CM-P00058797

Ref.TH.1296-CERN

Archives
FINITE ENERGY CURRENT ALGEBRA SUM RULES
John Ellis ^{*)} and P.H. Weisz ⁺⁾

CERN - Geneva

A B S T R A C T

Using the finite-contour method we derive the Adler-Weisberger relation in the form

$$\frac{1}{f_\pi^2} \left(1 - \frac{g_A^2}{g_v^2} \right) = \frac{2}{\pi} \int_{2m_\pi(m_p+m_{\pi/2})}^N \frac{d\nu}{\nu^2} (\sigma_{\pi-p}^{\text{tot}}(\nu) - \sigma_{\pi+p}^{\text{tot}}(\nu)) 2m_p p_\pi^{\text{lab}} + \frac{1}{2\pi i} \oint_{C_N} d\nu \frac{(T_{\pi-p}(\nu) - T_{\pi+p}(\nu))}{\nu^2}$$

where C_N is a circle radius N in the complex ν plane. A similar technique yields a sum rule from the charge-current divergence commutator. The sum rules are evaluated using Regge models with "unconventional" asymptotic behaviour. Models with weak absorptive cuts are not favoured.

^{*)} On leave of absence from DAMTP, Cambridge, England.

⁺⁾ NATO Fellow.

1. - INTRODUCTION

One of the most successful results of current algebra is the Adler-Weisberger ¹⁾ relation for the axial-vector coupling in β decay

$$\frac{1}{f_\pi^2} \left(1 - \frac{g_A^2}{g_V^2} \right) = \frac{2}{\pi} \int_{\bar{\nu} \equiv 2m_\pi(m_p + m_\pi/2)}^{\infty} \frac{d\nu}{\nu^2} (\sigma_{\pi^-p}^{\text{tot}}(\nu) - \sigma_{\pi^+p}^{\text{tot}}(\nu)) 2m_p p_\pi^{\text{Lab}} \quad (1)$$

Clearly, relation (1) cannot be correct if, as suggested by the Serpukhov data ²⁾, the Pomeranchuk theorem does not apply, and the $(\pi^\pm p)$ total cross-sections do not become asymptotically equal at high energies. This point has been observed by Thews ³⁾, who goes on to use experimental information on (πN) scattering lengths to show that the calculated value of $|g_A/g_V|$ is little changed for any reasonable behaviour of the high-energy cross-sections ⁴⁾. The reason for this is that even if unsubtracted dispersion relations formally diverge, their predictions agree with low-energy amplitudes. Thus a dispersion relation subtracted at threshold will predict extrapolations close to the result obtained by unsubtracted dispersion relations in a region close to threshold.

In this paper, we propose an alternative way of considering Adler-Weisberger-type relations: as in finite energy sum rules ⁵⁾, the dispersion integral for $1 - (g_A/g_V)^2$ is rewritten using a finite contour. In this way, Pomeranchuk-violating asymptotic forms for scattering amplitudes can be treated on the same footing as conventional ones. A similar technique is applied to deduce a sum rule from the charge-divergence commutator, and could in principle be used to obtain other sum rules. Numerical evaluations are made of the modified Adler-Weisberger and charge-divergence sum rules using tabulated ⁶⁾ $(\pi^\pm p)$ cross-section data, and figures from published ^{7),8),9)} unconventional asymptotic models. As argued by Thews, we find the calculated value of $1 - (g_A/g_V)^2$ is little changed in these models. The charge divergence sum rule does not give any useful estimate of the matrix elements of the charge divergence commutator (the so-called σ terms) but it does seem to prefer marginally the dipole ⁸⁾ and absorptive cut ^{7),9)} models over a non-absorptive cut model ⁹⁾.

2. - THE MODIFIED ADLER-WEISBERGER SUM RULE

Consider the forward crossing odd amplitude (we denote $\nu = 2pq$)

$$T_-(q^2, \nu) = i \frac{(m_\pi^2 - q^2)^2}{f_\pi^2 m_\pi^4} \int d^4x e^{iqx} \left\{ \begin{aligned} &\langle P(p) | T(\partial^\mu A_\mu^+(x), \partial^\nu A_\nu^-(0)) | P(p) \rangle \\ &- \langle P(p) | T(\partial^\mu A_\mu^-(x), \partial^\nu A_\nu^+(0)) | P(p) \rangle \end{aligned} \right\} \quad (2)$$

where

$$\partial^\mu A_\mu^\pm = \frac{1}{\sqrt{2}} (\partial^\mu A_\mu^1 \pm i \partial^\mu A_\mu^2)$$

Then, using standard current algebra techniques one obtains the low energy theorem

$$\frac{\partial T_-(0,0)}{\partial \nu} = \frac{1}{f_\pi^2} \quad (3)$$

It is clear that one can write a dispersion relation representation for $T_-(0, \nu)$ using the finite contour indicated in the Figure. Separating out the neutron pole term and using the crossing properties of $T_-(0, \nu)$ the dispersion relation is

$$T_-(0, \nu) - T_-^{\text{pole}}(0, \nu) = \frac{2\nu}{\pi} \int_{\bar{\nu}}^N \frac{\text{Im } T_-(0, \nu')}{\nu'^2 - \nu^2} d\nu' + \frac{1}{2\pi i} \oint_{C_N} \frac{T_-(0, \nu')}{\nu' - \nu} d\nu' \quad (4)$$

where C_N is a circle of radius N in the complex ν plane. Differentiating (4) with respect to ν , setting $\nu = 0$ and inserting the current algebra result (3) and the relation

$$\frac{\partial T_-^{\text{pole}}(0,0)}{\partial \nu} = \frac{1}{f_\pi^2} \left(\frac{g_A}{g_V} \right)^2$$

one obtains

$$\frac{1}{f_\pi^2} \left(1 - \left(\frac{g_A}{g_V} \right)^2 \right) = \frac{2}{\pi} \int_{\bar{\nu}}^N \frac{\text{Im } T_-(0, \nu) d\nu}{\nu^2} + \frac{1}{2\pi i} \oint_{C_N} \frac{T_-(0, \nu) d\nu}{\nu^2} \quad (5)$$

This equation is just an inverse moment finite energy sum rule ¹⁰⁾ for the off-shell amplitude $T_-(0, \nu)$ and it should be emphasized that it is an exact relation.

In order to make a connection with physics we now make the pion PCAC approximation ¹¹⁾ $T_-(0, \nu) \approx T_-(m_\pi^2, \nu)$ for $|\nu| \geq \bar{\nu}$. As emphasized in the original derivation, the PCAC approximation is made only for the two terms on the right-hand side of Eq. (5) which are both considerably smaller than $1/f_\pi^2$. Thus this derivation of the sum rule should be preferred over one working from an inverse moment sum rule for the on-shell amplitude where the cross-section and contour integrals would be exact but where both the current algebra term and the pole term [related to $(g_A/g_V)^2$ by the Goldberger-Treiman relation], which are both $\approx 1/f_\pi^2$, would have been subject to a priori uncorrelated PCAC approximation errors.

Then, using the optical theorem

$$\sigma_{\pi^- p}^{\text{tot}}(\nu) - \sigma_{\pi^+ p}^{\text{tot}}(\nu) = \frac{1}{2m_p p_\pi^{\text{lab}}} \text{Im } T_-(m_\pi^2, \nu)$$

where p_π^{lab} is the pion laboratory momentum, one obtains the modified Adler-Weisberger relation

$$\frac{1}{f_\pi^2} \left(1 - \left(\frac{g_A}{g_V} \right)^2 \right) \approx \frac{2}{\pi} \int_{\bar{\nu}}^N \frac{d\nu}{\nu^2} (\sigma_{\pi^- p}^{\text{tot}}(\nu) - \sigma_{\pi^+ p}^{\text{tot}}(\nu)) 2m_p p_\pi^{\text{lab}} + \frac{1}{2\pi i} \oint_{C_N} \frac{T_-(m_\pi^2, \nu) d\nu}{\nu^2} \quad (6)$$

The original Adler-Weisberger ¹⁾ relation (1) is now obtained if one takes $N \rightarrow \infty$ and assumes the integral around C_N to vanish.

In order to evaluate the sum rule (6), it is necessary to have a representation for the high-energy scattering amplitude which can be analytically continued to complex ν . Phenomenological Regge fits naturally provide such representations and the advantage of the sum rule (6) is that it can be used equally easily with Regge expressions which obey or disobey the conditions of the Pomeranchuk theorem.

3. - A CHARGE-CURRENT DIVERGENCE SUM RULE

We now discuss the possibility of getting a useful sum rule involving the charge-current divergence commutator. One could obtain such a sum rule by writing a once subtracted dispersion relation for the crossing-even amplitude $T_+(q^2, \nu)$ ¹²⁾, the subtraction being necessary since $\text{Im} T_+(m_\pi^2, \nu)$ does not vanish as $\nu \rightarrow \infty$. However, it is clear that if one uses a finite contour method, then one can obtain information without recourse to data on $\text{Re} T_+(m_\pi^2, \nu)$ at a particular ν .

One can write down the simple inverse moment sum rule for the on-shell crossing even amplitude $T_+(m_\pi^2, \nu)$

$$T_+(m_\pi^2, \nu) - T_+^{\text{pole}}(m_\pi^2, \nu) = \frac{2}{\pi} \int_{2m_p m_\pi}^N d\nu' \frac{\nu' \text{Im} T_+(m_\pi^2, \nu')}{\nu'^2 - \nu^2} + \frac{1}{2\pi i} \oint_{C_N} d\nu' \frac{T_+(m_\pi^2, \nu')}{\nu' - \nu} \quad (7)$$

Using the optical theorem and setting $\nu = 0$, one deduces

$$T_+(m_\pi^2, 0) - T_+^{\text{pole}}(m_\pi^2, 0) = \frac{2}{\pi} \int_{2m_p m_\pi}^N d\nu \frac{(\sigma_{\pi-p}^{\text{tot}}(\nu) + \sigma_{\pi+p}^{\text{tot}}(\nu)) 2m_p k_\pi^{\text{lab}}}{\nu} + \frac{1}{2\pi i} \oint_{C_N} \frac{T_+(m_\pi^2, \nu) d\nu}{\nu} \quad (8)$$

For $N \sim 20 \text{ GeV} \times 2\text{mp}$, the first term on the right-hand side of Eq. (8) is about 3000.

One now uses the charge-divergence commutator to estimate the magnitude of the left-hand side of (8). As usual, it is supposed that the principal variation in $T_+(q^2, 0)$ in the range $0 \leq q^2 \leq m_\pi^2$ comes from the pole contribution $T_+^{\text{pole}}(q^2, 0)$. Thus the magnitude of $T_+(m_\pi^2, 0) - T_+^{\text{pole}}(m_\pi^2, 0)$ can be estimated from the current algebra sigma term

$$\Sigma \equiv \frac{i}{f_\pi^2} \langle P | ([\bar{Q}^+(0), \partial^\mu A_\mu^-(0)] + [\bar{Q}^-(0), \partial^\mu A_\mu^+(0)]) | P \rangle$$

$T_+^{\text{pole}}(0, 0)$ is $O((m_N - m_P)/m_\pi)$ and negligible. In the $(3, \bar{3}) + (\bar{3}, 3)$ model of chiral symmetry breaking ¹³⁾

$$\Sigma = \frac{2(\sqrt{2} + c)}{3 f_\pi^2} \langle P | (\sqrt{2} u_0(0) + u_8(0)) | P \rangle$$

The two most thorough estimates of Σ are those due to Von Hippel and Kim ¹⁴⁾ : $\Sigma \simeq 10$, Cheng and Dashen ¹⁵⁾ : $\Sigma \simeq 45$. Hence the left-hand side of Eq. (8) is very much smaller than the cross-section integral on the right-hand side. In fact both estimates of Σ fall within the error (± 50) expected in the evaluation of the cross-section integral. For this reason, and also because the precise extrapolation from the current algebra point $T_+(0, 0)$ to the mass-shell is uncertain, Eq. (8) could not be used to estimate current algebra sigma terms.

However, the smallness of Σ does mean that there should be an almost exact cancellation between the cross-section and contour integrals in (8). Thus :

$$\frac{2}{\pi} \int_{2m_P m_\pi}^N \frac{d\nu}{\nu} (\sigma_{\pi^+ p}^{\text{tot}}(\nu) + \sigma_{\pi^- p}^{\text{tot}}(\nu)) 2m_P p_\pi^{\text{lab}} \approx \frac{i}{2\pi} \oint_{C_N} \frac{T_+(m_\pi^2, \nu) d\nu}{\nu} \quad (9)$$

Our derivation of the sum rule (9) has followed the principle that the PCAC approximation is made for the numerically small terms in the sum rule [see footnote 11]. If we had used an off-shell dispersion relation, a sum rule similar to (9) would have been obtained but the large cross-section and contour integrals would have been subject to apparently uncorrelated PCAC errors.

The right-hand side of sum rule (9) can be evaluated using various Regge fits to the high-energy elastic ($\pi^\pm p$) cross-sections. In the next Section we discuss the use of the charge-divergence sum rule (9) in discriminating between different Regge parametrizations.

It should be noted that because the sum rule (9) is an on mass-shell relation, one could obtain a similar sum rule using the phenomenological fact that T_+ is small at threshold, and without using any current algebra information.

The methods used in this and the previous Section could naturally be applied to obtain other sum rules from current algebra.

4. - THE SUM RULES AND REGGE MODELS

In this Section we discuss the evaluation of the sum rules (6) and (9). In Table I of the Appendix, various Regge asymptotic forms are tabulated together with the contributions they make to the contour integrals in the sum rules. In Table II, numerical evaluations of the sum rules are tabulated. The cross-section integrals $I_\pm$ were estimated by interpolating between a number ¹⁶⁾ of accurate cross-section measurements ¹⁷⁾ tabulated in the compilation of Giacomelli et al. ⁶⁾. The contour integrals $C_\pm$ were evaluated on a circle radius $(20 \text{ GeV}) \times m_P$ using the parameters of various Regge fits to the Serpukhov data ²⁾. These Regge models include dipoles ⁸⁾, absorptive cuts ^{7),9)} and non-absorptive cuts ⁹⁾.

The results of the evaluations are in Table II. It is apparent that as argued by Thews ³⁾, the contribution of the contour integral to the modified Adler-Weisberger relation is small for all the asymptotic models studied ¹⁸⁾. The calculated value of $|g_A/g_V|$ is reasonably stable, and we estimate

$$|g_A/g_V| \approx 1.17$$

The error in the right-hand side of Eq. (6) is hard to estimate, but ± 0.02 seems reasonable. Thus the modified Adler-Weisberger relation gives a value of $|g_A/g_V|$ within acceptable PCAC error of the experimental value ¹⁹⁾ $|g_A/g_V| = 1.23 \pm 0.01$.

Turning to the current-divergence sum rule, we find that it is satisfied by the dipole ⁸⁾ and absorptive ^{7),9)} cut models, but that there is a discrepancy with the weak non-absorptive cut model ⁹⁾ (see Table II). It should be noted that this latter model was only introduced by Martin and Wit to show the discriminatory power of a sum rule, and should not be regarded as a serious phenomenological proposal. It should also be pointed out that some experimental information on $\text{Re } T_+(\nu)$ was used by Barger and Phillips ⁸⁾ in choosing parameters of the dipole unconventional asymptotic model. However, they seem not to have considered $\text{Re } T_+(\nu)$ for $p_\pi^{\text{lab}} < 5 \text{ GeV}$.

It is unclear whether the discrepancy between the charge-divergence sum rule (9) and the non-absorptive cut model ⁹⁾ should be regarded as significant. However, if one alters the sum of cross-sections $(\sigma_{\pi^-p}^{\text{tot}} + \sigma_{\pi^+p}^{\text{tot}})$ by $\pm 1 \text{ mb}$ in the range $5 \text{ GeV} < p_\pi^{\text{lab}} < 20 \text{ GeV}$, the cross-section integral I_+ is altered by ± 50 , which is an order of magnitude less than the discrepancy. One might wonder whether the discrepancy between the weak cut model and the sum rule (9) could be removed by adding a constant to the parametrization of the amplitude T_+ which would contribute significantly to the contour integral C_+ . However, it can be easily checked that if such a constant were large enough to account for the discrepancy between the weak cut model and the sum rule (9), it would be so large as to completely destroy the reasonable agreement ⁸⁾ of $\text{Re } T_+$ of the present parametrization

with the available data. Thus it seems to us likely that the "charge-divergence sum rule" (9) may be useful in discriminating between some phenomenological fits to the Serpukhov data.

ACKNOWLEDGEMENTS

We would like to thank J.S. Bell, B. Renner and R. Wit for useful discussions and W. Klein for helping us with the calculations. We are also grateful for the hospitality of the CERN Theoretical Study Division. One of us (J.E.) thanks the Science Research Council for a grant.

APPENDIX

All the models we consider for the asymptotic (πp) elastic amplitudes have the form

$$T_{\pm} = T_{\pm}^{\text{Regge}} + T_{\pm}^A$$

where T^{Regge} is the conventional Regge term, and T^A represents additional terms particular to the models. In Table I below different parametrizations of $T_{\pm}$ are presented, together with the contributions they make to the contour integrals appearing in the sum rules (6) and (9).

We denote

$$I_- = \frac{2}{\pi} \int_{\bar{\nu}}^N \frac{d\nu}{\nu^2} (\sigma_{\pi-p}^{\text{tot}}(\nu) - \sigma_{\pi+p}^{\text{tot}}(\nu)) 2m_p p_{\pi}^{\text{lab}}$$

$$C_- = \frac{1}{2\pi i} \oint_{C_N} \frac{T_-(m_{\pi}^2, \nu)}{\nu^2} d\nu$$

$$I_+ = \frac{2}{\pi} \int_{2m_p m_{\pi}}^N \frac{d\nu}{\nu} (\sigma_{\pi-p}^{\text{tot}}(\nu) + \sigma_{\pi+p}^{\text{tot}}(\nu)) 2m_p p_{\pi}^{\text{lab}}$$

$$C_+ = \frac{1}{2\pi i} \oint_{C_N} \frac{T_+(m_{\pi}^2, \nu)}{\nu} d\nu$$

In Table II are presented numerical evaluations of terms in the sum rules (6) and (9), and estimates of $|g_A/g_V|$. Evaluations are made for strong absorptive cut, dipole and weak non-absorptive cut models, using $f_{\pi} = 93 \text{ MeV}$ and $N = 20 \text{ GeV} \times 2m_p$. All dimensional quantities are tabulated in units of GeV.

Table I

	Parametrizations of		Contributions to	
	T_-	T_+	C_-	C_+
Regge pole	$i\beta_{\alpha_-}^{\alpha_-} \nu^{\alpha_-} \exp(-\frac{1}{2} i\pi\alpha_-)$	$-\beta_{\alpha_+}^{\alpha_+} \nu^{\alpha_+} \exp(-\frac{1}{2} i\pi\alpha_+)$	$-\frac{2}{\pi} N^{\alpha_- - 1} \beta_{\alpha_-} \frac{\cos \frac{1}{2} \pi\alpha_-}{\alpha_- - 1}$	$-\frac{2}{\pi} N^{\alpha_+} \beta_{\alpha_+} \frac{\sin \frac{1}{2} \pi\alpha_+}{\alpha_+}$
Dipole	$\beta_{D_-}^{\alpha_-} \nu(\log \nu - b_- - \frac{1}{2} i\pi)$	$i\beta_{D_+}^{\alpha_+} \nu(\log \nu - b_+ - \frac{1}{2} i\pi) - \beta_s$	$\beta_{D_-} (\log N - b_-)$	$\frac{2}{\pi} \beta_{D_+} N(1 + b_+ - \log N) - \beta_s$
Crossing-even cut	0	$i\beta_c^{\alpha_c} \Gamma(\lambda + 1) \nu(\log \nu - b_c - \frac{1}{2} i\pi)^{-\lambda-1}$	0	$-\frac{1}{\pi} \beta_c \Gamma(\lambda + 1) N K_c(N) *$

*) where $K_c(N) = \text{Im} \int_0^\pi e^{i\theta} (\log N - b_c + i[\theta - \pi/2])^{-\lambda-1}$ and the cut is absorptive if $\beta_c < 0$.

Model	I_-	C_-	$ g_A/g_V $	I_+	C_+ (given to the nearest 50)
Strong cut ⁶⁾	-50.2	3.6	1.18	3350	-3400
Dipole ⁷⁾		5.0	1.17		-3350
Strong cut ⁸⁾		4.7 ²⁰⁾	1.17		-3450
Weak cut ⁸⁾		4.7 ²⁰⁾	1.17		-3800

- TABLE II -

REFERENCES AND FOOTNOTES

- 1) S.L. Adler - Phys.Rev.Letters 14, 1051 (1965) ;
W.I. Weisberger - Phys.Rev.Letters 14, 1047 (1965).
- 2) J.V. Allaby, Yu.B. Bushnin, S.P. Denisov, A.N. Diddens,
R.W. Dobinson, S.V. Donskov, G. Giacomelli, Yu.P. Gorin,
A. Klovning, A.I. Petrukhin, Yu.D. Prokoshkin,
R.S. Shuvalov, C.A. Stahlbrand and D.A. Stoyanova -
Phys.Letters 30B, 500 (1969).
- 3) R.L. Thews - Nuovo Cimento 68A, 497 (1970).
- 4) R.L. Thews estimates that the Adler-Weisberger result will be
essentially unchanged unless the $\pi^{\pm}p$ total cross-section
difference grows by a factor 10^5 in the region
 $40 \text{ GeV}^2 < \nu < 400 \text{ GeV}^2$.
- 5) A. Logunov, L.D. Soloviev and A.N. Tavkhelidze - Phys.Letters
24B, 181 (1967) ;
K. Igi and S. Matsuda - Phys.Rev.Letters 18, 625 (1967).
- 6) G. Giacomelli, P. Pini and S. Stagni - Report CERN-HERA 69-1 (1969).
- 7) V. Barger and R.J.N. Phillips - Phys.Rev.Letters 24, 291 (1970).
- 8) V. Barger and R.J.N. Phillips - Phys.Letters 31B, 643 (1970).
- 9) A.D. Martin and R. Wit - Nuclear Phys. B25, 85 (1971).
- 10) R. Dolen, D. Horn and C. Schmid - Phys.Rev. 166, 1768 (1968).
- 11) It may be argued that the PCAC assumption for complex ν is an
unphysical assumption. However, other methods of deriving
a modified Adler-Weisberger relation, e.g., writing an
unsubtracted dispersion relation for $T_-(0, \nu) - T_-^{\text{Regge}}(m_{\pi}^2, \nu)$
obviously involve equivalent assumptions.
- 12) B. Renner - "Current Algebras and their Applications", Pergamon
Press (1968).
- 13) M. Gell-Mann, R.J. Oakes and B. Renner - Phys.Rev. 175, 2195 (1968).
- 14) F. Von Hippel and J.K. Kim - Phys.Rev. D1, 151 (1970).
For a discussion on why one might expect to see this analysis
modified, see, e.g., :
J. Ellis - Phys.Letters 33B, 591 (1970).;
H. Kleinert, F. Steiner and P.H. Weisz - Phys.Letters, to
be published (1971).

FIGURE : The contour used in deriving the sum rules.