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PHENOMENOLOGICAL ACTIONS FOR
SPONTANEOUSLY BROKEN CONFORMAL SYMMETRY

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A B S T R A C T

The general form of invariant actions for spontaneously broken conformal symmetry is discussed using equations due to Callan, Coleman and Jackiw. Previously known rules for constructing conformally covariant derivatives are simply obtained, but not all conformal invariant actions can be obtained using fields and their covariant derivatives. A method due to Wess and Zumino is shown to generate all invariant actions not more than second order in the derivatives of spin 0, $\frac{1}{2}$ and 1 fields. Chiral and conformal invariant actions are discussed.

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1. - INTRODUCTION

There has recently been considerable interest ¹⁾⁻⁷⁾ in the possibility that scale and conformal invariance may be usefully regarded as approximate symmetries of physics. These symmetries may either be realized in the conventional way, in which all dimensional coupling constants including masses become zero, or in the spontaneously broken Goldstone way ^{2),3)}. Applications ⁴⁾ have been made of the idea that the conventional symmetry limit may be useful at high energies ⁵⁾, and the approximate Goldstone symmetry limit has been applied to deduce low energy theorems ^{6),7)}. Most work on the approximate Goldstone symmetry limit has used phenomenological actions, and rules for constructing conformal invariant actions have been given by Salam and co-authors ²⁾ and by Wess and Zumino ³⁾.

In this paper we investigate whether the rules given by these authors in fact generate all invariant actions for spontaneously broken conformal symmetry. Since phenomenological actions are used as a short-hand method of obtaining results from Ward identities and smoothness assumptions, it is important that there should be available the most general action consistent with the group theoretical and kinematical postulates.

We use necessary and sufficient conditions for scale and conformal invariance obtained by Callan, Coleman and Jackiw ⁸⁾. Although originally obtained with the conventional type of scale and conformal invariant theory in mind, their results are also directly applicable to the Goldstone theory. This is because the field of the Goldstone boson may be taken as a field of canonical dimensionality, albeit with a non-zero vacuum expectation value. The prescription of Salam et al. ²⁾ for constructing conformal invariant actions out of fields and their conformally covariant derivatives is easily re-obtained. However, it can also be shown that the Salam prescription is incomplete, as there are many conformal invariant actions not expressible in terms of fields and their covariant derivatives. An example of such a theory is the free massless scalar field, which is well known to be conformally invariant ^{8),9)}.

We also investigate the method for constructing conformal invariant actions due to Wess and Zumino ³⁾, which involves taking the restriction to flat space of an Einstein and Weyl invariant action. Their method is shown to generate all conformal invariant actions which contain spin 0, $\frac{1}{2}$ and 1 fields, and are at most second order in field derivatives. These are the actions of most phenomenological interest.

It is also shown that a chiral and conformal invariant action for pseudoscalar mesons and the Goldstone boson of conformal symmetry constructed recently ⁶⁾ using the prescription of Salam et al. is essentially unique. This is because the action is determined by the requirements of chiral and scale invariance alone.

2. - CONFORMAL INVARIANT ACTIONS

The conformal group in space-time is the largest group of transformations which leave angles invariant. In addition to the Poincaré transformations it contains :

$$\text{Scale transformations : } x_\mu \rightarrow \lambda x_\mu$$

Special conformal transformations :

$$x_\mu \rightarrow \frac{x_\mu + a_\mu x^2}{1 + 2a \cdot x + a^2 x^2}$$

In Lagrangian field theory one requires the transformation properties of fields ϕ_i which conventionally take the infinitesimal forms :

$$\text{Scale transformations : } \delta \phi_i = (\delta_{ij} x^\mu \partial_\mu + D_{ij}) \phi_j \quad (2.1)$$

Special conformal transformations :

$$\begin{aligned} \delta_\mu \phi_i = [& \delta_{ij} (2x_\mu x_\nu - \eta_{\mu\nu} x^2) \partial^\nu + 2x_\mu D_{ij} + \\ & + 2x^\nu (\Sigma_{\mu\nu})_{ij}] \phi_j \end{aligned} \quad (2.2)$$

where D_{ij} is the dimension matrix.

The action is invariant under scale transformations if the change in the Lagrangian induced by (2.1) is a total divergence. This means

$$\begin{aligned}\delta \mathcal{L} &= 4\mathcal{L} + x^\mu \partial_\mu \mathcal{L} \\ &= \partial^\mu (x_\mu \mathcal{L})\end{aligned}\quad (2.3)$$

This condition is obeyed if all dimensional coupling constants are zero including masses, and if we choose $D_{ij} = \delta_{ij}$ for boson fields, $= \frac{3}{2} \delta_{ij}$ for fermion fields. Callan, Coleman and Jackiw⁸⁾ have further established a necessary and sufficient condition that there be a conserved current S^μ whose associated charge generates scale transformations and which is expressible in the form

$$S^\mu = x_\lambda \Theta^{\mu\lambda} \quad (2.4)$$

where $\Theta^{\mu\lambda}$ is a possible energy-momentum tensor. The condition is that

$$\sum_{i,j} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} [\eta_{\nu\mu} D_{ij} + (\sum_{\nu\mu})_{ij}] \phi_j = \partial^\mu \sigma_{\nu\mu} \quad (2.5)$$

where $\sigma_{\nu\mu}$ is some tensor. The energy momentum tensor $\Theta^{\mu\lambda}$ obeying (2.4) is then given by

$$\Theta^{\mu\lambda} = T^{\mu\lambda} + \frac{1}{2} \partial_\nu \partial_\rho X^{\nu\rho\mu\lambda} \quad (2.6)$$

where $T^{\mu\lambda}$ is the symmetric Belinfante tensor, and

$$\begin{aligned}X^{\nu\rho\mu\lambda} &= \eta^{\nu\rho} \sigma_+^{\mu\lambda} - \eta^{\nu\mu} \sigma_+^{\rho\lambda} - \eta^{\nu\lambda} \sigma_+^{\mu\rho} + \eta^{\mu\lambda} \sigma_+^{\nu\rho} \\ &\quad - \frac{1}{3} \eta^{\nu\rho} \eta^{\mu\lambda} \sigma_{+\alpha}^\alpha + \frac{1}{3} \eta^{\nu\mu} \eta^{\rho\lambda} \sigma_{+\alpha}^\alpha\end{aligned}\quad (2.7)$$

where $\sigma_+^{\nu\mu} = \frac{1}{2}(\sigma^{\nu\mu} + \sigma^{\mu\nu})$.

Callan, Coleman and Jackiw ⁸⁾ have also shown that condition (2.5) is necessary and sufficient that there exists a current $C^{\mu\nu}$ obeying the conservation equation $\partial_\nu C^{\mu\nu} = 0$ which is expressible in the form

$$C^{\mu\nu} = (2x^\mu x^\nu - \eta^{\mu\nu} x^2) \Theta_\lambda \quad (2.8)$$

whose associated charge generates special conformal transformations. Thus conditions (2.3) and (2.5) are necessary and sufficient that a theory be conformal invariant.

In studying spontaneously broken conformal symmetry one introduces a scalar isoscalar Goldstone field with an anomalous transformation law. Salam et al. ²⁾ and Wess and Zumino ³⁾ both introduce a field σ with the infinitesimal transformation properties :

$$\text{Scale transformations : } \delta\sigma = \frac{1}{b} + x^\mu \partial_\mu \sigma \quad (2.9)$$

Special conformal transformations :

$$\delta_\mu \sigma = (2x_\mu x_\nu - \eta_{\mu\nu} x^2) \partial^\nu \sigma + \frac{2}{b} x_\mu \quad (2.10)$$

It is then immediate that the quantity $\frac{1}{b} e^{b\sigma}$ is a field which has the scale and conformal transformation properties of a scalar field of canonical dimension 1, and $\langle 0 | \frac{1}{b} e^{b\sigma} | \sigma \rangle = 1$. $\frac{1}{b} e^{b\sigma}$ also has the vacuum expectation value $\frac{1}{b}$. We conclude that a method of introducing a Goldstone field for spontaneously broken conformal symmetry, which is equivalent to that of previous authors, is to introduce a field χ with canonical scale and conformal transformation properties. χ will have a non-zero vacuum expectation value, V say, and $(\chi - V)$ will be a valid interpolating field for the Goldstone boson.

But now we can directly apply the conditions of Callan, Coleman and Jackiw to deduce the most general form of conformal invariant action for the theory of spontaneously broken symmetry. One must construct

the action such that its scale dimension is 4, and the conformal condition (2.5) holds, where the sum over particle fields is now understood to include χ . It is clear that in such a spontaneously broken theory dimensional coupling constants can be tolerated, as well as non-zero masses for particles other than the Goldstone boson.

As an application of the conditions (2.3) and (2.5) we show how one can easily rederive the prescription of Salam et al.²⁾ for constructing conformal invariant actions. It will also be clear how the Salam et al. prescription is incomplete. Let us consider as a special case of Eq. (2.5) the condition

$$\sum_{i,j} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} [\eta_{\nu\mu} D_{ij} + (\Sigma_{\nu\mu})_{ij}] \phi_j = 0 \quad (2.11)$$

In order to include the derivatives of non-Goldstone fields in the action one considers the possibility of constructing covariant derivatives^{*)}

$$D_\mu \phi_i = \partial_\mu \phi_i + (U_{\nu\mu})_{ij} \frac{\partial^\nu \chi}{\chi} \phi_j \quad (2.12)$$

where $(U_{\nu\mu})_{ij}$ is some tensor. Such an object will contribute to the left-hand side of (2.11) a term

$$\frac{\partial \mathcal{L}}{\partial (D_\mu \phi_i)} [(U_{\nu\mu})_{ij} + \eta_{\nu\mu} D_{ij} + (\Sigma_{\nu\mu})_{ij}] \phi_j$$

and it is clear that if we choose $(U_{\nu\mu})_{ij} = -\eta_{\nu\mu} D_{ij} - (\Sigma_{\nu\mu})_{ij}$ then we have a covariant derivative

$$D_\mu \phi_i = \partial_\mu \phi_i - (\eta_{\nu\mu} D_{ij} + (\Sigma_{\nu\mu})_{ij}) \frac{\partial^\nu \chi}{\chi} \phi_j \quad (2.13)$$

*) We will denote conformally covariant derivatives by $D_\mu \phi_i$
generally covariant derivatives by $\Delta_\mu \phi_i$
and chirally covariant derivatives by $\mathcal{D}_\mu \phi_i$.

Any action constructed out of the fields χ , ϕ_i and the covariant derivatives $D_\mu \phi_i$ which is Lorentz invariant and scale invariant will also be conformal invariant. Equation (2.13) is just the conformally covariant derivative of Salam et al. ²⁾, and we have just obtained their prescription for constructing conformal invariant actions ^{*}).

That there are other conformal invariant actions can be seen by comparing (2.5) and (2.11), and we give below explicit examples of actions not obtainable by the Salam et al. method. Consider a scalar field ϕ of canonical dimension 1, and consider the general form of Lagrangian of dimension 4 for this field in interaction with the Goldstone field, which is second order in the field derivatives :

$$\begin{aligned} \mathcal{L} = \sum_{m=0}^{\infty} \left[\alpha_m \partial_\mu \phi \partial^\mu \phi \phi^m \chi^{-m} + \beta_m \partial_\mu \phi \partial^\mu \chi \phi^{m+1} \chi^{-m-1} + \right. \\ \left. + \gamma_m \partial_\mu \chi \partial^\mu \chi \phi^{m+2} \chi^{-m-2} \right] \\ + \gamma_{-1} \partial_\mu \chi \partial^\mu \chi \phi \chi^{-1} + \gamma_{-2} \partial_\mu \chi \partial^\mu \chi \end{aligned} \quad (2.14)$$

where $\alpha_0 = \gamma_{-2} = \frac{1}{2}$. For this Lagrangian we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} \chi = \sum_{m=0}^{\infty} \left[(2\alpha_m + \beta_m) \partial^\mu \phi \phi^{m+1} \chi^{-m} + (2\gamma_m + \beta_m) \partial^\mu \chi \phi^{m+2} \chi^{-m-1} \right] \\ + 2\gamma_{-1} \partial^\mu \chi \phi + 2\gamma_{-2} \chi \partial^\mu \chi \end{aligned} \quad (2.15)$$

Equation (2.15) is of the required form $\partial_\nu \sigma^{\mu\nu}$ if and only if

$$\gamma_{-1} = 0, \quad -m(2\alpha_m + \beta_m) = (m+2)(2\gamma_m + \beta_m), \quad m \geq 0 \quad (2.16)$$

^{*}) Unless stated otherwise, we will always use fields of canonical dimensionality. This entails no loss of generality, as any particle interpolating field is related to a field of canonical dimension by the appropriate power of χ .

Equation (2.16) is the necessary and sufficient condition that a scale invariant theory involving ϕ and χ be conformal invariant. If we take the special solution

$$(2\alpha_m + \beta_m) = 0 = (2\gamma_m + \beta_m) \quad (2.17)$$

then the Lagrangians constructed out of covariant derivatives are recovered

$$\mathcal{L} = \sum_{m=0}^{\infty} \alpha_m \left(\partial_m \phi - \frac{\partial_m \chi}{\chi} \phi \right)^2 \phi^m \chi^{-m} : \alpha_0 = \frac{1}{2} \quad (2.18)$$

Amongst the other Lagrangians which are seen by Eq. (2.16) to be conformal is the free scalar field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \quad (2.19)$$

which has long been known ^{8),9)} to be a conformal invariant theory, but which could not have been obtained by the prescription of Salam et al.

It is amusing to observe that for two reasons covariant derivatives are of limited use in the study of conformal invariance. One is that there is no covariant derivative for the Goldstone field, and the other is that as shown above not all conformal invariant actions can be expressed in terms of covariant derivatives of other particle fields. This situation contrasts with the theory of spontaneously broken chiral symmetry, where covariant derivatives of pseudoscalar meson fields exist, and as far as is known all chiral invariant actions may be expressed in terms of chirally covariant derivatives.

3. - THE METHOD OF WESS AND ZUMINO

In this Section we show that the method proposed by Wess and Zumino ³⁾ for constructing conformal invariant actions can be used to obtain all such actions which are at most second order in field derivatives. The approach these authors propose to make an action conformal invariant is as follows :

- (a) make the action Einstein invariant, i.e., invariant under general co-ordinate transformations by introducing the gravitational and vierbein fields ;
- (b) subsequently make the action Weyl invariant by inserting χ fields where necessary.

The restriction to flat space of the Einstein and Weyl invariant action is conformally invariant. Wess and Zumino have found that if the gravitational field is introduced minimally the prescription of Salam et al. ²⁾ is reproduced. However, if the gravitational field is introduced non-minimally a wider class of actions can be obtained. For example one can extend the flat space action

$$A = \frac{1}{2} \int \partial_\mu \phi \partial_\nu \phi \eta^{\mu\nu} d^4x \quad (3.1)$$

to the general covariant action

$$\tilde{A} = \frac{1}{2} \int [\partial_\mu \phi \partial_\nu \phi g^{\mu\nu} + \frac{1}{6} R \phi^2] \sqrt{g} d^4x \quad (3.2)$$

where $g = -\det(g_{\mu\nu})$ and R is the double contraction of the curvature tensor. It is easy to check using the Weyl transformation properties tabulated in the Appendix that (3.2) is Weyl invariant. Also (3.2) reduces to (3.1) in flat space, so that one can conclude that the free scalar field action is conformally invariant.

We now show that the Wess-Zumino method generates all conformal invariant actions which are at most second order in the first derivatives of fields of spins 0, $\frac{1}{2}$ and 1. In making flat space actions Einstein invariant one must include possible non-minimal gravitational interaction terms of the form

$$\int R_{\mu\nu} \tilde{t}^{\mu\nu} \sqrt{g} d^4x$$

where $\tilde{t}_{\mu\nu}$ is some tensor, and $R_{\mu\nu}$ is the symmetric contraction of the curvature tensor.

Consider a general conformally invariant action $A = \int \mathcal{L} d^4x$ where

$$\sum_{i,j} \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi_i)} [\eta_{\mu\nu} D_{ij} + (\Sigma_{\mu\nu})_{ij}] \phi_j = \partial^\nu \sigma_{\mu\nu} \quad (3.3)$$

for some tensor $\sigma_{\mu\nu}$. If $\mathcal{L}$ is at most second order in field derivatives, then $\sigma_{\mu\nu}$ can be a function of the particle fields only. One can define the following general covariant extension of A :

$$\tilde{A} = \int (\tilde{\mathcal{L}} + R_{\mu\nu} \tilde{t}^{\mu\nu}) \sqrt{g} d^4x \quad (3.4)$$

where $\tilde{\mathcal{L}}$ is obtained from $\mathcal{L}$ by the minimal substitution, and $\tilde{t}_{\mu\nu}$ is obtained by the minimal substitution from

$$\tau^{\mu\nu} = \frac{1}{2} \left[\sigma_+^{\mu\nu} - \frac{1}{6} \eta^{\mu\nu} \text{Tr}(\sigma_+) \right] \quad (3.5)$$

where

$$\sigma_+^{\mu\nu} = \frac{1}{2} (\sigma^{\mu\nu} + \sigma^{\nu\mu})$$

We now calculate the Weyl variation of $\tilde{A}$ [Eq. (3.4)] using the results collated in the Appendix. We have

$$\delta_w \tilde{\mathcal{L}} = -4\Lambda \tilde{\mathcal{L}} - \sum_{i,j} \frac{\partial \tilde{\mathcal{L}}}{\partial (\Delta_\nu \phi_i)} \left[g_{\mu\nu} D_{ij} \phi_j + (\Sigma_{\mu\nu})_{ij} \phi_j \right] \partial^\mu \Lambda \quad (3.6)$$

using (A.2). We also have [Eq. (A.1)]

$$\delta_w \sqrt{g} = 4\Lambda \sqrt{g}$$

Because Λ is conformal invariant, $\mathcal{L}$ must have dimension 4. Hence $\sigma^{\mu\nu}$ and $\tau^{\mu\nu}$ must have dimension 2. Using then the fact that $\sigma^{\mu\nu}$ and hence $\tau^{\mu\nu}$ do not contain field derivatives, we can apply Eq. (A.4) to deduce

$$\delta_w \tilde{\tau}^{\mu\nu} = -4\Lambda \tilde{\tau}^{\mu\nu} \quad (3.7)$$

Using also Eq. (A.3), namely

$$\delta_w R_{\mu\nu} = -2\Delta_\mu \partial_\nu \Lambda - g_{\mu\nu} \Box \Lambda$$

we find that

$$\begin{aligned} \delta_w (R_{\mu\nu} \tilde{\tau}^{\mu\nu}) &= -4\Lambda (R_{\mu\nu} \tilde{\tau}^{\mu\nu}) - 2\tilde{\tau}^{\mu\nu} \Delta_\mu \partial_\nu \Lambda \\ &\quad - \text{Tr}(\tilde{\tau}) \Box \Lambda \end{aligned} \quad (3.8)$$

Combining Eqs. (3.6), (3.7), (3.8) and (A.1), we find

$$\begin{aligned} \delta_w \tilde{\mathcal{A}} &= \int \sqrt{g} \left[- \sum_{i,j} \frac{\partial \tilde{\mathcal{L}}}{\partial (\Delta_\nu \phi_i)} (g_{\mu\nu} D_{ij} \phi_j + (\Sigma_{\mu\nu})_{ij} \phi_j) \partial^\mu \Lambda - \right. \\ &\quad \left. - 2\tilde{\tau}^{\mu\nu} \Delta_\mu \partial_\nu \Lambda - \text{Tr}(\tilde{\tau}) \Box \Lambda \right] d^4x \end{aligned} \quad (3.9)$$

Making some partial integrations of terms involving second derivatives of Λ , (3.9) can be re-expressed as

$$\begin{aligned} \delta_w \tilde{\mathcal{A}} &= \int \sqrt{g} \left[- \sum_{i,j} \frac{\partial \tilde{\mathcal{L}}}{\partial (\Delta_\nu \phi_i)} (g_{\mu\nu} D_{ij} \phi_j + (\Sigma_{\mu\nu})_{ij} \phi_j) \partial^\mu \Lambda + \right. \\ &\quad \left. + 2\Delta_\nu (\tilde{\tau}^{\nu\mu} + \frac{1}{2} g^{\nu\mu} \text{Tr}(\tilde{\tau})) \partial_\mu \Lambda \right] d^4x \end{aligned} \quad (3.10)$$

But because $\tilde{\mathcal{L}}$ and $\tilde{\sigma}_{\mu\nu}$ are obtained from $\mathcal{L}$ and $\sigma_{\mu\nu}$ by the minimal substitution of the gravitational field, Eq. (3.3) implies

$$\sum_{i,j} \frac{\partial \tilde{\mathcal{L}}}{\partial (\Delta_r \phi_i)} (g_{\mu\nu} D_{ij} \phi_j + (\Sigma_{\mu\nu})_{ij} \phi_j) = \Delta^\nu \tilde{\sigma}_{\mu\nu} \quad (3.11)$$

Also

$$\tilde{\tau}^{\mu\nu} = \frac{1}{2} [\tilde{\sigma}_+^{\mu\nu} - \frac{1}{6} g^{\mu\nu} \text{Tr}(\tilde{\sigma}_+)]$$

where $\tilde{\sigma}_+^{\mu\nu}$ is the symmetric part of $\tilde{\sigma}^{\mu\nu}$ and hence

$$\tilde{\sigma}_+^{\mu\nu} = 2 \tilde{\tau}^{\mu\nu} + g^{\mu\nu} \text{Tr}(\tilde{\tau}) \quad (3.12)$$

Combining Eqs. (3.10), (3.11), (3.12), we find

$$\begin{aligned} \delta_w \tilde{A} &= \int \sqrt{g} [-\Delta^\nu \tilde{\sigma}_{\mu\nu} + \Delta^\nu \tilde{\sigma}_{+\mu\nu}] \partial^\mu \Lambda d^4x \\ &= - \int \sqrt{g} \Delta^\nu \tilde{\sigma}_{-\mu\nu} \partial^\mu \Lambda d^4x \end{aligned}$$

where $\tilde{\sigma}_{-\mu\nu}$ is the antisymmetric part of $\tilde{\sigma}_{\mu\nu}$.

Making a partial integration

$$\delta_w \tilde{A} = \int \sqrt{g} \Delta^\mu \Delta^\nu \tilde{\sigma}_{-\mu\nu} \Lambda d^4x = 0$$

Thus the Einstein invariant action $\tilde{A}$ of Eq. (3.4) is also Weyl invariant. Hence the flat space restrictions of Einstein and Weyl invariant actions include all conformal invariant actions composed of fields of spin 0, $\frac{1}{2}$ and 1 which are at most second order in field derivatives ^{*)}. Thus all such actions can be obtained by the method of Wess and Zumino ³⁾, as long as non-minimal substitutions of the gravitational field of the form

*) Actions greater than second order in field derivatives or involving second or higher field derivatives are not likely to be of interest in phenomenological applications.

$$A = \int \mathcal{L} d^4x \rightarrow \tilde{A} = \int \sqrt{g} (\tilde{\mathcal{L}} + R_{\mu\nu} \tilde{t}^{\mu\nu}) d^4x$$

are considered.

It is useful that this method generates all interesting conformal invariant actions, as it provides a convenient closed algorithm. To work directly from the conditions of Callan, Coleman and Jackiw⁸⁾ would in general be fatiguing.

4. - CHIRAL AND CONFORMAL INVARIANT ACTIONS

We have shown how conditions derived by Callan, Coleman and Jackiw⁸⁾ can be used to generate phenomenological actions for spontaneously broken conformal symmetry. It has been seen that the prescription of Salam et al.²⁾ for constructing conformal invariant actions, which involves the use of conformally covariant field derivatives, can be simply derived. However, the method of Salam et al. does not generate all conformal invariant actions.

A phenomenological action combining conformal symmetry and a non-linear realization of chiral symmetry has been obtained⁶⁾ using the prescription of Salam et al. The question arises whether this action is unique to within canonical transformations. As shown by Coleman, Wess and Zumino¹⁰⁾, any non-linear realization of chiral symmetry may be written in a form in which the transformation law of pseudoscalar meson fields does not involve any other fields :

$$[\bar{Q}_i, M_j] = i f_{ij}(M) \quad (i, j = 1, \dots, 8) \quad (4.1)$$

It is easy to see that the pseudoscalar mesons M_i thus chosen have dimension 0. Hence the kinetic energy term

$$\frac{1}{2} \mathcal{D}_\mu M_i \mathcal{D}^\mu M_i \quad (4.2)$$

where $\mathcal{D}_\mu$ denotes a chiral covariant derivative, has dimension 2. In order to include (4.2) in a scale invariant action which is not more than second order in field derivatives we multiply (4.2) by χ^2/V^2 where χ is the Goldstone field of dimension 1 with vacuum expectation value V introduced in Section 1. We then have

$$A = \int \frac{1}{2} \mathcal{D}_\mu M_i \mathcal{D}^\mu M_i (\chi^2/V^2) d^4x \quad (4.3)$$

The form of the action (4.3) is uniquely specified by chiral and scale invariance alone : it is easy to check using Eq. (2.5) that this action is also conformal invariant. Similar considerations apply to show that other chiral invariant action terms such as

$$\int (D_{ijk} + i F_{ijk}) \bar{\chi}_i \gamma^\mu \gamma_5 \chi_j \mathcal{D}_\mu M_k d^4x$$

also have a unique conformal invariant extension which has the minimum order in field derivatives.

As noted in a previous paper ⁶⁾, the existence of a derivative $M_i M_i \chi$ coupling in (4.3), required in a chiral and conformal invariant theory, affords an attractive way of reconciling the value of the $\epsilon \bar{N}N$ coupling constant with the large $\epsilon \rightarrow 2\pi$ decay width. This method seems more aesthetic than the postulation of three scalar isoscalar mesons.

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APPENDIX : WEYL TRANSFORMATION LAWS

For convenience we here list the Weyl transformation laws of fields of spin 0, $\frac{1}{2}$ and 1, and of other quantities commonly appearing in phenomenological actions.

$$\begin{aligned}
 \text{Spin } 0 & \quad \delta_w \phi = -\Lambda \phi \\
 \text{Spin } \frac{1}{2} & \quad \delta_w \psi = -\frac{3}{2} \Lambda \psi \\
 \text{Gravitational field} & \quad \begin{cases} \delta_w g_{\mu\nu} = 2\Lambda g_{\mu\nu} \\ \delta_w g^{\mu\nu} = -2\Lambda g^{\mu\nu} \end{cases}, \quad \delta \sqrt{g} = 4\Lambda \sqrt{g} \quad (\text{A.1}) \\
 \text{Spin } 1 & \quad \delta_w v_\mu = 0, \quad \delta_w v^\mu = -2\Lambda v^\mu
 \end{aligned}$$

$$\text{Vierbein field} \quad \delta_w e^a_\mu = \Lambda e^a_\mu$$

$$\begin{aligned}
 \text{Covariant derivative} \quad \delta_w (\Delta_\nu \phi_i) = & - \sum_j (g_{\mu\nu} D_{ij} + (\Sigma_{\mu\nu})_{ij}) \phi_j \partial^\mu \Lambda \\
 & - \sum_j D_{ij} \Delta_\nu \phi_j \Lambda \quad (\text{A.2})
 \end{aligned}$$

where ϕ_i may have spin 0, $\frac{1}{2}$ or 1, and D_{ij} is the canonical dimension matrix.

$$\begin{aligned}
 \text{Curvature tensor} \quad \delta_w R^\alpha_{\gamma\beta\delta} = & \Delta_\delta \partial_\gamma \Lambda \delta^\alpha_\beta - \Delta_\beta \partial_\gamma \Lambda \delta^\alpha_\delta \\
 & + \Delta_\beta \partial^\alpha \Lambda g_{\gamma\delta} - \Delta_\delta \partial^\alpha \Lambda g_{\beta\gamma}
 \end{aligned}$$

$$\begin{aligned}
 \text{Singly contracted curvature tensor} \quad \delta_w R_{\gamma\beta} = & -2\Delta_\gamma \partial_\beta \Lambda - g_{\gamma\beta} \square \Lambda \quad (\text{A.3})
 \end{aligned}$$

$$\text{Doubly contracted curvature tensor} \quad \delta_w R = -6\square\Lambda - 2\Lambda R$$

We may note that for particle fields and their covariant derivatives the coefficient of the term in the Weyl variation proportional to Λ is $(-d-u+l)$ where d is the canonical dimension of the object, u the number of upper, l the number of lower indices. Hence any tensor function of the fields, their covariant derivatives

and $g_{\mu\nu}$ will also have $(-d-u+l)$ as the coefficient of the Λ term in its Weyl variation.

It is then immediate that an Einstein and Weyl invariant action is scale and conformal invariant when restricted to flat space. Also it is clear that if a Lorentz tensor function of the particle fields $\tau^{\mu\nu}$ has dimension d , then the Einstein tensor $\tilde{\tau}^{\mu\nu}$ obtained from $\tau^{\mu\nu}$ by the minimal substitution of the gravitational field has the Weyl transformation law

$$\delta_w \tilde{\tau}^{\mu\nu} = - (d+2) \Lambda \tilde{\tau}^{\mu\nu} \quad (\text{A.4})$$

This result is useful in Section 3 [see Eq. (3.7)].

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