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SOME PROPERTIES OF A DUAL MULTILoop PLANAR OPERATOR
FREE OF SPURIOUS STATES

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A B S T R A C T

We want to study some properties of a dual multiloop planar operator free of spurious states in the loops. For this aim, we construct it by a simple iterative method in a form which allows us to prove directly some duality properties, such as the possibility of a tree to go around one tadpole, or in the two-loop case the transformation of one tadpole into a non-planar self-energy. The relation between some invariance properties of the integration measure and the integration domain is discussed.

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1. INTRODUCTION

In the last few months, much work has been devoted to the dual multiloop amplitude. A first and important work has been done by Kaku and Yu ¹⁾; it is very complete but leads to rather lengthy calculation in order to obtain the amplitude. A second by Lovelace ²⁾ and Alessandrini ³⁾ shows essentially that the part of the integrand which depends on the external momenta is given by the Neumann function associated with a sphere with N handles, but it does not give a determination of the integration measure. Both of these works do not prove the duality properties of the N loop amplitude.

Then, we want to study, in this paper, some duality properties of the N loop planar amplitude and how the duality restricts the natural integration range of Chan variables in the operatorial formalism. In particular, we want to know what is the change of variable associated with some duality transformation (what we can call the local duality). As it is well known, all planar diagrams can be written as a tree with an N loop tadpole on some leg. Then by factorization, we can define an N loop planar tadpole operator. We could start from the amplitude of Kaku and Yu and factorize it, but it is computed by means of an N Reggeon vertex ^{4),5),6),7)} in terms of Koba-Nielsen variables. Then, we prefer to construct this operator directly starting from the one-loop operator of Gross and Schwarz ⁷⁾; in this aim we present a simple iterative method. It is shown that the result is in agreement with the previous ones.

Before giving some more details on this paper, let us note that very similar problems have been studied independently in a very recent work by Alessandrini and Amati ⁸⁾. This work starts from the functional integration on the Riemann surface method and studies in a general and systematic way the integration measure, the domain of integration, the duality properties and various singularities of the multiloop amplitude. They treated on the same foot the planar and non-planar amplitudes. Their results are in agreement with ours.

In Part 2, we rewrite in a more transparent form the one-loop operators defined by Gross und Schwarz ⁷⁾. We prove directly the possibility of a tree (or more precisely of a Reggeon) to go around a loop.

The method to compute an N loop planar operator in the operatorial formalism is given in Part 3, when it is in a configuration of N consecutive tadpoles. The problem of periodicities is also exhibited.

It is shown in Part 4 how one can exhibit such periodicities when the operator is in a configuration of $(N-1)$ non-planar self-energy attached to a tadpole. That provides some restrictions on the integration domain which remove the infinite multiple-counting due to those periodicities. The proof of the identity of this operator with the previous one is explicitly given in the two-loop case.

Some details of computation are given in the various appendices. In particular, we refer the reader to Appendix A for all notations and some known properties of usual $SU(1,1)$ matrices. It is shown in Appendix B how to deal with 2×2 matrices with non-commutative elements (inversion of such matrix and computation of its determinant). We give in Appendix C some useful identities related to the combination of two groups in order to generate a greater group. The complete computation of the partition function is given in Appendix D. In Appendix E, it is shown how to pass from the Chan variables to the invariant points. Finally, the N loop amplitude is computed in Appendix F in order to compare with the previous results.

2. ONE-LOOP OPERATORS

We begin by putting the primitive one-loop operators of Gross and Schwarz ⁷⁾ in a more transparent form for their properties under projective transformation.

1. One-loop planar tadpole operator (Fig. 1)

It is constructed from the projected twisted propagator and the symmetric vertex, and is given by ⁷⁾

$$\begin{aligned} \langle 0_a | T = & \int_0^1 dw [f(w)]^{-4} \int d_4 k w^{-\alpha(k^2)-1} \times \\ & \times \langle 0_a | \exp \left\{ -\alpha(1-w)k + \frac{1}{2} \alpha(1-w) \left\{ M_+ \frac{[w]}{1-[w]} M_- + M_-^c \frac{[w]}{1-[w]} M_+ \right\} (1-w)k \right\} \end{aligned} \quad (2.1)$$

where $f(w)$ is the usual partition function. We recall that the integration on w went initially from 0 to ∞ , but is restricted to $[0,1]$ because $[f(w)]^{-1}$ has a natural boundary on the unit circle and is then defined as equal to zero for $|w| > 1$. (This prescription has been checked by Gross and Schwarz by attaching a tree to the tadpole and recovering the usual one-loop amplitude without spurious states.)

Using the results of Appendix A, it is easy to see that we can rewrite (2.1) in the form

$$\begin{aligned} \langle 0_a | T = & \int_0^1 dw w^{-\alpha_0-1} [f(w)]^{-4} \int d_4 k e^{-\frac{k^2}{2} \log w} \times \\ & \times \langle 0_a | \exp \left\{ \alpha \left[\left(\frac{1}{\beta} \right) - \left(\frac{1}{\alpha} \right) \right] k + \frac{1}{2} \alpha \Gamma(\mathcal{P}-1) | a \rangle \right\} \end{aligned} \quad (2.2)$$

with

$$\mathcal{P} = \sum_{n=-\infty}^{+\infty} P^n$$

where
$$P = \frac{1}{[1-w]} M_- [w] M_- [1-w]$$

or
$$\hat{P}(y) = 1 + wy \quad (2.3)$$

β and α are the invariant points of the associated projective transformation $\hat{P}$, i.e.,

$$\begin{aligned} \beta &= \widehat{\left(\frac{1}{[1-w]} M_- \right)} (\infty) = -\infty \\ \alpha &= \widehat{\left(\frac{1}{[1-w]} M_- \right)} (0) = \frac{1}{1-w} \end{aligned} \quad (2.4)$$

w being the multiplier of $\hat{P}$.

Under this form, the projective properties are very transparent. If we multiply on the right by $:e^{a^+(C-1)a}$: where C is a $SU(1,1)$ matrix⁵⁾, then (2.2) is formally invariant. We only replace P (and evidently α, β) by $P', (\alpha', \beta')$ using properties of Γ

$$P' = C^{-1} P C \quad (2.5)$$

and α', β' are now the invariant points of P' . [Let us note, however, that the relation between α', β', w has changed (2.4).]

2. Fundamental duality property of the planar tadpole

The fundamental property of the planar tadpole is that it can be put anywhere in a tree graph. We can also say that an arbitrary tree can go around a tadpole. After factorization of the trees, we can reduce all that to a fundamental property of the

planar tadpole with respect to the symmetric vertex. It is explicitated by the identity of Figs. 2a and 2b.

We shall now prove it directly. Let us attach a symmetric vertex and a twisted propagator ($\mathcal{D}(x)$) to a tadpole, i.e.,

$$\begin{aligned} \langle 0_{a,c} | X = \int \frac{d^2 z}{\pi} e^{-|z|^2} \langle 0_{a,b,c} | e^{ca|M_+|c\rangle + cc|M_-|b\rangle + (cb|M_+|a\rangle + (cb|-a|)|2p|z_b\rangle} \\ \times \langle z_b | \mathcal{D}(x) T(b^+) | 0_b \rangle \end{aligned} \quad (2.6)$$

which is easily shown to be equal to

$$\begin{aligned} \langle 0_{a,c} | X = \int \frac{dw}{w} \frac{dx}{x(1-x)} \left(\frac{wx}{1-x} \right)^{-\alpha_b} [f(w)]^{-4} \int d_4 k w^{-k^2/2} \\ \times \langle 0_{a,c} | \exp \left\{ \frac{1}{2} [ca|M_+^t + cc|M_- + p c 1] \Gamma (\mathcal{P} - I) [M_-|a\rangle + M_-^t|c\rangle + |1\rangle p] \right. \\ \left. + [ca|M_+^t + cc|M_- + p c 1] \left[\left| \frac{1}{\xi_2} \right\rangle - \left| \frac{1}{\xi_1} \right\rangle \right] + (ca|M_-|c\rangle - ca|1\rangle p) \right\} \end{aligned} \quad (2.7)$$

where the homographic transformation P has ξ_2 and ξ_1 as invariant points

$$P = [\xi_2] M_-^t \left[\frac{\xi_1}{\xi_1 - \xi_2} \right] M_- [w] M_- \left[\frac{\xi_1 - \xi_2}{\xi_1} \right] M_-^t \left[\frac{1}{\xi_2} \right] \quad (2.8)$$

or

$$\hat{P}(z) = \frac{z(\xi_1 + \xi_2 - 1) - \xi_1 \xi_2}{z - 1} \quad (2.9)$$

with

$$\xi_2 = \frac{1}{x}, \quad w = \frac{1-\xi_2}{1-\xi_1} \quad (2.10)$$

a similar expression is obtained for Y by changing $a \leftrightarrow c$, $p \leftrightarrow -p$, $x \leftrightarrow x'$ and w unchanged. We must then prove that there exists some α , β and $x' = f(x, w)$ such that

$$\langle 0_{a,c} | X \alpha^{A_a(p)-\alpha_0} \beta^{A_c(-p)-\alpha_0} = \langle 0_{a,c} | Y \quad (2.11)$$

The problem is easily solved by identifying the terms in $(a | \dots | a)$; $(c | \dots | c)$ and $(a | \dots | c)$ in the two members. We then obtain

$$\alpha = \frac{(1-\xi_1)(1-\xi_2)}{\xi_1 \xi_2} \quad (2.12)$$

$$\beta = \xi_1 \xi_2 \quad (2.13)$$

and

$$x' = \frac{1-x}{1-x+wx} \quad (2.14)$$

or

$$\begin{cases} 1-\xi_1 = \frac{1}{1-\xi_2'} \\ 1-\xi_2 = \frac{1}{1-\xi_1'} \end{cases} \quad (2.15)$$

We can then check that the other terms are also identical: for the terms in p or p^2 *) we can use the Ψ function introduced by Gross, Neveu, Scherk and Schwarz ⁹⁾. Let us remark that we can also express the other terms in function of Ψ , but the gauge and change of variables are more difficult to find. However, knowing them we can check that the identity (2.11) is true for any Ψ which satisfies

$$\left\{ \begin{array}{l} \Psi(\frac{1}{x}, w) = -\Psi(x, w) \\ \Psi(wx, w) = -\Psi(x, w) \\ \frac{\Psi(x, w)}{1-x} \xrightarrow{x \rightarrow 1} 1 \end{array} \right. \quad (2.16)$$

so that the counterterm introduced by Neveu and Scherk ¹⁰⁾ and Gross and Schwarz ⁷⁾ has automatically this duality property with the same gauge transformation.

Let us finally remark that we can do suitable gauge transformations on X , such that it satisfies the duality property without gauge transformations, more explicitly if we define

*) If we check before integration over $d_4 k$ we can see that we must do the change $k \rightarrow k+p$. We note also that the dependence in p^2 has a part of the form

$$\left[\Psi(w \frac{s_1}{s_2}, w) \right] p^2$$

But, contrary to the case of non-planar self-energy, it does not affect the divergence at $w \rightarrow 1$, because

$$w \frac{s_1}{s_2} = 1 - \frac{1-w}{s_2} \rightarrow 1 \quad \text{as} \quad w \rightarrow 1$$

so that

$$\Psi(w \frac{s_1}{s_2}, w) \underset{w \rightarrow 1}{\sim} - \frac{\log w}{\pi} \sin(\pi/s)$$

and combined with the other factor in $p^2 (\approx \frac{1}{1-w})$ gives a finite result as $w \rightarrow 1$.

$$\langle a_{a,c} | \tilde{X} = \langle a_{a,c} | X \left(\frac{1-x}{x} \right)^{\eta A_c(-p) + (1-\eta) A_a(p) - \alpha_0} \quad (2.17)$$

with

$$\eta = \frac{\text{Log } \xi_1 \xi_2}{\text{Log}[(1-\xi_1)(1-\xi_2)]} \quad \left(= \frac{\text{Log } \beta}{\text{Log}(\alpha\beta)} \right) \quad (2.18)$$

then

$$\langle a_{a,c} | \tilde{X}(a, c, p, x) = \langle a_{a,c} | \tilde{X}(c, a, -p, x') \quad (2.19)$$

with x' given by (2.14).

3. Non-planar self-energy operator

The non-planar self-energy operator, as shown in Fig. 3, has been computed by Gross and Schwarz ⁷⁾. Let us limit ourselves to the case $p = 0$. It is written

$$\begin{aligned} \langle 0_{1,2} | \Sigma_T(0) &= \int_0^1 \frac{dw}{w} \int_w^1 \frac{dx}{x} [f(w)]^{-4} \int d_4 k w^{-\alpha(k^2)} \times \\ &\times \langle 0_{1,2} | \exp \left\{ \frac{1}{2} C_{a_1} [1-w] \left\{ M_-^t \frac{[w]}{1-[w]} M_+ + M_+ \frac{[w]}{1-[w]} M_- \right\} [1-w] | a_1 \right\rangle \right. \\ &\quad + \frac{1}{2} C_{a_2} [1-w] \left\{ M_-^t \frac{[w]}{1-[w]} M_+ + M_+ \frac{[w]}{1-[w]} M_- \right\} [1-w] | a_2 \rangle \\ &\quad + C_{a_1} [1-w] \left\{ M_-^t \frac{[-x]}{1-[w]} M_- + M_+ \frac{[-x]}{1-[w]} M_+ \right\} [1-w] | a_2 \rangle \\ &\quad \left. + C_{a_1} [1-w] k - C_{a_2} [1-w] k \right\} \end{aligned} \quad (2.20)$$

Defining

$$P = \frac{1}{[1-w]} M_-[w] M_-[1-w] \quad \text{or} \quad \hat{P}(y) = 1 + wy \quad (2.21)$$

$$\Gamma_x = [1-w] M_-^t[-x] M_-[1-w] = \Gamma_x^t \quad (2.22)$$

which satisfies, for any x ,

$$\Gamma_x P \Gamma_x^{-1} = P^t \quad (2.23)$$

we can rewrite $\Sigma_T(0)$ in the more useful form

$$\begin{aligned} \langle 0_{1,2} | \Sigma_T(0) &= \int_0^1 \frac{dw}{w} \int_0^1 \frac{dx}{x} [P(w)]^{-4} \int d_4 k w^{-\alpha(k^2)} \\ &\times \langle 0_{1,2} | \exp \left\{ \frac{1}{2} (a_1 | \Gamma(P-I) | a_1) + \frac{1}{2} (a_2 | \Gamma(P-I) | a_2) \right. \\ &\quad \left. + (a_2 | \Gamma_x P | a_1) + [(a_2 | - (a_1 |)] \left[\left(\frac{1}{P} \right) - \left(\frac{1}{x} \right) \right] k \right\} \end{aligned} \quad (2.24)$$

Let us remark that primitively the integration over x went from 0 to ∞ , but as noted by Gross and Schwarz, the integrand is invariant under the change $x \rightarrow wx$ since

$$\Gamma_{wx} = \Gamma_x P \quad (2.25)$$

and

$$P P = P \quad (2.26)$$

and also the measure

$$\frac{d(wx)}{wx} = \frac{dx}{x}$$

so that the same contribution is counted an infinite number of times. This can also be seen from duality property if we write the self-energy in the form of Fig. 4, by dualizing one of the two internal lines, we recover the same configuration. Then, the range of integration is restricted to only one period, for instance, $[\bar{w}, 1]$.

4. Non-orientable tadpole operator

For completeness, we give some comments on the non-orientable tadpole operator.

a) As before, we can put it in the form

$$\begin{aligned} \langle 0 | T_N(a) = \int_0^1 \frac{dw}{w} \left(\frac{1+w}{1-w} \right) w^{-\alpha_0} [f(w)]^{-4} \int d_4 k w^{-k^2/2} \\ \times \langle 0 | \exp \left\{ \left(\alpha \left[\frac{1}{\beta_N} \right] - \frac{1}{\alpha_N} \right) k + \frac{1}{2} \alpha \Gamma (P_N - I) | a \right\} \end{aligned}$$

(2.27)

with

$$\begin{aligned} P_N &= \frac{1}{[1-w^2]} M_- [-w] M_- [1-w^2] \\ \alpha_N &= \frac{1}{1-w^2} \quad , \quad \beta_N = \infty \end{aligned}$$

(2.28)

Then, we have a very similar structure as in the planar case except essentially the change $w \rightarrow -w$. This is related to the fact that in the non-planar case, it is P_N^2 which is the basic transformation.

b) We can also attach it to a symmetric vertex, and then show exactly as for the planar case the fundamental duality property that a tree can go around this tadpole and the change of variable is now

$$x' = \frac{1-x}{1-x+w^2x} \quad (2.29)$$

The gauge parameters are the same except we replace w by w^2 .

- c) It can be seen in terms of the ψ_N function that this property depends only on the following conditions

$$\left\{ \begin{array}{l} \psi_N(xw^2, w) = -\psi_N(x, w) \\ \psi_N(\frac{1}{x}, w) = -\psi_N(x, w) \\ \frac{\psi_N(x, w)}{1-x} \xrightarrow{x \rightarrow 1} 1 \end{array} \right. \quad (2.30)$$

Then the counterterm constructed by Gross and Schwarz ⁷⁾ has also this duality property.

3. ITERATION OF N TADPOLE OPERATORS

Let us now show how to compute an N loop planar operator in the configuration of an iteration of N tadpoles. We shall study some properties in the special case of two-loop operators.

1. N loop planar operator

We construct it by iteration, or more precisely, by attaching a one-loop tadpole and vertex to an (N-1) loop operator.

Let us assume that the N loop operator, as shown in Fig. 5, is given by

$$\begin{aligned} \langle 0_b | T_n(b) = & \int \prod_{i=1}^{n-1} \left\{ \frac{dx_i}{x_i(1-x_i)} \frac{dy_i}{y_i(1-y_i)} \left[\frac{x_i y_i}{(1-x_i)(1-y_i)} \right]^{-\alpha_0} \right\} \int \prod_{i=0}^{n-1} \frac{dw_i}{w_i} w_i^{-\alpha_0} [F(p_0^{(n)}, \dots, p_{n-1}^{(n)})]^{-4} \\ & * \left(\prod_{i=0}^{n-1} d_4 k_i \right) \exp \frac{1}{2} \sum_{i,j=0}^{n-1} k_i \cdot k_j \left\{ \left[\frac{1}{\beta_j^{(n)}} - \frac{1}{\alpha_j^{(n)}} \right] \varphi_{ij}^{(n)} [\beta_j^{(n)}] - [\alpha_j^{(n)}] - \delta_{ij} \log w_i \right\} \\ & * \langle 0_b | \exp \left\{ \langle b | \Gamma \left(\sum_{j=0}^{n-1} \varphi_j^{(n)} [\beta_j^{(n)}] - [\alpha_j^{(n)}] \right) k_j \right\rangle + \frac{1}{2} \langle b | \Gamma (\Phi^{(n)} - I) | b \rangle \right\} \end{aligned} \quad (3.1)$$

The projective transformations $p_i^{(n)}$ ($i=0, n-1$) with invariant points $\beta_i^{(n)}$ and $\alpha_i^{(n)}$ and multiplier w_i generate the group $\mathcal{G}^{(n)}$. $\mathcal{G}^{(n)}$ is the sum of all the elements of $\mathcal{G}^{(n)}$: $Q_j^{(n)}$ the sum of those which do not end by $p_j^{(n)}$ or $p_j^{(n)-1}$; $Q_{ij}^{(n)}$ the sum of those which do not end by $p_j^{(n)}$ or $p_j^{(n)-1}$ and do not begin with $p_i^{(n)}$ or $p_i^{(n)-1}$. $F(p_0 \dots p_{n-1})$ is the usual partition function.

When attaching this operator to a tree $(\sum p_i | \mathcal{E}_i)$, we recognize easily the different terms of the N loop amplitude as given by Lovelace ²⁾, Alessandrini ³⁾, Alessandrini and Amati ⁴⁾

and Kaku and Yu¹⁾. The term in $k_i k_j$ is the matrix period; the term linear in k_j is the first Abelian integral, and the last is the third Abelian integral^{*)}.

Let us now show (3.1) by iteration. Assuming (3.1) for the N loop operator, we build the $N+1$ loop operator by attaching a vertex and tadpole, i.e.,

$$\begin{aligned} \langle 0_a | T_{n+1}(a) = \int \frac{d^2 z}{\pi} e^{-|z|^2} \langle 0_a, b | T_n(b) \mathcal{D}(x_n) | z \rangle \times \\ \times \langle z | V(a, b, b^+) \mathcal{D}(y_n) T_1^{(n)}(b^+) | 0_b \rangle \end{aligned} \quad (3.2)$$

Let us recall that the action of $\mathcal{D}(x_n)$ (resp. $\mathcal{D}(y_n)$) is only to replace the generator $P_i^{(n)}$ (resp. P_n) by $P_i'^{(n)}$ (resp. P_n')

$$P_i'^{(n)} = M_- \left[\frac{x_n - 1}{x_n} \right] n_-^t P_i^{(n)} M_-^t \left[\frac{x_n}{x_n - 1} \right] M_- \quad (3.3)$$

Then, using the well-known principal axis method we obtain (omitting for the moment the integration measure) for the vacuum expectation value

*) In the definition of the third Abelian integral, the identity is not missing, it comes from the integrand of the tree. The connection between the formal expression of Abelian integral as given by Alessandrini and those expressions which make use of invariant points is done by means of infinite cancellation as it can be seen in the paper of Kaku and Yu¹⁾.

$$\begin{aligned}
 & \int \frac{d^2 z}{\pi} e^{-|z|^2} \exp \left\{ \frac{1}{2} \langle z | \Gamma (\mathcal{G}'^{(n)} - I) | z \rangle + \langle z | \Gamma \sum_j Q_j'^{(n)} [|\beta_j'^{(n)}\rangle - |\alpha_j'^{(n)}\rangle] k_j \right. \\
 & \quad + \langle a | M_- | z^* \rangle + [\langle z^* | M_- + \langle a | M_-^t] \frac{1}{2} \Gamma (\mathcal{G}'_n - I) [M_- | a \rangle + M_-^t | z^* \rangle] \\
 & \quad \left. + [\langle z^* | M_- + \langle a | M_-^t] [|\frac{1}{\beta_n'}\rangle - |\frac{1}{\alpha_n'}\rangle] k_n \right\} \\
 & = [\det(iA)]^{-2} \exp - \frac{1}{2} B^t A^{-1} B \times \\
 & \quad \times \exp \left\{ \frac{1}{2} \langle a | M_-^t \Gamma (\mathcal{G}'_n - I) M_- | a \rangle + \langle a | M_-^t [|\frac{1}{\beta_n'}\rangle - |\frac{1}{\alpha_n'}\rangle] k_n \right\}
 \end{aligned}
 \tag{3.4}$$

with

$$A = \begin{pmatrix} \Gamma (\mathcal{G}'^{(n)} - I) & -I \\ -I & M_- \Gamma (\mathcal{G}'_n - I) M_-^t \end{pmatrix}
 \tag{3.5}$$

$$B = \begin{pmatrix} \Gamma \sum_{j=0}^{n-1} Q_j'^{(n)} [|\beta_j'^{(n)}\rangle - |\alpha_j'^{(n)}\rangle] k_j \\ [M_-^t + M_- \Gamma (\mathcal{G}'_n - I)] | a \rangle + M_- [|\frac{1}{\beta_n'}\rangle - |\frac{1}{\alpha_n'}\rangle] k_n \end{pmatrix}
 \tag{3.6}$$

We use the results of Appendix B in order to work with these 2×2 matrices and we define

$$P_j^{(n+1)} = M_-^t P_j'^{(n)} M_-^t \quad ; \quad \tilde{\mathcal{G}}^{(n)} = M_-^t \mathcal{G}'^{(n)} M_-^t
 \tag{3.7}$$

$$P_n^{(n+1)} = M_- P_n' M_- \quad ; \quad \tilde{P}_n = M_- P_n' M_- \quad (3.8)$$

We note that (3.7) leaves invariant the period matrix. We then obtain easily

$$\begin{aligned} -A^{-1} &= \begin{pmatrix} M_-^t & 0 \\ 0 & M_- \Gamma \end{pmatrix} \sum_{\ell=0}^{\infty} \begin{pmatrix} (\tilde{P}_n - I)(\tilde{P}^{(n)} - I)(\tilde{P}_n - I)^\ell & ; & [(\tilde{P}_n - I)(\tilde{P}^{(n)} - I)]^\ell \\ [(\tilde{P}^{(n)} - I)(\tilde{P}_n - I)]^\ell & ; & (\tilde{P}^{(n)} - I)[(\tilde{P}_n - I)(\tilde{P}^{(n)} - I)]^\ell \end{pmatrix}^* \\ &\quad \times \begin{pmatrix} \Gamma M_- & 0 \\ 0 & M_-^t \end{pmatrix} \end{aligned} \quad (3.9)$$

$$\begin{aligned} -\frac{1}{2} B^t A^{-1} B &= \frac{1}{2} \left(\sum_{j=0}^{n-1} k_j \left[\left\langle \frac{1}{\beta_j^{(n+1)}} \right\rangle - \left\langle \frac{1}{\alpha_j^{(n+1)}} \right\rangle \right] \tilde{Q}_j^{(n)} ; \langle a | \Gamma P_n + k_n \left[\left\langle \frac{1}{\beta_n^{(n+1)}} \right\rangle - \left\langle \frac{1}{\alpha_n^{(n+1)}} \right\rangle \right] \right) \\ &\times \sum_{\ell=0}^{\infty} \begin{pmatrix} \text{same as} \\ \text{above} \end{pmatrix} \begin{pmatrix} \sum_{j=0}^{n-1} \tilde{Q}_j^{(n)} [\langle \beta_j^{(n+1)} \rangle - \langle \alpha_j^{(n+1)} \rangle] k_j \\ \tilde{P}_n |a\rangle + [\langle \beta_n^{(n+1)} \rangle - \langle \alpha_n^{(n+1)} \rangle] k_n \end{pmatrix} \end{aligned} \quad (3.10)$$

and

$$\det(iA) = \det(I - (\tilde{P}^{(n)} - I)(\tilde{P}_n - I)) \quad (3.11)$$

Now, recollecting the different terms in (3.4) and using the results of Appendices C and D, we show easily that $\langle 0 | T_{n+1}$ has the same expression as $\langle 0 | T_n$ with one supplementary generator. This is also obvious for the integration measure since it comes from the integrand of the two propagators and of the tadpole T_n .

We have also the recurrence relations for the invariant points.

$$\begin{aligned} \beta_j^{(n)} &= \widehat{V}_{n-1}(\beta_j^{(n-1)}) \\ \alpha_j^{(n)} &= \widehat{V}_{n-1}(\alpha_j^{(n-1)}) \end{aligned} \quad (3.12)$$

with

$$\beta_j^{(j+1)} = \frac{\gamma_j - 1}{\gamma_j} \quad ; \quad \alpha_j^{(j+1)} = \frac{\gamma_j - 1}{\gamma_j w_j} \quad (3.13)$$

$$\beta_0^{(n)} = \infty \quad ; \quad \alpha_0^{(n)} = \frac{1}{1-w_0} \quad (3.14)$$

and

$$V_{n-1} = M_-^t M_- \left[\frac{x_{n-1} - 1}{x_{n-1}} \right] M_-^t \quad (3.15)$$

We can solve explicitly these relations

$$\frac{\beta_j^{(n)}}{\beta_j^{(n)} - 1} = \widehat{X}_j \left(\frac{\beta_j^{(j+1)}}{\beta_j^{(j+1)} - 1} \right) \quad (3.16)$$

with

$$X_j = \prod_{k=j+1}^{n-1} M_- \left[\frac{x_k - 1}{x_k} \right] \quad (3.17)$$

or

$$\hat{X}_j(z) = \prod_{i=j+1}^{n-1} \left(\frac{1-x_i}{x_i} \right) z + 1 + \sum_{k=j+2}^{n-1} \prod_{i=k}^{n-1} \left(\frac{1-x_i}{x_i} \right) \quad (3.18)$$

Instead of x_i and y_j we can choose the invariant points as variables as is shown in Appendix E. The measure is now written as

$$\begin{aligned} & \prod_{i=1}^{n-1} \left\{ \frac{dx_i}{x_i(1-x_i)} \frac{dy_i}{y_i(1-y_i)} \left[\frac{x_i y_i}{(1-x_i)(1-y_i)} \right]^{-\alpha_0} \right\} = \\ & \frac{d\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n)}} \frac{d\beta_{n-2}^{(n)} d\alpha_{n-2}^{(n)}}{\alpha_{n-2}^{(n)} - \beta_{n-2}^{(n)}} \left[\prod_{j=1}^{n-3} \frac{(1-w_j) d\alpha_j^{(n)} d\beta_j^{(n)}}{(\alpha_j^{(n)} - \beta_j^{(n)})^2} \left(\frac{\mu_j - \beta_j^{(n)}}{\beta_j^{(n)} - \mu_{j+1}} \right) \right] \frac{d\beta_0^{(n)}}{(1-\beta_0^{(n)}) (\beta_0^{(n)} - \mu_1)} \times \\ & \times \left\{ \frac{\beta_{n-1}^{(n)}}{\beta_{n-2}^{(n)} - \mu_{n-2}} \left[\prod_{j=1}^{n-3} \left(\frac{\beta_j^{(n)} - \mu_{j+1}}{\mu_j - \beta_j^{(n)}} \right) \right] \frac{\beta_0^{(n)} - \mu_1}{1 - \beta_0^{(n)}} \right\}^{\alpha_0} \end{aligned} \quad (3.19)$$

where

$$u_k = \overbrace{p_k^{(n)} p_{k+1}^{(n)} \dots p_{n-2}^{(n)}} (\infty) \quad (k \neq 0) \quad (3.20)$$

$\alpha_{n-1}^{(n)}$ and $\alpha_0^{(n)}$ are now given by

$$\alpha_{n-1}^{(n)} = \beta_{n-1}^{(n)} / w_{n-1} \quad (3.21)$$

$$\frac{\alpha_0^{(n)} - 1}{\alpha_0^{(n)} - u_1} = w_0 \frac{\beta_0^{(n)} - 1}{\beta_0^{(n)} - u_1} \quad (3.22)$$

We shall discuss later the integration domain.

We have also checked that by attaching this operator to a tree our result is in complete agreement with those of Kaku and Yu ¹⁾, and Alessandrini and Amati ⁸⁾, when the latter use the Lovelace vertex ⁴⁾. However, if they use the Olive vertex ⁵⁾, it differs by some extra factor $(1-w_j)$ which comes from the elimination of spurious states in the loops.

2. Properties of the N loop planar operator

We discuss here some properties of the integrand including the partition function, and not the measure.

- a) As for the one-loop case, if we multiply (3.1) on the right by $:e^{a^+(C-I)a}:$ where C is a $SU(1,1)$ matrix, then the only change is to replace $P_i^{(n)}$ by

$$P_i'^{(n)} = C^{-1} P_i^{(n)} C \quad (3.23)$$

and also the invariant points

$$\begin{aligned} \alpha_i'^{(n)} &= \widehat{C^{-1}} (\alpha_i^{(n)}) \\ \beta_i'^{(n)} &= \widehat{C^{-1}} (\beta_i^{(n)}) \end{aligned} \quad (3.24)$$

- b) If we keep the same group, but change the generators, the partition function is obviously invariant; the other part associated with the Neumann function is also invariant after integration on $d_4 k_i$.
- c) Let us now cut the generators' set $\{P_i\}$ of $\mathcal{G}$ into two disjoint subsets $\{P_{j_1}\}$ and $\{P_{j_2}\}$ which generate respectively the two subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$. We now make the change

$$\alpha'_k = \hat{P}(\alpha_k)$$

$$\beta'_k = \hat{P}(\beta_k) \quad \text{for } k \in \{j_1\}$$

$$P'_k = P P_k P^{-1}$$

(3.25)

and no change for $k \in \{j_2\}$, where P is an element of $\mathcal{G}_1$. From the previous remark, P can be chosen as a generator. Using the notation and the results of Appendix C we can then show that

$$P \mathcal{Q}^{(1)} P^{-1} = \mathcal{Q}^{(1)}$$

$$P \mathcal{Q}^{(1)} = \mathcal{Q}^{(1)}$$

$$\mathcal{Q}_{i_1}^{(1)} P = \mathcal{Q}_{i_1}^{(1)}$$

(3.26)

and therefore the Neumann part of the integrand is invariant; and using (D.1) we show that the partition function is invariant.

This property will lead to an invariance property of the multi-loop operator, if such a change can be realized by a change of variables which leaves invariant the integration measure.

In fact, we shall see later that if the integrand is invariant under an infinity of such transformation, only some particular P leaves invariant the measure.

3. Two-loop case. Periodicities

Let us now consider the special case of the two-loop operator.

a) Identity

It is obvious from duality that the two-loop operator (Fig. 6) must be symmetric in w_1 and w_2 up to a gauge transformation. Let us show it directly. It is easy if we use the fundamental duality property of the planar tadpole, (2.2). We recall that a gauge transformation changes the variable of the twisted projected propagator ⁶⁾. Applying (2.12), (2.13) and (2.14) to the tadpole (w_2) we obtain

$$\langle 0 | \mathcal{O}_2(x_1, w_1, x_2, w_2) \mu_2^{A(0)-d_0} = \langle 0 | \mathcal{O}_2(x'_2, w_2, x'_1, w_1) \quad (3.27)$$

with

$$\begin{aligned} \mu_2 &= \frac{(1-x_2)^2}{1-x_2(1-w_2)} \\ x'_2 &= \frac{1-x_2}{1-x_2(1-w_2)} \\ x'_1 &= \frac{x_1}{(1-x_1)v_2 + x_1} \end{aligned} \quad (3.28)$$

where v_2 is the gauge parameter

$$\nu_2 = \frac{1 - x_2(1-w_2)}{x_2^2 w_2} \quad (3.29)$$

then

$$x'_1 = \frac{x_1 x_2^2 w_2}{(1-x_1)(1-x_2(1-w_2)) + x_1 x_2^2 w_2} \quad (3.30)$$

In order to verify the invariance of the integration measure, we use the other form of these relations

$$\left\{ \begin{array}{l} \frac{x'_2}{1-x'_2} = \frac{x_2}{1-x_2} \mu_2 \nu_2 = \frac{1-x_2}{w_2 x_2} \\ \frac{x'_1}{1-x'_1} = \frac{1}{\nu_2} \frac{x_1}{1-x_1} \end{array} \right. \quad (3.31)$$

hence

$$\frac{dx_1}{x_1(1-x_1)} \frac{dx_2}{x_2(1-x_2)} = \frac{dx'_1}{x'_1(1-x'_1)} \frac{dx'_2}{x'_2(1-x'_2)} \quad (3.32)$$

and

$$\frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \mu_2 = \frac{x'_1}{1-x'_1} \frac{x'_2}{1-x'_2} \quad (3.33)$$

b) Periodicity

Let us now apply again this identity, but now with the tadpole w_1 . We then recover the same configuration, but from non-trivial duality transformation. This must then give some invariance (or periodicity) property. We have then (Fig. 7)

$$\begin{aligned} \langle 0 | \mathcal{G}_2(x_1, w_1, x_2, w_2) (\mu_2 \mu_1)^{A(0) - \alpha_0} &= \langle 0 | \mathcal{G}_2(x'_2, w_2, x'_1, w_1) \mu_1^{A(0) - \alpha_0} \\ &= \langle 0 | \mathcal{G}_2(x''_1, w_1, x''_2, w_2) \end{aligned} \quad (3.34)$$

with the relations

$$\left\{ \begin{aligned} \frac{x'_1 \mu_1 v_1}{1 - x'_1} &= \frac{x''_1}{1 - x''_1} & ; & \frac{x''_2}{1 - x''_2} = \frac{1}{v_1} \frac{x'_2}{1 - x'_2} \\ \frac{x_2 \mu_2 v_2}{1 - x_2} &= \frac{x'_2}{1 - x'_2} & ; & \frac{x'_1}{1 - x'_1} = \frac{1}{v_2} \frac{x_1}{1 - x_1} \end{aligned} \right. \quad (3.35)$$

$$\left\{ \begin{aligned} \mu_2 &= \frac{(1 - x_2)^2}{1 - x_2(1 - w_2)} & ; & v_2 = \frac{1 - x_2(1 - w_2)}{x_2^2 w_2} \\ \mu_1 &= \frac{(1 - x'_1)^2}{1 - x'_1(1 - w_1)} & ; & v_1 = \frac{1 - x'_1(1 - w_1)}{x'^2_1 w_1} \end{aligned} \right. \quad (3.36)$$

We have then shown that the two-loop operator is invariant (up to a gauge transformation) under the rather complicated change of variable.

$$x''_1 = \frac{(1-x_1)(1-x_2(1-w_2))}{(1-x_1)(1-x_2(1-w_2)) + x_1 x_2^2 w_2 w_1}$$

$$x''_2 = \frac{(1-x_2) x_1^2 x_2^4 w_2^2 w_1}{(1-x_2) x_1^2 x_2^4 w_2^2 w_1 + x_2 w_2 [(1-x_1)(1-x_2(1-w_2)) + x_1 x_2^2 w_2 w_1] [(1-x_1)(1-x_2(1-w_2)) + x_1 x_2^2 w_2]}$$

(3.37)

If from a fundamental region in the plane (x_1, x_2) we can, by power (>0 or <0) of the transformation (3.37) cover all the domain $(0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1)$, it means that we have counted an infinity times the same thing and then the result is trivially infinite. This divergence is easily removed by restricting the integration domain to the fundamental region. But the problem of finding this region is rather difficult in our case, and we shall see in the next part how exhibit this periodicity and then eliminate it.

c) Integration domain

We have seen previously that periodicity properties can restrict the integration region. There can also be some restrictions from the partition function because of its natural boundary, however, it has been shown by Kaku and Scherk^{11),12)} that it is not the case for the two-loop case, since $w_{12} \rightarrow 1$ if $1-x_1 \rightarrow 0$, $x_2 \rightarrow 0$ and $(1-x_1)/x_2 \rightarrow \sqrt{w_1 w_2}$.

4. ANOTHER FORM OF THE N LOOP PLANAR OPERATORS. PERIODICITIES

It is known that an iteration of planar tadpole can be set in an iteration of non-planar self-energy attached to a planar tadpole as shown in Fig. 8. This configuration clearly exhibits periodicity properties if we allow the variable x_1 of non-planar self-energy to vary in the range $[0, \infty]$. The consequent divergences are removed by restricting the range to $[w, 1]$. We shall show explicitly this identity in the two-loop case (Fig. 9) in order to study the integration domain. The results will be extended to the N loop case.

1. Two-loop case

We compute them using the operators defined in Chapter 2 (Fig. 10)

$$\langle 0 | T_2(a_1) = \int \frac{d^2 s}{\pi} e^{-|s|^2} \langle 0_{1,2} | T_1(a_2) \mathcal{D}(z) | s \rangle \langle s | \Sigma_T(a_1, a_2^+) | 0_2 \rangle \quad (4.1)$$

We perform the vacuum expectation value as in Chapter 3; we do not give here this calculation since it will be exactly the same in the N loop case (see Part 4.3). The result is

$$\begin{aligned} \langle 0 | T_2(a_1) = & \int_0^1 \frac{dw_1}{w_1} \int_0^1 \frac{dw_3}{w_3} \int_0^1 \frac{dx}{x} \int_0^1 \frac{dz}{z(1-z)} \left[\frac{w_1 w_3 z}{1-z} \right]^{-\alpha_0} [F(\bar{P}_1, P_3)]^{-4} \\ & \times \int d_4 k_1 d_4 k_3 \exp \frac{1}{2} \left\{ k_1^2 \left[\left(\frac{1}{\beta_1} \right) - \left(\frac{1}{\alpha_1} \right) \right] \bar{\Phi}_{11} [1|\beta_1\rangle - 1|\alpha_1\rangle] - \log w_1 + (1 \leftrightarrow 3) \right. \\ & \left. + 2 k_1 \cdot k_3 \left[\left(\frac{1}{\beta_1} \right) - \left(\frac{1}{\alpha_1} \right) \right] \bar{\Phi}_{13} [1|\beta_3\rangle - 1|\alpha_3\rangle] \right\} \\ & \langle 0 | \exp \left\{ c_{a_1} \Gamma [\bar{\Phi}_1 [1|\beta_1\rangle - 1|\alpha_1\rangle] k_1 + \bar{\Phi}_3 [1|\beta_3\rangle - 1|\alpha_3\rangle] k_3 \right. \\ & \left. + \frac{1}{2} c_{a_1} \Gamma (\bar{\Phi} - I) | a_1 \rangle \right\} \quad (4.2) \end{aligned}$$

where P_3 is given by (2.21)

$$\left(\beta_3 = -\infty, \alpha_3 = \frac{1}{1-w_3} \right)$$

and

$$\bar{P}_1 = \Gamma_x^{-1} P_1' \Gamma_x \quad (4.3)$$

P_1' being given by (3.3).

We note that we have always the same Neumann function for the integrand. The partition function is given by

$$F(\bar{P}_1, P_3) = F(w_1) F(w_3) [\det(I - (\bar{P}_1 - I)(P_3 - I))]^{\frac{1}{2}} \quad (4.4)$$

and using (4.3) and (2.23)

$$F(\bar{P}_1, P_3) = F(w_1) F(w_3) [\det(I - (P_1' - I)(P_3^t - I))]^{\frac{1}{2}} \quad (4.5)$$

Then it does not depend on x . Let us now study the transformation

$$\bar{P}_2 = \bar{P}_1^{-1} P_3 = \Gamma_x^{-1} (P_1'^{-1} P_3^t) \Gamma_x \quad (4.6)$$

Its multiplier w_2 is given by

$$\frac{1}{\sqrt{w_2}} + \sqrt{w_2} = \frac{1}{\sqrt{w_1 w_3}} \left| \frac{(1-z)^2 + z^2 w_1 w_3}{z(1-z)} \right| \quad (4.7)$$

This is easily solved and gives the two solutions in w_2

$$\sqrt{w_2} = \frac{1}{\sqrt{w_1 w_3}} \frac{1-z}{z} \quad \text{or} \quad \sqrt{w_2} = \frac{z \sqrt{w_1 w_3}}{1-z} \quad (4.8)$$

then $w_2 = 1$ if

$$z = \frac{1}{1 + \sqrt{w_1 w_3}} \equiv z_0 \quad (4.9)$$

Up to now we cannot say anything on the integration region of z . More precisely, when z reaches z_0 , we do not know if we must stop the integration or change the determination for w_2 . The solution of this problem will be given in the Part 4.2 in connection with the identity on two-loop tadpole: there is only one solution which is compatible with the identity

$$\sqrt{w_2} = \frac{z \sqrt{w_1 w_3}}{1-z} \quad (4.10)$$

and then we restrict the integration domain on z from $[0,1]$ to

$$\left[0, \frac{1}{1 + \sqrt{w_1 w_3}} \right]$$

We can now do the change of variables $(x, z) \rightarrow (w_2, \bar{\beta}_1)$

$$\begin{cases} \frac{z}{1-z} = \left(\frac{w_2}{w_1 w_3} \right)^{1/2} \\ \bar{\beta}_1 - \alpha_3 = \frac{1}{x(1-w_3)(1-z(1-w_3))} \end{cases} \quad (4.11)$$

Then

$$\frac{dw_1}{w_1} \frac{dw_3}{w_3} \frac{dx}{x} \frac{dz}{z(1-z)} = \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{dw_3}{w_3} \frac{d\bar{\beta}_1}{\bar{\beta}_1 - \alpha_3} \quad (4.12)$$

and

$$\left(\frac{w_1 w_3 z}{1-z} \right)^{-\alpha_0} = (w_1 w_2 w_3)^{-\alpha_0/2} \quad (4.13)$$

Let us now study how we can see the periodicity properties in this variable. They are now exhibited if we do the change of variable $(\bar{\beta}_1)$ associated with the transformation

$$\bar{P} \rightarrow P_3^{-1} \bar{P} P_3 \quad (4.14)$$

for $\bar{P} \in \mathcal{G}$ generated by $\bar{P}_1$ and P_3 , i.e.,

$$\bar{\beta}_1 - \alpha_3 \rightarrow \frac{1}{w_3} (\bar{\beta}_1 - \alpha_3) \quad (4.15)$$

the integrand is invariant under (4.14), as discussed in Chapter 3.2c, and the measure under (4.15). Then, the integration domain of $\bar{\beta}_1$ is instead of $[\alpha_3, \infty]$ if $x \in [0, \infty]$, reduced to $[B_0, \alpha_3 + \frac{1}{w_3} (B_0 - \alpha_3)]$, where B_0 is an arbitrary number $> \alpha_3$.

Note that we can also choose $\bar{\alpha}_1$ instead of $\bar{\beta}_1$ with

$$\bar{\alpha}_1 - \alpha_3 = \frac{1 - z(1 - w_1)}{x(1 - w_3)(1 - z(1 - w_1 w_3))} \quad (4.16)$$

2. Identity between two-loop operators

Let us now proceed to the proof of the identity of Fig. 10. We recall that the generators of the two-tadpole are given by

$$\left\{ \begin{array}{l} P_1(w_1) : \quad \alpha_1 = \frac{1 - x_1(1 - w_1)}{1 - x_1} \\ \quad \quad \quad \beta_1 = \frac{1}{1 - x_1} \\ \\ P_2(w_2) : \quad \alpha_2 = \frac{x_2 - 1}{x_2 w_2} \\ \quad \quad \quad \beta_2 = \frac{x_2 - 1}{x_2} \end{array} \right. \quad (4.17)$$

- a) We compute the gauge parameter γ which allows the identification of $\bar{P}_1$ and P_1 , i.e.,

$$P_1 = M_-^t[\gamma] M_-^t \bar{P}_1 M_-^t \left[\frac{1}{\gamma} \right] M_-^t \quad (4.18)$$

This gives

$$\gamma = \frac{(1 - \bar{\alpha}_1)(1 - \bar{\beta}_1)(1 - w_1)}{\bar{\alpha}_1 \bar{\beta}_1 (1 - w_1) + \bar{\alpha}_1 w_1 - \bar{\beta}_1} \quad (4.19)$$

$$\frac{1}{1-x_1} = \frac{\bar{\beta}_1 (1-\bar{\alpha}_1)(1-w_1)}{(\bar{\alpha}_1 - \bar{\beta}_1) w_1} \quad (4.20)$$

b) We identify

$$P_1 P_2 = M_-^t [\gamma] M_-^t P_3 M_-^t \left[\frac{1}{\gamma} \right] M_-^t \quad (4.21)$$

Then we obtain for w_2 the Eq. (4.7) and the relations

$$\frac{x_2}{1-x_2} = - \frac{1}{1+w_2} \frac{1 - \bar{\alpha}_1 - w_1(1-\bar{\beta}_1) + w_3(\bar{\beta}_1 - w_1\bar{\alpha}_1)}{(1-\bar{\alpha}_1)(1-\bar{\beta}_1)(1-w_1)} \quad (4.22)$$

and

$$\frac{(1-x_1)^2 (1-x_2)^2}{x_2^2 x_1^2 w_1 w_2} = \frac{\gamma^2}{w_3} \quad (4.23)$$

These two identifications show the equality of the two-integrand if we use the properties given in Chapter 3.2b.

c) Let us now show the equality of the measure. The equality of the α_0 dependent terms

$$\left[\frac{1-z}{z w_1 w_3} \gamma \right]^{\alpha_0} = \left[\frac{(1-x_1)(1-x_2)}{w_1 x_1 w_2 x_2} \right]^{\alpha_0} \quad (4.24)$$

using (4.23) implies as we say before

$$\sqrt{w_2} = \frac{z}{1-z} \sqrt{w_1 w_3}$$

We verify also after a rather tedious computation

$$\frac{dw_1}{w_1} \frac{dw_3}{w_3} \frac{dx}{x} \frac{dz}{z(1-z)} = \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{dx_1}{x_1(1-x_1)} \frac{dx_2}{x_2(1-x_2)} \quad (4.25)$$

This completes the proof. We have thus shown explicitly the way to exhibit the third divergence associated to the two-loop tadpole operator and eliminate a trivial divergence due to periodicity properties. This identity has been used in order to regularize the two-loop tadpole operator¹³⁾.

This shows also the important property that the different configurations which are identical by duality are related to different bases of generators of a same group.

3. N loop dual planar operator

As for the iteration of tadpoles, we construct it by recurrence as follows (see Fig. 11)

$$\langle 0_1 | T_{n+1} = \int \frac{d^2 s}{\pi} e^{-|s|^2} \langle 0_{1,2} | T_n(a_2) \mathcal{D}(z_n) | s \rangle \langle s | \Sigma_T^{(n)}(a_2^+, a_1) | 0_2 \rangle \quad (4.26)$$

and let us assume

$$\begin{aligned}
 \langle 0 | T_n(a) = & \int \prod_{k=1}^{n-1} \frac{d \left[\frac{\beta_{k-1}^{(n)} - \alpha_k^{(n)}}{\beta_{k-1}^{(n)} - \beta_k^{(n)}} \right]}{\left[\frac{\beta_{k-1}^{(n)} - \alpha_k^{(n)}}{\beta_{k-1}^{(n)} - \beta_k^{(n)}} \right]} \prod_{j=0}^{n-2} \frac{dw_{j,j-1}}{w_{j,j-1}} \prod_{i=0}^{n-1} \frac{dw_i}{w_i} \times \\
 & \times \left[w_0 w_{n-1} \prod_{i=0}^{n-2} w_{i,i-1} \right]^{-\alpha_0/2} \left[F(p_0^{(n)} \dots p_{n-1}^{(n)}) \right]^{-4} \\
 & \int \prod_{i=0}^{n-1} d_4 k_i \exp \frac{1}{2} \sum_{i,j=0}^{n-1} k_i \cdot k_j \left\{ \left[\left(\frac{1}{\beta_i^{(n)}} \right) - \left(\frac{1}{\alpha_j^{(n)}} \right) \right] \varphi_{ij}^{(n)} \left[|\beta_j^{(n)}\rangle - |\alpha_j^{(n)}\rangle \right] - \delta_{ij} \log w_i \right\} \\
 & \langle 0 | \exp \left\{ c_1 \Gamma \sum_{j=0}^{n-1} \varphi_j^{(n)} \left[|\beta_j^{(n)}\rangle - |\alpha_j^{(n)}\rangle \right] k_j + \frac{1}{2} c_1 \Gamma (P^{(n)} - I) | a \right\}
 \end{aligned}
 \tag{4.27}$$

where $w_{i,i-1}$ is the multiplier of $P_{i-1}^{(n)-1} P_i^{(n)}$.

a) We then obtain for $\langle 0_1 | T_{n+1}$ using the principal axis method

$$\begin{aligned}
 \langle 0_1 | T_{n+1} = & \int dV_n S_n^{\alpha_0} \frac{dw_n}{w_n} \frac{dx_n}{x_n} \frac{dz_n}{z_n(1-z_n)} \left(\frac{w_n z}{1-z} \right)^{-\alpha_0} \left[F(w_n) F(p_0^{(n)} \dots p_{n-1}^{(n)}) \right]^{-4} \\
 & \int \prod_{i=1}^n d_4 k_i \exp \left[\frac{1}{2} \sum_{i,j=0}^{n-1} k_i \cdot k_j \left\{ \left[\left(\frac{1}{\beta_i^{(n)}} \right) - \left(\frac{1}{\alpha_j^{(n)}} \right) \right] \varphi_{ij}^{(n)} \left[|\beta_j^{(n)}\rangle - |\alpha_j^{(n)}\rangle \right] - \delta_{ij} \log w_i \right\} \right] w_n^{-k_n^2/2} \\
 & (\det iA)^{-2} \langle 0_1 | \exp \left\{ \frac{1}{2} c_1 \Gamma (P_n - I) | a_1 \rangle + c_1 \Gamma \left[\left| \frac{1}{\beta_n} \right\rangle - \left| \frac{1}{\alpha_n} \right\rangle \right] k_n \right. \\
 & \quad \left. - \frac{1}{2} B^t A^{-1} B \right\}
 \end{aligned}
 \tag{4.28}$$

with

$$B = \begin{pmatrix} \Gamma \sum_{j=0}^{n-1} \varphi_j^{(n)} \left[|\beta_j^{(n)}\rangle - |\alpha_j^{(n)}\rangle \right] \\ \Gamma_{x_n} P_n | a_1 \rangle - \left[\left| \frac{1}{\beta_n} \right\rangle - \left| \frac{1}{\alpha_n} \right\rangle \right] k_n \end{pmatrix}
 \tag{4.29}$$

and

$$A = \begin{pmatrix} \Gamma(\mathcal{P}'^{(n)} - I) & -I \\ -I & \Gamma(\mathcal{P}_n - I) \end{pmatrix} \quad (4.30)$$

and the ' means the transformation of the $P_i^{(n)}$ ($i = 0, \dots, n-1$) by the propagator

$$P_i'^{(n)} = M_- \left[\frac{z_{n-1}}{z_n} \right] M_-^t P_i^{(n)} M_-^t \left[\frac{z_n}{z_{n-1}} \right] M_- \quad (4.31)$$

We define

$$P_i^{(n+1)} = \Gamma_{x_n}^{-1} P_i'^{(n)} \Gamma_{x_n} \quad (4.32)$$

and

$$\overline{\mathcal{P}}^{(n)} = \Gamma_{x_n}^{-1} \mathcal{P}'^{(n)} \Gamma_{x_n}$$

Using the properties of Γ and Γ_{x_n} , in particular

$$\Gamma_{x_n}^{-1} \Gamma[|\alpha_n\rangle - |\beta_n\rangle] = |\beta_n\rangle - |\alpha_n\rangle \quad (4.33)$$

and as before the results of Appendix B we obtain

$$-A^{-1} = \begin{pmatrix} \Gamma_{x_n} & 0 \\ 0 & \Gamma \Gamma_{x_n} \end{pmatrix} \sum_{\ell=0}^{\infty} \begin{pmatrix} (P_n - I)[(\bar{P}^{(n)} - I)(P_n - I)]^{\ell} & [(P_n - I)(\bar{P}^{(n)} - I)]^{\ell} \\ [(\bar{P}^{(n)} - I)(P_n - I)]^{\ell} & (\bar{P}^{(n)} - I)[(P_n - I)(\bar{P}^{(n)} - I)]^{\ell} \end{pmatrix} \times \begin{pmatrix} \Gamma_{x_n}^{-1} \Gamma & 0 \\ 0 & \Gamma_{x_n}^{-1} \end{pmatrix} \quad (4.34)$$

$$\det(iA) = \det(I - (P_n - I)(\bar{P}^{(n)} - I)) \quad (4.35)$$

Then following exactly the method of Chapter 2, we show that the integrand has the form indicated in (4.27).

b) It is easily seen that we can do exactly as in the two-loop case:

i) we consider the transformation

$$P_{n,n-1} = (P_{n-1}^{(n+1)})^{-1} P_n^{(n+1)} = \Gamma_{x_n}^{-1} \left[(P_{n-1}^{(n)})^{-1} (P_n^{(n+1)})^t \right] \Gamma_{x_n} \quad (4.36)$$

as previously, its multiplier is given by

$$\sqrt{w_{n,n-1}} + \frac{1}{\sqrt{w_{n,n-1}}} = \frac{1}{\sqrt{w_n w_{n-1}}} \left| \frac{(1-z_n)^2 + z_n^2 w_n w_{n-1}}{2z_n(1-z_n)} \right| \quad (4.37)$$

We assume that, as in the two-loop case, only one solution is retained

$$\sqrt{w_{n,n-1}} = \frac{z_n}{1-z_n} \sqrt{w_n w_{n-1}} \quad (4.38)$$

ii) we make the change of variables

$$\begin{cases} z_n \rightarrow w_{n,n-1} \\ x_n \rightarrow \beta_{n-1}^{(n+1)} - \alpha_n^{(n+1)} \text{ (or } \alpha_{n-1}^{(n+1)} - \alpha_n^{(n+1)}) \end{cases}$$

where

$$\alpha_n^{(n+1)} = \frac{1}{1-w_n}$$

as previously, we have

$$\begin{aligned} \left(\frac{w_n z}{1-z} \right)^{-\alpha_0} \frac{dw_n}{w_n} \frac{dx_n}{x_n} \frac{dz_n}{z_n(1-z_n)} &= \\ &= \left(\frac{w_{n,n-1} w_n}{w_{n-1}} \right)^{-\alpha_0/2} \frac{dw_n}{w_n} \frac{dw_{n,n-1}}{w_{n,n-1}} \frac{d\beta_{n-1}^{(n+1)}}{\beta_{n-1}^{(n+1)} - \alpha_n^{(n+1)}} \quad (4.39) \end{aligned}$$

We note that the change $P_i^{(n)} \rightarrow P_i^{(n+1)}$ neither affects the integrand nor the measure, since

$$\frac{d\left[\frac{x-\alpha}{x-\beta}\right]}{\left(\frac{x-\alpha}{x-\beta}\right)} = \frac{d\left[\frac{C(x)-C(\alpha)}{C(x)-C(\beta)}\right]}{\left(\frac{C(x)-C(\alpha)}{C(x)-C(\beta)}\right)} \quad (4.40)$$

if C is a homographic transformation which does not depend on x . $(dx)/(x-\alpha)$ is a particular case when β is infinite. We have then shown that the measure has the same form.

iii)

$$\alpha_{n-1}^{(n+1)} \text{ and } \beta_{n-1}^{(n+1)}$$

are given by

$$\alpha_{n-1}^{(n+1)} = \frac{1 - z_n(1 - w_{n-1}) + x_n(1 - z_n(1 - w_{n-1}w_n))}{x_n(1 - w_n)(1 - z_n(1 - w_{n-1}w_n))} \quad (4.41)$$

$$\beta_{n-1}^{(n+1)} = \frac{1 + x_n(1 - z_n(1 - w_n))}{x_n(1 - w_n)(1 - z_n(1 - w_n))} \quad (4.42)$$

so that we have the relation using (4.38)

$$\frac{\alpha_{n-1}^{(n+1)} - \alpha_n^{(n+1)}}{\beta_{n-1}^{(n+1)} - \alpha_n^{(n+1)}} = 1 - \frac{(1 - w_n)(1 - w_{n-1})}{\left(1 + \sqrt{\frac{w_n w_{n-1}}{w_{n,n-1}}}\right) \left(1 + \sqrt{w_n w_{n-1} w_{n,n-1}}\right)} \quad (4.43)$$

Noting that this is the cross-ratio $(\alpha_{n-1}, \beta_{n-1}, \alpha_n, \beta_n = \infty)$, it is invariant by homographic transformation and then we have also the relation which defines the invariant points

$$\frac{\alpha_{i-1}^{(n)} - \alpha_i^{(n)}}{\beta_{i-1}^{(n)} - \alpha_i^{(n)}} \frac{\beta_{i-1}^{(n)} - \beta_i^{(n)}}{\alpha_{i-1}^{(n)} - \beta_i^{(n)}} = 1 - \frac{(1-w_i)(1-w_{i-1})}{\left(1 + \sqrt{\frac{w_i w_{i-1}}{w_{i,i-1}}}\right) \left(1 + \sqrt{w_i w_{i-1} w_{i,i-1}}\right)}$$

(4.44)

with the condition

$$\alpha_{n-1}^{(n)} = \frac{1}{1-w_{n-1}}, \quad \beta_{n-1}^{(n)} = \infty$$

(4.45)

This relation defines completely the integrand of (4.27)
[note that (4.44) is quite general].

c) Let us remark that the invariant points are ordered as follows

$$\alpha_{n-1}^{(n)} = \frac{1}{1-w_{n-1}} < \alpha_{n-2}^{(n)} < \dots < \alpha_0^{(n)} < \beta_0^{(n)} < \dots < \beta_{n-2}^{(n)} < \beta_{n-1}^{(n)} = \infty \quad (4.46)$$

Note, however, that it does not mean that if $w_i \rightarrow 1$ then $w_j \rightarrow 1$ for $j < i$, it only means that $P_j^{(n)i}$ is of the form

$$P_j^{(n)} = C [w_j] C^{-1}$$

where C becomes singular.

d) The periodicity exhibited in the i^{th} non-planar self-energy is now seen as follows:

for $j \leq i$ we make the change of variables

$$\begin{aligned}\alpha_j^{(n)} &\rightarrow \hat{P}_i^{(n)}(\alpha_j^{(n)}) \\ \beta_j^{(n)} &\rightarrow \hat{P}_i^{(n)}(\beta_j^{(n)})\end{aligned}$$

and then

$$P_j^{(n)} \rightarrow P_i^{(n)} P_j^{(n)} P_i^{(n)-1}$$

and we do not change the α and β for $k \geq i$. The multipliers are evidently invariant under such a change, even $w_{i+i,i}$ since $P_i^{(n)}$ is invariant. The other part of the measure is invariant, because of (4.40).

We have seen in Chapter 3.2c that the integrand is invariant under such a transformation. Then, since primitively the integration domain of α_i can be $[\alpha_{i+1}, \beta_{i+1}]$, it is reduced only to one period.

5. SUMMARY AND CONCLUDING REMARKS

We have then shown in a simple way that the creation operator part of the integrand of the N loop tadpole operator is essentially related to the Neumann function N_n as is now well known. The method described here can be understood as the use of some convolution property of the Neumann function ($N_{n+m} \approx N_n * N_m$).

In that operator the loops are free of spurious states. We have proved directly the possibility of a tree to go around a loop. We have also seen that the various configurations of a same diagram are related to different choices of generators of a same group. We have there proved some local duality properties (i.e., before integration). This result is interesting since it has been argued that local duality is not automatically guaranteed in the absence of spurious states ⁸⁾.

We have shown that some periodicity properties are related to the invariance of the measure, and how they restrict the integration domain. This domain can also be restricted by the fact that the multipliers of the various elements of the group $g^{(n)}$ do not go further than 1. A general discussion of divergences associated with this limit can be found in Refs. 8) and 12). However, we must say that it is not clear that we have taken into account all the independent periodicities of the multi-loop amplitudes and therefore, perhaps, there can be some other restrictions on the integration domain, which can have some effects on the divergences associated with the partition function ⁸⁾.

This operator, perhaps, may be useful if one has some method in order to sum all the planar graphs since it has been recently shown by Frampton, Goddard and Wray ¹⁴⁾ that we must add all inequivalent diagrams with a coefficient 1 (once the coupling constant is included in the amplitude).

Let us now give some comments on the non-planar case. In this one, the problem cannot be reduced to only one operator as in the planar case. However, it has been shown ¹⁵⁾ that all non-planar

diagrams can be built by means of four basic operators, three with one-loop (those considered in Chapter 2) and one with two-loop, although the last is not constructed, we can argue that they have similar properties as the planar operator. In particular, it is obvious that the method developed here allows us to construct simply an iteration of N non-planar tadpoles.

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APPENDIX A - NOTATIONS, $SU(1,1)$ MATRICES

We list here our notations which are essentially those of Gross and Schwarz ⁷⁾ and some well-known properties of $SU(1,1)$ infinite representation matrices.

All matrices involved in the dual multiparticle theory can be expressed by means of infinite representation of a homographic transformation group, which is seen by acting these matrices on a vector noted $|\xi\rangle$ of components $\xi^n/\sqrt{n}$ ($n \geq 1$) ⁶⁾

$$C|\xi\rangle = |\hat{C}(\xi)\rangle - |\hat{C}(0)\rangle \quad (A.1)$$

with

$$\hat{C}(\xi) = \frac{a\xi + b}{c\xi + d} \quad (A.2)$$

We recall that the eigenvalues of C are of the form W^n where W is the ratio < 1 of the eigenvalues of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or also the multiplier of the homographic transformation

$$\left(\frac{\hat{C}(\xi) - \alpha}{\hat{C}(\xi) - \beta} = W \frac{\xi - \alpha}{\xi - \beta} \right.$$

if α and β are the invariant points). We have then

$$\det(I - C) = \prod_{n=1}^{\infty} (1 - W^n) \quad (A.3)$$

$$\text{Tr } C = \frac{W}{1 - W} \quad (A.4)$$

We give now some particular matrices which are often involved in the theory, by their correspondence (one to one) with a homographic transformation

$$\begin{aligned} M_- &\longleftrightarrow \widehat{M}_-(\gamma) = 1-\gamma \\ M_-^t &\longleftrightarrow \widehat{M}_-^t(\gamma) = \frac{\gamma}{1-\gamma} \\ M_+ &\longleftrightarrow \widehat{M}_+(\gamma) = \frac{1}{1-\gamma} \end{aligned} \quad (A.5)$$

They satisfy

$$\begin{aligned} M_-^2 &= (M_-^t)^2 = I \\ M_+ &= M_- M_-^t \end{aligned} \quad (A.6)$$

As shown in Ref. 5), if C is a $SU(1,1)$ matrix, then the transposed matrix C^t also and it is associated with the homographic transformation

$$\widehat{C}^t(\xi) = \frac{a\xi - c}{-b\xi + d} \quad (A.7)$$

We note by $\widehat{[x]}$ the diagonal matrix of element x^n ($n \gg 1$)

$$\widehat{[x]}(\gamma) = x\gamma$$

We use also intensively the matrix $\widehat{\Gamma}$ introduced by Alessandrini ³⁾ and which is associated to the homographic transformation

$$\widehat{\Gamma}(\xi) = 1/\xi \quad (A.8)$$

We see that it is not defined in the infinite space, but we can always use it if we define the multiplication of matrix in the 2x2 space. Formally, we can then write (∞ is not a singular point for the homographic transformations)

$$\Gamma |z\rangle = |\frac{1}{z}\rangle - |\infty\rangle \quad (\text{A.9})$$

$$\Gamma [|\gamma\rangle - |\alpha\rangle] = |\frac{1}{\gamma}\rangle - |\frac{1}{\alpha}\rangle \quad (\text{A.10})$$

In general, Γ has only a well-defined meaning when it acts on a vector of the form

$$\sum_i \alpha_i |x_i\rangle \quad \text{with} \quad \sum_i \alpha_i = 0$$

We note the properties of Γ

$$\Gamma = M_+ M_- \quad (\text{A.11})$$

$$\Gamma = \Gamma^t \quad (\text{A.12})$$

$$\Gamma^2 = I \quad (\text{A.13})$$

and the most useful, if C is a $SU(1,1)$ matrix, then

$$\Gamma C \Gamma = (C^t)^{-1} \quad (\text{A.14})$$

All these identities are trivially shown with the representation two (up to a multiplicative constant).

Let us also give the expression of the symmetric vertex, the projected twisted propagator and of the gauge transformations in these notations ⁷⁾

$$\langle 0 | V(p_1, p_2, p_3; a_1, a_2, a_3) =$$

$$\langle 0 | \exp \left\{ (a_1 | 1) p_2 + (a_2 | 1) p_3 + (a_3 | 1) p_1 + (a_1 | M_- | a_2) + (a_2 | M_- | a_3) + (a_3 | M_- | a_1) \right\} \quad (A.15)$$

$$\mathcal{D}(p) = \int_0^1 \frac{dz}{z(1-z)} \left(\frac{z}{1-z} \right)^{-\alpha(p^2)} (1-z)^{-p^2} \times e^{(a^+ | z) p} : e^{a^+ \{ M_-^t [\frac{z}{z-1}] M_- - I \} | a} : e^{-(a | z) p} \quad (A.16)$$

$$\lambda^{A(p)-\alpha_0} = : e^{a^+ \{ M_-^t [\lambda] M_- - I \} | a} : e^{-(a | 1-\lambda) p} \lambda^{-\alpha(p^2)} \quad (A.17)$$

APPENDIX B

We give here some methods of the way to work with 2×2 matrices of non-commutative elements. Especially in our case, these elements are infinite matrices. Let us take the matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (\text{B.1})$$

where a, b, c, d are non-commutative operators. We want to calculate its determinant and its inverse.

a) The multiplication of x by

$$\begin{pmatrix} I & -a^{-1}b \\ 0 & I \end{pmatrix}$$

on the right leads immediately to

$$\det X = \det (ad - aca^{-1}b) \quad (\text{B.2})$$

b) The multiplication by

$$\begin{pmatrix} c^{-1}dc & -b \\ -c & b^{-1}ab \end{pmatrix}$$

on the right leads to

$$X^{-1} = \begin{pmatrix} c^{-1}dc & -b \\ -c & b^{-1}ab \end{pmatrix} \begin{pmatrix} [ac^{-1}dc - bc]^{-1} & 0 \\ 0 & [db^{-1}ab - cb]^{-1} \end{pmatrix}$$

(B.3)

APPENDIX C

Let us note $G^{(n)}$ as a group generated by n basic elements $P_1 \dots P_n$ (and $P_1^{-1} \dots P_n^{-1}$). We define

- a) $Q^{(n)}$ as the sum of all the elements of $G^{(n)}$;
- b) Q_i (resp. ϕ_i) as the sum of all the elements of $G^{(n)}$ which do not end (resp. begin) by P_i or P_i^{-1} . It contains the identity;
- c) Q_{ij} as the sum of all the elements of $G^{(n)}$ which do not begin by P_i or P_i^{-1} and do not end by P_j or P_j^{-1} (it contains the identity if $i \neq j$).

Let us now consider a group $G^{(n)}$ generated by $P_1 \dots P_n$, a group $G^{(m)}$ generated by $P_1 \dots P_m$, and the group $G^{(n+m)}$ generated by $P_1 \dots P_n, P_1 \dots P_m$. It is easy to prove the following relations

$$\begin{aligned} \mathcal{P}^{(n+m)} &= \mathcal{P}^{(n)} \sum_{\ell=0}^{\infty} [(\mathcal{P}^{(m)} - I)(\mathcal{P}^{(n)} - I)]^{\ell} \mathcal{P}^{(m)} \\ &= \mathcal{P}^{(n)} + \mathcal{P}^{(n)} (\mathcal{P}^{(m)} - I) \sum_{\ell=0}^{\infty} [(\mathcal{P}^{(n)} - I)(\mathcal{P}^{(m)} - I)]^{\ell} \mathcal{P}^{(n)} \end{aligned} \quad (C.1)$$

$$Q_{ij}^{(n+m)} = Q_{ij}^{(n)} + \phi_i^{(n)} (\mathcal{P}^{(m)} - I) \sum_{\ell=0}^{\infty} [(\mathcal{P}^{(n)} - I)(\mathcal{P}^{(m)} - I)]^{\ell} Q_j^{(n)} \quad (C.2)$$

$$Q_{ij'}^{(n+m)} = \phi_i^{(n)} \sum_{\ell=0}^{\infty} [(\mathcal{P}^{(m)} - I)(\mathcal{P}^{(n)} - I)]^{\ell} Q_{j'}^{(m)} \quad (C.3)$$

$$\phi_i^{(n+m)} = \phi_i^{(n)} \sum_{\ell=0}^{\infty} [(\mathcal{P}^{(m)} - I)(\mathcal{P}^{(n)} - I)]^{\ell} \mathcal{P}^{(m)} \quad (C.4)$$

$$Q_k^{(n+m)} = P^{(m)} \sum_{\ell=0}^{\infty} [(P^{(n)} - I)(P^{(m)} - I)]^{\ell} Q_k^{(n)} \quad (C.5)$$

APPENDIX D - PARTITION FUNCTION

Let us consider, as previously, the groups $G^{(n)}$, $G^{(m)}$ and $G^{(n+m)}$. We will show that the partition function satisfies

$$F(G^{(n+m)}) = F(G^{(n)}) F(G^{(m)}) [\det(I - (P^{(n)} - I)(P^{(m)} - I))]^{\frac{1}{2}} \quad (D.1)$$

where $F(G)$ is defined by the usual formula

$$F(G^{(N)}) = \prod_{\{R\}} \prod_{q=1}^{\infty} (1 - w_R^q) \quad (D.2)$$

where $\{R\}$ is isomorphic to the set of all closed paths around N points without taking into account the origin and the direction along this path.

We compute the determinant involving in (D.1) by (we can also take that as a definition of this infinite determinant)

$$\det(I - A) = \exp\{\text{Tr} \log(I - A)\} = \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} A^n\right\} \quad (D.3)$$

Then

$$\begin{aligned} Z &= \det[I - (P^{(n)} - I)(P^{(m)} - I)] \\ &= \exp\left\{-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{\alpha_1 \dots \alpha_n \\ \beta_1 \dots \beta_n}} \text{Tr}(P_{\alpha_1} P_{\beta_1} \dots P_{\alpha_n} P_{\beta_n})\right\} \end{aligned} \quad (D.4)$$

where

$$P_{\alpha_i} \in G^{(n)} - \{I\} \quad \text{and} \quad P_{\beta_j} \in G^{(m)} - \{I\}$$

Using now the fact that these matrices products belong to an $SU(1,1)$ representation, we obtain

$$Z = \exp - \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{\alpha_1 \dots \alpha_n \\ \beta_1 \dots \beta_n}} \frac{w_{\alpha_1 \beta_1 \dots \alpha_n \beta_n}}{1 - w_{\alpha_1 \beta_1 \dots \alpha_n \beta_n}} \right\} \quad (D.5)$$

where $w(\dots)$ is the multiplier of the corresponding homographic transformation.

Since $w_p^n = (w_p)^n$ we recollect the terms of the form

$$\frac{1}{nq} (P_{\alpha_1 \beta_1} \dots P_{\alpha_n \beta_n})^q$$

and use the well-known formula

$$\prod_{q=1}^{\infty} (1 - x^q) = \exp - \sum_{q=1}^{\infty} \frac{x^q}{q(1 - x^q)} \quad (D.6)$$

Then Z becomes

$$Z = \prod_{\substack{\{\alpha_1 \dots \alpha_n\} \\ \{\beta_1 \dots \beta_n\}}} \prod_{q=1}^{\infty} (1 - w_{\alpha_1 \beta_1 \dots \alpha_n \beta_n}^q)^{\frac{1}{n}} \quad (D.7)$$

Using the cyclic invariance of the multiplier (or of the trace)
we cancel the power $1/n$. Using now $w_p^{-1} = w_p$ or more precisely

$$\begin{aligned} W(P_{\alpha_1} P_{\beta_1} \dots P_{\alpha_n} P_{\beta_n}) &= W(P_{\beta_n}^{-1} P_{\alpha_n}^{-1} \dots P_{\beta_1}^{-1} P_{\alpha_1}^{-1}) \\ &= W(P_{\alpha_n}^{-1} \dots P_{\beta_1}^{-1} P_{\alpha_1}^{-1} P_{\beta_n}^{-1}) \end{aligned} \quad (D.8)$$

we obtain a power two which cancels the square root and this shows
(D.1).

APPENDIX E

(E.1): Let us define

$$z_i = \frac{x_i - 1}{x_i} \quad (\text{E.1})$$

We recall that U_j is given by

$$U_j = \prod_{k=j+1}^{n-1} M_-^t M_- [z_k] M_-^t \quad (\text{E.2})$$

and then

$$\widehat{U}_j(\infty) = \widehat{M_-^t \prod_{k=j+1}^{n-1} M_- [z_k] M_-^t} (1) \quad (\text{E.3})$$

From (E.3) it is easy to resolve in z_i

$$z_i = \frac{\widehat{M_-^t U_i}(\infty) - \widehat{M_-^t U_{i-1}}(\infty)}{\widehat{M_-^t U_i}(\infty) - \widehat{M_-^t U_{i+1}}(\infty)} \quad (\text{E.4})$$

Noting that it is the unharmonic ratio of

$$\widehat{M_-^t U_i}(\infty), \infty, \widehat{M_-^t U_{i-1}}(\infty), \widehat{M_-^t U_{i+1}}(\infty)$$

we can rewrite it as

$$z_i = \frac{\hat{U}_i(\infty) - \hat{U}_{i-1}(\infty)}{\hat{U}_i(\infty) - \hat{U}_{i+1}(\infty)} \frac{1 - \hat{U}_{i+1}(\infty)}{1 - \hat{U}_{i-1}(\infty)} \quad (\text{E.5})$$

for $i \leq n-2$ and

$$z_{n-1} = \frac{1}{1 - \hat{U}_{n-2}(\infty)} \quad (\text{E.6})$$

Then, we have the following relations [defining $u_i = \hat{U}_i(\infty)$ and $u_{n-1} = \infty$]

$$\prod_{i=1}^{n-1} \frac{dz_i}{z_i} = \prod_{i=0}^{n-2} \frac{du_i}{u_{i+1} - u_i} \quad (\text{with } u_{-1} = 1) \quad (\text{E.7})$$

$$\begin{aligned} (-1)^{k+1} z_{n-k} \dots z_{n-1} &= \frac{u_{n-k-1} - u_{n-k}}{(1 - u_{n-k-1})(1 - u_{n-k})} \\ &= \frac{1}{1 - u_{n-k-1}} - \frac{1}{1 - u_{n-k}} \end{aligned} \quad (\text{E.8})$$

in particular

$$(-1)^n z_1 \dots z_{n-1} = \frac{1}{1 - u_0} - \frac{1}{1 - u_n}$$

Equation (3.18) is now written as

$$\hat{X}_j(z) = \left(\frac{1}{1-u_{j+1}} - \frac{1}{1-u_j} \right) z + 1 - \frac{1}{1-u_{j+1}} \quad (\text{E.9})$$

(E.2): It is now easy to compute $\beta_j^{(j+1)}$ in terms of $\beta_j^{(n)}$ and u_i we find using (E.9)

$$\beta_j^{(j+1)} = \frac{u_{j+1} - \beta_j^{(n)}}{u_j - \beta_j^{(n)}} \frac{1-u_j}{1-u_{j+1}} \quad j \leq n-2 \quad (\text{E.10})$$

Then we deduce immediately

$$\prod_{j=1}^{n-1} \frac{d\beta_j^{(j+1)}}{\beta_j^{(j+1)}} \prod_{i=0}^{n-2} \frac{du_i}{u_{i+1}-u_i} = \frac{d\beta_{n-1}^{(n)}}{\beta_{n-1}^{(n)}} \prod_{j=1}^{n-2} \frac{d\beta_j^{(n)} (u_{j+1}-u_j)}{(\beta_j^{(n)}-u_j)(\beta_j^{(n)}-u_{j+1})} \prod_{i=0}^{n-2} \frac{du_i}{u_{i+1}-u_i} \quad (\text{E.11})$$

and

$$\prod_{i=1}^{n-1} \beta_i^{(i+1)} = \left\{ \prod_{j=1}^{n-2} \frac{u_{j+1} - \beta_j^{(n)}}{u_j - \beta_j^{(n)}} \right\} \frac{\beta_{n-1}^{(n)} (1-u_1)}{\beta_{n-2}^{(n)} - u_{n-2}} \quad (\text{E.12})$$

(E.3): In order to take the $\alpha_i^{(n)}$ as variables instead of u_i , we used some properties of u_i . We recall that

$$P_j^{(n)} = U_j P_j^{(j+1)} U_j^{-1} \quad (\text{E.13})$$

and since

$$\beta_j^{(j+1)} = w_j \alpha_j^{(j+1)} \quad (j \neq 0) \quad (\text{E.14})$$

then

$$\widehat{(P_j^{(j+1)})^{-1}}(\infty) = 0 \quad (\text{E.15})$$

from (E.13)

$$\widehat{U}_j(\infty) = \widehat{P}_j^{(n)}[\widehat{U}_j(0)]$$

and from the definition of U_j and the fact that

$$\widehat{V}_j(0) = \infty \quad (\text{E.16})$$

we obtain

$$\widehat{U}_j(0) = \widehat{U}_{j+1}(\infty) \quad (\text{E.17})$$

Collecting these results we have the relations

$$u_j = \widehat{P_j^{(n)}}(u_{j+1}) \quad (\text{E.18})$$

and

$$u_j = \widehat{P_j^{(n)} P_{j+1}^{(n)} \dots P_{n-2}^{(n)}}(\infty) \quad (\text{E.19})$$

For u_0 since ∞ is an invariant point of $P_0^{(1)}$, then

$$u_0 = \beta_0 \quad (\text{E.20})$$

Now, from (E.18) taking $\beta_j^{(n)}$ as constant we obtain for $(1 \leq i \leq n-3)$

$$du_{i+1} du_i = \frac{(1-w_i)(u_{i+1}-\beta_i^{(n)})^2}{((1-w_i)u_{i+1} + \alpha_i^{(n)} w_i - \beta_i^{(n)})^2} d\alpha_i^{(n)} du_{i+1} \quad (\text{E.21})$$

and

$$u_j - \beta_j^{(n)} = \frac{(\alpha_j^{(n)} - \beta_j^{(n)})(u_{j+1} - \beta_j^{(n)})}{((1-w_j)u_{j+1} + \alpha_j^{(n)} w_j - \beta_j^{(n)})} \quad (\text{E.22})$$

so that

$$du_{i+1} du_i = \frac{(1-w_i)}{(\alpha_i^{(n)} - \beta_i^{(n)})^2} (u_i - \beta_i^{(n)})^2 d\alpha_i^{(n)} du_{i+1} \quad (\text{E.23})$$

We use also

$$u_{n-2} = \frac{\alpha_{n-2}^{(n)} - w_{n-2} \beta_{n-2}^{(n)}}{1 - w_{n-2}} \quad (\text{E.24})$$

APPENDIX F - N LOOP AMPLITUDE

In order to compare our result with the previous one, let us attach the operator defined by (3.1) and (3.19) to a tree as defined by Yu ⁴⁾ (Fig. 12)

$$\int \frac{\prod_{i=1}^S dy_i}{dy_a dy_b dy_c} (y_a - y_b)(y_b - y_c)(y_c - y_a) \prod_{i=0}^S (y_i - y_{i+1})^{\alpha_0 - 1} (y_\alpha - y_{\alpha+1})^{-\alpha_0} (y_{\alpha-1} - y_\alpha)^{-\alpha_0} (y_{\alpha-1} - y_{\alpha+1})^{\alpha_0} \prod_{i \neq j} (y_i - y_j)^{-\frac{1}{2} p_i \cdot p_j} \exp \sum_{\substack{i=0 \\ i \neq \alpha}}^S p_i (p_i | a^+) | 0_a \rangle \quad (F.1)$$

where

$$p_i = \widehat{R}_\alpha(y_i) = \frac{y_\alpha - y_{\alpha-1}}{y_{\alpha+1} - y_{\alpha-1}} \frac{y_{\alpha+1} - y_i}{y_\alpha - y_i} \quad (F.2)$$

and

$$p_\alpha = 0, \quad \sum_{i \neq \alpha} p_i = 0 \quad (F.3)$$

Using

$$\exp \sum_{i=0}^S p_i (p_i | a^+) | 0_a \rangle = : e^{a^+ (R_\alpha - I) a} : \exp \sum_{i=0}^S p_i (y_i | a^+) | 0_a \rangle \quad (F.4)$$

We define the generators of the multiloop amplitude as

$$\bar{P}_j^{(n)} = C^{-1} P_j^{(n)} C \quad (F.5)$$

where

$$C = M_-^t \left[\frac{z}{z-1} \right] M_- R_\alpha \quad (F.6)$$

if z is the Chan variable of the propagator. As previously, we take first as independent variables for the loop the $\bar{\alpha}_i$'s ($i = 1, \dots, n-2$) and $\bar{\beta}_j$ ($j = 0, \dots, n-1$), $\bar{\alpha}_0^{(n)}$ and $\bar{\alpha}_{n-1}^{(n)}$ are now given by

$$\frac{\bar{\alpha}_0 - \hat{C}^{-1}(1)}{\bar{\alpha}_0 - \bar{u}_1} = w_0 \frac{\bar{\beta}_0 - \hat{C}^{-1}(1)}{\bar{\beta}_0 - \bar{u}_1} \quad (F.7)$$

and the relation $\alpha_{n-1}^{(n)} = \beta_{n-1}^{(n)} / w_{n-1}$ becomes

$$\hat{C}^{-1}(\infty) = \hat{P}_{n-1}(\hat{C}^{-1}(0)) \quad (F.8)$$

Using

$$\left\{ \begin{array}{l} \bar{u}_{n-1} \equiv \hat{C}^{-1}(\infty) = R_\alpha^{-1}\left(\frac{1}{2}\right) \\ \hat{C}^{-1}(1) = y_\alpha \\ \hat{C}^{-1}(0) = y_{\alpha-1} \end{array} \right. \quad (F.9)$$

We do the change of variable $(y_\alpha, z) \rightarrow \bar{\alpha}_0^{(n)}, \bar{\alpha}_{n-1}^{(n)}$ and then obtain the following N loop amplitude

$$\begin{aligned}
 & \int \prod_{\substack{\mu=1 \\ \neq \alpha}}^S \frac{dy_\mu (y_a - y_b)(y_b - y_c)(y_c - y_a)}{dy_a dy_b dy_c} \prod_{j=0}^{n-1} \frac{dw_j}{w_j} \frac{d\bar{\alpha}_j^{(n)} d\bar{\beta}_j^{(n)} (1-w_j)}{(\bar{\alpha}_j^{(n)} - \bar{\beta}_j^{(n)})^2} \left(\frac{\bar{u}_j - \bar{\beta}_j^{(n)}}{\bar{\beta}_j^{(n)} - \bar{u}_{j+1}} \right) \\
 & \prod_{\substack{\mu=1 \\ \neq \alpha-1}}^S (y_\mu - y_{\mu+1})^{\alpha_0-1} \left\{ \prod_{j=0}^{n-1} \frac{1}{w_j} \left(\frac{\bar{\beta}_j^{(n)} - \bar{u}_{j+1}}{\bar{u}_j - \bar{\beta}_j^{(n)}} \right) \right\}^{\alpha_0} \\
 & \int \prod_{i=0}^{n-1} d_4 k_i w_i^{-k_i^2/2} \prod'_{i,j,\{\alpha\}} \left\{ \frac{(\bar{\beta}_j^{(n)} - \hat{P}_\alpha'(\bar{\beta}_i^{(n)}))(\bar{\alpha}_j^{(n)} - \hat{P}_\alpha'(\bar{\alpha}_i^{(n)}))}{(\bar{\beta}_j^{(n)} - \hat{P}_\alpha'(\bar{\alpha}_i^{(n)}))(\bar{\alpha}_j^{(n)} - \hat{P}_\alpha'(\bar{\beta}_i^{(n)}))} \right\}^{-\frac{1}{2} k_i \cdot k_j} \\
 & \prod''_{i,\mu \in \{\alpha\}} \left(\frac{y_\mu - \hat{P}_\alpha''(\bar{\beta}_i^{(n)})}{y_\mu - \hat{P}_\alpha''(\bar{\alpha}_i^{(n)})} \right)^{-k_i \cdot P_\mu} \quad (F.10) \\
 & \prod_{\substack{\mu \neq \nu \\ \mu, \nu \neq \alpha}} (y_\mu - y_\nu)^{-\frac{1}{2} P_\mu \cdot P_\nu} \prod_{\mu, \nu \in \{\gamma\}} (y_\mu - \hat{P}_\gamma(y_\nu))^{-\frac{1}{2} P_\mu \cdot P_\nu}
 \end{aligned}$$

With

$$\begin{aligned}
 \bar{u}_i &= \overbrace{\bar{\beta}_i \dots \bar{\beta}_{n-1}}^{\quad} (y_{\alpha-1}) \\
 \bar{u}_0 &= y_\alpha \\
 \bar{u}_n &= y_{\alpha-1}
 \end{aligned}$$

(F.11)

The different subsets of the infinite products have been defined in Chapter 3 and Appendix C.

It can be seen that this is exactly the result of Kaku and Yu ¹⁾ when we pay attention to the fact that the N loops are consecutive and then we must not use literally their formula, but follow carefully their derivation in order to obtain the correct result. It agrees also with that of Alessandrini and Amati ⁸⁾ when they use the Lovelace vertex ⁴⁾. However, it differs by a factor $\prod_j (1-w_j)$ when they use the Olive vertex ⁵⁾. This extra factor comes in fact from the elimination of spurious states in the loops.

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FIGURE CAPTIONS

Figure 1 : One-loop planar tadpole operator.

Figure 2a : Fundamental duality property of the planar tadpole
and 2b operator.

Figure 3 : Non-planar self-energy operator.
and 4

Figure 5 : Iteration of $N+1$ tadpole operators.

Figure 6 : Identity between two-loop tadpole operators.

Figure 7 : Periodicity in the two-loop tadpole operator.

Figure 8 : Identity between N loop tadpole operators.

Figure 9 : Identity between two-loop tadpole operators.
and 10

Figure 11 : Iteration of non-planar self-energy attached to
a planar tadpole operator.

Figure 12 : N loop amplitude.

FIG. 1

FIG. 2a

FIG. 2b

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7

FIG. 8

FIG.9

FIG.10

FIG.11

FIG.12