THE DIMENSION OF SCALE SYMMETRY BREAKING

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A B S T R A C T

Present theoretical understanding allows a c number δ term and any dimension for the chiral symmetry breaking term in the decomposition

$$\theta_{00} = \bar{\theta}_{00} + \epsilon(\delta + u).$$

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We consider the usual Gell-Mann ¹⁾ - Wilson ²⁾ decomposition of the energy-momentum tensor in the form :

$$\theta_{00} = \bar{\theta}_{00} + \epsilon (\delta + u) \quad (1)$$

where $\bar{\theta}_{00}$ is chiral invariant and of scale dimension 4,
 δ is chiral invariant and of scale dimension $\neq 4$,
 u breaks chiral invariance, and has scale dimension d .

We consider theories in which scale invariance is an approximate dynamical symmetry and a scalar meson dominates the trace of the energy-momentum tensor. It can be shown ³⁾ that in such theories a δ term is necessary. Several authors ^{1),2),4),5),6)} have argued that δ must be a q number, principally on the basis of the meson-baryon scattering analysis of von Hippel and Kim ⁷⁾. However, it has recently been shown ⁸⁾ that a q number δ term is not necessary for consistency with the work of von Hippel and Kim : accordingly we assume in the following that δ is a c number. Some authors ^{4),5),6)} have argued that if δ is a c number, then the dimension d of the chiral symmetry breaking term u can be deduced theoretically.

In this note we show that the assumption

$$\bar{\theta}_{00} = \lim_{\epsilon \rightarrow 0} \theta_{00} \quad (2a)$$

or

$$\frac{\partial \theta_{00}}{\partial \epsilon} = (\delta + u) \quad (2b)$$

used in some papers ^{4),5)} to obtain the value of d is incorrect in a theory in which scale invariance is an approximate dynamical symmetry. Modified appropriately, the equations of these papers just become identities which are satisfied by simple Lagrangian models ^{8),9),10)} with any value of d . We also show that other arguments ⁶⁾ on the form of the energy-momentum tensor, the dimension of scale

symmetry breaking and the form of the δ term are not valid when the trace of the energy-momentum tensor is dominated by a scalar meson. This is because the scalar meson pole must be taken into account in analyzing the Ward identities. We conclude that a decomposition of the form (1), with δ a c number and any value of d between 0 and 4, is compatible with present theoretical understanding. It seems that d can only be determined phenomenologically.

The coupling b of the scalar meson is introduced by ^{*)}

$$\langle 0 | \Theta_{\mu\nu} | \sigma(k) \rangle = -\frac{1}{3b} (g_{\mu\nu} m_\sigma^2 - k_\mu k_\nu) \quad (3)$$

Also we use the "virial" theorem, which for δ a c number is

$$\Theta^\mu_\mu = \epsilon(4-d)u + \epsilon 4\delta \quad (4)$$

for Lorentz invariance of the vacuum we require $\langle 0 | \Theta_{\mu\nu} | 0 \rangle = 0$, so that

$$\langle 0 | \Theta_{00} | 0 \rangle = 0 = \langle 0 | \bar{\Theta}_{00} | 0 \rangle + \epsilon \langle 0 | \delta | 0 \rangle + \epsilon \langle 0 | u | 0 \rangle \quad (5)$$

and

$$\langle 0 | \Theta^\mu_\mu | 0 \rangle = 0 = (4-d)\epsilon \langle 0 | u | 0 \rangle + 4\epsilon \langle 0 | \delta | 0 \rangle \quad (6)$$

We also have the low-energy theorem ³⁾

$$i \int d^4x \langle 0 | T(\Theta^\mu_\mu(x), \Theta^\nu_\nu(x)) | 0 \rangle = -d(4-d)\epsilon \langle 0 | u | 0 \rangle \quad (7)$$

^{*)} We will denote covariantly normalized states by $|s\rangle$,
unit normalized states by $|s\rangle\rangle$.

from which we see that $\langle 0 | u | 0 \rangle \neq 0$ in a theory with σ meson dominance. Indeed saturating the spectral function in (7) by the intermediate σ state, we get the mass formula

$$m_\sigma^2 = -b^2 d(4-d)\epsilon \langle 0 | u | 0 \rangle \quad . \quad (8)$$

We now see from (6) that $\langle 0 | \delta | 0 \rangle = -(1-d/4) \langle 0 | u | 0 \rangle$. We take δ to be the c number $\delta = -(1-d/4) \langle 0 | u | 0 \rangle$. From (5) we see that

$$\langle 0 | \bar{\theta}_\infty | 0 \rangle = -\frac{\epsilon d}{4} \langle 0 | u | 0 \rangle \quad . \quad (9)$$

We can in fact see from (1), (3), (4) and (5) that

$$\langle 0 | \bar{\theta}_\infty | \sigma(k) \rangle = -bd\epsilon \langle 0 | u | 0 \rangle + \frac{k^2}{3b} \quad . \quad (10)$$

From (9) and (10) it follows that $\bar{\theta}_\infty$ contains ϵ dependent terms, and that assumption (2) is invalid ^{*)}.

In the case of spontaneously broken scale invariance (2b) is replaced by

$$\frac{\partial \theta_\infty}{\partial \epsilon} = \frac{\partial \bar{\theta}_\infty}{\partial \epsilon} + \delta + u \quad (11)$$

which can be used to derive a modified version of an equation due to Carruthers ⁴⁾. Taking matrix elements of (11) between unit normalized states $|p, \epsilon\rangle$ we get

$$\begin{aligned} \langle\langle p, \epsilon | \frac{\partial \theta_\infty}{\partial \epsilon} | p, \epsilon \rangle\rangle &= \langle\langle p, \epsilon | (\frac{\partial \bar{\theta}_\infty}{\partial \epsilon} + \delta + u) | p, \epsilon \rangle\rangle \\ &= \langle\langle p, \epsilon | \frac{\partial \bar{\theta}_\infty}{\partial \epsilon} | p, \epsilon \rangle\rangle_{\text{conn}} + \langle\langle p, \epsilon | u | p, \epsilon \rangle\rangle_{\text{conn}} \quad . \quad (12) \end{aligned}$$

^{*)} Hence, the argument of certain authors ⁵⁾ that d must equal one is incorrect.

Because matrix elements of the Hamiltonian between unit normalized states depend only on the form of the Hamiltonian ¹¹⁾, we can rewrite the left-hand side of (12) to obtain

$$\frac{\partial}{\partial \epsilon} \langle\langle p, \epsilon | \theta_{00} | p, \epsilon \rangle\rangle = \langle\langle p, \epsilon | \frac{\partial \bar{\theta}_{00}}{\partial \epsilon} | p, \epsilon \rangle\rangle_{\text{conn}} + \langle\langle p, \epsilon | u | p, \epsilon \rangle\rangle_{\text{conn}} .$$

Using $E^2(\epsilon) = m^2(\epsilon) + \underline{p}^2$, and converting to covariantly normalized states $|p, \epsilon\rangle$, we get, using the "virial" theorem (4) and Eq. (7) :

$$\frac{\partial m^2(\epsilon)}{\partial \epsilon} = \langle p, \epsilon | \frac{\partial \bar{\theta}_{00}}{\partial \epsilon} | p, \epsilon \rangle_{\text{conn}} + \frac{2m^2(\epsilon)}{(4-d)\epsilon} . \quad (13)$$

Equation (13) has been previously derived by Carruthers ⁴⁾, but he omitted the first term on the right-hand side. This term is in general non-zero, because of a σ pole in the matrix element of $\partial \bar{\theta}_{00} / \partial \epsilon$ [see Eq. (10)]. Neglecting this term he found by considering matrix elements between pseudoscalars that $d = 2$. However, the general equation (13) is identically satisfied in simple Lagrangian models, whatever the value of d .

Consider the simple Lagrangian theory ^{8),9),10)}

$$\mathcal{L} = \frac{1}{2} D_\mu M_i D^\mu M_i e^{2b\sigma} + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma e^{2b\sigma} - \epsilon \left[-\frac{d}{4} \langle 0 | u_0 | 0 \rangle e^{4b\sigma} - \left(\frac{4-d}{4} \right) \langle 0 | u_0 | 0 \rangle + (u_0 + c u_8) e^{db\sigma} \right] \quad (14)$$

where the M_i are pseudoscalar meson fields of dimension zero with the non-linear chiral transformation law $[Q_i^5, M_j] = i f_{ij}(M)$ and

σ is chiral invariant and has the anomalous scale transformation law $i[D, \sigma(x)] = +1/b + x^\mu \partial_\mu \sigma(x)$. The derivative terms in (14) are then of dimension 4. The chiral symmetry breaking terms have the series expansions :

$$\begin{aligned}
 u_0 &= \langle 0 | u_0 | 0 \rangle \left(1 - \frac{M_i M_i}{3f^2} + \dots \right) \\
 u_8 &= \sqrt{\frac{2}{3}} \langle 0 | u_0 | 0 \rangle \left(-d_{8ij} \frac{M_i M_j}{2f^2} + \dots \right)
 \end{aligned} \tag{15}$$

and belong to a $(3, \bar{3}) + (\bar{3}, 3)$ representation of $SU(3) \times SU(3)$, with $c \approx -1.25$ as in Gell-Mann, Oakes and Renner ¹²⁾. The coefficient of $e^{4b\sigma}$ in (14) is fixed by requiring $\mathcal{L}$ to have no term linear in the field σ : the c number δ term is then determined by requiring $\mathcal{L}$ to have no over-all constant. f is the pseudoscalar meson decay constant:

$$\partial_\mu A_{\pi, \kappa}^\mu = f m_{\pi, \kappa}^2 M_{\pi, \kappa}$$

In the Lagrangian theory (14) we have

$$\delta = -\left(\frac{4-d}{4}\right) \langle 0 | u_0 | 0 \rangle, \quad \frac{\partial \bar{\Theta}_{00}}{\partial \epsilon} = -\frac{d}{4} \langle 0 | u_0 | 0 \rangle e^{4b\sigma}. \tag{16}$$

Using Eqs. (14) to (16) it is easy to show that for pseudoscalar mesons ^{*)}

$$\frac{\partial m^2}{\partial \epsilon} = \frac{m^2}{\epsilon}$$

$$\langle p, \epsilon | \frac{\partial \bar{\Theta}_{00}}{\partial \epsilon} | p, \epsilon \rangle_{\text{comm}} = \left[b(d-2)m^2 \right] \left[\frac{-1}{m_\sigma^2} \right] [-bd \langle 0 | u_0 | 0 \rangle]. \tag{17}$$

Using the σ mass formula (8) we find

$$\langle p, \epsilon | \frac{\partial \bar{\Theta}_{00}}{\partial \epsilon} | p, \epsilon \rangle_{\text{comm}} = \left(\frac{2-d}{4-d} \right) \left(\frac{m^2}{\epsilon} \right) \tag{18}$$

^{*)} Note that Gell-Mann ¹⁾ assumes the low-energy theorem ¹²⁾ $\langle p | (u_0 + cu_8) | p \rangle = m^2$, which does not hold in general in spontaneously broken scale theories.

and hence using (17) and (18) we see that Eq. (13) is identically satisfied for all values of d . Equation (13) is also satisfied if one considers matrix elements of baryons, and uses the natural extension ⁸⁾ of the simple Lagrangian (14) with any value of d . We conclude that Eq. (13) does not give useful restrictions on the dimension d of chiral symmetry breaking.

In recent papers ⁶⁾, Jackiw asserts that, once the chiral limit has been reached, it is no longer possible to go to the scale invariant limit. He also argues that $d = 1$. His derivation is based on the relations ^{*})

$$\begin{aligned} T(0,0,0) &= 2m_\pi^2 (2-d) \\ T(m_\pi^2, 0, 0) &= m_\pi^2 (4-d) \\ T(m_\pi^2, m_\pi^2, m_\pi^2) &= 2m_\pi^2 \end{aligned} \tag{19}$$

which are satisfied, as a consequence of well-known Ward identities ¹³⁾, by the truncated amplitude

$$T(p^2, q^2, p \cdot q) = \frac{-(p^2 - m_\pi^2)(q^2 - m_\pi^2)}{f_\pi^2 m_\pi^4} \int d^4x d^4y e^{ip \cdot x} e^{-iq \cdot y} \langle 0 | T(\partial_\lambda A_\pi^\lambda(x), \partial_\mu A_\pi^\mu(y), \theta_\nu^v | 0) | 0 \rangle$$

If one parametrizes T as a polynomial in the momenta, one obtains immediately

$$T = (4-2d)m_\pi^2 + (d-1)(p^2 + q^2) + t + \dots \tag{20}$$

^{*}) The variation in the pion propagator between the origin and the pion mass is neglected.

where $t = (p-q)^2$ and the dots denote terms of higher order. Clearly, in the chiral limit, $m_\pi \rightarrow 0$, it is no longer possible to pass to the scale invariant limit, in which T should vanish identically. This difficulty, together with the observation that (20) exhibits relatively strong variations of the amplitude T with the momenta, has led Jackiw to change the definition of the energy momentum tensor. His "modified" energy momentum tensor satisfies, instead of (19), corresponding relations which can be satisfied by an approximately constant amplitude provided $d = 1$.

The modified energy momentum tensor proposed by Jackiw does not correspond to the case in which the σ pole is the dominant low energy singularity. It is therefore not appropriate in a theory in which scale invariance is broken only by the σ mass and all other particles can have non-vanishing masses in the limit of scale invariance. Consequently his argument that $d = 1$ also does not apply in theories in which a would-be Goldstone boson σ dominates the trace of the energy momentum tensor. In such theories one should rather use the parametrization ¹³⁾

$$T \approx \frac{\alpha + \beta(p^2 + q^2) + \gamma t}{t - m_\sigma^2} \quad (21)$$

(we have assumed that there is only one subtraction in each particle channel). The relations (19) can now be used to determine the constants α , β and γ . One obtains

$$\alpha = 2m_\pi^2 m_\sigma^2 (d-2)$$

$$\beta = -m_\sigma^2 (d-1)$$

$$\gamma = (4-d)m_\pi^2 - m_\sigma^2$$

Observe that the above expression (20) follows from $m_\sigma \rightarrow \infty$. On the other hand, keeping m_σ finite, one sees that in the chiral limit $m_\pi \rightarrow 0$, T has the form

$$T = -m_\sigma^2 \frac{(d-1)(p^2+q^2) + t}{t - m_\sigma^2}$$

This expression vanishes identically, as it should, in the scale invariant limit $m_\sigma \rightarrow 0$.

By taking the residue of the pole on the mass shell, (21) gives the $\sigma\pi\pi$ coupling constant. Remembering that

$$\langle 0 | \theta^\mu_\mu | \sigma \rangle = - \frac{m_\sigma^2}{b}$$

one finds 9),10),13)

$$g_{\sigma\pi\pi} = -\frac{b}{2} [m_\sigma^2 + (d-2)m_\pi^2]$$

The numerator of (21) describes the effective $\sigma\pi\pi$ coupling. Its relatively rapid variation with the momenta is actually a desirable feature, since it gives rise to the dependence of $g_{\sigma\pi\pi}$ on the mass and consequently eliminates the discrepancy with experiment which would arise if $g_{\sigma\pi\pi}$ were proportional to m_π^2 only. Furthermore, the momentum dependence in the $\sigma\pi\pi$ coupling seems to be required in phenomenological analyses of π nucleon scattering 14).

The linear dependence of the fully truncated amplitude $(t - m_\sigma^2)T$ on the square momenta of the particles corresponds to the usual kind of ansatz familiar in hard meson or effective Lagrangian calculations. In the case of the $\sigma\pi\pi$ vertex we have seen that the Ward identities determine completely the coefficients α, β and δ , a situation which we consider satisfactory. Jackiw interprets smoothness as meaning approximate constancy, a property which is not necessarily desirable and cannot always be required. Observe that the dependence on the pion momentum in the off-shell amplitude, if $d \neq 1$, only

modifies the Goldberger-Treiman relation for $\langle \sigma | \partial_\mu A_\pi^\mu | \pi \rangle$ by order m_π^2/m_σ^2 , which is acceptable. If one requires the form factor to vanish at infinity one would obtain $d = 1$, however, we do not find this requirement compelling.

The case of the σ baryon coupling can be treated in a similar manner, or equivalently by means of an effective Lagrangian⁸⁾. Here one can also make the one-subtraction ansatz. The coefficients of the linear ansatz for the effective vertex can be determined provided one assumes that a single q number term of dimension d breaks both $SU(3) \times SU(3)$ and scale invariance. From the examples of the $\sigma\pi\pi$ and σBB coupling we are led to believe that in general one should not neglect subtractions in using θ_μ^μ as an interpolating field for the scalar meson.

In conclusion, it seems that no theoretical reasons are known why the decomposition (1) should not hold with δ a c number and u having any scale dimension d between 0 and 4 in a theory with σ dominance of the trace of the energy momentum tensor.

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