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UNCORRELATED JET MODEL FOR PARTICLE SPECTRA AND ELASTIC
PROTON-PROTON SCATTERING AT VERY HIGH ENERGY

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A B S T R A C T

Assuming that the particles produced in a high energy collision are uncorrelated, except for energy and momentum conservation, we derive a very simple formula, which exhibits the energy dependence of the relevant phase space integral. Using this result it is a simple matter to derive the magnitude and the energy dependence of the average multiplicity and the total inelastic cross-section. A relation between these two quantities is established and compared with the prediction of Feynman's parton model. A few single particle spectra are calculated and compared with experiment. Choosing a simple form for the phases of the nucleon and meson wave functions we calculate the overlap function and hence the diffraction peak for elastic proton - proton scattering. In addition to the well-known effect of the meson phase we find that also the nucleon phase has a considerable influence on the form of the diffraction peak.

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1. INTRODUCTION AND CONCLUSIONS

It seems plausible that the dynamics of high energy interactions will be very hard to discover by studying high multiplicity processes. Recent attempts to separate the phase space effects from the dynamical ones are reviewed by Krzywicki ¹⁾. Much emphasis at present is on reactions at finite energy and there the problem of disentangling the data is formidable indeed.

At very high energies, however, it is known that the statistical method of Lurçat and Mazur ²⁾ for the calculation of phase space integrals becomes exact. Still, this method involves the solution of a complicated integral equation for a parameter α and it is in general not possible to find an analytic expression for the energy dependence of this parameter and hence for the energy dependence of the phase space integrals. In the present paper, however, we will restrict ourselves to the (realistic) case where the high transverse momenta of the mesons are strongly suppressed. For this case a closed expression can be derived, which clearly exhibits the energy dependence of the relevant phase space integral. Assuming the uncorrelated production of all particles in the final state, it is then a simple matter to calculate the average multiplicity, the single particle spectra and the overlap function. Since for very high energies the phase space problem is now solved, it becomes possible to test the effect on the above mentioned quantities of different assumptions about the dynamics of the interaction.

In considering the final state for inelastic nucleon-nucleon scattering we will assume that only symmetric jets occur, i.e., that in the c.m. system the final nucleons have momenta $\vec{p}_f$ and $-\vec{p}_f$. This assumption is not essential and is only made to simplify the further calculations.

The invariant amplitude for the production of n mesons with four-momenta $k_1, \dots, k_n$ and two nucleons is assumed to be of the product form, so that

$$|T(p_f, k_1, \dots, k_n)|^2 = |\rho_N(p_f, p_i)|^2 \delta(p_f^2 - M^2) \theta(p_f) \prod_{j=1}^n |\rho(k_j)|^2 \delta(k_j^2 - \mu^2) \theta(k_j). \quad (1.1)$$

The total cross-section for the production of n mesons is then given by

$$\sigma_n = \frac{1}{\sqrt{\lambda(s, M^2, M^2)}} \frac{1}{n!} \int dp_f dk_1 \dots dk_n |T|^2 \delta_4(P_i - P_f - \sum_{j=1}^n k_j), \quad (1.2)$$

where P_i and P_f are the four-momenta of the two nucleons in the initial and in the final state respectively and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$.

The functions $\rho_N(p_f, p_i)$ for the nucleon and $\rho(k)$ for the mesons will be specified later. For the moment we will restrict ourselves to a description of their general characteristics. With Feynman³⁾ we take the longitudinal momentum distribution of the partons (we call them mesons), which constitute the right moving proton, in such a way that equal momentum intervals contain the same energy. This gives a distribution which for $k_z > 0$ is proportional to $1/\omega(k)$ with $\omega(k) = \sqrt{\vec{k}^2 + \mu^2}$. We do the same for the left moving mesons. Since

$$\int \frac{d\vec{k}}{2\omega(k)} F(k) = \int F(k) \delta(k^2 - \mu^2) \theta(k) dk$$

this justifies, at least for mesons which are not wee, our assumption that $|\rho(k)|^2$ in (1.1) and (1.2) only depends on the transverse component $k_\perp$ of $\vec{k}$. For the mesons of very small momentum in the c.m. system we make the same assumption, since it simplifies matters. This can, however, not be justified at all, because these wee mesons are supposedly the ones which participate in the interaction, about which we know next to nothing.

In the rest of the paper it will furthermore be essential that for the dependence on the transverse component $k_\perp$ of $\vec{k}$ we

assume a cut-off for large $k_{\perp}$. We again follow Feynman ³⁾ in this respect as well as in assuming that for high energies and in the c.m. system the wave function for the mesons, i.e., $\rho(k)$, no longer depends on this energy.

For the nucleon wave function $\rho_N(p_f, p_i)$ we make quite different assumptions. We have already built in a strong correlation by allowing only symmetric jets. Furthermore, since experiment ⁴⁾ shows that there is an exponential dependence on the square of the momentum transfer $t' = (p_i - p_f)^2$, we include a factor $e^{Bt'}$ in $|\rho_N(p_f, p_i)|^2$. In order to fix the energy dependence we chose a form which gave a total cross-section for $NN \rightarrow NN\pi$ proportional to

$$\sigma_1(s) \sim \left(\frac{s}{s_0}\right)^{2\alpha-2} \ln \frac{s}{s_0}, \quad (1.3)$$

i.e., the result of a multi-Regge model ⁵⁾. By taking

$$|\rho_N(p_f, p_i)|^2 \sim \left(\frac{s}{s_0}\right)^{2\alpha-\frac{1}{2}} \ln \frac{s}{s_0} e^{Bt'}, \quad (1.4)$$

substituting this into the expression (1.2) for $\sigma_1(s)$ and taking

$$|\rho(k)|^2 \sim e^{-\frac{1}{2}\beta^2 k_{\perp}^2}$$

we indeed arrive at the form (1.3). The steep t' dependence in (1.4) is the reason why one need not take $2\alpha-1$ in the exponent of (1.4), but that $2\alpha-\frac{1}{2}$ suffices to produce the energy dependence as required in $\sigma_1(s)$. Another reason, which actually forces us to include a logarithmic factor in $|\rho_N|^2$, is the requirement that in the asymptotic limit the total inelastic cross-section be energy independent. A logarithmic factor in the phase space integral $C(P)$ [see Eq. (2.24)] must therefore be cancelled and this is achieved by our choice (1.4) for $|\rho_N|^2$. For most processes $NN \rightarrow NN\pi$ the cross-section $\sigma_1(s)$ decreases with increasing energy, so that we will choose $\alpha < 1$. In some cases, however, the production of a final state $NN\pi$ via the formation of an isobar in $NN \rightarrow NN^*$ has a finite

limiting cross-section at high energies. From our choice (1.3) for the cross-section $\sigma_1(s)$ it is clear that this formation of isobars is not taken into account in our model. This is not to say that the pions come from "pionization", that is from a mesonic cloud which in the c.m. system is arrested and then decays. As a matter of fact our model may well be in agreement with the hypothesis of limiting fragmentation⁶⁾. This is shown by the fact that the width Δ of the angular distribution of the pions, plotted as a function of

$$x = \log \cot \frac{\theta}{2},$$

increases logarithmically with the energy, see Fig. 1. Depending on the coefficient in $\Delta = c \log E$, the distribution in the laboratory system could therefore consist of an energy independent part and a part which, for each finite x , vanishes as E goes to infinity. With the main characteristics of the nucleon and meson wave functions specified, we now consider the total inelastic cross-section

$$\sigma_{\infty} = \sum_{n=1}^{\infty} \sigma_n.$$

It can be written as

$$\sigma_{\infty} = \frac{1}{\sqrt{\lambda(s, M^2, M^2)}} \int d p_f |p_N(p_f, p_i)|^2 \delta(p_f^2 - M^2) \theta(p_f) C(P_i - P_f), \quad (1.5)$$

where we have introduced the abbreviations

$$C(P) = \sum_{n=1}^{\infty} \frac{1}{n!} \Omega_n(P) \quad (1.6)$$

and

$$\Omega_n(P) = \int d k_1 \dots d k_n \prod_{j=1}^n |p(k_j)|^2 \delta(k_j^2 - \mu^2) \theta(k_j) \cdot \delta(k_1 + \dots + k_n - P). \quad (1.7)$$

In the next section we will show that for very high energy the function $C(P)$ is given by the expression (2.24)

$$C(P) = \frac{\sigma_0^2}{\ln\left(\frac{P^2}{m^2}\right)} \left(\frac{P^2}{m^2}\right)^{\frac{n_0}{2}-1}.$$

It contains $|\rho(k_\perp)|^2$ only in the form of a few simple integrals, one of which is the dimensionless number

$$n_0 = 2\pi \int_0^\infty |\rho(k_\perp)|^2 k_\perp dk_\perp. \quad (1.8)$$

The same integral occurs when the energy dependence of the total inelastic cross-section is calculated. We find

$$\sigma_\infty \sim s^{\frac{n_0}{2} + 2\alpha - \frac{5}{2}}. \quad (1.9)$$

Since, experimentally, this cross-section certainly does not vary with a power of s we conclude that the following relation must hold

$$n_0 = 5 - 4\alpha. \quad (1.10)$$

This number n_0 also appears as the only a priori unknown factor in the formula for the average multiplicity. For the average number of pions we obtain

$$\bar{n} = \frac{1}{2} n_0 \ln \frac{s}{s_0} = \left(\frac{5}{2} - 2\alpha\right) \ln \frac{s}{s_0}. \quad (1.11)$$

The assumptions which were made in the derivation of this formula are the same as Feynman³⁾ used to derive his formula $\bar{n}_F = (2 - 2\alpha) \ln(s/s_0)$. The difference of $\frac{1}{2}$ in the coefficient can be traced back to the steep t' dependence of our $|\rho_N(p_f, p_i)|^2$.

Feynman took $\alpha = 0.43$ and obtained a good agreement with the average experimental multiplicity if in addition he assumed that each parton gave two pions. In this way he got $\bar{n}_F(\pi) = 2.28 \ln(s/s_0)$. With the same α our formula (1.11) gives $\bar{n} = 1.64 \ln(s/s_0)$. Although there is no compelling reason to do so we should like to work with a definite α in what follows. A good choice seems to be to take α such that for high energies also the nucleon wave function comes as close to a limiting function as possible. Formula (1.4) shows that this principle leads to $\alpha = 0.25$. The number n_0 becomes

$$n_0 = 4, \quad (1.12)$$

which gives for the multiplicity

$$\bar{n} = 2 \ln \frac{s}{s_0}. \quad (1.13)$$

The corresponding energy-dependence of the cross-section for $NN \rightarrow NN\pi$ becomes $\sigma_1 \sim (s/s_0)^{-1.5} \ln(s/s_0)$, which, experimentally, can probably not be distinguished from Feynman's choice $\sigma_1 \sim (s/s_0)^{-1.14}$.

The rest of the paper is organized as follows. Section 2 is devoted to a discussion of the phase space integrals. In Section 3 we calculate the average multiplicity, the angular distribution of a pion, (Fig. 1), the distribution of the inelasticity (Fig. 2), and the distribution of the momentum of the final nucleon for a fixed scattering angle (Fig. 3). In the last section the overlap function is calculated and it is shown that not only the meson phase but also the phase of the nucleon wave function has an important effect on the form of the diffraction peak.

2. PHASE SPACE INTEGRALS

The method of Lurçat and Mazur ²⁾ for the calculation of the phase space integral $\Omega_n(P)$ [Eq. (1.7)] consists of the following recipe [for a short exposé we refer to the Appendix of Ref. 7].

a) Take $\vec{P}=0$ and $P_0 > 0$.

b) Define the function

$$\varphi(\alpha) = \int \frac{|\rho(k)|^2}{2k_0} e^{-\alpha k_0} d\vec{k}. \quad (2.1)$$

c) For each n solve $\alpha = \alpha_n$ from the equation

$$\frac{P_0}{n} = - \frac{1}{\varphi(\alpha)} \frac{d\varphi}{d\alpha}. \quad (2.2)$$

d) Define the probability distribution

$$u_n(k) = \frac{1}{\varphi(\alpha_n)} \frac{|\rho(k)|^2}{2k_0} e^{-\alpha_n k_0} \quad (2.3)$$

and calculate the averages occurring in

$$B_{\mu\nu}(n) = n \left[\overline{k_\mu k_\nu} - \overline{k_\mu} \overline{k_\nu} \right]. \quad (2.4)$$

e) For large energy transfer $\Omega_n(P)$ is now well approximated by

$$\Omega_n(P) \simeq \frac{1}{(2\pi)^2 \sqrt{\det B(n)}} [\varphi(\alpha_n)]^n e^{\alpha_n P_0}. \quad (2.5)$$

Following the scaling hypothesis of Feynman we now assume that in $|\rho(k)|$ the $k_\perp$ dependence factors out and that the $k_\parallel$ dependence occurs only in the combination $x = k_\parallel / p_0$. We actually assume that $|\rho(k)|$ does not depend on $k_\parallel$ at all. A separate calculation has shown that multiplying our $|\rho(k_\perp)|$ by a factor $e^{-a(k_\parallel / p_0)^2}$, as was done in Ref. 8), hardly changed the average multiplicity and the phase space integral $C(P)$ and has only some effect on the overlap function for large t .

In considering the function $\varphi(\alpha)$ of Eq. (2.1) we can now perform the integrations over the azimuthal angle and the longitudinal component of $\vec{k}$ and we find

$$\varphi(\alpha) = 2\pi \int_0^\infty k_\perp |\rho(k_\perp)|^2 K_0(\alpha \sqrt{k_\perp^2 + \mu^2}) \alpha k_\perp dk_\perp, \quad (2.6)$$

where $K_0(z)$ is a modified Bessel function [Ref. 9), (9.6.23)] and μ is the mass of the meson. Since $\varphi(\alpha)$ will be needed only for $\alpha \rightarrow 0$ [see Eq. (2.12)] this Bessel function may be replaced by

$$K_0(z) \approx -\gamma - \ln \frac{z}{2} + O(z^2 \ln z), \quad (2.7)$$

where $\gamma = 0.5772156$ is Euler's constant. We find for $\alpha \rightarrow 0$

$$\varphi(\alpha) \approx -n_0 \left(\gamma' + \ln \frac{\alpha \mu}{2} \right), \quad (2.8)$$

with n_0 given by equation (1.8) and

$$\gamma' = \gamma + \frac{2\pi}{n_0} \int_0^\infty |\rho(k_\perp)|^2 k_\perp \ln \sqrt{\frac{k_\perp^2}{\mu^2} + 1} dk_\perp \approx \gamma + \ln \sqrt{\frac{k_\perp^2}{\mu^2} + 1}. \quad (2.9)$$

Equation (2.8) can also be written as

$$\varphi(\alpha) \approx -n_0 \ln \frac{\alpha \bar{\mu}}{2}, \quad (2.10)$$

where

$$\bar{\mu} = \mu e^{\gamma'}. \quad (2.11)$$

It is seen that the integrals in (1.8) and (2.9) only exist if the high transverse momenta are cut off. In the same limit $\alpha \rightarrow 0$ the determinant of $B_{\mu\nu}(n)$ can be calculated.

Using the relation

$$\alpha_n = \frac{n_0 n}{P_0 \varphi_n}, \quad (2.12)$$

which follows from (2.2) and (2.10) and writing $\varphi_n = \varphi(\alpha_n)$, we find

$$\det B(n) = \frac{1}{4} P_0^4 (\overline{k_1^2})^2 \left(1 - \frac{n_0}{\varphi_n}\right) \left(\frac{\varphi_n}{n_0}\right)^2, \quad (2.13)$$

with

$$\overline{k_1^2} = \frac{2\pi}{n_0} \int_0^\infty |\varphi(k_1)|^2 k_1^3 dk_1. \quad (2.14)$$

The expression for the phase space integral now becomes

$$\Omega_n(P) = \frac{n_0}{2\pi^2} \frac{1}{P_0^2 \overline{k_1^2}} \tilde{\Omega}_n(P), \quad (2.15)$$

with

$$\tilde{\Omega}_n(P) = \left(1 - \frac{n_0}{\varphi_n}\right)^{-1/2} \frac{1}{\varphi_n} \left(\varphi_n e^{\frac{n_0}{\varphi_n}}\right)^n. \quad (2.16)$$

From (2.10) and (2.12) follows that φ_n must satisfy the equation

$$\varphi_n = \bar{n} + n_0 \ln \frac{\varphi_n}{n}, \quad (2.17)$$

where $\bar{n}$ is defined by

$$\bar{n} = n_0 \ln \frac{2P_0}{n_0 \mu}. \quad (2.18)$$

From (2.17) it is seen that in first approximation φ_n becomes constant and equal to $\bar{n}$. For $\tilde{\Omega}_n$ one then obtains the value $\bar{n}^{-n-1}$. In this case the distribution of the multiplicity, i.e.,

$$W(n) = \frac{1}{\tilde{C} n!} \tilde{\Omega}_n \quad (2.19)$$

with

$$\tilde{C}(P) = \sum_n \frac{1}{n!} \tilde{\Omega}_n, \quad (2.20)$$

is a Poisson distribution with both the average and Δ_p^2 , i.e., the mean square deviation from this average, equal to $\bar{n}$. By expanding around $\bar{n}$ we find for $n_0 \neq 0$ that $W(n)$ deviates from a Poisson distribution in that the width has decreased. In Table 1 we give the ratio $r = \Delta / \Delta_p$ for a number of values of $\bar{n}$.

The position of the maximum is not changed and the average is still given by

$$\langle n \rangle = \bar{n} + O\left(\left(\frac{n_0}{\bar{n}}\right)^2\right). \quad (2.21)$$

The function $\tilde{C}(P)$ of (2.20) becomes

$$\tilde{C}(P) = \frac{e^{\bar{n} + n_0}}{\bar{n}} \left[1 + O\left(\left(\frac{n_0}{\bar{n}}\right)^2\right) \right]. \quad (2.22)$$

For $n_0 = 4$ we have calculated the relative difference d between this $\tilde{C}(P)$ and the exact $\tilde{C}(P)$ as obtained via Eqs. (2.16), (2.17) and (2.20). Table 2 shows that the exact phase space integral can be very well approximated by formula (2.22).

One should realize, of course, that for smaller multiplicities bigger discrepancies than shown in this table will occur, because the statistical method approaches the exact phase space integrals only for infinite energy.

Since the average multiplicity $\bar{n}$ and the phase space factor $C(P)$ are Lorentz invariant quantities they can be written as functions of the square of the invariant mass P^2 of the meson field and we find

$$\bar{n} = \frac{1}{2} n_0 \ln \left(\frac{P^2}{m^2} \right) \quad (2.23)$$

and

$$C(P) = \frac{\sigma_0^2}{\ln \left(\frac{P^2}{m^2} \right)} \left(\frac{P^2}{m^2} \right)^{\frac{n_0}{2} - 1}. \quad (2.24)$$

In these formulae the mass m and the cross-section σ_0 are defined by

$$m = \frac{1}{2} n_0 / \bar{n} \quad (2.25)$$

and

$$\sigma_0^2 = \frac{e^{n_0}}{\pi^2 m^2 k_{\perp}^2}. \quad (2.26)$$

In order to proceed we must now specify the functions $\rho_N(p_f, p_i)$ and $\rho(k)$. For the latter we choose a form such that $|\rho(k)|^2$ fits the experimental distribution¹⁰⁾ of the transverse momentum of a single meson. This leads us to

$$|\rho(k_{\perp})| = \frac{\beta}{1+f} \sqrt{\frac{n_0}{2\pi}} \left[e^{-\frac{1}{4}\beta_1^2 k_{\perp}^2} + f e^{-\frac{1}{4}\beta_2^2 k_{\perp}^2} \right]. \quad (2.27)$$

The factor $\beta/1+f$ can be determined by the requirement that (1.8) be satisfied. We find in this way

$$\frac{\beta}{1+f} = \left[\beta_1^{-2} + f^2 \beta_2^{-2} + 4f(\beta_1^2 + \beta_2^2)^{-1} \right]^{-1/2} \quad (2.28)$$

The fit to the data ¹⁰⁾ gives

$$\beta_1^2 = 6 \left(\frac{GeV}{c} \right)^{-2}, \beta_2^2 = 30 \left(\frac{GeV}{c} \right)^{-2} \text{ and } f = +2. \quad (2.29)$$

Equation (2.28) then gives

$$\beta^2 = 17.2 \left(\frac{GeV}{c} \right)^{-2} \quad (2.30)$$

So far the only undetermined parameter is the dimensionless number n_0 , which measures the strength of the meson-nucleon coupling. For $n_0 = 4$, the previously introduced quantities take the following values

$$m = 1.5 \text{ GeV}/c^2 \quad \sigma_0 = 3.75 \text{ GeV}^{-2} \triangleq 1.46 \text{ mb}$$

$$\overline{k}_1 = 0.34 \text{ GeV}/c \quad \sqrt{\overline{k}_1^2} = 0.42 \text{ GeV}/c \quad (2.31)$$

For the nucleon factor we take (as was discussed in Section 1)

$$p_N(p_f, p_i) = m_1 e^{\frac{\omega t'}{4M^2}} \sqrt{\ln \frac{s}{s_0}} e^{i \frac{1}{2} i \vec{\sigma} \cdot \vec{p}_f}, \quad (2.32)$$

where the parameter m_1 has the dimension of a mass, M is the nucleon mass, ω is a dimensionless number and $t' = (p_f - p_i)^2$. The effect of the phase factor will be studied in the last section. The logarithmic term will cancel the logarithm in $C(P)$ [Eq. (2.24)] so that a total inelastic cross-section (1.5) is obtained, which for $s \rightarrow \infty$ approaches a constant value. The specific form of the t' dependence in (2.32) not only agrees with experiment ⁴⁾ but also simplifies later

integrations. It favours large p_{f0} and, therefore, makes the nucleon the leading particle of the jet. It will turn out that for $-t$ not too small the overlap function and thus the elastic cross-section shows a considerable dependence on the form of $\rho_N(p_f, p_i)$. In this way it may therefore be possible to distinguish between different amplitudes, like the electromagnetic form factor used by Chou and Yang¹¹⁾ and the Regge amplitude employed in Refs. 8) and 12). This will not be done in the present paper.

Our choices for $\rho(k_\perp)$ and $\rho_N(p_f, p_i)$ can now be substituted in the expression (1.5) for the total inelastic cross-section at very high energy. Using equation (2.24) for $C(P)$ we obtain

$$\sigma_\infty = \frac{\pi m_1^2 \sigma_0^2 M^2}{\omega m^2} e^{\omega \left(\frac{s}{m^2} \right)^{\frac{n_0}{2} + 2\alpha - \frac{5}{2}}} \int_0^1 (1-x)^{n_0-2} e^{-\frac{\omega}{2}(x+x^{-1})} dx. \quad (2.33)$$

It is seen that for $\alpha = 0.25$ this cross-section approaches a constant value for increasing s only if $n_0 = 4$. This is the value of n_0 which will henceforth be assumed. An expression similar to (2.33) can be derived for the average transverse momentum of the nucleon. Using the experimental values $\sigma_\infty = 31$ mb and $\langle p_{f\perp} \rangle = 0.41$ GeV/c the two remaining parameters m and ω can be adjusted and we find

$$m_1 = 8.5 \frac{\text{GeV}}{c^2} \quad \text{and} \quad \omega = 5. \quad (2.34)$$

With the amplitude for inelastic processes completely known we can now calculate any quantity of interest. In the next section a few single particle distributions are calculated. We restrict ourselves to the case of very high energy, i.e., to the case where the average multiplicity is not less than $2n_0 = 8$, so that the asymptotic expression (2.24) for the phase space integral can be used. In the last section the t dependence of the elastic cross-section will be studied in the limit of infinite energy.

3. SINGLE PARTICLE DISTRIBUTIONS

The inelasticity of an event in which the final nucleons have a total energy $2p_{fo}$ (in the c.m. system) is defined as $1-x$ with $x = 2p_{fo}/2p_{io}$. In our model the distribution of the inelasticity is given by the integrand in (2.33), i.e., by

$$(1-x)^2 e^{-\frac{1}{2}\omega(x+x')} \quad (3.1)$$

For $\omega = 5$ it is plotted in Fig. 2, together with the experimental results for 4π production quoted by Turkot¹³⁾. The area under the two curves is made equal. Although our distribution seems to be a little bit too broad, the average inelasticity $1-\bar{x} = 0.4$ is in excellent agreement with the experimental value.

Averaging the multiplicity (2.18) over the nucleon distribution $|\rho_N|^2$ and multiplying by $2/3$ gives for the average number of charged mesons the formula

$$\bar{n}_{ch} = \frac{4}{3} \ln \frac{s}{s_0} \quad (3.2)$$

(we took $n_0 = 4$).

A recent derivation⁸⁾, also based on scaling, gave

$$\bar{n}_{ch} = 0.89 \ln \frac{s}{s_0} + \text{constant}. \quad (3.3)$$

This formula seems to be in reasonable agreement with recent data from cosmic ray experiment, also quoted in Ref. 8). McCusker and Peak¹⁴⁾ obtain for the best fit of the cosmic ray data the formula

$$\bar{n}_{ch}^{exp} = 4.1 \frac{\log E}{\log 16}, \quad (3.4)$$

where E is the laboratory energy expressed in GeV. Our formula (3.2), when written in the same way, becomes

$$\bar{n}_{ch} = 3.7 \frac{\log E}{\log 16} . \quad (3.5)$$

For cosmic ray energies the angular distribution of a single meson in the c.m. system, i.e., $d\sigma/d(\cos\theta)$, is usually plotted as a function of the variable $x = \log^{10} \cotg \theta/2$. The function

$$I(x) = \frac{1}{\sigma_{\infty}} \frac{d\sigma}{dx} = \frac{2.30}{\sigma_{\infty}} \frac{d\sigma}{d(\cos\theta)} \sin^2\theta \quad (3.6)$$

is symmetric around $x=0$ and normalized to unity, which means that

$$\int_{-\infty}^{\infty} \frac{d\sigma}{dx} dx = \sigma_{\infty} .$$

For a number of values of the energy this function is calculated and plotted in Fig. 1, (only the $x>0$ part is given). We find a structure with two maxima, although no two-fireball was used. This result was also obtained by Ranft¹⁵⁾ in the thermodynamical model of strong interactions. The width Δ , defined by

$$\Delta^2 = \frac{1}{\sigma_{\infty}} \int_{-\infty}^{\infty} x^2 \frac{d\sigma}{dx} dx , \quad (3.7)$$

increases logarithmically with the laboratory energy and we find for high energies

$$\Delta \simeq 0.24 \log \frac{E}{0.078} . \quad (3.8)$$

In Table 3 we give the results of this formula for different laboratory energies, expressed in GeV.

Both the logarithmic increase and the numerical value of the width agree with other recent calculations ¹²⁾. The agreement with the experimental results [see Refs. 8) and 12)] seems to be good.

Using the Reggeized multiperipheral model of Caneschi and Pignotti ¹⁶⁾ it was shown by Caneschi, Lyon and Risk ¹⁷⁾ that at 175 GeV the experimental angular distribution of pions could be quite well reproduced. These authors emphasize that the most important elements in their calculations are

- a) a sharp drop-off in the distribution of the transverse pion momentum $k_{\perp}$;
- b) Lorentz invariance in $k_{\parallel}$ and
- c) phase space effects.

These same elements play a dominant role also in the present paper and are used to explain the main features of highly inelastic collisions. The details can only be varied by changing the nucleon wave function $\rho_N(p_f, p_i)$. With a simple choice for $\rho_N(p_f, p_i)$ [see (2.32) and its justification in Section 1] we calculated the laboratory momentum distribution of the final nucleon for scattering over a fixed angle $\theta = 10$ mrad. For initial momenta ranging from 70 to 500 GeV/c the results are shown in Fig. 3. For comparison we also give the curves calculated by Caneschi and Pignotti ¹⁶⁾ at 70 and 200 GeV/c. Up to 30 GeV/c they obtained good agreement with experiment. Integrating our nucleon distribution over the azimuthal angle and the longitudinal component of $\vec{p}_f$ we obtain a formula for the distribution of the transverse momentum of the nucleon, which can be very well approximated by

$$e^{-\frac{1}{2} \lambda^2 p_{f\perp}^2} p_{f\perp} dp_{f\perp} \quad (3.9)$$

with

$$\lambda^2 = g \left(\frac{g}{c} \right)^{-2} \quad (3.10)$$

This distribution reproduces $\langle p_{\perp} \rangle = 0.41$ GeV/c and thereby serves as a check on the computations. The form (3.9) and the value of $\lambda^2 = 9$ are in reasonable agreement with the experimental results ¹⁸⁾. This can therefore be considered as an extra justification of our choice (2.32) for the t' dependence of ρ_N .

4. THE DIFFRACTION PEAK

It was observed by Mischejda ¹⁹⁾ that the slope of the diffraction peak for elastic nucleon-nucleon scattering, as calculated via the overlap function in Van Hove's ²⁰⁾ uncorrelated jet model, is much too small. Later it was shown ²¹⁾ that the incorporation of the phase of the amplitude, predicted by a Regge pole model, changed the slope considerably, so that agreement with the experimental value could be obtained. The effect on the shape of the diffraction peak has not been studied, however. In order to do so we make a simple assumption about the momentum dependence of this phase factor. For this factor in $\rho(k)$ we take $e^{\frac{1}{2}i\vec{\eta} \cdot \vec{k}}$ and we assume that $\vec{\eta}$ is a constant vector of length η , which is perpendicular to $\vec{p}_i$ and in the plane of $\vec{p}_i$ and $\vec{p}_f$. Assuming that for high energies the amplitude for elastic proton-proton scattering is purely imaginary its absolute value can be expressed in terms of the inelastic amplitudes by requiring that the S matrix be unitary. For the uncorrelated jet model this elastic amplitude becomes, apart from a normalization factor,

$$F(t) \sim \sum_{n=1}^{\infty} \frac{1}{n!} \int d\vec{p}_f d\vec{k}_1 \dots d\vec{k}_n \rho_N^*(\vec{p}_f, \vec{p}_+) \rho_N(\vec{p}_f, \vec{p}_-) \delta(\vec{p}_f^2 - M^2) \theta(\vec{p}_f) \cdot$$

$$\prod_{j=1}^n \rho^*(\vec{k}_j, \vec{p}_+) \rho(\vec{k}_j, \vec{p}_-) \delta(\vec{k}_j^2 - \mu^2) \theta(\vec{k}_j) \cdot \delta_4(\vec{P} - \vec{p}_f - \sum_{j=1}^n \vec{k}_j).$$

(4.1)

Actually the contribution of the elastic scattering to this overlap integral should also be included, but this can, according to Mischejda ¹⁹⁾ be neglected. The momenta $\vec{p}_+$ and $\vec{p}_-$ represent the

initial c.m. momentum of the nucleon, but in reference systems, which are rotated over $+\theta/2$ and $-\theta/2$ around an axis perpendicular to p_i . The scattering angle θ and the invariant square of the momentum transfer t are related in the usual way. Similarly to the expression for the total cross-section (1.5) we write the overlap function as

$$F(t) \sim \int d p_f \rho_N^*(p_f, p_+) \rho_N(p_f, p_-) \delta(p_f^2 - M^2) \theta(p_f) C(P_i, P_f, \theta), \quad (4.2)$$

with

$$C(P, \theta) = \sum_{n=1}^{\infty} \frac{1}{n!} \int d k_1 \dots d k_n \prod_{j=1}^n \rho^*(k_j, p_+) \rho(k_j, p_-) \delta(k_j^2 - \mu^2) \theta(k_j) \delta(P - \sum_{j=1}^n k_j). \quad (4.3)$$

Writing the δ function for energy and momentum conservation as a Fourier integral and introducing the function

$$\begin{aligned} \psi(x, \theta) &= \int d k \rho^*(k, p_+) \rho(k, p_-) \delta(k^2 - \mu^2) \theta(k) = \\ &= \int d k \{ |\rho(k)|^2 + \Delta(k, \theta) \} \delta(k^2 - \mu^2) \theta(k), \end{aligned} \quad (4.4)$$

we can perform the summation over n in (4.3). Expanding in powers of

$$\Delta(k, \theta) = \rho^*(k, p_+) \rho(k, p_-) - |\rho(k)|^2 \quad (4.5)$$

and neglecting terms of the same order as the elastic overlap terms, one can show that $C(P, \theta)$ of Eq. (4.3) can be written as

$$\begin{aligned} C(P, \theta) &= C(P) + \sum_{m=1}^{\infty} \frac{1}{m!} \int d k_1 \dots d k_m C(P - \sum_{j=1}^m k_j, \theta) \cdot \\ &\cdot \prod_{j=1}^m \Delta(k_j, \theta) \delta(k_j^2 - \mu^2) \theta(k_j). \end{aligned} \quad (4.6)$$

We now neglect the dependence of the integrand on the transverse components of $k_1, \dots, k_m$, except as far as they occur in $\Delta(k_j, \theta)$. This leads us to define a function $\bar{\Delta}(w)$, which for high energies and fixed t depends only on $w = k_z \sin \theta/2$, by

$$\bar{\Delta}(w) = n_0 + \int_{-\infty}^{\infty} \alpha k_x \alpha k_y \Delta(k, \theta). \quad (4.7)$$

Substitution of (4.5) gives, with $u = k_y$ and $v = k_x \cos \theta/2$,

$$\bar{\Delta}(w) = e^{-i\eta w} \int_{-\infty}^{\infty} du dv \rho(u^2 + (v-w)^2) \rho(u^2 + (v+w)^2). \quad (4.8)$$

Instead of k_z or w , we now introduce $x_0 = 2k_0/P_0$ as integration variable, where $P_0 = \sqrt{s'} - \sqrt{s'}$ and $s' = 4p_{f0}^2$. With the inelasticity χ defined as $\chi = 1 - \sqrt{s'/s}$ we then get the relation (valid for high energies)

$$w^2 = -\frac{1}{4} \chi^2 t' x_0^2. \quad (4.9)$$

Since only the real part of $\bar{\Delta}(w)$ contributes to the integral in (4.6) we expand this part in powers of w^2 and obtain

$$\bar{\Delta}(w) = n_0 \sum_{\ell=0}^{\infty} a_{\ell} \left(\frac{w^2}{-t'_0} \right)^{\ell}, \quad (4.10)$$

where the dimensionless expansion parameters a_{ℓ} can be evaluated for any $\rho(k)$. From $\bar{\Delta}(0) = n_0$ one gets $a_0 = 1$. The quantity t'_0 is defined by the requirement $a_1 = 1$. It can be calculated explicitly and its inverse is given by

$$\frac{1}{t'_0} = \frac{1}{2} \eta^2 + \frac{\pi}{n_0} \rho^2(0) - \frac{2\pi}{n_0} \int_0^{\infty} z \rho^2(z) \frac{d^2 \ln \rho(z)}{dz^2} dz, \quad (4.11)$$

where $z = k_{\perp}^2$. With the asymptotic form (2.24) for $C(P - \sum_{j=1}^m k_j)$ the remaining integrations in (4.6) can now be performed. It is not quite straightforward, but the details of this calculation will be reported elsewhere. Equation (4.9) shows that the function

$$G(x) = \frac{C(P, \theta)}{C(P)} \quad (4.12)$$

depends only on the variable $x = -\kappa^2 t = -(1 - \sqrt{s'/s})^2 t$. It was calculated numerically for different values of the meson phase η . We tried to describe these numerical results by a simple formula and found that in the range $x < 4$ (GeV/c)² the function $G(x)$ could be very well approximated by

$$G'(x) = e^{-\frac{1}{12}\beta^2 x} \left(\frac{\sin \frac{1}{2}\eta \sqrt{x}}{\frac{1}{2}\eta \sqrt{x}} \right)^2. \quad (4.13)$$

Also the positions where $G(x)$ and $G'(x)$ vanished in this region differed only by a few per cent. We notice that for a fixed inelasticity $\kappa = 0.4$ and $\eta = 15(\text{GeV/c})^{-1}$ (see later) the overlap function $G'(x)$ has its first zero at $t = -1.1(\text{GeV/c})^2$. This is very close to the break observed in the elastic proton-proton scattering²²⁾.

With the simple form (4.13) for $G(x)$ the p_f integration in (4.2) can now be very much reduced. For $s \rightarrow \infty$ we find [with $\vec{\delta} // \vec{\eta}$ in (2.32)]

$$\rho_N^*(p_f, p_+) \rho_N(p_f, p_-) \sim e^{\frac{\omega}{2M^2} (t' + \frac{1}{4} t \sqrt{\frac{s'}{s}})} \cos \frac{1}{2} \delta \sqrt{-\frac{s'}{s} t}, \quad (4.14)$$

with $t' = (p_i - p_f)^2$. The p_f integration can be written as an integration over s' and t' . Since t' occurs only in (4.14) the t' integration can be performed and $F(t)$ can finally be written as an integral over s' or over $y = \sqrt{s'/s}$. We find

$$F(t) \sim \int_0^1 (1-y)^2 e^{-\frac{\omega}{2}(y+\bar{y})} G'(y, t) \cos \frac{1}{2} \delta y \sqrt{-t} e^{\frac{\omega y t}{8M^2}} dy. \quad (4.15)$$

The square of this function, normalized to one at $t=0$, should be identified with $(d\sigma/dt)/(d\sigma/dt)_{t=0}$. A few years ago Bialas²³⁾ has remarked that the nucleons may give an important contribution to the slope of this diffraction peak at $t=0$. For the present model we derived the following expression for this slope

$$A = \frac{1}{4} (\lambda^2 + \delta^2) \overline{(1-\kappa)^2} + \frac{1}{6} (\beta^2 + \eta^2) \overline{\kappa^2}, \quad (4.16)$$

where β (2.30) and λ (3.10) are the parameters of the transverse momentum distribution of the mesons and the nucleon and where η and δ are the phase parameters. For our distribution of the inelasticity with $\omega = 5$ we find $\kappa^2 = 0.180$ and $(1-\kappa)^2 = 0.376$. For $\delta = 0$ we get $A = 0.9 + 10.3 = 11.2 \text{ (GeV/c)}^{-2}$ if $\eta^2 = 325$ and $A = 0.9 + 14.8 = 15.7 \text{ (GeV/c)}^2$ if $\eta^2 = 475$. These numbers illustrate that the nucleon contribution is small as compared with the meson contribution and also, since $\eta^2 \gg \beta^2$, that the meson phase is all important. For these two cases we have plotted the diffraction peaks in Fig. 4. For comparison we also give the experimental curve at 19.3 GeV/c and the form of the diffraction scattering without averaging over the nucleon momentum and with a fixed inelasticity $\kappa = 0.4$. It is seen that the averaging has a dramatic effect. A detail of the region $-t < 2 \text{ (GeV/c)}^2$ shows, however, that agreement with the experimental points is not very good (Fig. 5). There is not even an indication of a break around $-t \simeq 1.3 \text{ (GeV/c)}^2$. Taking the nucleon phase different from zero produces again a zero point in $d\sigma/dt$. The parameters η and δ can now be chosen in such a way that the slope in $t=0$ is around $A = 9.3 \text{ (GeV/c)}^{-2}$ and the first zero in $d\sigma/dt$ occurs around $t \simeq -1.3 \text{ (GeV/c)}^2$. The diffraction peaks obtained in this way, together with the experimental points²²⁾ at 19.3 GeV/c are shown in Figs. 6 and 7. The two curves correspond with $\eta_1 = 15.0 \text{ (GeV/c)}^{-1}$ and $\delta_1 = 3.6 \text{ (GeV/c)}^{-1}$ and with $\eta_2 = 15.2 \text{ (GeV/c)}^{-1}$

and $\delta_2 = 3.4 \text{ (GeV/c)}^{-1}$. Half of this value of η , that is $\frac{1}{2}\eta = 1.48 \text{ fm}$, should be compared with the coefficient a_1 used in the phase factor by Koba and Namiki²⁴⁾. They obtain $a_1 = 1.96 \text{ fm}$.

The main conclusion we should like to draw from our calculations is that the most important characteristics of high energy proton-proton scattering can be explained with an uncorrelated jet model, combined with scaling and a cut-off for the high transverse momenta of the mesons. For some distributions of secondary particles this was already shown by Sosnowski and Wojcik²⁵⁾. To this we have added - a) the angular distribution of a single meson, - b) the average multiplicity - c) the inelasticity distribution - d) predictions about the momentum distribution of a final nucleon at fixed scattering angle and - e) the form of the diffraction peak for elastic proton-proton scattering. We do not find a shrinkage of this peak with increasing energy, but this could easily be obtained by omitting the factor $\sqrt{\ln s/s_0}$ in the formula (2.32) for the nucleon wave function. Notice, however, that this would imply not only a logarithmic decrease of the elastic, but also of the total inelastic cross-section. The phase of the nucleon wave function has almost no effect on the slope of the diffraction peak at $t=0$, but it can have a drastic influence on the form of $d\sigma/dt$ for larger values of t . Although the momentum dependence of this phase factor is probably more complicated than we assumed, it may nevertheless account for the observed break in the diffraction peak.

Finally it may be remarked that we have shown that within the uncorrelated jet model calculations for very high energies become simple and do not require lengthly Monte-Carlo calculations. It is therefore not a major task to investigate the effects of changes in $\rho_N(p_f, p_i)$ and $\rho(k)$.

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FIGURE CAPTIONS

- Figure 1 : Angular distribution of mesons in c.m. system at various laboratory energies for the incoming proton.
- Figure 2 : Distribution of the inelasticity $K = 1-X$ compared with the experimental results quoted in Ref. 13).
- Figure 3 : Laboratory spectrum of the momentum of the final nucleon at fixed scattering angle $\theta_{\text{lab}} = 10$ mrad and at various laboratory momenta in GeV/c for the incoming proton. The dotted lines give the predictions of Ref. 16) at the same angle.
- Figure 4 : Logarithmic plot of $(d\sigma/dt)/(d\sigma/dt)_{t=0}$ for $\delta = 0$, $\eta_1^2 = 325$ and $\eta_2^2 = 475$ compared with the experimental results at 19.2 GeV/c. No averaging over the final nucleon momenta gives the dotted curve for $\eta^2 = 325$.
- Figure 5 : Detail of Fig. 4 with $\eta_3^2 = 275$ and the experimental data for 21.1 GeV/c indicated by dots.
- Figure 6 : Same as Fig. 4, but with $\eta_1 = 15.0$, $\delta_1 = 3.6$ and $\eta_2 = 15.2$, $\delta_2 = 3.4$. The averaging over the final nucleon momenta has been performed. The experimental curve gives the data at 19.2 GeV/c.
- Figure 7 : Detail of Fig. 6. The experimental data for 19.2 and 21.1 GeV/c have been indicated by dots.

Table 1

$\bar{n}$	10	11	12	13	14	20	30	40	50	100	1000
r	0.752	0.783	0.805	0.820	0.836	0.892	0.920	0.937	0.950	0.970	0.995

Ratio of actual width of $W(n)$ to Poisson width for different average multiplicities.

Table 2

$\bar{n}$	8	9	10	11	12	14	16
d	5%	4%	3%	2%	1%	0.3%	0.1%

Error in phase space integral by using Eq. (2.22).

Table 3

E	500	10^3	10^4	10^5	10^6
Δ	0.89	0.96	1.21	1.48	1.74

Width of angular distribution for different laboratory energies.

FIG.1

FIG.3

FIG. 4

FIG.5

FIG.7