CM-P00058681

Archives

Ref.TH.1270-CERN

PERTURBATIVE UNITARITY OF DUAL LOOPS

P.H. Frampton, P. Goddard ^{*)} and D.A. Wray
CERN - Geneva

A B S T R A C T

The N loop dual graphs are classified and enumerated in terms of integration contours on a sphere with N handles. Perturbative unitarity at the leading Landau singularity dictates that all inequivalent N loop diagrams (unrelated by duality) be added with equal weight. A proof of perturbative unitarity for the resulting expansion is hindered by the presence of complex anomalous thresholds.

^{*)} On leave of absence from Trinity College, Cambridge.

Ref.TH.1270-CERN

18 December 1970

In this paper, our object is to discuss certain properties of a dual multiloop expansion of the S matrix ¹⁾. The properties which we shall discuss are :

- a) the classification and enumeration of the inequivalent dual amplitudes;
- b) the relative weighting of the inequivalent amplitudes in order that perturbative unitarity at the leading singularity be satisfied;
- c) perturbative unitarity at the non-leading singularities where the derivation is found to be identical to that for the usual Feynman diagrams, except for one complication which is pointed out.

a) Classification and Enumeration of Inequivalent Dual Amplitudes

A complete topological classification of dual graphs has been given by Kikkawa, Klein, Sakita and Virasoro ²⁾ (KKSv) who consider the triangulation of an arbitrary Feynman-like diagram ¹⁾ (FLD) with loops and assign to it a dual surface with windows, bordered by the external momenta, and dots which correspond to the simple tadpoles. A family (or equivalence class) of FLD's have a common dual surface, and correspond to different triangulations of that surface. Starting from the KKSv approach, Gross et al. ³⁾ were able to show that an arbitrary FLD can be written in terms of four primitive graphs inserted in a tree diagram; certain identities ³⁾ between the primitive graphs then further reduce the number of inequivalent amplitudes.

Since these classifications were given, it has been found by Lovelace ⁴⁾ and Alessandrini ⁵⁾ that the N loop dual amplitude can be very conveniently expressed in terms of an integral on a Riemann surface comprising a sphere with N handles. Alessandrini and Amati ⁶⁾ have further shown that a general dual amplitude can be written as a particular partitioning of the external lines on the integration boundaries of such a Riemann surface, and that at a boundary the positions

of the external lines must be integrated once around the boundary, preserving the order. It is now therefore instructive to classify and enumerate the inequivalent N loop graphs in terms of topologically inequivalent sets of integration boundaries on a sphere with N handles. We do this in two steps by considering : i) the integration boundaries; ii) the partitioning of external lines among the boundaries.

The first step is to look at the N loop vacuum fluctuations. The Riemann surface appropriate to an orientable N loop vacuum fluctuation is designated by $\beta_n(h)$ and is shown in Fig. 1.a. This surface is an n -fold iteration of a fundamental unit, and there are h holes (not shown in Fig. 1) in the surface. In this notation $\beta_0(0)$ is a disc. The double of $\beta_n(h)$ is a sphere $\mathcal{S}_N$ with N handles, where $N=2n+h$. The boundaries of $\beta_n(h)$, of which there are $b=N-2n+1$, become the integration boundaries on $\mathcal{S}_N$. The number of inequivalent orientable N loop vacuum fluctuations is the number of different solutions of $N=2n+h$, for $n=0,1,2,\dots$. This number is given by $\Delta_{0,N}^{(0)} = \lfloor (N/2)+1 \rfloor$ where $\lfloor x \rfloor$ is the largest integer not greater than x . The subscripts in $\Delta_{e,N}$ refer to the numbers of external lines and loops, respectively.

For a non-orientable N loop vacuum fluctuation, the Riemann surface will be denoted by $\gamma_n(h)$ and is shown in Fig. 1.b. This surface is an $(n+1)$ -fold iteration of a Möbius strip, and has h holes in it (again not shown in Fig. 1). Its double is $\mathcal{S}_N$, where now $N=(n+h+1)$. The number of integration boundaries is $b=(N-n)$. The number of inequivalent non-orientable N loop vacuum fluctuations is $\Delta_{0,N}^{(NO)} = N$, the number of different solutions of $N=(n+h+1)$.

The total number of inequivalent graphs is thus

$$\Delta_{0,N}^{(tot)} = \Delta_{0,N}^{(o)} + \Delta_{0,N}^{(NO)} = \left[\left(\frac{3N}{2} \right) + 1 \right] ;$$

this is of course very much less than the corresponding number for Feynman diagrams.

We now turn to the second step, the partitioning of external lines among the boundaries. The number of ways of partitioning e external lines on b boundaries will be denoted by $\delta_{e,b}$. Since all boundaries are equivalent, and since we must take into account the different cyclic permutations (also, for an orientable surface, the anticyclic permutations) within a boundary the definition of $\delta_{e,b}$ is as follows.

Partition e different objects into b identical boxes. Assign a weight to each partition of $\frac{1}{2} \prod_i (n_i - 1)!$ (for an orientable surface) or $\prod_i \frac{1}{2} (n_i - 1)!$ (for a non-orientable surface) where the product is over all boxes with $n_i (\geq 3)$ objects; if all $n_i \leq 2$ assign unit weight to the partition. The total is then $\delta_{e,b}$.

In terms of $\delta_{e,b}$ we can write formal expressions for $\Delta_{e,N}^{(tot)}$, the total number of inequivalent dual graphs with N loops and e external lines, as follows

$$\Delta_{e,N}^{(tot)} = \Delta_{e,N}^{(o)} + \Delta_{e,N}^{(No)} \quad (1)$$

where

$$\Delta_{e,N}^{(o)} = \sum_{n=0}^{[N/2]} \delta_{e,N-2n+1} \quad (2)$$

and

$$\Delta_{e,N}^{(No)} = \sum_{n=0}^{(N-1)} \delta_{e,N-n} \quad (3)$$

are the numbers of inequivalent orientable and non-orientable graphs, respectively.

For general e and b the expression $\delta_{e,b}$ is very complicated. Note that we should take $\delta_{0,b} = 1$ for the vacuum fluctuations. For $e=4$ the values are $\delta_{4,1}=3$, $\delta_{4,2}=10$, $\delta_{4,3}=16$ and $\delta_{4,b}=17$ (all $b \geq 4$); the corresponding numbers of inequivalent graphs are given in the Table.

b) The Leading Singularity

The relative weighting of the inequivalent dual amplitudes for any N is readily fixed by a straightforward generalization of the procedure of Refs. 7) and 8), where it is pointed out that for $N=1$ perturbative unitarity obtains at the leading singularity (all internal lines on mass shell) when all inequivalent amplitudes are added with equal weight.

Consider an arbitrary N loop FLD with all internal lines, of which there are $i=3N+e-3$, untwisted. To obtain perturbative unitarity at the leading singularity of this graph, with correctly signaturized intermediate states included in the unitarity sum, we must add (with equal weight) the 2^i FLD's obtained by distributing all possible combinations of trusts on the internal lines of the graph. Now it is a result of Ref. 6) that the singularities corresponding to each related FLD come from distinct regions of the integration. Thus to obtain the correct discontinuity for the leading singularity, it is clear that all the inequivalent dual amplitudes must be added with equal weight in order that the 2^i FLD's mentioned above be added with equal weight.

The different orders in N must be weighted ⁹⁾ by a factor f^{2N+e-2} where f is the dual coupling constant. This then leads to a formal S matrix expansion with three parameters : in addition to f there are the intercept $\alpha(0)$ and slope α' of the input Regge trajectory.

c) Perturbative Unitarity

Basing our discussion on a rule for the general discontinuity of a dual loop contribution, we now show how the standard S matrix methods ¹⁰⁾ (applicable equally to a $\lambda\phi^3$ field theory) can be used to confirm that perturbative unitarity is satisfied also for all the non-leading singularities. There is, however, one complication which occurs in the proof, namely the existence of a proliferation of complex anomalous thresholds as pointed out below.

The discontinuity rule for dual loops, which is expected as a natural consequence ^{*)} of the method of construction by sewing reggeons ⁴⁾⁻⁶⁾, is as follows. The discontinuity of a particular dual loop contribution across a given Landau curve is the sum of all inequivalent contributions formed from FLD's by putting appropriate lines on mass shell; here two contributions are equivalent if they are formed from two FLD's which are related by duality transformations only on the lines left off mass shell. As an example, consider the three particle threshold of the diagram in Fig. 2.a in the $12 \rightarrow 34$ channel. Figures 2.b and 2.c are equivalent, while Fig. 2.d is equivalent to Fig. 2.b and so its discontinuity is not added to that of Fig. 2.b.

From this rule it is not difficult to see that the discontinuity of the total N loop amplitude (formed by adding all inequivalent N loop amplitudes with equal weight) across a particular singularity is obtained by inserting on-mass shell signaturized propagators for the lines of the diagram corresponding to the discontinuity and total lower order amplitudes (again formed by adding all inequivalent amplitudes with unit weight, and with the appropriate power of f) at the vertices. This is illustrated by the example of a normal threshold discontinuity in Appendix A. Because the discontinuity formula is similar to that in ordinary perturbation theory, the proof of perturbative unitarity then can proceed as in non-dual S matrix theory ¹⁰⁾, except for the complication of complex thresholds.

Here we will assume $\alpha(0) < 0$ to avoid tachyons. For the purposes of perturbative unitarity, we include in the unitarity sum all excited particles which are taken to be stable (of zero width) despite satisfying mass inequalities favourable to their decay. Since the masses of the excited states are arbitrarily large this means that even for the lowest non-trivial order in f (that is, $N=1$) there is already an infinite number of complex singularities to be considered. If we restrict the energy then the number is finite and we expect that the complex singularities are then consistent with perturbative unitarity as in the case of ordinary Feynman graphs. The first complex

^{*)} The rule is shown to hold for the two-particle threshold at the $N=1$ level in Ref. 8); for the general case it remains a conjecture.

singularity in general occurs above the first three particle normal threshold [unless $\alpha(0)$ is an integer multiple of $(-1/3)$ whereupon they coincide], so that we can prove perturbative unitarity at least in a large triangle of the Mandelstam plane which includes $s, t, u < 9m_0^2$ where m_0 is the ground state mass.

Finally, we must acknowledge the presence of the peculiar discontinuity discovered in non-planar graphs [speculatively associated with the Pomeranchukon¹²⁾] whose contribution to the discontinuity has been ignored in the above discussion. At present, we are unable to shed any light on the relationship between this new contribution and unitarity.

The authors would like to thank Dr. V. Alessandrini and Professor D. Amati for useful discussions. One of us (D.A.W.) also wishes to thank Dr. J.M. Kosterlitz for discussions.

A P P E N D I X

We give details about how it comes about that, on taking for instance a normal threshold discontinuity of the total N loop amplitude (every inequivalent discontinuity counted once), one obtains a split into two total amplitudes A and B , joined by on mass shell signaturized propagators as in Fig. 3. A and B themselves contain every inequivalent surface once with the appropriate number of loops.

For a particular dual loop contribution, after putting the appropriate lines on mass shell, we have a sum of (inequivalent) terms with various contributions for A and B and combinations of crosses (twists) on the on mass shell lines, but not all possibilities occur. Before the lines were put on mass shell these terms were all dually equivalent. If we now take the sum of all inequivalent dual N loop contributions, we get all the possible combinations of contributions for A and B exactly once. This is because the same contribution cannot come from two inequivalent dual N loop contributions, and given any contribution, we can construct a dual N loop from which it comes. Further we also get every combination of crosses on the intermediate state lines and so will obtain signaturized on mass shell propagators. This is clearly the expression obtained by joining total amplitudes for A and B by signaturized propagators on mass shell, as in Fig. 3.

It remains to verify that the criterion of equivalence used in the separate parts A , B of the diagram and in the rule are consistent. The point is illustrated in Fig. 4. It shows that it is implicit in the discontinuity rule and what has followed that the operations of twisting and putting on mass shell commute. This is needed to establish the equality of Figs. 4a and 4b. Given this, any duality transformation on A or B corresponds to a duality transformation on the full Fig. 3.

Identical particles, or rather identical levels, are taken care of in this scheme. If between two vertices in the diagram of the Landau curve n lines are put at the same energy level, an extra $n!$ symmetries come into play, because of the total symmetry of the amplitudes. So by the rule, a factor $1/n!$ should be applied to each contribution to the discontinuity. This is just the factor required by the statistics of identical particles.

Finally, we note that because amplitudes with excited particles occur there may be some ambiguity with the position of the dots ¹³⁾. We adopt the convention in this paper that for any dual loop contribution with n spinning reggeons, we consider the sum of all contributions with all orientations of the dots included, dividing by 2^n . This signaturizes the external particles.

R E F E R E N C E S

- 1) K. Kikkawa, B. Sakita and M.A. Virasoro, Phys.Rev. 184, 1701 (1969).
- 2) K. Kikkawa, S. Klein, B. Sakita and M.A. Virasoro, Phys.Rev. D1, 3258 (1970).
- 3) D.J. Gross, A. Neveu, J. Scherk and J.H. Schwarz, Phys.Letters 31B, 592 (1970).
- 4) C. Lovelace, CERN preprint TH.1203 (1970).
- 5) V. Alessandrini, CERN preprint TH.1215 (1970).
- 6) V. Alessandrini and D. Amati, private communication, and to be published.
- 7) V. Alessandrini, D. Amati, M. Le Bellac and D.I. Olive, CERN preprint TH.1160 (1970).
- 8) D.J. Gross, A. Neveu, J. Scherk and J.H. Schwarz, Phys.Rev. D2, 697 (1970).
- 9) For a possible objection to this power counting, see :
A. Di Giacomo, S. Fubini, L. Sertorio and G. Veneziano, Phys.Letters 33B, 171 (1970).
- 10) R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, The Analytic S Matrix, Cambridge University Press (1966).
- 11) G. Frye and L. Susskind, Phys.Letters 31B, 589 (1970).
- 12) P.G.O. Freund and R.J. Rivers, Phys.Letters 28B, 510 (1969).
- 13) S. Sciuto, Nuovo Cimento Letters 2, 411 (1969).

TABLE : Numbers of inequivalent graphs for $e = 4$.

	$\Delta_{4,N}^{(o)}$	$\Delta_{4,N}^{(NO)}$	$\Delta_{4,N}^{(tot)}$
$N = 0$	3	0	3
$N = 1$	10	3	13
$N = 2$	19	13	32
$N \geq 3$	$\lfloor (17N/2) + 2 \rfloor$	$17N - 22$	$\lfloor (51N/2) - 20 \rfloor$

* * * * *

FIGURE CAPTIONS

Figure 1 : Riemann surface for a) orientable dual amplitude (β_n)
b) non-orientable dual amplitude (γ_n).

Figure 2 : Equivalent and inequivalent contributions to the discontinuity.

Figure 3 : Normal threshold discontinuity.

Figure 4 : Two equal discontinuities.

(a)

(b)

FIG.1

FIG. 2

FIG. 4