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ON THE KSFR SUM RULE

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ABSTRACT

Necessary and sufficient conditions for the validity of the KSFR sum rule $C_V = 2F_\pi^2$ are derived from current algebra, unitarity and analyticity. Some consequences for ρ pole dominance of vector current spectral function integrals and for smoothness of proper vertices are discussed.

The so-called KSFR relation $g_\rho^2 = 2F_\pi^2 m_\rho^2$, which relates the ρ vector current coupling constant g_ρ to the pion decay constant F_π , has played a somewhat mysterious role among current algebra predictions. The original derivations¹⁾ from current algebra, PCAC and ρ dominance of the isovector current were subsequently shown to be highly ambiguous²⁾. The prevailing opinion at present is that at least two additional assumptions must be added to current algebra in order to obtain the KSFR relation^{2),3),4)}. A very interesting derivation has recently been given by Schnitzer⁴⁾ within his approach to construct an approximately unitary $\pi\pi$ scattering amplitude which fulfils the current algebra constraints. One of the main features of Schnitzer's result is that it does not depend on vector meson dominance, but rather has the form of a spectral function sum rule. In this letter the equation $C_V = 2F_\pi^2$, where C_V is the vector current Schwinger term, will be called the KSFR sum rule, which in the narrow width approximation for the ρ reduces to the original KSFR relation.

In a recent investigation⁵⁾ of unitarity and analyticity constraints in the hard pion current algebra method, the KSFR sum rule was obtained under certain assumptions about the asymptotic behaviour of the vector current spectral function $\rho_V(m^2)$. It is the main purpose of this letter to present a theorem which gives necessary and sufficient conditions for the validity of the KSFR sum rule and which stresses in particular its close connection to unitarity. For the proof of the theorem we will use results of Ref. 5) and a method of Lehmann, Symanzik and Zimmermann⁶⁾ applied to the proper vertex of the vector current between pion states.

From the assumptions of $SU(2) \times SU(2)$ current algebra, conservation of the vector current, finite Schwinger terms C_V, C_A and the technical assumptions of no $I = 1$ operator Schwinger terms and pion pole dominance of the propagator of the divergence of the axial-vector current, one derives Weinberg's first sum rule⁷⁾ and five Ward identities⁸⁾ for the proper, diagonalized VAA vertex functions $\Gamma_{\mu\nu\lambda}(q,p)$, $\Gamma_{\nu\lambda}(q,p)$, $\Gamma_\lambda(q,p)$ ^{*)}.

If one defines a function $A(k^2)$ by

$$q^\mu p^\nu \Gamma_{\mu\nu\lambda}(q,p) \Big|_{p^2=q^2=m_\pi^2} = \frac{1}{2C_A^2} A(k^2) (p_\lambda + q_\lambda) \quad (1)$$

and assumes for convenience^{**)} that $\text{Im } A(k^2)$ is not indefinitely oscillating for $k^2 \rightarrow \infty$, it can be shown that $A(k^2)$ is of the form

*) The nomenclature is the same as in Ref. 5).

**) This assumption allows one to interchange limits and integrations in dispersion integrals; however, the method of Ref. 5) is also applicable in the case of an oscillating imaginary part.

$$A(k^2) = a k^2 + \frac{k^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im } A(x)}{x(x-k^2)} dx \quad (2)$$

and the pion form factor $F(t)$ is given by

$$F(t) = 1 - \frac{C_V}{2F_\pi^2} + \frac{1}{2F_\pi^2} (A(t) - 1) \Delta_V(t) \quad (3)$$

with

$$\Delta_V(t) = \int \frac{dm^2 \rho_V(m^2)}{t - m^2}, \quad C_V = -\Delta_V(0)$$

Furthermore, the proper scalar $V\pi\pi$ vertex $\Gamma(k^2)$ is defined by

$$\Gamma_\lambda(q, p) \big|_{p^2=q^2=m_\pi^2} = \Gamma(k^2) (p_\lambda + q_\lambda) \quad (4)$$

Since $F(t) = \Gamma(t) \Delta_V(t)$, we have

$$\Gamma(t) = \left(1 - \frac{C_V}{2F_\pi^2}\right) \Delta_V^{-1}(t) + \frac{1}{2F_\pi^2} (A(t) - 1) \quad (5)$$

Theorem

1)

$$\lim_{t \rightarrow \infty} t \int \frac{\text{Im } A(x) dx}{x(t-x)} \cdot \Delta_V(t)$$

exists and is a real number b ; b can be non-zero only if both $\int \frac{\text{Im } A(x)}{x} dx$ and $\int \rho_V(m^2) dm^2$ diverge.

2) If $b = 0$ and $\int \rho_V(m^2) m^2 dm^2 = \infty$, then

$$C_V = 2F_\pi^2$$

3) If $\frac{1}{\pi} \int \frac{\text{Im } A(x) dx}{x} = -1$, then

$$C_V = 2F_\pi^2$$

4) If $\lim_{t \rightarrow \infty} \frac{A(t)}{t} = 0$ (i.e., $a = 0$) and $b = 0$, then

$$C_V = 2F_\pi^2$$

5) If $C_V = 2F_\pi^2$, then

a) $\lim_{t \rightarrow \infty} \frac{A(t)}{t} = 0, b = 0$

b) $\frac{1}{\pi} \int \frac{\text{Im } A(x) dx}{x} = -1$

c) $\Gamma(t) = \frac{1}{2F_\pi^2} (A(t) - 1)$

Proof

In order to prove the theorem we need some information on the asymptotic behaviour of $\Gamma(t)$, which is provided by a method of Lehmann, Symanzik and Zimmermann.

The vector current spectral function $\rho_V(t)$ may be written

$$\begin{aligned} \rho_V(t) &= \frac{1}{48\pi^2 \sqrt{t}} (t - 4m_\pi^2)^{3/2} |F(t)|^2 \Theta(t - 4m_\pi^2) + \hat{\rho}_V(t) = \\ &= |\Delta_V(t)|^2 \left\{ \frac{1}{48\pi^2 \sqrt{t}} (t - 4m_\pi^2)^{3/2} |\Gamma(t)|^2 \Theta(t - 4m_\pi^2) + \right. \\ &\quad \left. + \tilde{\rho}_V(t) |\Delta_V^{-1}(t)|^2 \right\} \equiv |\Delta_V(t)|^2 G(t) \end{aligned} \quad (6)$$

where $\tilde{\rho}_V(t) \geq 0$. Thus

$$\rho_V(t) = G(t) \left\{ \pi^2 \rho_V^2(t) + \left[\mathcal{P} \int \frac{dm^2 \rho_V(m^2)}{t - m^2} \right]^2 \right\}$$

or

$$G(t) = \frac{1}{\pi} \text{Im} \frac{1}{\int \frac{dm^2 \rho_V(m^2)}{t - m^2 + i\varepsilon}} \quad (7)$$

For any complex numbers x, y, z not on the cut $4m_\pi^2 \leq t < \infty$, we therefore have

$$\int_{4m_\pi^2}^{\infty} \frac{G(t) dt}{(t-x)(t-y)(t-z)} = \frac{1}{2\pi i} \int_C \frac{dt}{(t-x)(t-y)(t-z) \int \frac{dm^2 \rho_V(m^2)}{t-m^2}} \quad (8)$$

where the contour C encircles the cut in a clockwise manner (in the Figure).

Closing the contour C by a circle at infinity and applying Cauchy's theorem, one finds

$$\int_{4m_\pi^2}^{\infty} \frac{G(t) dt}{(t-x)(t-y)(t-z)} = \frac{1}{(z-x)(z-y) \int \frac{dm^2 \rho_V(m^2)}{z-m^2}} + \\ + \text{cyclic perm. } (x, y, z) \quad (9)$$

Consider the special case $x \rightarrow 0, y \rightarrow 0, z \rightarrow -\infty$; from Eq. (9)

$$\int_{4m_\pi^2}^{\infty} \frac{G(t)}{t^2} dt = - C_V^{-2} \int \frac{dm^2 \rho_V(m^2)}{m^4} \quad \text{for } \int dm^2 \rho_V(m^2) = \infty \\ \left[C_V^2 \int dm^2 \rho_V(m^2) \right]^{-1} \left[\int \frac{dm^2 \rho_V(m^2)}{m^4} \cdot \int dm^2 \rho_V(m^2) - C_V^2 \right] \\ \text{for } \int dm^2 \rho_V(m^2) < \infty \quad (10)$$

Thus $G(t)/t \xrightarrow[t \rightarrow \infty]{} 0$ and since

$$G(t) \geq \frac{1}{48\pi^2 \sqrt{t}} (t - 4m_\pi^2)^{3/2} |\Gamma(t)|^2 \quad (11)$$

we finally obtain

$$\lim_{t \rightarrow \infty} \Gamma(t) = 0 \quad (12)$$

With the help of Eqs. (2), (3), (5), (12) and the asymptotic constraints on $F(t)$ ⁵⁾, the proof of the theorem is now straightforward.

One interesting result is the equivalence of the KSFR sum rule to a sum rule for the absorptive part of $q^\mu p^\nu \Gamma_{\mu\nu\lambda}(q, p)$, which in this case also equals the absorptive part of $\Gamma_\lambda(q, p)$ (both for $p^2 = q^2 = m_\pi^2$). This sheds some light on the applicability of the usual smoothness assumptions by which the absorptive

parts of the proper vertices are neglected in some domain. In particular, because of parts 2) and 5b) of the theorem, the absorptive part can certainly not be neglected completely, if the first moment of the vector current spectral function diverges ^{*}).

As there are good reasons to believe that $\text{Im } A(k^2)$ is negligible in the region of elastic unitarity ^{4),5),9)}, the absorptive part must be sufficiently large in the non-elastic region if the KSFR sum rule is to hold. If this is not the case, i.e., if $|\text{Im } A(k^2)|$ is too small, one finds as immediate consequences of our theorem and Eqs. (3.18) and (3.19) of Ref. 5) $\int \rho_V(m^2) m^2 dm^2 < \infty$ and $C_V > 2F_\pi^2$. Indeed, the old Orsay data ¹⁰⁾ on g_ρ indicated $C_V > 2F_\pi^2$. However, with the new Orsay measurements ¹¹⁾ the situation is completely open again.

Some final remarks concern the consequences of the Lehmann-Symanzik-Zimmermann theorem. This method provides rigorous upper bounds for the integrals

$$\int_{4m_\pi^2}^{\infty} \frac{(t - 4m_\pi^2)^{3/2}}{\sqrt{t}} \frac{|\Gamma(t)|^2}{t^n} dt, \quad n \geq 2$$

where these bounds are given by certain negative moments of $\rho_V(m^2)$. To obtain the bounds, one simply has to take into account Eqs. (10), (11) and

$$\int_{4m_\pi^2}^{\infty} \frac{G(t)}{t^n} dt = \frac{1}{2\pi i} \int_C \frac{dt}{t^n \left(\frac{dm^2 \rho_V(m^2)}{t - m^2} \right)} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dt^{n-1}} \left. \frac{1}{\left(\frac{dm^2 \rho_V(m^2)}{t - m^2} \right)} \right|_{t=0} \quad (13)$$

for $n \geq 3$

The interesting thing about these bounds is that they are zero for simple ρ pole dominance, i.e., for $\rho_V(m^2) = g_\rho^2 \delta(m^2 - m_\rho^2)$. In other words, any non-trivial proper vertex function $\Gamma(t)$ automatically implies that the spectral function integrals $\int \rho_V(m^2) dm^2 / m^{2n}$ differ from their ρ dominance values. The proof of this statement follows directly from Eqs. (10) and (13) for $\rho_V(m^2) = g_\rho^2 \delta(m^2 - m_\rho^2)$.

Finally, it is clear that Eq. (10) not only implies the asymptotic vanishing of the proper $V\pi\pi$ vertex $\Gamma(t)$, but also gives strong asymptotic constraints on the proper vertices of the form factors $\langle \pi | V_\mu | A_1 \rangle$ and $\langle A_1 | V_\mu | A_1 \rangle$. These constraints may be useful for further investigations ⁵⁾ of the Ward identities for the VAA vertices.

^{*}) In Ref. 5) a polynomial model for the proper vertex $\Gamma(t)$ was examined also for $\int \rho_V(m^2) dm^2 = \infty$; however, in this case such a model is incompatible with the Lehmann-Symanzik-Zimmermann theorem [Eq. (12)].

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REFERENCES

- 1) K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters 16, 255 (1966);
Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966).
- 2) D. Geffen, Phys. Rev. Letters 19, 770 (1967);
S.G. Brown and G.B. West, Phys. Rev. Letters 19, 812 (1967).
- 3) F. Gilman and H.J. Schnitzer, Phys. Rev. 150, 1362 (1966);
J.J. Sakurai, Phys. Rev. Letters 17, 552 (1966);
M. Ademollo, Nuovo Cimento 46, 156 (1966);
L.S. Brown and R.L. Goble, Phys. Rev. Letters 20, 346 (1968).
- 4) H.J. Schnitzer, Phys. Rev. Letters 24, 1384 (1970) and Phys. Rev. D2, 1621 (1970).
- 5) G. Ecker, CERN Preprint TH.1405 (1971), submitted to Nuclear Physics B.
- 6) H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 2, 425 (1955).
- 7) S. Weinberg, Phys. Rev. Letters 18, 507 (1967).
- 8) H.J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967).
- 9) P. Ford and H.J. Schnitzer, Brandeis University Preprint (April 1971).
- 10) J.E. Augustin et al., Phys. Letters 28B, 503 (1969).
- 11) J. Le François, Results presented at the International Symposium on Electron and Photon Interactions at High Energies, Ithaca, N.Y. (August 1971).

FIGURE CAPTION

Figure : Integration contour in the complex t plane for the integral in Eq. (8).