and the proof would proceed as before if the difficulties mentioned earlier in this Section could be overcome.

5. - ESTIMATE OF t CHANNEL SCATTERING LENGTHS

As discussed earlier one can give rigorous bounds to π^A cross-sections from (4.19) once one has a reliable estimate to $h_2^{(+)}(t)$ (the coupling constants in the pole terms are known from forward dispersion relations). We will in this Section make such an estimate for πN scattering and discuss the extension to the cases of scattering on other nuclei when there are no anomalous thresholds.

If there were such a threshold starting at $t = t_0$, then one could rewrite (4.19) in the form

$$\begin{aligned} h_2^{(+)}(t) + (\text{pole contribution}) &= \frac{1}{2\pi q_t q_t} \int_{t_0}^{(M+\mu)^2} ds' \hat{A}^{(+)}(s', t) Q_{2j+2} \left[\frac{s' + q_t^2 + Q_t^2}{2q_t q_t} \right] \\ &= \int_{(M+\mu)^2}^{\infty} ds' \hat{A}^{(+)}(s', t) Q_{2j+2} \left[\frac{s' + q_t^2 + Q_t^2}{2q_t q_t} \right] \quad (5.1) \end{aligned}$$

One can then derive bounds to cross-sections for scattering in the s channel if a reasonable estimate of the l.h.s. of the above were known.

For πN scattering we have from axiomatic field theory the necessary analyticity to show that (4.19) gives the correct analytic continuation of the $\pi\pi \rightarrow N\bar{N}$ partial waves to unphysical energies. However, for πA scattering $\hat{F}(s,t)$ has from axiomatic field theory only the domain of analyticity shown in Fig. 1. We will have to postulate analyticity in the full cut t plane as indicated in Fig. 2 before being sure that $h_2^{(+)}(t)$ given by (4.19) for $0 < t < t_0$ is the correct analytic continuation of the $\pi\pi \rightarrow A\bar{A}$ partial wave from physical energies.

The best bounds for the cross-sections will presumably be obtained by working at $t = 4\mu^2$ as we have seen previously in the case of $\pi\pi$ scattering. Thus for πN scattering we use (3.14) and need an estimate for

$$a_2^{(+)} = \frac{15\pi M \alpha_2^{(+)}}{2(M^2 - \mu^2)} - \frac{G^2 \pi}{\mu^4} \quad (5.2)$$

where

$$\alpha_2^{(+)} = \lim_{t \rightarrow 4\mu^2} \frac{h_2^{(+)}(t)}{(4\mu^2 - t)} \quad (5.3)$$

To calculate it we approximate the left-hand cut of $h_2^{(+)}(t)$ by the contribution of the nucleon pole plus that of all the nucleon resonances (Fig. 3). The contribution of the nucleon pole to $a_2^{(+)}$ cancels of course with the second term on the r.h.s. of (5.2) and the N_{33}^* gives the remaining dominant contribution.

The right-hand cut of $h_2^{(+)}(t)$ is approximated by the f^0 and all its "nieces" which fortunately contribute practically nothing. The last approximation may be justified by the well-known fact that the phases of the D waves for $\pi\pi \rightarrow N\bar{N}$ and for $\pi\pi \rightarrow \pi\pi$ stay the same, rigorously for $4\mu^2 - (\mu^4/M^2) < t \leq 16\mu^2$, and approximately to the right of $16\mu^2$ up to 1.5 GeV^2 where the process $\pi\pi \rightarrow \pi\pi$ no longer dominates over all others. Then an Omnès equation is written for the right-hand cut contribution to $h_2^{(+)}(t)$, normalizing at the f^0 position and remembering that higher contributions are very small since $|h_2^{(+)}(t)| < 1$ for $t > 4M^2$ and $h_2^{(+)}(4M^2) = 0$.

To calculate the f^0 contribution a knowledge of the fNN coupling constants is needed. We use the value recently given by Engels and Höhler ²¹⁾ and get the estimate ^{*)} (in units where $\mu = 1$)

$$a_2^{(+)} = 0.058$$

The details of the calculation are given in Appendix A.

The case of $\pi^4\text{He}$ scattering is in some respects easier to deal with than πN scattering since ^4He is a spin zero particle ^{**)}. If $F_{\text{He}}(s, t)$ is the scattering amplitude for this process, then we proceed as before to define

$$h_2^{(+)}(t) = \frac{1}{2} \int_{-1}^1 d\cos\theta_t P_2(\cos\theta_t) F_{\text{He}}^{(+)}(s, t) \quad (5.4)$$

where

$$F_{\text{He}}^{(+)}(s, t) = \frac{1}{2} [F_{\text{He}}(s, t) + F_{\text{He}}(u, t)] \quad (5.5)$$

^{*)} This estimate is consistent with values for $\pi\pi \rightarrow N\bar{N}$ partial waves obtained by R. Strauss using the Ball-Wong method. We would like to thank G. Höhler for informing us of these results prior to their publication.

^{**)} The spin problem does not exist in this case and we can proceed as in $\pi\pi$ scattering.

Then with the assumed cut-plane analyticity we can prove *)

$$h_2^{(+)}(t) + (\text{pole term}) = \frac{1}{\pi Q_t q_t (M_{He} + \mu)^2} \int_{(M_{He} + \mu)^2}^{\infty} ds' A_{He}^{(+)}(s', t) Q_2 \left[\frac{s' + q_t^2 + Q_t^2}{2 q_t Q_t} \right] \quad (5.6)$$

where q_t and Q_t are as defined in (3.12) but with M_{He} replacing M . We will as previously consider

$$\begin{aligned} a_2^{(+)}(\pi He) &= \lim_{t \rightarrow 4\mu^2} [h_2^{(+)}(t) + (\text{pole term})] / (4\mu^2 - t) \\ &= \frac{4(M_{He}^2 - \mu^2)}{15\pi} \int_{(M_{He} + \mu)^2}^{\infty} ds' \frac{A_{He}^{(+)}(s', 4\mu^2)}{(s' + \mu^2 - M_{He}^2)^3} \end{aligned} \quad (5.7)$$

Bounds to $\pi^4 He$ cross-sections may be obtained from (5.7) using the standard methods so long as we have an estimate for $a_2^{(+)}(\pi^4 He)$. Plausibility arguments will be given in Appendix A why we take the estimate,

$$a_2^{(+)}(\pi^4 He) \leq 16 a_2^{(+)}(\pi N) \quad (5.8)$$

For the case of scattering of π mesons on any nucleus A where there are no anomalous thresholds, then we would again get a relation like (5.7) with M_{He} replaced by the mass of the nucleus. The above plausibility arguments would then lead to the estimate

$$a_2^{(+)}(\pi A) \leq A^2 a_2^{(+)}(\pi N) \quad (5.9)$$

from which we can get bounds to πA cross-sections in the standard way.

*) Following Ref. 22), we assume that we can combine all the pole terms due to 4He and its excited states into one effective pole term and that the effect of cuts due to pion absorption can be neglected.

6. - BOUNDS USING DIRECT CHANNEL QUANTITIES

The bounds described above have the disadvantage that they depend on the quantity $a_2^{(+)}$ for which we can only make phenomenological estimates. In this Section we will describe other bounds which although necessarily weaker from the abstract point of view have the advantage of depending on the scattering amplitude at or slightly below the physical s channel threshold. Good estimates for these quantities can be obtained from fixed t dispersion relations as shown in the case of πN scattering by Höhler et al. ²³⁾.

We will only consider the case of πN scattering, the generalization to other processes being straightforward. Substituting from (3.9) in the twice subtracted dispersion relation for $F^{(+)}(s,t)$, the sum over helicities of the πN amplitudes, we get the inequality

$$\begin{aligned} & \frac{F^{(+)}(s,t)}{[s-(M+\mu)^2][s+t-(M-\mu)^2]} - \frac{F^{(+)}[(M+\mu)^2,t]}{[s-(M+\mu)^2][4M\mu+t]} + \frac{F^{(+)}[-t+(M-\mu)^2,t]}{[s+t-(M-\mu)^2][4M\mu+t]} + \\ & + (\text{nucleon pole terms}) \geq \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \text{Im} F^{(+)}(s',t) [1+e^{-i\chi}] \times (6.1) \\ & \times \left\{ \frac{1}{[s'-s][s'-(M+\mu)^2][s'+t+(M-\mu)^2]} + \frac{1}{(s'-u)[s'+t-(M-\mu)^2][s'-(M+\mu)^2]} \right\} ds' \end{aligned}$$

where χ is the s - u crossing angle defined in (3.8).

Now

$$F^{(+)}(s,t) = A^{(+)}(s,t) + B^{(+)}(s,t) \left\{ \frac{1}{2M}(s-\mu^2-M^2) - \frac{M(s-u)}{4M^2-t} \right\} \quad (6.2)$$

where

$$A^{(+)}(s,t) = A^{(+)}(s,t) + \frac{M(s-u)}{4M^2-t} B^{(+)}(s,t) \quad (6.3)$$

From Ref. 23), we have the following expansions for $\tilde{A}'$ and $\tilde{B}$ which correspond to A' and B with the nucleon pole contribution removed ^{*}),

$$\begin{aligned} \tilde{A}'^{(+)} &= (25.60 + 1.11t) + [1.31 + 0.22t]v^2 + \dots \\ \tilde{B}^{(+)} &= (-3.20 + 0.19t)v + (-1.43 + 0.11t)v^3 + \dots \end{aligned} \quad (6.4)$$

where $v = (s-u)/4M$. These expansions are expected to be reasonably accurate for $|t| \leq 2\mu^2$ and up to threshold in the s and u channels.

We use (6.4) to obtain an estimate for the l.h.s. of (6.1) and we will presumably get the best bound to the corresponding πN cross-sections by taking t as large as possible. Therefore in our calculations we have set $t = 2\mu^2$. Finally we have to take a value for s and we choose it to be midway between s and u channel thresholds, i.e., we set $s = M^2$.

Since the weight function multiplying $\text{Im } F^{(+)}(s',t)$ is positive for all $s' \geq (M+\mu)^2$ for the above values of s and t we can proceed in the standard way to derive bounds to the averaged total cross-sections. The results will be presented in the following Section.

^{*}) The values given for the coefficients are for the case when units are chosen so that $\mu = 1$.

7. - IMPLICATION OF THE BOUNDS AND COMPARISON WITH EXPERIMENT

A. - $\pi\pi$ scattering

We have calculated the bounds to $\overline{\sigma_6^{\text{tot}}}$ defined in (1.2) for the case when $m = 1$ assuming the value $\alpha_2^t = 1.6 \times 10^{-4} \mu^{-5}$. The results are displayed in Fig. 4, for various values of T_π , the lab. kinetic energy of one of the pions in the rest frame of the other, where they are compared with the "experimental" values. To obtain the latter, we have written

$$\sigma_{\text{TOT}}^{\text{exp}}(s) = \frac{32\pi}{s-4\mu^2} \{ \text{Im} f_0(s) + 5 \text{Im} f_2(s) \} + b(s) \quad (7.1)$$

where we have saturated the f_0 and f_2 waves with ϵ and f^0 resonances respectively, and $b(s)$ is a background "Pomeranchuk" term that rises from zero at $\sqrt{s} \sim 0.7$ GeV to give a constant total cross-section of 16 mb at infinity as suggested by factorization and the quark model.

The exact form we use is therefore

$$\sigma_{\text{TOT}}^{\text{exp}}(s) = \frac{32\pi}{s-4\mu^2} \text{Im} \left[\frac{m_\epsilon \Gamma_\epsilon}{s-(m_\epsilon - i\Gamma_\epsilon/2)^2} - \frac{5m_f \Gamma_f}{s-(m_f - i\Gamma_f/2)^2} \right] + b(s) \quad (7.2)$$

where $\Gamma_\epsilon \simeq 3\mu$, $\Gamma_f \simeq \mu$. $b(s) = 0$ for $\sqrt{s} < 0.7$ GeV, $b(s) = 0.8/\mu^2$ for $\sqrt{s} > 1.5$ GeV and between these two values of $\sqrt{s}$ a linear interpolation is used to give $b(s)$.

Although a very conservative estimate, it is not difficult to see that our bounds put a lower limit on the mass of possible resonances. For example the bounds would be violated if the mass of $f^0 < 5$. In Regge language this means that the slope of the Pomeron is bounded by $\alpha_P'(0) < 1/25 \mu^2$ to be compared with the experimental slope of the ρ , $\alpha_\rho'(0) \simeq 1/40 \mu^2$.

Of course we are aware that the argument is somewhat circular in the sense that the determination of α_2^t depends on t channel resonances ⁶⁾. Alternatively one can view this result as support for a type duality or bootstrap mechanism as the position of the t channel resonances puts a constraint on those in the s channel.

B. - πN scattering

The theoretical bounds for $\overline{\sigma}_{\text{tot}}(s)$ defined in (3.17) with $m = 1$ obtained from the constraint (3.14) are given in Fig. 5 along with the experimental values for $\overline{\sigma}_{\text{tot}}(s)$ which can be calculated from the well-known ^{*)} experimental values for $\sigma_{\text{tot}}(s')$. They are plotted against the lab. kinetic energy of the pion, T_π .

The bounds are again reasonably tight in the resonance region and one can again view our result as a type of bootstrap mechanism. For the $\pi\pi$ resonances in the t channel help to determine $a_2^{(+)}$ and hence our bound to the s channel cross-sections so finally they constrain the positions and strengths of the πN resonances.

We also plot on Fig. 5 the bounds given by the direct channel method described in the previous Section. It is seen that the bounds obtained with the two different methods are of more or less the same order, though it is interesting to note that for some energies the second method is better. In the direct channel method we do not have of course a "bootstrap mechanism" working. However, one could think of our result as a type of "sum rule" as we have derived a connection between the πN scattering amplitude at (and below) threshold with its behaviour at higher energies.

*) See for example pp. 72, 73 of Ref. 24).

C. - πA scattering

Theoretical bounds for $\pi^4\text{He}$ cross-sections are given in Fig. 6, where the values of $\overline{\sigma}_{\text{tot}}(s)$ have again been calculated with $m = 1$ in (1.2) and under the assumption that $a_2^{(+)}(\pi\text{He})$ is given by (5.8). Also given are experimental values for $\overline{\sigma}_{\text{tot}}(s)$ obtained from the corresponding experimental values for $^{22}) \overline{\sigma}_{\text{tot}}(s')$. As previously these values are plotted against the lab. kinetic energy of the pion. Once again our bounds are more stringent in the resonance region than in the asymptotic region. In fact in the former region our bounds are only four times the experimental values which is appreciably tighter than in the case of πN scattering. We return to this point below.

For the general case of πA scattering we run into the problem of anomalous thresholds as mentioned earlier. Actually the need for anomalous thresholds is strongly suggested by our analysis. For total cross-sections $^{22})$ are scaled up by a factor $A^{2/3}$ while our bounds to these cross-sections depend on A through $a_2^{(+)}(\pi A)$, i.e., only logarithmically $^{*)}$ as can be seen in the explicit asymptotic forms (3.18), (3.19).

Therefore for sufficiently large enough A the bound would be violated. Our numerical results already suggest this since our bounds for $\pi^4\text{He}$ scattering are much tighter than those for πN scattering. The only escape is to recognize that nuclear forces must saturate around a binding energy sufficiently small to produce anomalous thresholds.

*) An intuitive way to understand this is to recall that the unitarity limit reads the same, $\text{Im } f_l \leq 1$, for any process. Therefore the coefficient of the bound is independent of the target and the influence of the latter is only manifest through the effective number of waves which occur (because of its size) in the "tail", which is what gives the logarithmic term in the bound.

If for simplicity we consider a model world in which the binding energy is a linearly increasing function of A , one can check as shown in Appendix B that anomalous thresholds appear as soon as the binding energy per nucleon is less than approximately 10 MeV. Thus we predict for large A such binding energies are < 10 MeV which compares very nicely with the experimental values of 8 MeV. We would like to stress that the above expression is based on the assumption that $a_2^{(+)}(\pi A)$ increases no faster than A^2 .

For the so-called "magic nuclei" ²⁵⁾ (which include ^4He , $^{16}\text{O}, \dots$), however, it can be proved that no anomalous thresholds exist ^{*)} and in these cases our analysis can be applied.

8. - DISCUSSIONS AND COMMENTS

We have generalized the analysis of Refs. 1), 2) to pion scattering on any target as far as spin is concerned to any process, giving also new bounds which concern only direct channel quantities.

From a qualitative point of view, the bounds are fairly tight at intermediate energies, but too large at high energies and close to threshold. In fact our bounds tend to infinity at threshold and this is to be expected because we do not use in our analysis the correct threshold behaviour of the partial waves.

At medium energies the situation is quite satisfactory since our bounds permit a rudimentary bootstrap game in which the position and strength of t channel resonances constrain the location and strengths of resonance in the s channel. The only weak point of our analysis is of course the estimation of the quantity $a_2^{(+)}$ defined in Section 5 : unlike the $\pi\pi$ case where totally parameter-independent bounds can be deduced ²⁾, we have in other cases to rely on more or less phenomenological estimations. The direct channel bounds given in Section 6 were aimed at remedying this situation by using quantities on which we have more reliable estimates.

^{*)} See Appendix B for details.

It should also be noticed that our results can be turned around in that if there are reliable estimates for cross-sections in the s channel as for example in πN scattering then we can put lower bounds ^{*)} on the partial waves in the t channel as for example on $a_2^{(+)}$. This is easy to see since if $a_2^{(+)}$ is decreased our bound to σ_{tot} will decrease and eventually will become inconsistent with the experimental values. Hence $a_2^{(+)}$ cannot be made too small. We are investigating further this matter.

Finally we would like to express regret that the present state of the art does not allow us to derive bounds for, e.g., pp scattering, where one and two pion exchange models give fairly good results and where therefore one would expect quite tight medium energy bounds.

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^{*)} We have been informed by G. Höhler in a private communication that such a bound may be useful in a phase-shift analysis of $\pi\pi \rightarrow NN$ scattering.

APPENDIX A

We consider first of all the computation of $a_2^{(+)}$ in the case of πN scattering. To do this we have to know the contribution of the nucleon resonances to $h_2^{(+)}(t)$. The contribution of the N_{33}^* is easily computed in the narrow resonance approximation and is equal to

$$\frac{2\pi R [\beta_1^2 \beta_4 - \beta_3 \beta_1 + \beta_2]}{2 q_r q_t^2 Q_t^2} Q_2 \left[\frac{s_r - M^2 - \mu^2 + \frac{t}{2}}{2 q_t Q_t} \right] \quad (A.1)$$

where q_t, Q_t are defined in (3.12).

$s_r = M^{*2}$ with M^* = mass of N_{33}^* and q_r is the c.m. momentum corresponding to $s = s_r$. $R = q_r^3 (g^{*2}/4\pi)(E_r + M)/3$ where $^{26)} (g^{*2}/4\pi) = 0.264$ and E_r is the nucleon energy in the c.m. frame. Finally

$$\beta_1 = [s_r - M^2 - \mu^2 + \frac{t}{2}]/2 q_t Q_t; \beta_4 = \frac{3 q_t^2 Q_t^2}{M(E_r + M) q_r^2} \quad (A.2)$$

$$\beta_2 = 3 \left(\frac{W_r + M}{E_r + M} \right) \left[1 + \frac{1}{2 q_r^2} (M^2 + \mu^2 - s_r + \frac{t}{2}) \right] + \frac{W_r - M}{E_r - M} + \quad (A.3)$$

$$+ \frac{3t}{4M(E_r + M)} \left[1 + \frac{1}{2 q_r^2} (M^2 + \mu^2 - s_r + \frac{t}{2}) \right] - \frac{t}{4M(E_r - M)}$$

$$\beta_3 = -3 \left[\frac{W_r + M}{E_r + M} \right] \frac{q_t Q_t}{q_r^2} - \frac{3}{2M(E_r + M)} \left\{ 2 q_t Q_t \left[1 + \frac{1}{2 q_r^2} (M^2 + \mu^2 - s_r + \frac{t}{2}) \right] + \frac{t q_t Q_t}{2 q_r^2} \right\} \quad (A.4)$$

It is a simple matter to evaluate the limit of the above contribution to $h_2^{(+)}(t)$ as $t \rightarrow 4\mu^2$ i.e., $q_t \rightarrow 0$. Near $q_t = 0$, $Q_2 \sim (2q_t Q_t)^3 / [M^{*2} + \mu^2 - M^2]^3$ so that the contribution of a resonance with mass M_0 to $h_2^{(+)}(t)$ is $\sim [M_0^2 + \mu^2 - M^2]^{-3}$. One would then expect the contribution of higher resonances to $h_2^{(+)}(t)$ to be $\sim 7\%$ of the contribution of the N_{33}^* and so we feel justified in neglecting them.

If $h_{2L}^{(+)}(t)$ is the left-hand cut contribution to $h_2^{(+)}(t)$ with the nucleon pole term omitted then in units where $\mu = 1$,

$$\lim_{t \rightarrow 4\mu^2} \frac{h_{2L}^{(+)}(t)}{4\mu^2 - t} = 0.028 \quad (A.5)$$

Defining $h_{2R}^{(+)}(t)$ to be the right-hand cut contribution to $h_2^{(+)}(t)$ which we approximate by just taking the f^0 resonance contribution (Fig. 3b), we find that

$$h_{2R}^{(+)}(t) = \frac{2\pi q_k^2 g_{\pi\pi f}^{(+)} g_{N\bar{N}f}^{(+)}}{t - m_f^2} - \left(\frac{\sqrt{2}\pi q_k^2}{3\sqrt{3}M} \right) \frac{g_{\pi\pi f}^{(-)} g_{N\bar{N}f}^{(-)}}{t - m_f^2} \quad (A.6)$$

where m_f is the mass of the f^0 resonance and $g_{\pi\pi f}$ is the well-known $f \pi\pi$ coupling constant.

The $g_{N\bar{N}f}^{(\pm)}$ are the coupling constants of the f^0 to nucleon, antinucleon when the two baryons have the (same) helicities. We use a recent estimation of these coupling constants by Engels and Höhler ²⁶⁾ and find that ^{*})

$$g_{\pi\pi f}^{(+)} g_{N\bar{N}f}^{(+)} = \frac{1}{15\pi m_f} G_{\pi\pi f}^{(1)} G_{N\bar{N}f}^{(1)} ; g_{\pi\pi f}^{(-)} g_{N\bar{N}f}^{(-)} = \frac{1}{60\pi M} G_{\pi\pi f}^{(1)} G_{N\bar{N}f}^{(1)} \quad (A.7)$$

where $G_{\pi\pi f}^{(1)} G_{N\bar{N}f}^{(1)} = 600$.

Substituting from (A.7) in (A.6) we get

$$\lim_{t \rightarrow 4\mu^2} \frac{h_{2R}^{(+)}(t)}{4\mu^2 - t} = 0.030 \quad (A.8)$$

From (A.5) and (A.8) we get the estimate for $a_2^{(+)}$ quoted in Section 5.

^{*}) In the notation of Ref. 26) we have taken $G_{N\bar{N}f}^{(2)} = 0$ which is within the quoted errors.

When considering the general case of π nucleus scattering the following modifications have to be taken into account :

- (i) the nucleon pole is replaced in general by a continuum of nuclear excited states ; this does not cause any difficulty because first of all as shown in Ref. 22), one can lump all these poles together and replace them by a single effective pole with coupling constant $G_A = A^2 G$ and secondly the contributions of these poles to $a_2^{(+)}$ drops out anyway ;
- (ii) the coupling constant g^* is to be replaced by g_A^* , this being the coupling constant between π , A and the nuclear "resonance" corresponding to the N_{33}^* . Experimentally ^{**)} we find that $g_A^{*2} \simeq A g^{*2}$;
- (iii) the coupling constant g_{NNf} has to be replaced by g_{AAf} ; we have no experimental information on this quantity but we assume it obeys the same scaling law as the coupling constants $G_A \equiv g_{\pi A \bar{A}}$, i.e., we assume $g_{AAf} = A^2 g_{NNf}$.

From the above scaling laws for the coupling constants we get, neglecting any other effects depending on the mass of the nucleus A, the relation (5.9) which gives us an upper bound to $a_2^{(+)}(\pi A)$ in terms of the known quantity $a_2^{(+)}(\pi N)$.

*) This relation follows from Eq. (3.8) of Ref. 22) which is a theoretical estimate of the π nuclear pole term and is expected to be reliable for $A \leq 16$. However, we will assume that this scaling law also holds for $A \geq 16$.

***) This scaling law follows from the magnitudes of the relevant πA cross-sections as taken from Section 4.2 of Ref. 22).

APPENDIX B

We recall here the calculation of anomalous thresholds [see Ref. 27) for further details]. Diagrams that may give anomalous thresholds are shown in Fig. 7a. Drawing the dual diagram (Fig. 7b), where m' is the mass of particle A' and m that of particle x , then t_0 is the location of the threshold. (A necessary condition for the existence of an anomalous threshold is that the point O lies inside the triangle.)

Therefore defining $B = m + m' - M$, $2M' = m + m' + M$, where M is the mass of the nucleus A , we see from elementary geometrical considerations that

$$t_0 = \frac{2BM'[m^2 - (m - m')^2]}{m^2} \quad (B.1)$$

It is easy to check that in the model described at the end of Section 7C, B is independent of the different possibilities for particle A' and it follows that t_0 is a minimum when A' is a single nucleon. This minimum value must satisfy

$$4\mu^2 > t_0 = 4BA(2A-2)M/(A-1)^2 \approx 56B\mu \quad (B.2)$$

Therefore anomalous thresholds start to appear when $B < 10$ MeV.

For the so-called ²⁵⁾ magic nuclei ${}^4\text{He}$, ${}^{16}\text{O}$, etc., the binding energies of the last nucleon, i.e., the values of B when A' is a single nucleon are much larger than 10 MeV ²⁸⁾ and one expects no anomalous thresholds to occur. However, since in these cases B is not independent of the different choices for A' the minimum of t_0 may occur when A' is more than one nucleon. Therefore to be absolutely certain that no anomalous thresholds occur one has to calculate t_0 from (B.1) for all possible decompositions of A . For ${}^4\text{He}$ at least it is a simple matter to check that $t_0 > 4\mu^2$ so that there are no anomalous thresholds.

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FIGURE CAPTIONS

- Figure_1 Analyticity domain for the $\pi\pi \rightarrow A\bar{A}$ amplitude from axiomatic field theory (no anomalous thresholds).
- Figure_2 Analyticity domain for the $\pi\pi \rightarrow A\bar{A}$ amplitude from the Mandelstam representation with a possible anomalous threshold at t_0 .
- Figure_3 (a) Crossed channel contributions to $a_2^{(+)}$.
(b) Direct channel contributions to $a_2^{(+)}$.
- Figure_4 $\pi\pi$ cross-sections.
(a) Theoretical bound to $\overline{\sigma_{\text{tot}}}$.
(b) "Experimental" values for $\overline{\sigma_{\text{tot}}}$.
- Figure_5 πN cross-sections.
(a) Theoretical bound to $\overline{\sigma_{\text{tot}}}$ using $a_2^{(+)} = 0.058$ ($\mu=1$).
(b) Theoretical bound to $\overline{\sigma_{\text{tot}}}$ using direct channel quantities.
(c) Experimental values for $\overline{\sigma_{\text{tot}}}$.
- Figure_6 $\pi^4\text{He}$ cross-sections.
(a) Theoretical bound to $\overline{\sigma_{\text{tot}}}$ using $a_2^{(+)}(\pi\text{He}) = 0.928$ ($\mu=1$).
(b) Experimental values for $\overline{\sigma_{\text{tot}}}$.
- Figure_7 (a) Diagram giving possible anomalous threshold in $\pi\pi \rightarrow A\bar{A}$ scattering.
(b) The dual diagram.

FIG.1

FIG.2

FIG.3

FIG.4

FIG. 6

(a)

(b)

FIG.7