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DUAL AMPLITUDES FOR CURRENTS

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A B S T R A C T

Dual amplitudes for scalar currents which exhibit consistent factorization properties are constructed. This is true both of internal channels and external legs. However, the spectrum necessary for factorization requires an enlargement of the hadron S matrix to include more channels than appear in the original Veneziano theory.

The "photoproduction" amplitudes have fixed poles in the J plane as well as Regge poles so FESR of the usual kind cannot be derived. These amplitudes also have interesting behaviour in the scaling limit.

The photon propagator, however, has a somewhat unsatisfactory structure.

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1. - INTRODUCTION

The construction of dual amplitudes for currents has been considered by a number of authors ^{1),2)}. An important reason for investigating this problem is that the answer may provide a stronger guide than Regge theory to the understanding of electromagnetic processes where the photon can have an arbitrary mass. There is also the hope of providing a solution to current algebra equations. Although this motivation emphasizes vector mesons, all angular momenta are equally important in a dual model. We present in this paper, therefore, an amplitude for "scalar photons" as a first step realizing the above goals.

Most previous attempts to create such amplitudes were based on the assumption that they comprised exchange-degenerate pieces. Although our model of strong interactions exhibits planar duality we abandon this assumption for off-mass-shell amplitudes. This permits us to overcome the factorization difficulties of previous models. However, we find that we need two sets of harmonic oscillators to achieve factorization of the many-photon amplitudes. As we show later this conclusion is very difficult to avoid. It implies in turn that the hadronic spectrum to be associated with off-mass-shell continuations is considerably bigger than originally envisaged. We are able to show that the factorizing states obey gauge conditions of a standard kind ³⁾.

Assuming our model is a satisfactory guide it shows that the off-shell-amplitudes have fixed wrong-signature poles in the J plane so that the application of duality to such processes must be made with care.

2.- THE n "SCALAR PHOTON" AMPLITUDE

The physical idea in our approach was proposed by Nielsen and Susskind ⁴⁾ in the context of a parton interpretation of dual models. They suggested that while hadrons are emitted from the edge of the Kobayashi-Nielsen ⁵⁾ disc photons emerge from its interior, with an amplitude proportional to a Möbius invariant density ⁶⁾. The invariance group comprises those Möbius transformations R , which map the boundary of the disc into itself.

The defect of the approach used by Nielsen and Susskind is that its intuitive character does not permit the consistency of the idea to be tested. Furthermore they were not able to discuss many photon amplitudes. These difficulties are not present in the model presented below, nor is it tied to the parton interpretation of these authors.

The amplitude for n scalar photons of momentum q_i ($i = 1, 2, \dots, n$) with arbitrary q_i^2 is, then

$$A_n = \int dI_n(z) F_n(z, q) \quad (2.1)$$

where the integrand is

$$F_n(z, q) = \prod_{i=1}^n \left| \frac{\lambda_0}{2} (z_i - z_i^*) \right|^{q_i^2} \prod_{1 \leq i < j \leq n} |z_i - z_j^*|^{2q_i \cdot q_j} \quad (2.2)$$

and the measure is

$$\left. \begin{aligned} dI_n(z) &= \left[\prod_{i=1}^n \frac{d^2 z_i}{\rho(z_i)} \right] / dR \\ \rho(z_i) &= \left| \frac{\lambda_0}{2} (z_i - z_i^*) \right|^2 \end{aligned} \right\} \quad (2.3)$$

The invariant density is $(\rho(z))^{-1}$ and z_i^* is the electrostatic image of z_i in the K-N disc. The quantity dR divided out of the integration measure is the volume element of the invariance group of the integrand. Apart from this the d^2z integrations are performed over the complex plane lying inside the K-N disc. This means that the various factors in the integrand cannot vanish or become infinite except on the edge of the K-N disc. In fact it is convenient to add the equal contribution which comes when all K-N variables lie outside the disc.

For convenience of discussion we shall choose the K-N circle to be the real axis. The invariance group of the integrand is that of real Möbius transformations and z^* is just the complex conjugate of z .

We have included a parameter λ_0 in the integrand of (2.1) to indicate which factors produce in the language of Ref. 4), the cut-off of high frequency modes. In our approach these factors are there to guarantee real Möbius invariance and hence duality.

For future reference we note that if under a real Möbius transformation R

$$z \rightarrow Rz = \frac{\alpha z + \beta}{\gamma z + \delta} \quad (2.4)$$

then $z^* \rightarrow Rz^*$ and

$$F_n(z, q) \rightarrow F_n(Rz, q) = \prod_{i=1}^n \left| \frac{\alpha\delta - \beta\gamma}{(\gamma z_i + \delta)^2} \right|^{q \cdot q_i} F_n(z, q) \quad (2.5)$$

where $q = \sum q_i$. When momentum is conserved $q = 0$ and the integrand is invariant.

We see immediately from the form of the integrand and measure Eqs. (2.2) and (2.3) that the amplitude has poles in the external mass variables q_i^2 , coming from the divergence of the $dI_m z_i$

integrations on the real axis. The poles are at $q_i^2 = \alpha_0 - n$ ($n=0,1,2,\dots$) with $\alpha_0 = 1$. Our theory as presently formulated has a tachyon and indeed turns out to have a leading trajectory with intercept $\alpha_0 = 1$. When all external photons are on the tachyon mass shell, the amplitude becomes ($x = \text{Re } z$)

$$A_n \cong \prod_i \left(\frac{\lambda_0^{-1}}{q_i^2 - 1} \right) \int \frac{\pi dx_i}{dR} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2q_i \cdot q_j} \quad (2.6)$$

which can be split up into the $\frac{1}{2}(n-1)!$ different exchange-degenerate contributions to the standard scalar particle amplitude with $\alpha_0 = 1$. Our theory seems, therefore, to have external mass poles of the right kind though the exact structure of the higher poles remains to be investigated (see Section 5). The "cut-off" parameter λ_0^{-1} is the photon-tachyon transition amplitude.

3. - THE INTEGRATION MEASURE

The problem of dividing the integration measure by dR can be solved by choosing three complex integration variables (z_1, z_2, z_3) say, and parametrizing them in the following way ⁷⁾. First we place them on three suitable one-dimensional arcs such as three concentric circles or three vertical lines. Then we boost them into general position by a real Möbius transformation R which contains three more real parameters making up six. Finally we calculate the Jacobian of the transformation. A general theorem tells us and we can check explicitly that we always obtain a result of the form

$$\prod_{i=1}^3 \frac{d^2 z_i}{\rho(z_i)} = dR d\mu(\theta_1, \theta_2, \theta_3) \quad (3.1)$$

or

$$\left[\prod_{i=1}^3 \frac{d^2 z_i}{\rho(z_i)} \right] / dR = d\mu(\theta_1, \theta_2, \theta_3) \quad (3.2)$$

where $d\mu(\theta)$ is a measure over the space of parameters (θ_i) which determine the positions of the z_i on their respective arcs before boosting.

If we choose our basic arcs to be circles of fixed radii (r_1, r_2, r_3) concentric about the origin and $(\theta_1, \theta_2, \theta_3)$ to be the three polar angles then the transformation of variables

$$(z_k) \longrightarrow (R, \theta) \quad (3.3)$$

is given by

$$z_k = \frac{\alpha r_k e^{i\theta_k} + \beta}{\gamma r_k e^{i\theta_k} + \delta} \quad |\alpha\delta - \beta\gamma| = 1 \quad (3.4)$$

When we calculate the Jacobian we find

$$d\mu(\theta) = 2 \prod_{i=1}^3 \frac{d\theta_i}{\sin^2 \theta_i} \left| \cos \theta_1 \cos \theta_2 \left(\frac{r_1}{r_2} - \frac{r_2}{r_1} \right) + \text{cyclic perms.} \right| \quad (3.5)$$

We must also sum over cyclically permuted orders of the radii.

If we choose the arcs to be the three vertical lines

$$z_k = \xi_k + i \eta_k \quad (\eta_k \text{ arbitrary}) \quad (3.6)$$

then we find

$$d\mu(\eta) = \prod_{i=1}^3 \frac{d\eta_i}{\eta_i^2} \left| \xi_1 (\xi_2^2 - \eta_2^2) - \xi_2 (\xi_1^2 - \eta_1^2) + \text{cyclic perms.} \right| \quad (3.7)$$

We could solve this problem in other ways. For example when a number of particles are on-mass-shell, they have real K-N variables. If we choose three of them (x_1, x_2, x_3) then we can write

$$dR = \frac{dx_1 dx_2 dx_3}{|(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)|} \quad (3.8)$$

and then fix (x_1, x_2, x_3) at $(0, 1, \infty)$ as in the standard case.

Another possible method is to choose two complex integration variables. Fix one of them in the complex plane (not on the real axis) place the other on a vertical line (one parameter) then boost into general position and calculate the Jacobian. We find that in order to cover the entire integration region the fixed variable must lie on the arc of the moving one. If we choose this to be

$$z_2 = \xi + i\eta \quad (3.9)$$

with ξ fixed so that $z_1 = \xi + i$ say, then

$$d\mu(\eta) = \left[\frac{d^2 z_1}{\rho(z_1)} \frac{d^2 z_2}{\rho(z_2)} \right] / dR = \frac{d\eta |1 - \eta^2|}{\eta^2} \quad (3.10)$$

4. - FACTORIZATION

Because of the complexity of our integrand we are forced to adopt general methods for exhibiting the factorization properties of A_n . These methods are independently interesting and can be applied also to the original K-N expression for the on-shell amplitude. We hope to develop the discussion of them in another paper.

We consider the amplitude for $n + m$ "photons" with momenta q_i ($i = 1, \dots, n$) and k_j ($j = 1, 2, \dots, m$) in the form

$$A_{n+m} = \int \left[\prod_i \frac{\pi d^2 z_i \rho(z_i) d^2 t_i / \rho(t_i)}{dR} \right] F_n(z, q) F_m(t, k) \prod_i |z_i - t_i^*|^{2q_i \cdot k_i} \quad (4.1)$$

The poles in the variable $q^2 = (\sum q_i)^2$ occur when all the z 's (for example) come together on the boundary of the integration region. We want to show that these poles have factorizing residues.

First we note we can change the integration variables by parametrizing the z 's as

$$z_i = M \bar{z}_i \quad (4.2)$$

where the $(\bar{z}_i)$ are restricted to a basic set with three of them confined to arcs as explained in Section 3 and M is a real Möbius transformation. We have then

$$\prod_i \left[\frac{d^2 z_i}{\rho(z_i)} \right] = dI_n(\bar{z}) dM \quad (4.3)$$

Similarly we can write

$$t_i = N \bar{t}_i$$

$$\prod_i \left[\frac{d^2 t_i}{\rho(t_i)} \right] = dI_m(\bar{t}) dN \quad (4.4)$$

If we make these substitutions in Eq. (4.1) and then replace $\bar{z}$ by z and $\bar{t}$ by t (dummy variables) we get

$$A_{n+m} = \int dI_n(z) dI_m(t) \left(\frac{dM dN}{dR} \right) \times$$

$$\times F_n(Mz, q) F_m(Nt, k) \prod_i |Mz_i - Nt_i^*|^{2q_i \cdot k_i} \quad (4.5)$$

In order to remove dR in a suitable way we make another change of variables $(M, N) \rightarrow (N, P)$ where $P = N^{-1}M$. The invariance of the group volume element yields

$$dM dN = dN dP \quad (4.6)$$

Now we parametrize P in a special way, thus

$$P = (B_2^\dagger)^{-1} \wedge B_1 \quad (4.7)$$

where

$$\left. \begin{aligned} B_1 &= \begin{pmatrix} 1-x_1 & 0 \\ 1 & -x_1 \end{pmatrix} \\ \wedge &= \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \\ B_2^\dagger &= \begin{pmatrix} 1-x_2 & -1 \\ 0 & -x_2 \end{pmatrix} \end{aligned} \right\} \quad (4.8)$$

The invariant measure is

$$dP = dB_2 d\wedge dB_1 \quad (4.9)$$

where

$$\left. \begin{aligned} dB_i &= \frac{dx_i}{ix_i(x_i-1)} \quad (i=1,2) \\ d\wedge &= d\lambda / \lambda^2 \end{aligned} \right\} \quad (4.10)$$

The above choice of parametrization is the least obvious step in the argument. It is essentially determined by requiring $d\wedge$ to have the form in Eq. (4.10) which yields the simplest factorization spectrum. Other choices are possible and their equivalence is due on the one hand to the Möbius invariance of the theory and on the other to the "gauge conditions" which this implies for the physical intermediate states. The above form for P is also particularly suited to proving these conditions since $\{B_1\}$ is the single parameter group that leaves $(0,1)$ invariant and $\{B_2^+\}$ that leaves $(1,\infty)$ invariant.

We introduce the redundant group integration R by the definition

$$N = R B_2^\dagger \quad M = R \wedge B_1 \quad (4.11)$$

so that

$$dN dP = dR dP$$

or

$$\left(\frac{dM dN}{dR} \right) = dP = dB_2 d\wedge dB_1 \quad (4.12)$$

Having removed dR , R is an arbitrary constant transformation which can be chosen to be the identity. The $n + m$ photon amplitude becomes then

$$A_{n+m} = \int_{-\infty}^{\infty} d\lambda |\lambda|^{q^2-2} \int dB_1 dI_n(z) \int dB_2 dI_m(t) \times \\ \times F_n(B_1 z, q) F_m(B_2^\dagger t, k) \prod_i \pi |B_2^\dagger t_i - B_1 z_i|^{2q \cdot k_i} \quad (4.13)$$

where we have used the result [see Eq. (2.5)]

$$F_n(\wedge B_1 z, q) = |\lambda|^q F_n(B_1 z, q) \quad (4.14)$$

The poles arise from divergences when $\lambda \approx 0$ which is also the point at which all the original (z_i) are squeezed down together on the $K-N$ boundary. In order to examine the residues we use Eq. (2.5) to see that

$$F_m(B_2^\dagger t, k) = \prod_i \pi |B_2^\dagger t_i|^{-2k_i \cdot q} F_m\left(\frac{-1}{B_2^\dagger t}, k\right) \quad (4.15)$$

Now

$$-(B_2^\dagger t)^{-1} = B_2 (-t^{-1}) \quad (4.16)$$

where B_2 is got from B_1 by $x_1 \rightarrow x_2$. This is really the definition of the dagger operation. Since $dI_m(-t^{-1}) = dI_m(t)$ we can replace t by $(-t^{-1})$ and obtain finally

$$\begin{aligned} A_{n+m} = & \int_{-\infty}^{\infty} d\lambda |\lambda|^{q^2-2} \int dB_1 dI_n(z) F_n(B_1 z, q) \cdot \\ & \cdot \int dB_2 dI_m(t) F_m(B_2 t, k) \\ & \cdot \prod_{i,j} |1 + \lambda (B_1 z_i)(B_2 t_j^*)|^{2q \cdot k_j} \end{aligned} \quad (4.17)$$

When λ is small this last factor can be expanded as

$$\exp \left\{ - \sum_i \frac{(-\lambda)^n}{n} [q^{(n)}(\xi) \cdot k^{(n)}(\eta^*) + q^{(n)}(\xi^*) \cdot k^{(n)}(\eta)] \right\} \quad (4.18)$$

where

$$\left. \begin{aligned} q^{(n)}(\xi) &= \sum_i q_i \xi_i^n \\ k^{(n)}(\eta) &= \sum_j k_j \eta_j^n \\ \xi_i &= B_1 z_i \quad \eta_j = B_2 t_j \end{aligned} \right\} \quad (4.19)$$

Now introduce two sets of harmonic oscillator operators ⁸⁾

$$\begin{aligned} [a_{\mu}, a_{\nu}^\dagger] &= \delta_{\mu\nu} g_{\mu\nu} \\ [b_{\mu}, b_{\nu}^\dagger] &= \delta_{\mu\nu} g_{\mu\nu} \\ [a, b] &= 0 \end{aligned} \quad (4.20)$$

If $H = \sum_1^{\infty} n(a_n^+ \cdot a_n + b_n^+ \cdot b_n)$ then expression (4.18) becomes

$$\langle 0 | e^{\sum (a_n \cdot q^{(n)}(\xi) + b_n \cdot q^{(n)}(\xi^*)) / \sqrt{n}} (-\lambda)^H e^{-\sum (a_n^+ \cdot k^{(n)}(\eta^*) + b_n^+ \cdot k^{(n)}(\eta)) / \sqrt{n}} | 0 \rangle \quad (4.21)$$

The poles of A_{n+m} are contained in the contribution coming from $\int_{-1}^1 d\lambda$, that is the term

$$A'_{n+m} = \langle q | \frac{1 + (-1)^H}{H + q^2 - 1} | k \rangle \quad (4.22)$$

where

$$\langle q | = \int dI_n(z) dB, F_n(\xi, q) \langle 0 | e^{\sum (a_n \cdot q^{(n)}(\xi) + b_n \cdot q^{(n)}(\xi^*)) / \sqrt{n}} \quad (4.23)$$

The pole residues have the factorizing form

$$\sum_{\lambda} \langle q | \lambda \rangle \langle \lambda | k \rangle$$

where $|\lambda\rangle$ is an occupation number state involving both a^+ and b^+ .

Because two sets of oscillators are required for factorization the situation is in a way similar to that which occurs in the Virasoro-Shapiro model⁹⁾. However, because the invariance group of the integrand in our model is only $SL(2, R)$ against $SL(2, C)$ in the V-S model we do not have the restriction that physical states obey

$$\sum n a_n^+ \cdot a_n = \sum n b_n^+ \cdot b_n \quad (4.24)$$

This means that the states on the leading trajectory are increasingly degenerate unless there are as yet undiscovered gauge conditions satisfied by the many-photon states $\langle q |$. These states do, however, satisfy equations which are obvious generalizations of the standard ones.

If we define the operators

$$L_0 = q^2 + H$$

$$L_+ = q \cdot (a_1^\dagger + b_1^\dagger) + \sum \sqrt{(n+1)n} (a_{n+1}^\dagger \cdot a_n + b_{n+1}^\dagger \cdot b_n) \quad (4.25)$$

then it is easy to check that

$$\begin{aligned} \langle q | (1-\nu)^{L_0 - L_+} \\ = \int dI_n(z) dB_1 F_n(c\xi, q) \langle 0 | e^{\sum (a_n \cdot q^{(n)}(c\xi) + b_n \cdot q^{(n)}(c\xi^*)) / \sqrt{n}} \end{aligned} \quad (4.26)$$

where

$$C\xi = \frac{(1-\nu)\xi}{1-\nu\xi} \quad (4.27)$$

Hence C is an element of the group $\{B_1\}$. The invariance of the group measure gives $dB_1 = d(CB_1)$ and as a result, Eq. (4.26) implies

$$\langle q | (1-\nu)^{L_0 - L_+} = \langle q | \quad (4.28)$$

or in differential form

$$\langle q | (L_0 - L_+) = 0 \quad (4.29)$$

Finally it is interesting to calculate the amplitude $\langle q | 0 \rangle$ for the coupling of n photons to a tachyon. On-mass-shell ($q^2 = +1$) it yields

$$\langle q | 0 \rangle = \int dx_1 dI_n(z) F_n(z, q) \prod_{i=1}^n |x_1 - z_i|^{-2q \cdot q_i} \quad (4.30)$$

where the parameter x_1 of the group element B_1 has become the K-N variable for the tachyon.

5. - EXTERNAL MASS POLES

By arguments similar to those used in the previous Section, it is possible to examine factorization in the channel corresponding to an external leg. Using the same notation as before we write the amplitude for $n + 1$ photons with momenta $(q_1 \dots q_n, p)$ as

$$A_{n+1} = \lambda_0^{p^2-2} \int dI_n(z) F_n(z, q) d^2w |g_{mw}|^{p^2-2} \prod_i |w - z_i|^{2p \cdot q_i} \quad (5.1)$$

we now parametrize w as

$$w = (B_1^{-1} \Lambda^{-1}) \zeta^{-1} = \frac{x_1}{1 + \lambda \zeta (x_1 - 1)} \quad (5.2)$$

where ζ is an arbitrary complex number with $\text{Im } \zeta \neq 0$. The Jacobian is

$$\frac{\partial(\text{Re } w, \text{Im } w)}{\partial(\lambda, x_1)} = \frac{|x_1(x_1 - 1)| \text{Im } \zeta}{|1 + \lambda \zeta (x_1 - 1)|^4} \quad (5.3)$$

Substituting into (5.1) and retaining the notation of the previous Section one finds

$$A_{n+1} = \lambda_0^{p^2-2} |g_m \zeta|^{p^2-1} \int dI_n(z) dB_1 F_n(\xi, q) \cdot \int d\lambda |\lambda|^{p^2-2} \prod_{i=1}^n |(1 - \lambda \zeta \xi_i^*)|^{2p \cdot q_i} \quad (5.4)$$

The part generating the poles can now be isolated in the form

$$A'_{n+1} = \lambda_0^{p^2-2} |g_m \zeta|^{p^2-1} \langle q | \frac{1+(-1)^H}{1+p^2-1} e^{-p \cdot \sum (a_m^\dagger \zeta^{*n} + b_n^\dagger \zeta^n)/\sqrt{n}} | 0 \rangle \quad (5.5)$$

We see that the factorization has exactly the same form as before. The coupling of states in the $2m$ level to the photon is

$$f_{2m} = \lambda_0^{-2m-1} |g_m \zeta|^{-2m} \langle 2m | e^{-p \cdot \sum (a_n^\dagger \zeta^{*n} + b_n^\dagger \zeta^n)/\sqrt{n}} | 0 \rangle \quad (5.6)$$

The most curious feature of this result is that it depends on ζ which is arbitrary. However, the "gauge" invariance of the theory which permits ζ to be arbitrary also restricts the physical states to lie in a subspace of the harmonic oscillator excitations. Presumably then physical states are those whose couplings to the photon are independent of ζ . We can examine this explicitly for the case $m = 1$. There are five possible states in this level. There are two which couple in a ζ independent way, namely, $[c_n(\pm) = (a_n \pm b_n)/\sqrt{2}]$

- (i) $\langle 0 | [2p \cdot c_2(+)] - (p \cdot c_1(+))^2 |$,
- (ii) $\langle 0 | [p \cdot c_1(-)]^2 |$,

one which has zero coupling,

- (iii) $\langle 0 | [p \cdot c_2(-)] - p \cdot c_1(+) p \cdot c_1(-) |$,

and two which couple in a ζ dependent way,

- (iv) $\langle 0 | [p \cdot c_2(+)] + (p \cdot c_1(+))^2 |$,
- (v) $\langle 0 | [p \cdot c_2(-)] + p \cdot c_1(+) p \cdot c_1(-) |$.

States (i) and (ii) have positive norm and satisfy the condition for physical states,

$$\langle i) \text{ or } ii) | L_+ (-p) = 0 \quad (5.7)$$

which is derived exactly in the manner of Del Guidice and Di Vecchia ¹⁰⁾ from Eq. (4.29). States (iv) and (v) are zero norm spurious states and are given by

$$\langle 0 | c_1(\pm) L - (p) \quad (5.8)$$

The state (iii) also satisfies (5.7). However, it contains an odd number of $c(-)$ excitations so we should not expect it to couple since the expression for $\langle q |$ is actually invariant under the interchange

$$a_n \longleftrightarrow b_n$$

that is $c_n(-) \rightarrow -c_n(-)$. States which change sign under this operation are therefore spurious.

6. - SPECTRUM

The discussion at the end of the previous Section shows that the leading trajectory states are generated by even numbers of (spacelike) $c_1(+)$ and $c_1(-)$ excitations. Both occupation numbers are even since the total is even and we have shown that the $\{c_n(-)\}$ must appear in even numbers. The levels on the leading trajectory become increasingly degenerate, therefore, with increasing mass. This is in a way an unsatisfactory result since it implies that the weak interaction causes excitations not present in the original hadron S matrix whose levels are generated by $\{c_n(+)\}$ operators alone. Consistency between internal and external states requires therefore that we enlarge the hadron S matrix to include the interactions of these new levels. The amplitudes describing these interactions could be calculated by multiple factorization of the weak amplitudes.

Since this enlargement of the S matrix was not envisaged at the beginning of the investigation we can ask what happens if we attempt to exclude the $\{c_n(-)\}$ excitations. That the result is unsatisfactory can be seen as follows ¹¹⁾.

The reason for requiring two sets of oscillators is, basically, so that we can write

$$|z_1 - z_2^*|^2 q_1 \cdot q_2 = |z_2|^2 q_1 \cdot q_2 \langle 0 | e^{q_1 \cdot \sum_n (a_n z_1^n + b_n z_1^{*n}) / \sqrt{n}} e^{-q_2 \cdot \sum_n (a_n^\dagger z_2^{*-n} + b_n^\dagger z_2^{-n}) / \sqrt{n}} | 0 \rangle \quad (6.1)$$

That is

$$|z_1 - z_2^*|^2 q_1 \cdot q_2 = |z_2|^2 q_1 \cdot q_2 \langle 0 | e^{q_1 \cdot \sum_n \left[c_n(+)\frac{(z_1^n + z_1^{*n})}{\sqrt{2}} + c_n(-)\frac{(z_1^n - z_1^{*n})}{\sqrt{2}} \right] / \sqrt{n}} \times e^{-q_2 \cdot \sum_n \left[c_n(+)\frac{(z_2^{*-n} + z_2^{-n})}{\sqrt{2}} + c_n(-)\frac{(z_2^{*-n} - z_2^{-n})}{\sqrt{2}} \right] / \sqrt{n}} | 0 \rangle \quad (6.2)$$

If now we eliminate the $\{c_n(-)\}$ operators from the right-hand side of Eq. (6.2) we find that we have replaced $|z_1 - z_2^*|^2 q_1 \cdot q_2$ by

$$|(z_1 - z_2)(z_1 - z_2^*)|^2 q_1 \cdot q_2 = |z_2|^2 q_1 \cdot q_2 \langle 0 | e^{q_1 \cdot \sum_n \left[c_n(+)\frac{z_1^n + z_1^{*n}}{\sqrt{2}} \right] / \sqrt{n}} e^{-q_2 \cdot \sum_n \left[c_n(+)\frac{z_2^{*-n} + z_2^{-n}}{\sqrt{2}} \right] / \sqrt{n}} | 0 \rangle \quad (6.3)$$

It is easy to see that this replacement is unsatisfactory since its use in Eq. (2.2) gives rise to poles in the variables $(q_i \cdot q_j)$ coming from divergences which occur when $z_i = z_j$. We are therefore left with the situation described above in which we accept the enlarged S matrix which contains the original one as a self-consistent sector. The implications, if any, of these results for the physical world remain obscure for the moment.

7. - FORM FACTOR AND PHOTOPRODUCTION

It is easy to check by calculating A_3 with two particles on-mass-shell or by performing a suitable double factorization that the scalar form factor for a tachyon is the same as found in Ref. 4), namely

$$F(q^2) = \lambda_0^{q^2-1} \int_0^{\pi/2} d\theta (\sin \theta)^{q^2-2} \quad (7.1)$$

where q is the photon momentum. That is

$$F(q^2) = \lambda_0^{q^2-1} B(\frac{1}{2}, \frac{q^2-1}{2}) \sim \lambda_0^{q^2-1} / |q| \quad q^2 \rightarrow \infty \quad (7.2)$$

So if we put $\lambda_0 = 1$, $F(q^2)$ dies away like $1/|q|$ and obeys a dispersion relation.

The simplest non-trivial amplitude we can calculate is the photoproduction of a tachyon T , by an off-mass-shell photon γ .

$$\gamma(q) + T(k_1) \rightarrow T(-k_2) + T(-k_3) \quad (7.3)$$

If we put the K-N variables for the tachyon at $(0, 1, \infty)$ then the amplitude is (we put $\lambda_0 = 1$)

$$A = 2 \int_0^\infty dr \, r^{-\alpha_1-1} \int_0^{\pi/2} d\theta (\sin \theta)^{q^2-2} |1 - re^{i\theta}|^{2q \cdot k_2} \quad (7.4)$$

where $\alpha_i = 1 - (q + k_i)^2$. The amplitude in Eq. (7.4) can be split into two pieces :

- (i) the part $\int_0^1 dr$ which has poles in α_1 and α_2 ,
- (ii) the part $\int_1^\infty dr$ which has poles in α_2 and α_3 .

We can show by deforming contours that this represents the division of the amplitude into right and left singularities in the sense of dispersion theory.

It turns out that each piece has a fixed pole at $J = -1$ in the angular momentum plane in addition to a Regge pole at $J = \alpha_2$. Hence finite energy sum rules of the usual kind cannot be obtained ¹²⁾. Of course the fixed pole disappears when we go on-mass-shell.

We shall study the term (i) above. The important asymptotic contribution as $\alpha_1 \rightarrow -\infty$ comes from $r \approx 1$. Furthermore, we can examine the relationship of the fixed and moving poles in good approximation by considering the region of small θ .

That is we put $r = 1 - u$ and require $0 < u < K$ (small) and $0 < \theta < H$ (small). The relevant contribution to A is then

$$A = \int_0^H d\theta \theta^{q^2-2} \int_0^K du (1-u)^{-\alpha_1-1} (u^2 + \theta^2)^{q \cdot k_2} \quad (7.5)$$

To examine the complex J plane we take a Mellin transform of A .

$$A_J = \int_0^\infty d(-\alpha_1) (-\alpha_1)^{-J-1} A \quad (7.6)$$

Dropping an irrelevant factor of $\Gamma(-J)$ and using the smallness of u we find

$$A_J = \int_0^H d\theta \theta^{q^2-2} \int_0^K du u^J (u^2 + \theta^2)^{q \cdot k_2} \quad (7.7)$$

Finally we change to $\phi = \theta/u$ whereupon the integration region divides naturally into two parts, thus

$$A_J = \left\{ \int_0^{H/K} d\phi \int_0^K du + \int_{H/K}^\infty d\phi \int_0^{H/\phi} du \right\} \phi^{q^2-2} (1+\phi^2)^{q \cdot k_2} u^{J-\alpha_2-1} \quad (7.8)$$

In the first term we approximate $(1 + \phi^2)^{q \cdot k_2} \approx 1$ and in the second $\approx \phi^{2q \cdot k_2}$. The u and ϕ integrations can be performed explicitly and the result is

$$A_J \cong \frac{H/K}{(q^2-1)(J-\alpha_2)} + \frac{K/H}{(J-\alpha_2)(J+1)} \quad (7.9)$$

Equation (7.9) shows the essential features of the amplitude namely that it has a Regge pole at $J = \alpha_2$ and a fixed pole at $J = -1$. In the term which contains both they appear multiplied together ¹³⁾. Also the residue of the external mass pole does not contain the fixed J plane pole.

The Compton scattering amplitude exhibits similar structure.

In fact our photoproduction amplitude can be evaluated explicitly to give

$$A = \sqrt{\pi} \frac{\Gamma(\frac{q^2-1}{2}) \Gamma(-\frac{\alpha_1}{2}) \Gamma(-\frac{\alpha_2}{2}) \Gamma(-\frac{\alpha_3}{2})}{\Gamma(-\frac{\alpha_1+\alpha_2}{2}) \Gamma(-\frac{\alpha_2+\alpha_3}{2}) \Gamma(-\frac{\alpha_3+\alpha_1}{2})} \quad (7.10)$$

We can see the external mass poles explicitly in the factor $\Gamma(q^2/2 - 1/2)$. Apart from this A is simply the Virasoro ⁹⁾ amplitude off-mass-shell. It is now possible to examine the scaling limit, α_2 fixed

$$\begin{aligned} \alpha_1 &= -\omega q^2 \\ \alpha_3 &= (1-\omega) q^2 \\ q^2 &\rightarrow \infty \end{aligned} \quad (7.11)$$

We have

$$A \sim \Gamma(-\frac{\alpha_2}{2}) \left(\frac{q^2}{2}\right)^{\frac{\alpha_2-1}{2}} [\omega(\omega-1)]^{\frac{\alpha_2}{2}} \quad (7.12)$$

This result implies that the amplitude scales in an unconventional way with a power dependent on the momentum transfer. We can contrast such behaviour with that predicted by other types of models ¹⁴⁾. It is rather similar to a suggestion of Müller and Rühl ¹⁵⁾.

8. - PROPAGATOR

It is interesting to calculate the simplest two-photon amplitude, namely

$$A_2(q^2) = \int \left[\frac{d^2z \, d^2z'}{\rho(z)\rho(z')} \right] / dR \quad \left| \frac{g_{mz} \, g_{mz'}}{(z-z')^2} \right|^{q^2} \quad (8.1)$$

If we use the method described at the end of Section 3 for removing dR we find

$$\begin{aligned} A_2(q^2) &= \int_0^\infty d\eta \, \frac{|1-\eta^2|}{\eta^2} \quad \left| \frac{\eta}{(1-\eta)^2} \right|^{q^2} \\ &= \frac{2}{q^2-1} \end{aligned} \quad (8.2)$$

Since $A(q^2)$ is supposed to be the Fourier transform of a time-ordered product of two currents it should have poles at all the mass-values of states into which the photon can make a transition. This result is therefore unsatisfactory for a field theoretic interpretation of the amplitudes. It could be simply that A_2 is not correctly given by Eq. (8.1). Alternatively the whole theory may require a deep modification. We leave this question open for the moment.

CONCLUSIONS

We have examined a model for scalar currents which, by construction, exhibits duality. It is based on the idea of Nielsen and Susskind that the off-shell particles are emitted from the interior of the K-N disc. The resulting amplitudes factorize but with a spectrum which requires two sets of harmonic oscillators. This means, in contrast to the Virasoro-Shapiro model, that the states on the leading trajectory have increasing degeneracy. However, the factorization is consistent between external and internal lines. For the theory to be completely consistent the hadron S matrix has to be enlarged, the original one (which factorized with only one set of oscillators) becoming simply a self-consistent sector of the bigger one. The most obvious way of avoiding the necessity for enlargement leads to difficulties with analyticity properties.

The off-mass-shell amplitudes while dual nevertheless contain wrong-signature nonsense fixed poles which disappear on-mass-shell. This means that finite energy sum rules which assume planar duality cannot be derived for off-mass-shell processes. The scaling properties of the amplitude are of some interest.

Finally our calculation of the photon propagator yielded a result unsatisfactory for a field theory interpretation of the model.

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