For internal circulation only

NP Internal Report 70-11
13 April 1970

PION-NUCLEON ELASTIC SCATTERING

G. Giacomelli

University of Bologna, Italy
INFN, Sezione di Bologna, Italy
CERN, Geneva, Switzerland

G E N E V A

1970

ERRATA TO NP INTERNAL REPORT 70-11

PION-NUCLEON SCATTERING

G. Giacomelli

<u>Page</u>	<u>Line</u>	<u>Instead of</u>	<u>Please read</u>
5	Table 2.3		$\vec{k}, -\vec{k}, \vec{k}', -\vec{k}'$
13	Eq. (2.28b)	$2\mu/(\vec{k} - \vec{k}') \int_0^\infty$	$2\mu/\hbar^2(\vec{k} - \vec{k}') \int_0^\infty$
14	Eq. (2.35)	$ a_\ell ^2 + f_\ell ^2 + \dots = 1$	$ a_\ell ^2 + \frac{K}{K'} f_\ell ^2 + \dots = 1$
17	Eq. (2.45b)	$\{1 - \eta(E, b^-)\}^2$	$\{1 - \eta^2(E, b^-)\}$
16	Eq. (2.40a)	$\left(\frac{d\sigma}{dt}\right)_{\theta=0}$	$\left(\frac{d\sigma}{d\Omega}\right)_{\theta=0}$
29	After Eq. (2.82)		Eq. (2.82) is approximately verified
30	Eq. (3.3)		$- \left[\frac{a}{b}(e^{bt} - 1) + \frac{c}{d}(e^{dt} - 1) \right]$
36	5 from bottom	resonances	resonates
40	Eq. (4.13)	ct^2	$\frac{c}{2} t^2$
53	Eq. (8.4a)	$\{1 - a(E, b)\}$	$\{1 - \text{Re } a(E, b)\}$
54	line 24	$a_\ell = 0$	$a_\ell = 1$
55	Eq. (8.15b)	$f(E, \theta)$	$g(E, \theta = 0)$
55	Eq. (8.17)		$\frac{d\sigma}{d t } \approx (1 - a)^2 \pi R^4 \left[\frac{J_1(R\sqrt{ t })}{R\sqrt{ t }} \right]^2$

<u>Page</u>	<u>Line</u>	<u>Instead of</u>	<u>Please read</u>
65	Eq. (10.3)	$[\ell - n/\alpha'(E_n)]$	$[(\ell - n)/\alpha'(E_n)]$
71	Eq. (10.23)	$\pm \rho$	$\mp \rho$
72	Eq. (10.25)	$\frac{\sqrt{2}}{16\pi}$	$\frac{1}{8\pi s_0^2}$
82	Eq. (13.11)	$\text{Re } \rho_+(m_\pi, 0)$	$\text{Re } \rho_\mp(m_\pi, 0)$
85	Table 13.1	Several numbers are in wrong places	
86	Eq. (14.1)	$\sqrt{2} \left(\frac{d\sigma}{d\Omega} \right)_0 -$	delete
88	line 3	$\ell_0 < 10^{15} \text{ cm}$	$\ell_0 < 10^{-15} \text{ cm}$
87	Eq. (14.4)	$+ \frac{K^2}{16\pi} (\sigma_- - \sigma_+)^2$	$\frac{K^2}{16\pi^2} (\sigma_- - \sigma_+)^2$

PLUS MANY OTHERS

CONTENTS

	Page
1. INTRODUCTION	1
2. GENERAL CONSIDERATIONS	4
2.1 Notations and general formulae	4
2.2 Scattering amplitudes and cross-sections	7
2.3 Unitarity, crossing and analyticity	11
2.4 Potential scattering -- Born approximation	13
2.5 Partial wave expansion	13
2.6 Optical theorem	16
2.7 Impact parameter expansion of the scattering amplitude	16
2.8 Isospin analysis of the pion-nucleon system	18
2.9 Exchange amplitudes	19
2.10 Resonances	20
2.11 Coulomb and nuclear scattering	24
2.12 S and T matrices	25
2.13 High-energy theorems	26
2.14 Implications of unitary symmetries for total cross-sections	28
3. TOTAL CROSS-SECTIONS	29
3.1 Experimental method	29
3.2 Analysis of total cross-section data	31
4. ELASTIC CROSS-SECTIONS	32
4.1 Some experimental examples	33
4.2 Total elastic cross-sections	35
4.3 Angular distributions -- low energies	36
4.4 Angular distributions -- high energies	38
5. CHARGE EXCHANGE SCATTERING	43
5.1 Experimental methods	43
5.2 Analysis of the charge exchange data	44
6. POLARIZATION MEASUREMENTS	45
6.1 Experimental methods in elastic scattering	45
6.2 Experimental methods in charge exchange scattering	46
6.3 Elastic scattering polarization data	46
6.4 Polarization data in charge exchange scattering	48

7. PHASE-SHIFT ANALYSIS	48
7.1 The methods	49
7.2 The results	50
8. OPTICAL MODELS	52
8.1 Comparison with classical optics	52
8.2 Optical model with constant opacity	54
8.3 Optical model with Gaussian opacity	56
8.4 Optical models with more than one potential	57
8.5 The radius of interaction	57
8.6 The opacity	59
8.7 The backward peak	59
8.8 Some considerations on the validity of the optical models	60
9. THE STATISTICAL MODEL	61
9.1 The classical statistical model	61
9.2 Limitations of the statistical interpretation	63
9.3 The inclusion of collective motions	63
10. REGGE POLES	64
10.1 Regge poles in potential scattering	65
10.2 Regge trajectories for the nucleon resonances	67
10.3 The high-energy Regge scattering amplitude	69
10.4 High-energy total cross-sections	71
10.5 High-energy differential cross-sections	71
10.6 High-energy polarization	73
10.7 Further considerations on Regge poles	74
11. ELASTIC SCATTERING AS THE SHADOW OF INELASTIC PROCESSES	76
11.1 Intermediate energies	76
11.2 Very high energies	77
12. MIXED MODELS	77
12.1 Interference model	78
12.2 Diffraction and resonant scattering	79
13. DISPERSION RELATIONS FOR THE FORWARD SCATTERING AMPLITUDE	79
13.1 The basic formulae for πN scattering	80
13.2 The parameters of low-energy pion-nucleon physics	84
15. CONCLUSIONS	88
REFERENCES	89

1. INTRODUCTION

In this review the present phenomenological situation of pion-nucleon elastic scattering will be discussed.

Since the pioneering work of Fermi and co-workers in the early 1950's²³⁰⁻²³³), the pion-nucleon interaction, being fundamental to the understanding of Strong Interactions, has been extensively studied both experimentally and theoretically, as is shown by the large number of references at the end of this review. The experimental techniques, the amount of data, and the theoretical analyses have improved tremendously since those early days. Nevertheless, much more experimental and theoretical work is needed in order to reach a real understanding of the subject.

At present the experimentally accessible quantities are the following: a) π^+p and π^-p total cross-sections; b) differential distributions, integrated values and polarization parameters for the three following elastic and charge exchange processes: $\pi^+p \rightarrow \pi^+p$, $\pi^-p \rightarrow \pi^-p$ and $\pi^-p \rightarrow \pi^0n$. Only the P polarization parameter has been measured until now, but experiments are in progress for the R and A polarization parameters.

Total cross-sections are generally measured by the absorption technique, which is capable of high precisions. Recent measurements at the Brookhaven and Rutherford Laboratories achieved statistical precisions of $\pm 0.05\%$, with absolute scale errors of about 0.3-1.0%.

Differential cross-sections have been measured with counters at synchro-cyclotron energies, with bubble chambers, spark chambers, and counters at proton synchrotron energies. Large counter hodoscopes directly on-line with computers, complicated arrays of spark chambers with automatic analysis, and large bubble chambers have become more and more popular. While bubble chamber data are usually limited by statistics, counter and spark chamber data are limited to a precision of about $\pm 5\%$ by systematic errors. The advent of polarized targets allowed systematic polarization experiments. Their precision is nevertheless still poor (generally not much better than $\pm 15\%$).

Phenomenologically one may divide the pion-nucleon interaction into two energy regions:

- i) the resonance region, from zero to about 5 GeV pion laboratory kinetic energy;
- ii) the high-energy region, for energies higher than 5 GeV.

In the first region the total, elastic, and total charge exchange cross-sections are characterized by structures, a number of bumps usually interpreted as resonances. The cross-sections are, moreover, strongly isospin dependent. The heights of the structures decrease with increasing energy, and then the resonance region smoothly joins the high-energy region, where the cross-sections are slowly decreasing functions of the energy and do not exhibit appreciable structures. At high energies the charge exchange cross-section is approaching zero, as is the polarization.

The resonance region may be further subdivided: the lower energy part, where sophisticated phase-shift analyses were performed, may be called the phase-shift analysis region; at the moment it stops at 2 GeV. The phase-shift analyses have shown that the number of resonances is much larger than the number of peaks observed in total cross-sections; sometimes a single peak may correspond to several resonances in different angular momentum states.

In the first energy region the angular distributions involve a small number of orbital angular momenta. The introduction of isospin, together with the assumption of charge independence, simplifies the analysis. At the very low energy, the phenomenology is dominated by the large $I = J = 3/2$ phase shift, which resonates at a laboratory kinetic energy of 198 MeV. At higher energies, inelastic processes become important; therefore the phase shifts have a large imaginary part and soon the number of parameters for a phase-shift analysis becomes very large. The elastic scattering angular distribution starts to present some of the features that will become dominant at higher energies, i.e. a diffraction peak at forward angles, a bump or structure at $|t| \approx 1.2 \text{ (GeV/c)}^2$ and, in π^+p scattering, a narrow backward bump.

Also the high-energy region may be further subdivided. One may define the asymptotic or very high energy region as that region where: a) the number of open inelastic channels is very large, or b) the pion multiplicity is large, or c) the asymptotic theorems come true, or d) the charge exchange total cross-section is zero, or e) the summation over orbital angular momenta may be replaced with good approximation by an integral.

The asymptotic region may start at pion kinetic energies as large as 1000-10,000 GeV and is therefore outside the experimental possibilities, since available high-energy accelerators allow experiments up to 70 GeV. In the asymptotic region the effects of symmetry-breaking interactions become minimal, and the properties of the underlying symmetries may be revealed.

In the experimentally accessible high-energy region, the elastic angular distribution is dominated by a large diffraction peak that contains most of the elastic cross-section. In order to simplify the discussion, we may divide the high-energy elastic angular distributions into four angular ranges:

- i) The very small angles [$|t| \lesssim 0.01 \text{ (GeV/c)}^2$], where the Coulomb scattering contribution is important and where a Coulomb-nuclear interference can be observed.
- ii) The diffraction region proper [$\sim 0.01 < |t| < 2 \text{ (GeV/c)}^2$]. For $|t| \lesssim 1 \text{ (GeV/c)}^2$, the π^+p diffraction patterns expressed as $(d\sigma/dt)/(d\sigma/dt)_{t=0}$ versus t remain quite invariant from a few GeV up to the highest energies where measurements have been performed.
- iii) The large-angle region [$|t| \gtrsim 2 \text{ (GeV/c)}^2$], where the cross-section is very small and is poorly known experimentally.
- iv) The backward region. A backward peak is present in $\pi^+p \rightarrow \pi^+p$ elastic scattering; for $\pi^-p \rightarrow \pi^-p$ the peak is much less prominent.

At high energy, the charge exchange cross-section is much smaller than the elastic one; its angular distribution is known only in the forward region, where it resembles the elastic one. Contrary to elastic scattering, the charge exchange amplitude has a large contribution from the real part and from spin-flip.

At high energies the polarization of the recoil proton in elastic scattering is of opposite sign for π^+p and π^-p and seems to behave as the logarithmic derivative of the differential cross-section. The amount of polarization is at most 20% and seems to decrease with increasing energy. In charge exchange scattering a 10% polarization is found.

For the theoretical interpretation of pion-nucleon data we may quote some remarks of Van Hove¹⁹²): there is no theory in the real sense of the word, i.e. there is no mathematical

formalism that can give numerical predictions of the experimental quantities via well-defined and consistent procedures. The most comprehensive mathematical formalism would be the S-matrix formalism, which unfortunately has not developed into a theory with real predictive power of the measurable quantities. Since a complete theory for strong interaction does not exist, one is often forced to use plausibility arguments and guided guesses. The interpretation of the experimental results is phenomenological and is carried out with the aid of models.

Most of the theoretical works on pion-nucleon scattering combine specific dynamical models with the general principles of symmetry and analyticity. With a minimum of assumptions these models have been able to derive relations between observables, which generally agree with the experimental results. An example of this is the dispersion relation computation of the real parts of the forward scattering amplitude. It is more difficult to interpret the successes and failures of dynamic models. The validity of a model may rest first on its predicting correctly the experimental observables with the minimum number of parameters, and secondly on its simplicity and the ability of incorporating in a natural way the general principles of symmetry and analyticity. Field theoretical models, like the one-meson exchange models, satisfy the second requirement well, but not the first. In contrast, optical models have more freedom to fit the data, but they do not easily incorporate the symmetry and analyticity principles. The Regge-pole model for high-energy collisions may be a good compromise between these two tendencies. This model has had highs and lows in popularity, and a number of problems are still open. Consequences of unitary symmetries and of the quark model approach to high-energy scattering are still in their development stage.

In a certain sense the situation is similar to that existing in nuclear physics: we have a number of models, which are born in a completely uncorrelated way. Each of these models explains or emphasizes one aspect of the physical situation. Later on, one tries to correlate the models or to explain a process as being due to two or more different mechanisms.

In choosing a logical order for this review, one had the usual problem arising from the interconnections existing among the different subjects. I have chosen to discuss first some general ideas, then the experimental data, and finally their interpretation. This may not be the best approach.

In Section 2 we give some general considerations, define notations, and state some High-Energy Theorems. In general we shall not prove any theorem, unless the proof is necessary for the understanding of the rest of the report. All theoretical considerations are expressed in as simple a form as possible.

In Sections 3 to 6 we shall review the experimental results and discuss preliminary phenomenological analyses.

Section 7 is devoted to phase-shift analysis. In Sections 7 to 12, various models frequently used in the literature are described. In particular we shall discuss the optical, statistical and Regge-pole models. Also mixed or hybrid models will be considered.

The application of forward dispersion relations to pion-nucleon scattering is taken up in Section 13.

We shall assume that the conservation laws usually accepted for strong interactions are valid for pion-nucleon scattering. In particular, we shall make extensive use of isospin conservation. In Section 14 we shall discuss how well these conservation laws are tested in pion-nucleon scattering.

In the concluding remarks we shall try to summarize the present knowledge on pion-nucleon scattering, and shall point out a number of open problems.

We shall assume that the pion has baryonic number $B = 0$, strangeness $S = 0$, isospin $I = 1$, spin $|\vec{S}| = 0$. The nucleon has $B = 1$, $S = 0$, $I = \frac{1}{2}$, $|\vec{S}| = \frac{1}{2}$.

We shall normally use natural units with $\hbar = c = 1$. Notations, constants, and units are tabulated in Tables 2.1 to 2.3.

The survey of the literature was stopped at the end of May 1968. I have tried to be as complete as possible, but if I have failed in this, I apologize for the omissions. The available pion-nucleon data, as well as the bibliography, were collected on punched cards, and arranged as explained in the bibliography section²¹).

2. GENERAL CONSIDERATIONS

2.1 Notations and general formulae

Consider the elastic scattering process $\pi p \rightarrow \pi p$ in the centre-of-mass system (c.m.s.) and in the laboratory system (l.s.) (Fig. 2.1);

Table 2.1
General notations

c.m.s.	centre of mass system (all quantities are normally in the c.m.s.)
l.s.	laboratory system (laboratory quantities have a subscript lab)
$P_{\pi \text{lab}}$	laboratory pion momentum
$T_{\pi \text{lab}}$	" " kinetic energy
$E_{\pi \text{lab}}$	" " total energy
$\theta_{\pi \text{lab}}$	" " scattering angle
$\vec{L}, \ell$	orbital angular momentum
$\vec{S} = \vec{\sigma}/2, s$	spin angular momentum
$\vec{J} = \vec{L} + \vec{S}, j$	total angular momentum
$f = g + i\vec{\sigma} \cdot \vec{n} \hbar$	non-relativistic scattering amplitudes
F, G, H	relativistic scattering amplitudes
$N(\text{mass})$	$I = \frac{1}{2}$ pion-nucleon resonance
$\Delta(\text{mass})$	$I = 3/2$ pion-nucleon resonance
N_{α}	Regge trajectory for nucleon isobars with $I = \frac{1}{2}, P = +, \tau = -$
N_{γ}	" " " " " " $I = \frac{1}{2}, P = -, \tau = +$
Δ_{δ}	" " " " " " $I = 3/2, P = +, \tau = -$
S, P, D, F, G, H	states with orbital angular momenta $\ell = 0, 1, 2, 3, 4, 5$
δ_{2I}, η_{2I}	phase shifts and absorption coefficients for $\ell = 0, j = \frac{1}{2}$ states
$\delta_{2I}, {}_2J, \eta_{2I}, {}_2J$	" " " " " " $\ell \neq 0$ states

Table 2.2
Constants and units

$m_{\pi^{\pm}}$	$(139.579 \pm 0.014) \text{ MeV}$
m_{π^0}	$(134.975 \pm 0.014) \text{ MeV}$
m_p	$(938.256 \pm 0.005) \text{ MeV}$
m_n	$(939.550 \pm 0.005) \text{ MeV}$
m_N	$\frac{1}{2}(m_p + m_n) = 938.903 \text{ MeV}$
$\hbar$	$6.5819 \times 10^{-22} \text{ MeV sec}$
c	$2.997925 \times 10^{10} \text{ cm/sec}$
1 f	$10^{-13} \text{ cm} = (0.197)^{-1} (\text{GeV}/c)^{-1}$
1 mb	$10^{-27} \text{ cm}^2 = 0.1 \text{ f}^2 = (2.57)^{-2} (\text{GeV})^{-2}$
	$(0.1 \text{ GeV})^{-1} = 7 \times 10^{-24} \text{ sec}$
r_{π}	$\hbar/m_{\pi}c \approx \sqrt{2} \text{ f}$
σ_g	$\pi r_{\pi}^2 = 62.8 \text{ mb}$

$\vec{p}_1, \vec{p}_2$ are the three-momenta of the incident particles, and $\vec{p}_3, \vec{p}_4$ are the corresponding outgoing ones. Energy-momentum conservation implies in the c.m.s.: $|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3| = |\vec{p}_4| = |\vec{K}|$. Table 2.3 illustrates the notations that will be used.

Table 2.3
Notations for kinematical quantities

Particle	Mass	l.s.	c.m.s.	Four vectors in arbitrary frame
π_{in}	m_{π}	$(E_{\pi_{\text{lab}}}, \vec{p}_{\pi_{\text{lab}}})$	$(E_{\pi}, -\vec{K})$	p_1
p_{in}	m_p	$(m_p, 0)$	$(E_p, \vec{K})$	p_2
π_{out}	m_{π}	$(E'_{\pi_{\text{lab}}}, \vec{p}'_{\pi_{\text{lab}}})$	$(E'_{\pi}, -\vec{K})$	p_3
N_{out}	m_N	$(E'_{N_{\text{lab}}}, \vec{p}'_{N_{\text{lab}}})$	$(E'_N, \vec{K})$	p_4

Convenient kinematical parameters for describing scattering processes are the Lorentz-invariant quantities (Mandelstam variables):

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \quad (2.1a)$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \quad (2.1b)$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \quad (2.1c)$$

s is the square of the c.m. total energy; t is the square of the four-momentum transfer; and u is the square of the crossed four-momentum transfer. The metric is chosen such that $p_i^2 = m_i^2$. s, t, u are related by

$$s + t + u = \sum_{i=1}^4 m_i^2 . \quad (2.2)$$

If the target and recoil particles are both protons, then:

$$t = -2 m_p T_{p\text{lab}} \quad (2.3a)$$

where m_p is the proton mass and $T_{p\text{lab}}$ is the recoil proton laboratory kinetic energy. For elastic scattering ($m_1 = m_3, m_2 = m_4$)

$$t = -2 k^2 (1 - \cos \theta) , \quad (2.3b)$$

where θ is the c.m. scattering angle. Formula (2.3a) may be obtained immediately when t is written in the lab. system; analogously formula (2.3b) is easily derived in the c.m. frame. For small scattering angles, formula (2.3b) reduces to

$$t \underset{\theta \ll 1}{\approx} -k^2 \theta^2 \approx -p_{\pi\text{lab}}^2 \theta_{\pi\text{lab}}^2 . \quad (2.3c)$$

[This approximate formula for t differs from the correct one by 1% at $\theta = 20^\circ$ and by 4% at $\theta = 40^\circ$. At $p_{\pi\text{lab}} = 12 \text{ GeV/c}$, $\theta = 20^\circ$ corresponds to $t = 0.67 (\text{GeV/c})^2$, while $\theta = 40^\circ$ corresponds to $t = 2.6 (\text{GeV/c})^2$. Therefore this approximation is quite good in the high-energy diffraction region]. The analogous formulae for u are:

$$u = m_\pi^2 + m_p^2 - 2m_p E_{p\text{lab}} \quad (2.4a)$$

$$u = -2 k^2 (1 + \cos \theta) + \frac{(m_p^2 - m_\pi^2)^2}{s} \quad (2.4b)$$

$$u \underset{\theta \ll 1}{\approx} -k^2 (4 - \theta^2) + \frac{(m_p^2 - m_\pi^2)^2}{s} . \quad (2.4c)$$

Formula (2.4a) is valid when target and recoil particles are both protons, while formula (2.4b) is valid only for elastic scattering, and formula (2.4c) is only an approximation of formula (2.4b), valid for small scattering angles.

All kinematical variables may be expressed in terms of two of them. We may choose the two independent variables as s and t or $\sqrt{s}$ and θ , etc., according to convenience.

In the general study of the interactions among four particles, one may form different combinations corresponding to different two-body reactions, for each of which one may define the invariants s, t, u, which assume a different meaning in every channel. Figure 2.2 illustrates the situation with a Feynman diagram.

For the s-channel the reaction described by Fig. 2.2 is the elastic reaction:

$$\pi^- p \rightarrow \pi^- p . \quad (2.5a)$$

We may denote the physical quantities with a subscript to indicate the different channels.

In the s-channel: s_s is the square of the total c.m. energy; t_s is the square of the four-momentum transfer, and u_s is the square of the crossed momentum transfer.

For the t-channel the reaction described by Fig. 2.2 is the annihilation reaction:

$$p\bar{p} \rightarrow \pi^+\pi^- . \quad (2.5b)$$

Its energy and momentum transfer variables are s_t, t_t, u_t . Their relations with the s-channel variables are: $s_t = t_s$, $t_t = s_s$ and $u_t = u_s$.

For the crossed or u-channel, the reaction described by Fig. 2.2 is the elastic reaction:

$$\pi^+p \rightarrow \pi^+p . \quad (2.5c)$$

Its energy and momentum variables are s_u, t_u, u_u . Their relations with the s-channel variables are: $s_u = u_s$, $t_u = t_s$, and $u_u = s_s$.

The variables for the s-, t-, and u-channels display the following exchange of roles (in terms of the s-channel variables denoted as s,t,u without the s-subscripts:

Table 2.4
Exchange of roles of s,t,u in terms of the s-channel variables

	s-channel	t-channel	u-channel
Energy variable	$s > 0$	$t > 0$	$u > 0$
Momentum variable	$t < 0$	$s < 0$	$t < 0$
Cross-momentum variable	$u \geq 0$	$u < 0$	$s \leq 0$

With reference to the two body-reactions described by Fig. 2.2, the physical regions for the three channels can be drawn in two dimensions. The three axes s, t, and u can be oriented mutually 120° out of phase with units appropriate to satisfy Eq. (2.2). The boundaries for the physical regions for each channel are found by the physical conditions:

- a) energy variable $\geq (\text{sum of incoming masses})^2$
  - b) $-1 \leq \cos \theta \leq +1$.
- (2.6)

The physical regions are shown in Fig. 2.3 as shaded areas. Note that the three physical regions do not overlap.

2.2 Scattering amplitudes and cross-sections

We shall treat the collision process of a spin-zero particle from a spin $\frac{1}{2}$ one in the centre-of-mass system. The frame of reference is chosen as in Fig. 2.4: the collision takes place at the origin, the z-axis is chosen along the incident particle momentum, and the x-axis in the plane of the drawing which is made to coincide with the plane of scattering. We shall assume: a) that the scattering process has cylindrical symmetry; therefore the measurable quantities do not depend on the azimuthal angle ϕ ; b) that the interaction

is of finite range; therefore the particles can be considered as free outside the interaction region, of radius R .

Let us specify the reactions initiated by π^- 's on protons:

By neglecting the spin, under the previous assumptions the asymptotic wave function is:

$$\psi(\vec{r}) \underset{r \gg R}{\approx} e^{ikz} + f(E, \theta) \frac{e^{ikr}}{r} + f_{ce}(E, \theta) \frac{e^{ik'r}}{r} + \dots, \quad (2.7)$$

where:

- i) e^{ikz} represents the incoming plane wave, which passes the scattering centre almost unchanged.
- ii) $[f(E, \theta) \exp(ikr)/r]$ represents an outgoing spherical wave, which describes elastic scattering; $f(E, \theta)$ is the elastic scattering amplitude, which is a function of two variables, for instance the c.m. total energy E and the c.m. scattering angle θ . Notice that for elastic scattering the magnitude of the momentum $|k|$ remains unchanged during the collision.
- iii) $[f_{ce}(E, \theta) e^{ik'r}/r]$ represents an outgoing spherical wave, which describes the charge exchange process. $f_{ce}(E, \theta)$ is the scattering amplitude for charge exchange. Because of the $\pi^- - \pi^0$ and $n - p$ mass differences, the magnitudes of the momenta of the particles change in the scattering: k' is the momentum of the π^0 after scattering.
- iv) Other outgoing spherical waves represent the various inelastic processes.

The nucleon has spin $s_p = \frac{1}{2}$ with components $m_s = \pm \frac{1}{2}$. The total angular momentum is the sum of the orbital and the spin angular momenta:

$$\vec{J} = \vec{L} + \vec{S}. \quad (2.8)$$

$\vec{S}$ may be written as $\vec{S} = \vec{\sigma}/2$, where $\vec{\sigma}$ represents the Pauli spin matrices:

$$\begin{aligned}\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma_{\pm} &= \sigma_x \pm i\sigma_y.\end{aligned} \quad (2.9)$$

The spin function is described by the Pauli spinors:

$$\xi\left(\frac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi\left(-\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.10)$$

In the presence of the nucleon spin, the asymptotic wave function (2.7) is modified as follows: the incident plane wave is multiplied by the spinor $\xi(m_s)$, which in general will

be a linear combination of the two possible spin states. The outgoing spherical waves have one part where the spin of the nucleon is unchanged (spin-non-flip) and another where the spin is reversed (spin-flip):

$$\psi(\vec{r}) \underset{r \gg R}{\approx} e^{ikz} \xi(m_s) + \left[g(E, \theta) \xi(m_s) + h(E, \theta) \xi(-m_s) \right] \frac{e^{ikr}}{r} + \dots \quad (2.11)$$

The spin-non-flip and spin-flip amplitudes g and h are found to be independent of m_s . Moreover, $\xi(-m_s)$ may be changed into $\xi(m_s)$ by applying to $\xi(-m_s)$ the $\vec{\sigma} \cdot \vec{n}$ operator (where $\vec{n}$ is the normal to the scattering plane):

$$2m_s i \xi(m_s) = \xi(-m_s) \quad (2.12)$$

In conclusion, formula (2.11) becomes:

$$\psi(\vec{r}) \underset{r \gg R}{\approx} \left[e^{ikz} + f(E, \theta) \frac{e^{ikr}}{r} + \dots \right] \xi(m_s) + \dots, \quad (2.13)$$

where

$$f(E, \theta) = g(E, \theta) + i \vec{\sigma} \cdot \vec{n} h(E, \theta). \quad (2.14)$$

This form is applicable also to charge exchange scattering.

The differential cross-section for elastic scattering is given by

$$\left(\frac{d\sigma}{d\Omega} \right)_{el} = |f_{el}(E, \theta)|^2, \quad (2.15)$$

and for charge exchange:

$$\left(\frac{d\sigma}{d\Omega} \right)_{ce} = \frac{k''}{k} |f_{ce}(E, \theta)|^2. \quad (2.16)$$

The connection with the cross-section expressed as $(d\sigma/d|t|)$ is:

$$\left(\frac{d\sigma}{d|t|} \right)_{el} = \frac{\pi}{k^2} \left(\frac{d\sigma}{d\Omega} \right)_{el}. \quad (2.17)$$

The previous formulae contain invariant variables, such as t , and c.m. variables, such as θ and E . If we use only relativistic invariants and redefine the normalization of the scattering amplitude, we have:

$$F_{el}(s, t) = 8\pi \sqrt{s} f_{el}(E, \theta) \quad (2.18)$$

$$\left(\frac{d\sigma}{d|t|} \right)_{el} = \frac{1}{64\pi s k^2} |F_{el}(s, t)|^2. \quad (2.19a)$$

At high energies $k \approx \sqrt{s}/2$, therefore:

$$\left(\frac{d\sigma}{d|t|} \right)_{el} \underset{s \gg}{\approx} \frac{1}{16\pi s^2} |F_{el}(s, t)|^2. \quad (2.19b)$$

We shall denote with capital letters the relativistic expressions of the scattering amplitudes; that is F , G , and H are the relativistic expressions analogous to the non-relativistic expressions f , g , and h .

The explicit calculation of Eq. (2.15) in terms of g and h requires writing the spinor $\xi(m_s)$ in terms of the eigenvectors $X(\pm\frac{1}{2})$ of the operator $\sigma_y = \vec{\sigma} \cdot \vec{n}$. After tedious calculations one obtains the formulae for the following types of measurements:

1. Unpolarized differential cross-section:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unp}} = |f(E, \theta)|^2 = |g|^2 + |h|^2. \quad (2.20)$$

2. The polarization parameter P is defined as:

$$P = \frac{2 \operatorname{Im} (g h^*)}{|g|^2 + |h|^2}. \quad (2.21a)$$

P may be measured in two ways, by using either polarized or unpolarized targets:

- a) In the elastic scattering off an unpolarized target, the recoil nucleon may be polarized only in the direction perpendicular to the scattering plane. The state of polarization of the recoil proton is measured through a second scattering off a spin-zero nucleus, such as carbon (Fig. 2.5a). If P_C is the carbon analysing power, the measurement of the right-left scattering asymmetry yields:

$$P P_C = \frac{N_r - N_\ell}{N_r + N_\ell}. \quad (2.21b)$$

where N_r is the number of events for which, in the proton-nucleus collision, the proton recoils at an angle θ to the right, and N_ℓ is the number of events for which the proton recoils at the same angle θ , but to the left. In analogy to classical optics, we may call polarizer the first scattering and analyser the second scattering off carbon.

- b) P may also be measured by scattering pions on a hydrogen target, polarized perpendicular to the scattering plane (Fig. 2.5b). One measures the number of events with a scattering angle θ when the target polarization is up, along the positive y -axis (N_u), and when it is down, reversed (N_d). The measured asymmetry is given by:

$$P P_{Ty} = \frac{N_u - N_d}{N_u + N_d}, \quad (2.22)$$

where P_{Ty} is the hydrogen target polarization in the y direction. The polarized differential cross-section $(d\sigma/d\Omega)_p$ is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_p = \left(\frac{d\sigma}{d\Omega}\right)_{\text{unp}} (1 + P P_{Ty}). \quad (2.23)$$

On the grounds of general symmetry principles, one may prove that in the experiments (a) and (b) the same P -parameter is being measured.

3. The A polarization parameter is defined non-relativistically as:

$$A \approx \frac{2 \operatorname{Re} (g h^*)}{|g|^2 + |h|^2} . \quad (2.24a)$$

It may be measured in an experiment where the pion is scattered off a target proton polarized in the scattering plane, perpendicular to the incident pion (Fig. 2.5c). The transverse polarization of the recoil proton, in the scattering plane is then measured with a second scattering off a carbon nucleus. The measured asymmetry is given by:

$$A P_{Tx} P_C = \frac{N_u - N_d}{N_u + N_d} , \quad (2.24b)$$

where P_{Tx} is the target polarization in the x-direction, P_C is the carbon analysing power, and N_u, N_d are the number of events at an angle θ , when the target proton is polarized along the positive or negative x-axis.

4. The R polarization parameter is defined non-relativistically as:

$$R \approx \frac{|g|^2 - |h|^2}{|g|^2 + |h|^2} . \quad (2.25a)$$

It may be measured in an experiment where the pion incident in the z-direction is diffused by a target proton, polarized in the same z-direction (Fig. 2.5d). The transverse polarization of the recoil proton is measured with a second scattering off a carbon nucleus. The measured asymmetry is given by:

$$R P_{Tz} P_C = \frac{N_u - N_d}{N_u + N_d} , \quad (2.25b)$$

where P_{Tz} is the target polarization in the z-direction; N_u and N_d are the number of events at an angle θ , when the target proton is polarized along the positive or negative z-axis.

The three parameters, A, P, R, are not independent. It is in fact easy to prove that

$$P^2 + A^2 + R^2 = 1 . \quad (2.26)$$

At a certain energy and scattering angle, the scattering amplitude (2.14) is determined by four real numbers (the real and imaginary parts of g and h). Therefore the three independent measurements, which are in principle possible [for instance $(d\sigma/d\Omega)_{\text{unp}}$, P , and A] leave undetermined one parameter, for instance the relative phase of g and h . In practice, the scattering amplitudes for different processes are connected by charge independence, and therefore there will be an over-determination.

2.3 Unitarity, crossing and analyticity

The general properties that are usually assumed to be valid for the scattering amplitude are in decreasing order of likely validity: a) unitarity, b) crossing, and c) analyticity. We shall now briefly discuss the meaning and some of the consequences of these properties.

2.3.1 Unitarity

The unitarity condition expresses the general law of conservation of probability. Mathematically it is best expressed in terms of unitary operators, such as the S and T matrices (Section 2.13). An important consequence of unitarity is the optical theorem (Section 2.7), which is solely the consequence of the conservation of probability.

2.3.2 Crossing

Crossing symmetry relates the interactions in the s-, t- and u-channels by telling us that the same analytic function of s, t, u describes the three processes. Since the physical regions for the three processes do not overlap, one must have a way of connecting the scattering amplitudes in the three regions: that is, the scattering amplitudes must satisfy some analyticity properties, in particular analytic continuation.

Let us denote by F_s , F_t , and F_u the relativistic scattering amplitudes for the s-, t-, and u-channels. The crossing relations connect these three amplitudes by exchanging the roles of the variables. They are best expressed in terms of particle four-momenta by using the following substitution rule: change the sign of antiparticle momenta, but not of the particle momenta. For instance, with respect to diagram 2.2, the s- and t-channel amplitudes are connected by:

$$F_t(P_3, \bar{P}_1; \bar{P}_4, P_2) = F_s(P_3, -P_1; -P_4, P_2). \quad (2.27a)$$

In terms of the Mandelstam variables:

$$F_t(s_t, t_t) = F_s(s_s = t_t, t_s = s_t). \quad (2.27b)$$

Analogous formulae are valid for the F_u amplitude. The presence of spin and isospin complicates the situation.

The crossing relation (or crossing symmetry) is a powerful theoretical tool, since it is sufficient to determine the scattering amplitude in one channel in order to know the scattering amplitudes in the other channel. Practical applications in the comparison of experimental data are of a much more limited use since direct comparison is not possible. The comparison requires an hypothesis or a model of the analytic continuation; moreover, because of the extrapolations involved, small errors in the input can be magnified out of proportion in the output.

2.3.3 Analyticity

One can list different kinds of analyticity properties of the scattering amplitude, depending on the general postulates from which one starts: 1) analytic continuation, as discussed in connection with crossing symmetry; 2) analyticity in the upper half of the complex energy plane; this condition is necessary for the validity of forward dispersion relations (Section 13); 3) maximal analyticity properties: the scattering amplitude possesses only those singularities that are required by the unitarity condition, which must be satisfied in all three channels (s, t, u) simultaneously. This condition allows the writing of the Mandelstam representation of the scattering amplitude, for which we refer to specialized articles.

2.3.4 Singularities of the scattering amplitude

For each energy threshold allowing a new channel of interaction, the scattering amplitude has a singularity in the energy variable. These singularities are simple poles in the case of bound states; they are a string of singularities (branch cuts) above the threshold for elastic scattering.

Figure 2.6 shows diagrammatically, in the complex s -plane, the location of the singularities in the case of the s - and u -channels. In the s -channel one has a simple pole for $s = m_N^2$ corresponding in perturbation theory to the diagram of Fig. 2.7a. The elastic cut starts at $s = (m_p + m_\pi)^2$, and is usually called the right-hand cut. A similar situation exists for the u -channel: a pole for $u = m_N^2$ (corresponding to Fig. 2.7b) and a cut for $u \geq (m_p + m_\pi)^2$. For a fixed t , one may obtain the equivalent s -values from Eq. (2.2). In Fig. 2.6 the crossed pole and branch cut (left-hand cut) are shown in the s -plane for small negative values of t ($t < 0$ is a physical condition for both s - and u -channels).

2.4 Potential scattering -- Born approximation

In solving a classical low-energy scattering problem, we may use the Born approximation if the scattering potential represents a small perturbation with respect to the total energy of the particles, that is if $V(r) \ll \text{kinetic energy} \approx p^2/2m \approx p^2/m$. Or, by using the uncertainty relation $p \approx \hbar/R$, one should have $V(r) \ll (\hbar^2/mR^2) = (\hbar/R)v$. R is the radius of interaction; m and v are the mass and velocity of the particle. The first-order Born approximation gives for the scattering amplitude:

$$f(K, \theta) = - \frac{\mu}{2\pi \hbar^2} \int V(r) \exp [i(\vec{K} - \vec{K}') \cdot \vec{r}] d^3r \quad (2.28a)$$

$$= \frac{2\mu}{|\vec{K} - \vec{K}'|} \int_0^\infty r V(r) \sin(|\vec{K} - \vec{K}'|r) dr, \quad (2.28b)$$

where μ is the reduced mass and $|\vec{K} - \vec{K}'| = 2K \sin(\theta/2)$.

As an example, a Gaussian potential

$$V = V_0 \exp [-3/2(r^2/R^2)], \quad (2.29)$$

where R^2 is the mean square radius of the potential, yields:

$$\frac{d\sigma}{d|\mathbf{t}|} = \frac{2\pi^2 \mu^2 V_0^2 R^6}{27 p^2 \hbar^4} \exp [-1/3 t R^2]. \quad (2.30)$$

Many of the expressions we shall obtain for the scattering amplitude contain the first-order Born approximation as their first-order approximation.

2.5 Partial wave expansion

In a scattering process, incoming and outgoing waves are eigenstates of L_z , with eigenvalue zero, and of L^2 with eigenvalues $\ell(\ell+1)$. Consequently they can be expanded in terms of the Legendre polynomials $P_\ell(\cos \theta)$. For spinless particles, the elastic scattering amplitude $f(E, \theta)$ may be written as:

$$f(E, \theta) = \frac{1}{K} \sum_{\ell=0}^{\infty} (2\ell + 1) f_{\ell}(E) P_{\ell}(\cos \theta), \quad (2.31)$$

where the ℓ^{th} partial wave amplitude $f_{\ell}(E)$ is given by:

$$f_{\ell}(E) = \frac{a_{\ell}(E) - 1}{2i} = \frac{\exp \{2i[\delta_{\ell}(E) + i \epsilon_{\ell}(E)]\} - 1}{2i} = \frac{\eta_{\ell}(E) \exp [2i\delta_{\ell}(E)] - 1}{2i}. \quad (2.32)$$

$a_{\ell}-1$ is sometimes called the opacity of the ℓ^{th} wave; $\delta_{\ell}(E)$ and $\eta_{\ell}(E) = \exp [-2i\epsilon_{\ell}(E)]$ are the ℓ^{th} phase shift and the absorption parameter, respectively; both $\delta_{\ell}(H)$ and $\eta_{\ell}(E)$ are real quantities. In the absence of absorption, that is if only the elastic channel is open, one has $\eta_{\ell}(E) = 1$; in the general case $0 < \eta_{\ell} \leq 1$.

A formula similar to (2.31) may be written for the charge exchange amplitude.

The scattering process may be seen from a different point of view. To this end let us expand at large distances the incoming plane wave in terms of Legendre polynomials:

$$e^{ikz} \xrightarrow{r \gg} \frac{1}{2iK} \sum_{\ell=0}^{\infty} (2\ell + 1) \left[-(-1)^{\ell} \frac{e^{-ikr}}{r} + \frac{e^{ikr}}{r} \right] P_{\ell}(\cos \theta). \quad (2.33)$$

Equation (2.33) tells us that, at large distances, an incoming plane wave may be written as a superposition of all possible orbital angular momenta, and that each partial wave is a linear combination of an incoming and an outgoing spherical wave.

The total wave function in the presence of a scattering centre may now be written as:

$$\psi_{\text{tot}}(\vec{r}) \xrightarrow{r \gg} \frac{1}{2iK} \sum_{\ell=0}^{\infty} (2\ell + 1) \left[-(-1)^{\ell} \frac{e^{-ikr}}{r} + a_{\ell}(E) \frac{e^{ikr}}{r} + \sqrt{\frac{K}{K''}} f_{\ell,ce}(E) \frac{e^{ik''r}}{r} + \dots \right] P_{\ell}(\cos \theta). \quad (2.34)$$

In formula (2.34) each partial wave is the sum of an incoming spherical wave and of many outgoing spherical waves describing elastic scattering, charge exchange, etc. The square of the coefficient for the incoming spherical wave gives the probability of finding an incoming system with orbital angular momentum ℓ . In the same way, the square of the coefficient in front of every outgoing wave gives the probability of finding that particular final state. As a consequence of the conservation of probability and angular momentum, the sum of the probabilities for the various final states with orbital angular momentum ℓ should be equal to the probability of finding the initial system with the same orbital angular momentum ℓ ; therefore from Eq. (2.34) one has:

$$|a_{\ell}(E)|^2 + |f_{\ell}(E)_{ce}|^2 + \dots = 1. \quad (2.35)$$

Formula (2.35) expresses the conservation of probability for the ℓ^{th} partial wave and is usually referred to as the partial wave unitarity condition. Inelastic channels contribute to formula (2.35) in a form similar to the charge exchange contribution.

The physical picture underlying Eqs. (2.33) to (2.35) is the following: the incoming spherical wave is modified by the interaction, and appears as an outgoing spherical wave having in general a different phase, represented by the phase shift $\delta_\ell(E)$, and a smaller amplitude, given by the absorption coefficient.

We shall now sketch the procedure used in deriving the partial wave expansion for the scattering amplitude in the presence of spin. In this case, for each ℓ -value of the orbital angular momentum, we have two values ($j = \ell \pm \frac{1}{2}$) for the total angular momentum. Both the total and orbital angular momenta are conserved, the conservation of the orbital angular momentum following from the conservation of the total angular momentum and the conservation of parity. If we expand the wave function in terms of eigenfunctions of both the total and orbital angular momenta with eigenvalues j and ℓ , then each partial wave may be treated separately, as was done in the spinless case.

We may arrive at the partial wave expansion in two steps:

1) by expansion of ψ_{in} in terms of eigenfunctions of j and ℓ ; 2) by considering that each incoming wave, with a definite ℓ and j , is modified by the interaction so as to have a phase shift and an absorption. One may either use the technique of Clebsch-Gordan coefficients or the projection operator technique. Formulae (2.31), (2.33), and (2.34) will now have to be multiplied by the spinor wave function. With the projection operator technique, one introduces in the ingoing spherical wave of Eq. (2.34) the sum of the projection operators, which project out of a state with eigenvalue ℓ the two eigenstates with $j = \ell \pm \frac{1}{2}$. These two states then appear as outgoing waves with a phase shift and an absorption.

The final result is (for elastic scattering only):

$$g(E, \theta) = \frac{1}{K} \sum_{\ell=0}^{\infty} [(\ell + 1) f_{\ell+}(E) + \ell f_{\ell-}(E)] P_\ell(\cos \theta) \quad (2.36a)$$

$$h(E, \theta) = \frac{\sin \theta}{K} \sum_{\ell=1}^{\infty} [f_{\ell+}(E) - f_{\ell-}(E)] \cdot \frac{dP_\ell(\cos \theta)}{d \cos \theta}, \quad (2.36b)$$

where the subscript $\ell^+(\ell^-)$ indicates the state with $j = \ell + \frac{1}{2}$ ($j = \ell - \frac{1}{2}$). A similar equation would apply to the charge exchange reaction, replacing $1/K$ with $1/\sqrt{KK'}$.

In terms of the partial wave amplitudes $f_{\ell+}$ and $f_{\ell-}$, the elastic, inelastic, and total cross-sections, σ_{el} , σ_{in} , σ_{tot} , may be written as:

$$\sigma_{el}(E) = \frac{4\pi}{K^2} \sum_{\ell=0}^{\infty} [(\ell + 1) |f_{\ell+}|^2 + \ell |f_{\ell-}|^2] \quad (2.37a)$$

$$\sigma_{in}(E) = \frac{4\pi}{K^2} \sum_{\ell=0}^{\infty} [(\ell + 1)(1 - \eta_{\ell+}^2) + \ell(1 - \eta_{\ell-}^2)] \quad (2.37b)$$

$$\sigma_{tot}(E) = \sigma_{el}(E) + \sigma_{in}(E) = \frac{4\pi}{K^2} \sum_{\ell=0}^{\infty} [(\ell + 1) \text{Im } f_{\ell+} + \ell \text{Im } f_{\ell-}]. \quad (2.37c)$$

2.6 Optical theorem

Since $P_\ell(1) = 1$, Eqs. (2.36a,b) for $\theta = 0$ give:

$$g(E,0) = \frac{1}{K} \sum_{\ell=0}^{\infty} [(\ell+1) f_{\ell+} + \ell f_{\ell-}]$$

$$h(E,0) = 0.$$
(2.38)

A comparison of Eq. (2.38) with Eq. (2.37c) yields the relation:

$$\text{Im } g(E,0) = \frac{K}{4\pi} \sigma_{\text{tot}}.$$
(2.39)

This equation is the expression of the optical theorem, which is a way of expressing the attenuation of the incoming beam in the forward direction. In the most general proof, Eq. (2.39) may be obtained from the unitarity of the S-matrix.

As a consequence of Eq. (2.39) we have^{*)}:

$$\left(\frac{d\sigma}{dt} \right)_{\theta=0} = |\text{Re } g(E,0)|^2 + \frac{K^2}{16\pi^2} \sigma_{\text{tot}}^2$$
(2.40a)

$$\left(\frac{d\sigma}{d|t|} \right)_{t=0} = \frac{\pi}{K^2} |\text{Re } g(E,0)|^2 + \frac{\sigma_{\text{tot}}^2}{16\pi}.$$
(2.40b)

2.7 Impact parameter expansion of the scattering amplitude

At high energies many partial waves must be considered in Eqs. (2.36a,b). Semiclassically, $\ell_{\text{max}} \approx KR$. If $R = 1 \text{ f}$, an incoming pion with a laboratory momentum of 50 GeV/c has $K = 4.8 \text{ GeV/c}$ and $\ell_{\text{max}} \approx 24$: at least 24 waves must be considered in Eqs. (2.36a,b). In this case it is often more convenient to use an integral representation of the scattering amplitude, known as the impact parameter expansion. We now proceed to derive this expansion from the partial wave formulae, though the equation we shall obtain could in reality be defined independently.

For large values of K , the wavelength of the incident wave $\lambda = 1/K$ is much smaller than the range of the forces responsible for the scattering. We approach the limit of geometrical optics and a picture such as Fig. 2.8 is now valid. The impact parameter b is defined as the minimum distance from the line of flight of the incident particle to the scattering centre. We may define:

$$\ell + \frac{1}{2} = Kb$$

$$d\ell = Kdb.$$
(2.41)

If we put Eqs. (2.41) into Eqs. (2.36a,b), and replace the summation over ℓ with an integral over b , we have:

$$g(E,\theta) = \int_0^\infty db \left[\left(Kb + \frac{1}{2} \right) f(E,b^+) + \left(Kb - \frac{1}{2} \right) f(E,b^-) \right] P_b(\cos \theta)$$
(2.42a)

---

*) Numerically $(d\sigma/dt)_{t=0} [\text{mb}/(\text{GeV}/c)^2] = 5.095 \times 10^{-2} \sigma_{\text{tot}}^2 (\text{mb})^2$.

$$h(E, \theta) = \sin \theta \int_0^{\infty} db [f(E, b^+) - f(E, b^-)] \frac{dP_b(\cos \theta)}{d \cos \theta}. \quad (2.42b)$$

In order to write Eqs. (2.42a,b), we must assume: a) that the Legendre polynomials can be generalized for all positive values of ℓ , not only integers; in other words, the Legendre polynomials become the Legendre functions; b) that the partial wave amplitudes f_ℓ could be generalized or interpolated for all positive values of ℓ , not only integers; we express this condition by writing $f(E, b)$ instead of $f_\ell(E)$.

The first two terms of the power series expansions of $P_\ell(\cos \theta)$ and $J_0(\ell\theta)$ are the same. The higher-order terms are negligible at small angles and high energies, where one may write:

$$P_\ell(\cos \theta) \underset{\substack{\theta \gg \\ \ell \ll}}{\approx} J_0(\ell\theta) \approx J_0(Kb\theta) \quad (2.43a)$$

$$\frac{dP_\ell(\cos \theta)}{d(\cos \theta)} \underset{\substack{\theta \gg \\ \ell \ll}}{\approx} \frac{\ell}{2 \sin(\theta/2)} J_1\left(2\ell \sin \frac{\theta}{2}\right) \approx \frac{Kb}{\theta} J_1(Kb\theta). \quad (2.43b)$$

Substitution of these relations into Eqs. (2.42a,b), neglecting $\frac{1}{2}$ compared to Kb , yields:

$$g(E, \theta) \underset{\substack{\theta \gg \\ \ell \ll}}{\approx} K \int_0^{\infty} b db [f(E, b^+) + f(E, b^-)] J_0(Kb\theta) \quad (2.44a)$$

$$h(E, \theta) \underset{\substack{\theta \gg \\ \ell \ll}}{\approx} K \cos \frac{\theta}{2} \int_0^{\infty} b db [f(E, b^+) - f(E, b^-)] J_1(Kb\theta). \quad (2.44b)$$

These formulae represent the impact parameter expansions of the scattering amplitude. Their validity is greater than what our derivation shows. Mathematically, formulae (2.44a,b) constitute a Fourier-Bessel integral representation of the scattering amplitude.

The elastic, inelastic, and total cross-sections (2.37a,b,c) become:

$$\sigma_{ei}(E) \approx 4\pi \int_0^{\infty} b db [|f(E, b^+)|^2 + |f(E, b^-)|^2] \quad (2.45a)$$

$$\sigma_{in}(E) \approx 4\pi \int_0^{\infty} b db [1 - \eta^2(E, b^+)] + [1 - \eta(E, b^-)]^2] \quad (2.45b)$$

$$\sigma_{tot}(E) \approx 4\pi \int_0^{\infty} b db [\text{Im} \{f(E, b^+) + f(E, b^-)\}]. \quad (2.45c)$$

In order to obtain explicit values from the above formulae, we need a model for the strong interaction structure of the target particle. One may make as many models as those used for the Electromagnetic Structure of Nuclei⁵). In Chapter 8 we shall discuss some of these models. They usually assume that the interaction is spin-independent: that is $f(E, b^+) = f(E, b^-)$.

2.8 Isospin analysis of the pion-nucleon system

In pion-nucleon scattering the following 10 two-body interactions are in principle allowed:

$$\begin{aligned}
 \pi^+p &\rightarrow \pi^+p & (a) \\
 \pi^+n &\rightarrow \pi^+n & (b) \\
 &\rightarrow \pi^0p & (c) \\
 \pi^0p &\rightarrow \pi^0p & (d) \\
 &\rightarrow \pi^+n & (e) \\
 \pi^0n &\rightarrow \pi^0n & (f) \\
 &\rightarrow \pi^-p & (g) \\
 \pi^-p &\rightarrow \pi^-p & (h) \\
 &\rightarrow \pi^0n & (i) \\
 \pi^-n &\rightarrow \pi^-n & (l)
 \end{aligned} \tag{2.46}$$

Reactions (c), (e), (g) and (i) are simply one the inverse of the other, so that one is left with eight independent scattering processes. The principle of charge independence for the strong interactions applied to the above reactions implies that only two of them are independent; the scattering amplitudes for reactions (2.46a-l) may be expressed as linear combinations of two of them.

Only three of the above scattering processes are accessible experimentally (a,h,i). The other scatterings involve either the π^0 , which is too-short-lived, or a target neutron, which is found only bound in nuclei, and this complicates the analysis.

In the isotopic-spin formalism, the principle of charge independence means that strong interactions depend on the total isospin, and not on the z-components. Since the pion has $I_\pi = 1$ and the nucleon has $I_N = \frac{1}{2}$, the total isospin for the pion-nucleon system may be only $\frac{1}{2}$ or $\frac{3}{2}$. The six possible pion-nucleon states may be written in terms of the $I = \frac{1}{2}$ and $\frac{3}{2}$ isospin states, denoted by $|\frac{1}{2}\rangle$ and $|\frac{3}{2}\rangle$, using the addition rules of angular momenta and the Clebsch-Gordan coefficients:

$$\begin{array}{ccc}
 \underline{I_z(\pi)} & \underline{I_z(N)} & \underline{I_z(\pi + N)} \\
 +1 & +\frac{1}{2} & |\pi^+p\rangle = |\frac{3}{2}\rangle
 \end{array} \tag{2.47a}$$

$$\begin{array}{ccc}
 & -\frac{1}{2} & |\pi^+n\rangle = \sqrt{\frac{1}{3}} |\frac{3}{2}\rangle + \sqrt{\frac{2}{3}} |\frac{1}{2}\rangle
 \end{array} \tag{2.47b}$$

$$\begin{array}{ccc}
 0 & +\frac{1}{2} & |\pi^0p\rangle = \sqrt{\frac{2}{3}} |\frac{3}{2}\rangle - \sqrt{\frac{1}{3}} |\frac{1}{2}\rangle
 \end{array} \tag{2.47c}$$

$$\begin{array}{ccc}
 & -\frac{1}{2} & |\pi^0n\rangle = \sqrt{\frac{2}{3}} |\frac{3}{2}\rangle + \sqrt{\frac{1}{3}} |\frac{1}{2}\rangle
 \end{array} \tag{2.47d}$$

$$\begin{array}{ccc}
 -1 & +\frac{1}{2} & |\pi^-p\rangle = \sqrt{\frac{1}{3}} |\frac{3}{2}\rangle - \sqrt{\frac{2}{3}} |\frac{1}{2}\rangle
 \end{array} \tag{2.47e}$$

$$\begin{array}{ccc}
 & -\frac{1}{2} & |\pi^-n\rangle = |\frac{3}{2}\rangle
 \end{array} \tag{2.47f}$$

The scattering amplitudes f_+ , f_- , and f_0 for the three measurable processes are:

$$f_+(E, \theta) = \langle \pi^+ p | H | \pi^+ p \rangle = f_{3/2} \quad (2.48a)$$

$$f_-(E, \theta) = \langle \pi^- p | H | \pi^- p \rangle = \frac{1}{3} f_{3/2} + \frac{2}{3} f_{1/2} \quad (2.48b)$$

$$f_0(E, \theta) = \langle \pi^0 n | H | \pi^- p \rangle = \frac{\sqrt{2}}{3} (f_{3/2} - f_{1/2}), \quad (2.48c)$$

where $f_{3/2}$ and $f_{1/2}$ are the scattering amplitudes for the isospin states $I = 3/2$ and $I = 1/2$, respectively. Only two of the three scattering amplitudes f_+ , f_- , and f_0 are independent. The relation between them is:

$$\sqrt{2} f_0(E, \theta) = f_+(E, \theta) - f_-(E, \theta). \quad (2.49)$$

The elastic scattering cross-sections are:

$$\frac{d\sigma}{d\Omega}(\pi^+ p \rightarrow \pi^+ p) = |f_+|^2 = |f_{3/2}|^2 \quad (2.50a)$$

$$\frac{d\sigma}{d\Omega}(\pi^- p \rightarrow \pi^- p) = |f_-|^2 = \left| \frac{1}{3} f_{3/2} + \frac{2}{3} f_{1/2} \right|^2 \quad (2.50b)$$

$$\frac{d\sigma}{d\Omega}(\pi^- p \rightarrow \pi^0 n) = |f_0|^2 = \frac{2}{9} |f_{3/2} - f_{1/2}|^2 = \frac{1}{2} |f_+ - f_-|^2. \quad (2.50c)$$

The scattering amplitudes (2.48a,b,c) are, in general, complex. Therefore each relation (2.48) consists of two equations, one for the real part and one for the imaginary part. From the imaginary parts of f_+ and f_- , by using the optical theorem, one obtains relations among the total cross-sections:

$$\sigma_{\text{tot}}(\pi^+ p) = |\langle \text{sum over all final states } |H| \pi^+ p \rangle|^2 = \sigma_{3/2} \quad (2.51a)$$

$$\sigma_{\text{tot}}(\pi^- p) = \frac{1}{3} \sigma_{3/2} + \frac{2}{3} \sigma_{1/2}, \quad (2.51b)$$

where $\sigma_{3/2}$ and $\sigma_{1/2}$ are the total cross-sections for the pure isospin states $I = 3/2$ and $I = 1/2$. From formulae (2.51a,b) one has:

$$\sigma_{1/2} = \frac{1}{2} [3 \sigma_{\text{tot}}(\pi^- p) - \sigma_{\text{tot}}(\pi^+ p)]. \quad (2.52)$$

Experiments on pion-nucleon scattering are consistent with the hypothesis of charge independence, but do not offer a good test of it. As we shall discuss later, a test using the relation (2.49) at $\theta = 0$ checks charge independence to about $\pm 10\%$.

2.9 Exchange amplitudes

The decomposition (2.47a) of the physical pion-nucleon states in terms of pure isospin states may be expressed diagrammatically as in Fig. 2.9a; the intermediate baryon state, whatever its complexity, carries total isospin I with third component I_z .

Alternately we may decompose the physical pion-nucleon states in terms of the "exchanged isospin" states. The situation is represented diagrammatically in Fig. 2.9b: the incoming

pion π_i emits the final pion π_f and a boson which carries isospin I' , with third component I'_3 . This object, which may be a very complex dynamical system, combines with the initial nucleon N_i to give the final nucleon state N_f . From isospin conservation, at the pion vertex, one must have $I' = 0, 1, 2$, while from isospin conservation at the nucleon vertex $I' = 0, 1$ only. Therefore we may now decompose the physical pion states in terms of the independent exchange amplitudes $v_0(s, t)$ for $I' = 0$ and $v_1(s, t)$ for $I' = 1$. A calculation using the Clebsch-Gordan coefficients at the two vertices of Fig. 2.9b gives:

$$\langle \pi^+ p | H | \pi^+ p \rangle = v_0 - \frac{1}{\sqrt{6}} v_1 \quad (2.53a)$$

$$\langle \pi^- p | H | \pi^- p \rangle = v_0 + \frac{1}{\sqrt{6}} v_1 \quad (2.53b)$$

$$\langle \pi^0 n | H | \pi^- p \rangle = \frac{v_1}{\sqrt{3}}. \quad (2.53c)$$

In pion-nucleon scattering there are just enough measurements to compute the exchange amplitudes from relations (2.53) (for instance, in the form $|v_0|, |v_1|, |v_0 - v_1|$; the relative phase could be obtained from the interference of nuclear and Coulomb scattering).

It can be proved that if one exchange amplitude dominates the other, and if it is not purely real at $t = 0$, then the dominating amplitude has the quantum number of the vacuum. At high energy the charge exchange reaction becomes small; therefore $v_1 \rightarrow 0$ and $f_{1/2} \approx f_{3/2}$.

It may be worth while to point out that the exchanged isospin in the process $\pi^+ p \rightarrow \pi^+ p$ is the normal isospin in reaction $\pi^+ \pi^- \rightarrow \bar{p} p$.

A third decomposition of the physical pion-nucleon states is in terms of the "cross-exchanged isospin" states, corresponding to the diagram of Fig. 2.9c, which represents a nucleon or nucleon-isobar exchange. In this case the isospin of the exchanged object can only be $I'' = \frac{1}{2}$ or $3/2$. A calculation using the Clebsch-Gordan coefficients at the two vertices gives:

$$\langle \pi^+ p | H | \pi^+ p \rangle = \sqrt{\frac{2}{3}} v_{1/2} - \frac{1}{2\sqrt{6}} v_{3/2} \quad (2.54a)$$

$$\langle \pi^- p | H | \pi^- p \rangle = -\frac{\sqrt{3}}{2\sqrt{2}} v_{3/2} \quad (2.54b)$$

$$\langle \pi^0 n | H | \pi^- p \rangle = \sqrt{\frac{1}{3}} v_{1/2} - \frac{1}{2\sqrt{3}} v_{3/2}, \quad (2.54c)$$

where $v_{1/2}$ and $v_{3/2}$ are the cross-exchanged amplitudes for $I'' = \frac{1}{2}$ and $3/2$, respectively.

2.10 Resonances

Basically there are two methods for detecting the presence of a resonance: formation and production. In a formation experiment the resonance is produced directly at the resonant energy as a kind of compound nucleus. It manifests itself in a bump in a total and/or partial cross-section, or in a more subtle way in a phase-shift analysis. In a production experiment the resonance is produced together with other particles. The resonance decays, and is observed as a peak in an effective mass plot of two or more particles or in a missing-mass distribution.

In pion-nucleon elastic scattering, the resonances are detected in formation. In a formation experiment a resonance may still be defined in several ways, which are all equivalent for a classical, purely elastic resonance. The situation becomes more complex for inelastic resonances.

Let us first consider the case of a resonance in a purely elastic partial wave ($\eta_\ell = 1$). We shall say that at an energy $E = E_r$ a resonance with quantum numbers ℓ, J^P, I occurs if the total cross-section contribution from that partial wave has a maximum at that energy. Since this contribution is proportional to $\sin^2 \delta(E)$, this implies $\delta(E_r) = \pi/2 + n\pi$. In this case the partial wave amplitude with total angular momentum

$$f_j(E) = \frac{e^{2i\delta_j} - 1}{2i} = e^{i\delta_j} \sin \delta_j = \frac{1}{\text{ctg } \delta_j - i} \quad (2.55a)$$

may be expanded in a Taylor series around the resonance energy; in fact we shall expand $\text{ctg } \delta_j$:

$$\text{ctg } \delta_j(E) = \text{ctg } \delta_j(E_r) + (E - E_r) \left[\frac{d}{dE} \text{ctg } \delta_j(E) \right]_{E=E_r} + \dots ; \quad \text{ctg } \delta_j(E_r) = 0. \quad (2.55b)$$

By defining:

$$\frac{2}{\Gamma} = - \left[\frac{d}{dE} \text{ctg } \delta_j(E) \right]_{E=E_r} \quad (2.56)$$

one has

$$f_j(E) \approx \frac{\Gamma/2}{E_r - E - i(\Gamma/2)}. \quad (2.57)$$

This is the non-relativistic Breit-Wigner expression for a resonating partial wave amplitude. Many simple properties of a resonance follow directly from the assumption that they are described by a Breit-Wigner form. The insertion of (2.57) into (2.37a) or (2.37c) yields:

$$(\sigma_{\text{tot}})_j = (\sigma_{\text{el}})_j = \frac{4\pi}{K^2} \left(j + \frac{1}{2} \right) |f_j(E)|^2 = \frac{\pi}{K^2} \left(j + \frac{1}{2} \right) \frac{\Gamma^2}{(E_r - E)^2 + (\Gamma^2/4)} \quad (2.58)$$

The meaning of Γ is found immediately to be the full width at half-height of the resonance. Γ must be positive; from Eq. (2.56) this implies that in order to have a resonance, the phase shift δ must increase through 90° as the energy increases. One may also show that the rate of change of the phase is greatest at the resonance position.

We now have to generalize these considerations to inelastic resonances. Again we shall say that a resonance occurs in a certain partial wave if that wave contributes a "bump" to the total cross-section. In other words, a resonance is a state that has definite quantum numbers, a finite energy width, and therefore a lifetime. A typical width of the pion-nucleon resonances is $\Gamma \approx 100$ MeV, which corresponds to a lifetime of about $2/3 \times 10^{-23}$ sec, a time comparable to the time it takes for a light-ray to cross a nucleon.

We shall define the total width (Γ) of a resonance as the sum of the elastic (Γ_{el}) and inelastic (Γ_{in}) widths:

$$\Gamma = \Gamma_{\text{el}} + \Gamma_{\text{inel}}. \quad (2.59)$$

The elasticity of a resonance is defined as:

$$x = \frac{\Gamma_{el}}{\Gamma} . \quad (2.60)$$

$(1 - x)$ will be called the inelasticity.

The generalization of Eq. (2.58) may be obtained qualitatively as follows: the probability that the resonance will decay into the elastic channel is proportional to x [to $(1 - x)$ for the inelastic channel]; the probability of formation of the resonance in the elastic channel will again be proportional to x . Therefore the elastic, inelastic, and total cross-sections will be proportional to x^2 , $x(1 - x)$, and x respectively, or to Γ_{el}^2 , $\Gamma_{el}\Gamma_{in}$, $\Gamma_{el}\Gamma$. This qualitative reasoning may justify the following non-relativistic generalizations of the Breit-Wigner formula:

$$\sigma_{el}(E) = \frac{\pi}{K^2} \left(j + \frac{1}{2} \right) \frac{\Gamma_{el}^2}{(E_r - E)^2 + (\Gamma^2/4)} = 4\pi\lambda^2 \left(j + \frac{1}{2} \right) \frac{x^2}{[(E_r - E)(2/\Gamma)]^2 + 1} \quad (2.61a)$$

$$\sigma_{in}(E) = \frac{\pi}{K^2} \left(j + \frac{1}{2} \right) \frac{\Gamma_{el}\Gamma_{in}}{(E_r - E)^2 + (\Gamma^2/4)} = 4\pi\lambda^2 \left(j + \frac{1}{2} \right) \frac{x(1 - x)}{[(E_r - E)(2/\Gamma)]^2 + 1} \quad (2.61b)$$

$$\sigma_{tot}(E) = \frac{\pi}{K^2} \left(j + \frac{1}{2} \right) \frac{\Gamma_{el}\Gamma}{(E_r - E)^2 + (\Gamma^2/4)} = 4\pi\lambda^2 \left(j + \frac{1}{2} \right) \frac{x}{[(E_r - E)(2/\Gamma)]^2 + 1} . \quad (2.61c)$$

The elastic partial wave amplitude becomes:

$$f_j(E) = \frac{(1/2) \Gamma_{el}}{E_r - E - i(\Gamma/2)} . \quad (2.62)$$

The relativistic generalizations of Eqs. (2.61a,b,c) are of the type ⁶¹⁾:

$$\sigma_{el}(E) = 4\pi\lambda^2 \left(j + \frac{1}{2} \right) \frac{x^2}{\left(\frac{E_r^2 - E^2}{E_r\Gamma} \right)^2 + 1} . \quad (2.63)$$

and so on for σ_{in} and σ_{tot} . The width Γ turns out not to be constant, but has an energy dependence which at low energy may be given as

$$\Gamma(E) \approx \Gamma_0 \left(\frac{K}{K_r} \right)^{2\ell+1} , \quad (2.64a)$$

where Γ_0 is the intrinsic (reduced) width and K_r is the value of the c.m. momentum at the resonant energy. At higher energies the following empirical dependence is often used for pion-nucleon resonances:

$$\Gamma(E) \approx \Gamma_0 \left(\frac{K}{K_r} \right)^{2\ell+1} \frac{E_r(A^2 + K_r^2)}{E(A^2 + K^2)} , \quad (2.64b)$$

where $A = 350 \text{ MeV}$ ^{61, 194)}.

The Breit-Wigner form (2.57) or (2.62) implies that the amplitude has a pole at the energy $E = E_r - i(\Gamma/2)$ which, of course, is not a physical energy. It may be shown that the partial wave amplitude is an analytic function of the energy, except for a) a cut along the positive real axis starting at $m_\pi + m_N$, and b) isolated poles (Fig. 2.10). A pole may be a) a bound state if it occurs in the real axis in the range $0 < E < m_\pi + m_p$; b) a resonance if $E = E_r - i(\Gamma/2)$, with $E_r > m_\pi + m_N$ and $\Gamma > 0$. A resonance may therefore be defined also as a pole of the partial wave scattering amplitude.

From Eq. (2.57), letting $\epsilon = (E_r - E)(2/\Gamma)$ one has:

$$f = \frac{1}{\epsilon - i} = \frac{\epsilon}{\epsilon^2 + 1} + i \frac{1}{\epsilon^2 + 1} = \text{Re } f + i \text{Im } f, \quad (2.65a)$$

from which:

$$(\text{Re } f)^2 + (\text{Im } f - \frac{1}{2})^2 = \frac{1}{4} \quad (2.65b)$$

which, in the plane $[\text{Re } f, \text{Im } f]$, is the equation of a circle of radius $\frac{1}{2}$ and centre $(0, \frac{1}{2})$. The circle is described counter-clockwise with increasing energy. For inelastic resonances, the diameter of the circle becomes equal to the elasticity and therefore smaller than 1. The circle with $R = 0,5$ is the maximum circle allowed and is called the unitary circle. Figure 2.11 illustrates the situation for three values of the elasticities¹⁹⁴): the diagram is usually referred to as the Argand diagram. In the figure are shown also the connections between the partial wave amplitude, its phase and absorption coefficient, and also the locations on the circle of the points corresponding to the energies $[E_r - (\Gamma/2)]$, E_r , and $[E_r + (\Gamma/2)]$.

Figure 2.12 shows the energy dependence of the phase shifts and of the absorption coefficients for three different values of the elasticity¹⁹⁴): at the resonant energy the phase has a rapid variation around 0° or 90° , while the absorption coefficient has a minimum, which is very deep if the elasticity is close to 0.5. The discontinuity in the phase at the resonant energy, for elasticities smaller than 0.5, cannot be seen from experiments, since the observable quantities remain unchanged for a change in the phase of 180° .

If the partial wave f_ℓ contains a resonance contribution, described by a Breit-Wigner formula, plus a "non-resonant elastic background", the resonating circle in the Argand diagram may be translated, rotated, and distorted. Figure 2.13 shows two types of distortions. In these cases one may have a resonance for practically any value of the phase shift. Under these circumstances one can decompose the phase shift and the absorption coefficient as follows

$$\delta(E) = \delta_r(E) + \delta_B(E) \quad (2.66a)$$

$$\eta(E) = \eta_r(E) \cdot \eta_B(E), \quad (2.66b)$$

where R and B refer to the resonant and background contributions. δ_B may change rapidly with energy, while η_B does not. This makes it advisable to determine the resonant masses and widths from η_B . The resonant energy will correspond to a minimum in η , $\eta_{\min}$; the half-width at

$$\eta\left(E_r - \frac{\Gamma}{2}\right) = \sqrt{\frac{\eta_{\min}^2 + \eta_B^2}{2}} \quad (2.66c)$$

and the elasticity

$$x = \frac{1}{2}(\eta_B - \eta_{\min}) . \quad (2.66d)$$

The energy dependence of the phase shifts is theoretically unknown. Several forms have been used in the literature: for energies below 100 MeV, non-relativistic expressions have been used³¹⁸⁾ (see Section 13.2); polynomial dependences^{50, 52)} and more complicated forms suggested by dispersion relations^{174, 203)} and other methods¹⁸³⁾ have been used over a wider energy range.

The energy dependences of the phase shifts and the absorption parameters are interconnected by the requirement of analyticity: Fig. 2.14 shows such connections²⁰³⁾.

In short: one may define a resonance in several ways:

- a) as a state with definite quantum numbers for which one may define a lifetime and branching ratios that are independent of the method of production of the resonance;
- b) as a pole of the partial wave scattering amplitude in the complex energy plane;
- c) as a peak in a cross-section arising from a single partial wave;
- d) as a closed curve, described counter-clockwise, in the Argand diagram;
- e) under certain circumstances the resonance corresponds to a phase shift of 0° or 90°;
- f) under certain circumstances, the resonance corresponds to a minimum in the absorption coefficient.

The parameters of a resonance depend considerably on the definition used for the resonant energy, as is shown in Table 2.5⁶¹⁾.

Table 2.5

Resonant energy of the $\Delta(1236)$ according to the different definitions

Definition of the resonant energy	Mass (MeV)
Peak position in total cross-sections	1225
Peak position in a mass distribution in production reactions	1211
Energy at which the phase shift equals 90°	1236

2.11 Coulomb and nuclear scattering

The Coulomb interaction is a long-range interaction. Therefore its contribution is important at small momentum transfers: it is as large as the nuclear scattering contribution at $|t| \approx 0.002 \text{ (GeV/c)}^2$. In this region the simultaneous presence of Coulomb and nuclear scattering may be expressed as:

$$\frac{d\sigma}{d\Omega} = |g_N + g_C|^2 + |h_N + h_C|^2 , \quad (2.67)$$

where C and N denote Coulomb and nuclear amplitudes, respectively. In the non-relativistic limit, the Coulomb spin-flip amplitude is zero: $h_C = 0$. Moreover, the Coulomb non-spin-flip amplitude, g_C , is mainly real. Hence, if the nuclear amplitude has a real part, then an interference term arises between Coulomb and nuclear scattering. The study of the interference allows an experimental determination of the sign and magnitude of the real part of the forward nuclear scattering amplitude.

We shall distinguish two cases: a) the low-energy case, and b) the high-energy case.

At low energies (~ 100 MeV) the interference in $\pi^+p \rightarrow \pi^+p$ is destructive; therefore the nuclear interaction, which still has a real amplitude only, is attractive, and the dominant $I = 3/2$, $J = 3/2$ phase shift δ_{33} is positive.

At high energies the interference in $\pi^+p \rightarrow \pi^+p$ is constructive (destructive in $\pi^-p \rightarrow \pi^-p$); therefore that part of the interaction which gives rise to the real part of the nuclear scattering amplitude is repulsive. In Section 4.6 we shall analyse in detail the interference in the multi-GeV region.

The presence of the Coulomb interaction complicates the phase-shift analysis at very low energies, where the presence of the Coulomb interaction is felt up to c.m. scattering angles of several tens of degrees. The procedure that is usually adopted in the analysis is that of subtracting from the data a computed Coulomb scattering term, so as to obtain nuclear cross-sections only. From these one computes the phase shifts, which in turn are the sum of a nuclear phase shift plus a small Coulomb correction.

In the approximation of Van Hove¹⁾, one assumes that there is a sphere of radius r_0 , such that for $r < r_0$ the Coulomb interaction can be ignored, and for $r > r_0$ the nuclear interaction can be ignored. At $r = r_0$ he joins the "nuclear wave function", valid for $r < r_0$, to the "Coulomb wave function", valid for $r > r_0$.

The small correction to the phases arises from the interference between Coulomb and nuclear scattering, both for $r > r_0$ (outer Coulomb correction) and for $r < r_0$ (inner Coulomb correction). The relations between the observed phases, δ^+ and δ^- for π^+p and π^-p elastic scattering, respectively, the nuclear phase δ^N , and the Coulomb correction δ^C , are:

$$\pi^+p \rightarrow \pi^+p \quad \delta^+ = \delta^N \pm \delta^C. \quad (2.68)$$

For charge exchange scattering, in first approximation, there are no Coulomb corrections. The corrections δ^C are given by Van Hove¹⁾, and in a more complete form by Hamilton et al.¹⁵⁾ and Schnitzer²⁹⁾. They are different for S- and P-waves³¹⁸⁾. The phases δ^N are the phases to which the charge independence hypothesis applies.

It is customary to take $r_0 = \hbar/m_\pi c$ on the grounds that this is roughly the radius of the nuclear forces. The uncertainty on the value of r_0 gives automatically an uncertainty of a few per cent on the very low energy phase shifts.

2.12 S and T matrices

Let us consider the particles both long before and long after the scattering process (i.e. for times $\rightarrow \pm\infty$); the particles may be considered as non-interacting, and the initial and final states can be described by a linear superposition of a properly symmetrized free

particle wave functions. The S-matrix is defined as the operator that transforms the initial state $|\phi_i\rangle$ into the final state $|\phi_f\rangle$:

$$|\phi_f\rangle = S|\phi_i\rangle. \quad (2.69)$$

S is linear, hermitian, and unitary:

$$SS^\dagger = S^\dagger S = 1. \quad (2.70)$$

It is often convenient to define the transition matrix T as:

$$S = 1 + iT. \quad (2.71)$$

The conditions (2.70) for S give for T:

$$TT^\dagger = T^\dagger T = i(T^\dagger - T). \quad (2.72)$$

This relation may be written for all the matrix elements in terms of a complete set of intermediate states $|K_n\rangle$, which are eigenstates of the four momentum:

$$i \langle K|(T^\dagger - T)|K\rangle = \sum_n \langle K|T^\dagger|K_n\rangle \langle K_n|T|K\rangle. \quad (2.73)$$

We shall later make use of this relation.

The most important properties of the T-matrix are:

- a) it is relativistic;
- b) it satisfies the unitarity condition (2.73);
- c) it satisfies the postulate of maximum analyticity;
- d) it has crossing symmetries;
- e) it may be expressed in terms of dispersion relations;
- f) for our purpose the T-matrix and the relativistic scattering amplitude F, defined in Eq. (2.18), are equivalent.

2.13 High-energy theorems

Several high-energy theorems, valid in the limit $s \rightarrow \infty$, have been derived as a consequence of the general properties of unitarity, crossing and analyticity. We shall briefly review their implications for high-energy pion-nucleon scattering.

2.13.1 First Pomanchuk theorem

In the limit of high energies, the total cross-sections on the same target particle, of particles belonging to the same isospin multiplet, are equal if they have the same real parts of the forward scattering amplitudes^{6,9}:

for $s \Rightarrow \infty$

$$\begin{aligned} \sigma_{\text{tot}}(\pi^-p) &= \sigma_{\text{tot}}(\pi^+p) = \sigma_{\text{tot}}(\pi^0p) \\ \sigma_{\text{tot}}(pp) &= \sigma_{\text{tot}}(np), \end{aligned} \quad (2.74)$$

The justification of this theorem goes as follows. As the energy increases, the number of open channels increases and the cross-section for each channel decreases, with the possible exception of the diffraction cross-section, which is the shadow of all inelastic processes. In particular, the elastic charge exchange cross-section will tend to zero as $s \rightarrow \infty$; thence from formula (2.48c):

$$\frac{f_{3/2}}{f_{1/2}} \xrightarrow{s \rightarrow \infty} 1, \quad (2.75)$$

and from formulae (2.48a,b):

$$\frac{d\sigma}{d\Omega}(\pi^+p) \xrightarrow{s \rightarrow \infty} \frac{d\sigma}{d\Omega}(\pi^-p), \quad (2.76)$$

i.e. at high energies the scattering becomes independent of isospin. From formula (2.76) if $\text{Re } f(s,0) = 0$, or if $[\text{Re } f(s,0)]_{\pi^+p} = [\text{Re } f(s,0)]_{\pi^-p}$, then the imaginary parts of the forward scattering amplitudes are also equal. From the optical theorem there then follows the equality of the total cross-sections.

If the charge exchange cross-section tends to zero, then from formulae (2.53a,c) follows:

$$\frac{V_1}{V_0} \xrightarrow{s \rightarrow \infty} 0. \quad (2.77)$$

This relation expresses a more general theorem, which affirms the dominance at high energies of the exchange amplitude with the quantum numbers of the vacuum.

2.13.2 Second Pomanchuk theorem

If: a) at high energies the total cross-sections become constant, independent of energy, and b) dispersion relations are valid, then for $s \rightarrow \infty$ the total cross-sections on the same target particle of particles and antiparticles are equal^{6,9}):
for $s \Rightarrow \infty$:

$$\sigma_{\text{tot}}(pp) = \sigma_{\text{tot}}(\bar{p}p), \text{ etc.} \quad (2.78)$$

For pions this theorem yields again the equality of π^+p and π^-p total cross-sections.

2.13.3 Upper bounds on the high-energy scattering amplitude

Always using the general principles of unitarity, crossing, and analyticity, a number of upper bounds have been established for the scattering amplitude, as $s \rightarrow \infty$ ²⁰²):

$$\text{Greenberg-Low:} \quad T(s, \theta = 0) < C s^2 \log^2 s \quad (2.79a)$$

$$T(s, \theta \neq 0) < C s^{\frac{3}{2}} \log^{\frac{3}{2}} s \quad (2.79b)$$

$$\text{Froissart:} \quad T(s, \theta = 0) < C s \log^2 s \quad (2.79c)$$

$$\sigma_{\text{tot}}(s) < C \log^2 s \quad (2.79d)$$

$$T(s, \theta \neq 0) < C s^{\frac{3}{4}} \log^{\frac{3}{4}} s \quad (2.79e)$$

$$\text{Martin-Kinoshita:} \quad T(s, \theta \neq 0) < C (\log^{\frac{3}{2}} s) / \sin^2 \theta, \quad (2.79f)$$

Where C is a multiplicative constant. Khuri and Kinoshita were able to improve the Froissart bound under different conditions, and showed the interconnections between the ratio $\text{Re}(s,0)/\text{Im}(s,0)$ and the total forward scattering amplitude²⁰²).

2.13.4 Lower bounds on the high-energy scattering amplitude

For $s \rightarrow \infty$ ²⁰²):

Jin-Martin: The amplitude cannot decrease faster than $1/s^2$ (2.80a)

Martin:
$$\sigma_{\text{el}}(s) > C \frac{\sigma_{\text{tot}}^2(s)}{\log^2(s)} \quad (2.80b)$$

Gribov: At backward angles the amplitude cannot decrease faster than s^{-n} (n integer). (2.80c)

Cerulus-Martin:
$$T(s, \theta) > \exp [-C \sqrt{s} \log s] \quad (2.80d)$$

None of the upper and lower bounds have been violated until now. The total cross-section behaviour is close to the Froissart upper bound, while the large-angle pp elastic cross-section is close to the Cerulus-Martin lower bound.

In conclusion, it is generally believed that in the high-energy asymptotic region the following are valid:

- i) the scattering process becomes independent of spin and isospin;
- ii) the real part of the scattering amplitude becomes small;
- iii) there is dominance of the exchange amplitude with the quantum numbers of the vacuum; the charge exchange cross-section tends to zero;
- iv) the total cross-sections of the particles belonging to the same isospin multiplet are equal;
- v) the total cross-sections of particle and antiparticle are equal;
- vi) the ratios between various total cross-sections are governed by some internal symmetry.
- vii) the total cross-sections as function of energy may become constant;

In other words, in the limit of very high energies, hadron collisions should not depend on the nature of the interacting particles nor on the specific mechanism of strong interactions.

Where does "Asymptopia" start? Presently available experiments indicate that we are approaching it, but also that we are far away from it.

2.14 Implications of unitary symmetries for total cross-sections

2.14.1

The observed forward peaking in elastic scattering processes is suggestive of meson exchange in the t -channel. Within this dynamical picture of peripheral single-particle exchanges, symmetries for the couplings of the exchanged mesons lead to predicted relations among hadron scattering amplitudes and therefore, via the optical theorem, among total cross-sections. For pion-nucleon scattering, the ρ meson is the lightest boson system which can be exchanged. If its coupling obeys $SU(3)$ symmetry, then one predicts simple relations among total cross-section differences for meson-baryon scatterings, where the mesons are

members of the 0^- octet and the baryons of the $\frac{1}{2}^+$ octet. We shall discuss some of these relations in the context of Regge poles (Section 10.7). They are reasonably well verified by the experiments.

If more than one meson is exchanged, the situation becomes more complex; one needs to assume universality of the couplings for all the exchanged mesons. The combination of SU(3) and universality for the ρ and ω couplings leads to another series of predictions such as, for instance, the Johnson-Treiman relations:

$$\left[\frac{\sigma_{\text{tot}}(K^+p) - \sigma_{\text{tot}}(K^-p)}{2} \right] = \left[\sigma_{\text{tot}}(K^0p) - \sigma_{\text{tot}}(\bar{K}^0p) \right] = \left[\sigma_{\text{tot}}(\pi^+p) - \sigma_{\text{tot}}(\pi^-p) \right], \quad (2.81)$$

which disagree somewhat with the experiments. Part of the disagreement may be due to the fact that the formulae should be valid in the limit of mass equality among the different members of the meson octet. This is essentially correct for the first relation involving only kaons, but it is not correct for the second one involving K's and π 's. Therefore, while the first relation of (2.81) could be used at all energies, the second one should be used only at reasonably high energies, where mass differences become unimportant. In any case one has to give up some part of the universality hypothesis or SU(3) symmetry for the couplings.

2.14.2 The quark model

In the quark model the pion is assumed to be made by a quark-antiquark pair and the nucleon by three quarks¹⁹⁹). It is usually assumed that the hadron-hadron amplitude is obtained by taking the sum of independent quark-antiquark and quark-quark amplitudes, as illustrated in Fig. 2.15. The quark model, together with the Pommeranchuk theorems for the quarks total cross-sections, yields in the limit of large energies:

$$\sigma_{\text{tot}}(\pi N) \underset{s \gg}{\approx} \frac{2}{3} \sigma_{\text{tot}}(NN). \quad (2.82)$$

The relation does not seem to be verified too well; this is not too surprising in view of the theoretical and experimental limitations.

3. TOTAL CROSS-SECTIONS

The total cross-section measurement is, at the moment, the simplest and the most precise cross-section measurement. In the last few years, most of the systematic total cross-section measurements have been performed by using the transmission method, in "good geometry". The experimental results of experiments in the same energy range show the same structures, but are often in systematic disagreement one with the other, well outside their quoted statistical error; the disagreements result from relatively large systematic errors arising from beam contamination, beam optics, experimental geometry, and counting equipment.

3.1 Experimental method

We shall take as an example of the method the recent measurements performed at Brookhaven National Laboratory^{221,287}).

The classical geometry used for a total cross-section measurement (Fig. 3.1) consists of: a) a system of counters to monitor the incoming beam; b) the target, where the beam is absorbed; and c) a series of transmission counters, which detect the transmitted particles.

The partial cross-section σ_i seen by the i^{th} transmission counter is given by:

$$\sigma_i = \frac{A}{Nm} \log \frac{R_{0i}}{R_i}, \quad (3.1)$$

where N is Avogadro's number, A is the atomic weight of the target material, $m = \rho l$ is the target thickness in g cm^{-2} , R_{0i} is the fraction of beam reaching the i^{th} transmission counter when the dummy target is in place, and R_i the same fraction for the full target.

Each of the transmission counters measures the total cross-section minus the cross-section for having a charged secondary in the solid angle subtended by the i^{th} transmission counter:

$$\sigma_i = \sigma_{\text{tot}} - \int_0^{\Omega_i} \left(\frac{d\sigma}{d\Omega} \right) d\Omega, \quad (3.2)$$

where σ_{tot} is the total cross-section, Ω_i is the mean solid angle subtended by the counter i , and $(d\sigma/d\Omega)$ is the differential cross-section for the emission of at least one charged particle inside the solid angle Ω_i . Most of this cross-section will be elastic diffraction.

Using the square of the momentum transfer t as a variable, and assuming that the differential cross-section has an inelastic contribution, which is constant in t , and an elastic contribution, which is exponential in t , one has:

$$\sigma(t) \approx \sigma_{\text{tot}} - \int_0^t \left(a e^{bt} + c e^{dt} \right) dt = \sigma_{\text{tot}} - \left[a \left(\frac{e^{bt}}{b} - 1 \right) + c \left(\frac{e^{dt}}{d} - 1 \right) \right] \quad (3.3)$$

where $a = (d\sigma/dt)_{t=0}$ for the elastic part, and b is the slope of the elastic contribution; c and d are the corresponding terms for the inelastic contribution. Equation (3.3) may be checked experimentally by plotting $\sigma(t)$ versus t : one should have a curve exponential in t (quadratic in second approximation). Several experiments have been extrapolated to $t = 0$ using a polynomial linear in t ; this procedure is expected to give a systematic, slowly momentum-dependent, underestimate of σ_{tot} . At very small angles multiple scattering will become important, while at larger angles the approximations in formula (3.3) start to break down.

Several improvements in the technique were made during the last years, and it is now relatively easy to achieve precisions of $\pm 0.1\%$ in the relative values of measurements at different energies, and $\pm 1\%$ in the absolute values of the cross-sections.

Some of the improvements which made such precisions possible are:

a) the construction of beams of good optical quality and of high intensity. At intermediate energies, partially separated beams reduce the contamination of unwanted particles. Usable beam intensities are typically 10^5 pions per machine pulse, with a momentum resolution of typically $\leq \pm 1\%$.

- b) Refined differential Čerenkov counters separate electronically the wanted from the unwanted particles in the beam.
- c) Hydrogen and deuterium targets of a double-jacketed design provide long-term density stability to $\pm 0.03\%$. The vapour pressure is well fixed and continuously monitored so that hydrogen density is known in absolute value to about $\pm 0.05\%$.
- d) The use of a large number of transmission counters, each subtending a different solid angle, allows an easy extrapolation of the data to zero solid angle.
- e) The data taking, which is automated, usually follows a well-defined pattern designed to minimize and check fluctuations. At each energy the following sequence has often been used:

$$B \beta \epsilon E H H E D D E H H E D D C E \epsilon , \quad (3.4)$$

where B means beam tuning, β means measurement of beam contamination before and after electronics separation, ϵ is an efficiency measurement of the transmission counters, and E, H, and D correspond to Empty, Hydrogen, Deuterium and Carbon targets, respectively. The deuterium target is not really needed for measurements of pion cross-sections, though measurements of π^+ and π^- absorption in deuterium give an immediate check of charge symmetry. The carbon target is used to complement the data and to provide cross-sections for corrections arising from the absorption of the beam in materials with mostly carbon content.

The main corrections on the data are: a) a correction due to the increased decay rate of mesons after having lost energy in the targets; b) corrections from single Coulomb scattering and Coulomb-nuclear interference; c) correction for beam contamination. Above 1 GeV it is very difficult to separate muons from pions. One has to use threshold gas Čerenkov counters and/or muon filters, which let through only muons. At intermediate energies the uncertainties in the muon contamination usually give the largest source of systematic error.

3.2 Analysis of total cross-section data

Figures 3.2 and 3.3 show the published total, elastic, and charge exchange cross-sections. The line indicating the inelastic cross-section was obtained as the difference

$$\sigma_{in} = \sigma_{tot} - (\sigma_{el} + \sigma_{ce}) . \quad (3.5)$$

Most of the total cross-sections come from transmission measurements. Most of the elastic and charge exchange data are obtained by integration of angular distributions.

The $I = \frac{1}{2}$ cross-sections, shown in Fig. 3.4, have been computed with the formula (2.52). The $(\pi^- p) - (\pi^+ p)$ total cross-section differences are shown in Fig. 3.5.

Table 3.1 gives a list of interpolated total cross-sections for $\pi^- p$ and $\pi^+ p$, together with their differences. The errors shown are quadratically compounded statistical plus systematic standard deviations. The systematic error was taken from the literature; when missing, it was guessed. When combining data from different experiments, it is very important to include the systematic error, since there are a number of systematic differences.

In the resonance region, the pure isospin total cross-sections as function of the c.m. total energy may be fitted by a superposition of Breit-Wigner formulae plus a non-resonant background described mathematically by a polynomial of the c.m. energy:

$$\sigma_{\text{tot}}(E) = \sum_i (\text{B.W.})_i + b_1 + b_2 E + b_3 E^2. \quad (3.6)$$

From such fits one may obtain values of the resonant energies, of the widths, and of the resonant cross-section

$$\sigma_R = 4\pi\lambda^2 \left[\left(j + \frac{1}{2} \right) x \right], \quad (3.7)$$

from which one has the product $\left[\left(j + \frac{1}{2} \right) x \right]^2$.

For energies larger than about 5 GeV, one may fit the data with simple formulae:

$$\sigma(P_{\pi \text{ lab}}) = \sigma(\infty) + \frac{a}{P_{\pi \text{ lab}}^b}, \quad (3.8a)$$

$$\sigma(P_{\pi \text{ lab}}) = c P_{\pi \text{ lab}}^{-d} \quad (3.8b)$$

and one may examine possible asymptotic behaviours of the total cross-sections.

Foley et al.²⁹²) fitted several cross-section combinations to Eq. (3.8). First they fitted simultaneously $\pi^- p$ and $\pi^+ p$ total cross-sections, with the requirement, arising from the Pomeranchuk theorem, that $\sigma_{\pi^+ p}(\infty) = \sigma_{\pi^- p}(\infty)$. This fit involves five arbitrary parameters; its results are shown in Table 3.2. The second fit of Table 3.2 is a fit to Eq. (3.8a) of the sum and the difference of the measured cross-sections, requiring that $[\sigma_{\pi^-}(\infty) - \sigma_{\pi^+}(\infty)] = 0$. The third fit is a fit to Eq. (3.8a) of the pure isospin cross-sections, requiring that $\sigma_{3/2}(\infty) = \sigma_{1/2}(\infty)$. The fits have reasonable χ^2 . Table 3.3 gives simpler fits of each individual cross-section to formula (3.8) with $b = 1$. By taking the most convergent fit, one may estimate that the Pomeranchuk theorem will be valid to $\sim 0.1\%$ at a laboratory momentum of 20,000 GeV/c.

The fact that the cross-sections seem to stay finite as the energy goes to infinity, is an indication that the range R of the forces responsible for the interaction is finite. In fact, in the crudest of models, one expects a total cross-section of the order of the geometrical one:

$$\sigma_{\text{tot}} \sim \pi R^2. \quad (3.9)$$

A value $\sigma_{\text{tot}} \sim 25 \text{ mb}$ implies $R \sim 1 \text{ f}$.

4. ELASTIC CROSS-SECTIONS

Differential elastic cross-sections have been measured by using a number of experimental methods; at present only three techniques are employed:

- i) scintillation counters, either detecting both scattered particles in coincidence or only the scattered pion;
- ii) spark chambers, triggered by a system of scintillation counters;
- iii) hydrogen bubble chambers.

Nuclear emulsions, cloud chambers, and heavy-liquid spark chambers have also been employed. The experiments of a systematic type have been performed with counters and/or spark chambers. They usually cover many energies over a large angular interval. There are several elastic scattering bubble chamber measurements, most of which are really by-products of other experiments and suffer from limited statistics.

Low-energy experiments have been performed for many years at cyclotron laboratories. In the pioneering stage, the beams were usually of poor optical quality, and the experimental equipment consisted of a small number of counters. Then synchrotron beams of very good optical properties were designed and built, and slowly the complexity of the experiments increased until hodoscopes with several hundred scintillation counters were employed, and a computer on-line became a necessity^{286,291}). The number of spark chambers also increased, and refined data-handling facilities had to be used³⁴⁶).

4.1 Some experimental examples

A large number of different experimental arrangements have been used for measurements of angular distributions, so that it is not really possible to speak of a typical experiment.

In the very low energy range (less than 45 MeV), several experiments have been performed at Rochester^{317,319}). Only five counters were used (Fig. 4.1): counters 1 and 2 detected the incoming pion, 3 and 4 the scattered pion; 5 was an anticoincidence counter. Moreover, counter 3 was pulse-height analysed: an event was defined as a coincidence-anticoincidence $e = (12)(34)5$, with the extra requirement that the scattered pion had to give a count in counter 3 in the proper channel of the pulse height analyser. The beam intensity was of the order of 1000 pions per sec. Muons and electrons in the beam were partly rejected by a proper bias in counters 1 and 2; they were also measured in separate runs. The recoil proton could not be counted because it did not have enough energy to escape the hydrogen target; this fact does not pose a background problem, since at these low energies no inelastic process is possible. The geometry was a tight semicylindrical symmetry, no-wall scattering geometry, typical of these Rochester measurements. The large azimuthal acceptance was needed for counting rates, since beams of these energies were of low intensities and since the cross-section is small. The tight geometry was necessary because of the short range and large decay rates of the particles to be detected; the no-wall scattering reduced the background considerably. The components used in the experiments were essentially miniaturized, and compressed in a small space; this created several geometrical problems, for instance for light collection, and complicated hydrogen target shapes. The statistical standard deviation was typically $\pm 5\%$, while the systematic scale error may be estimated at $\pm 4\%$.

At higher energies the experiments became physically larger; the presence of a large inelastic cross-section forced stricter requirements. The beam was defined in a more complex way, and beam contamination was eliminated with the use of time-of-flight between two or more counters and/or Čerenkov counters. Usually both recoil particles were observed, except in the very forward scattering region where the recoil proton does not escape from the target. The requirement of observing the angles of both scattered particles at the kinematically conjugate angles, reduced the background at acceptable levels for measurements at intermediate energies. At angles such that the kinematics does not allow a sepa-

ration between a pion and proton, a Čerenkov counter of the threshold type had to be used in order to obtain an electronic separation. Figure 4.2 shows the layout employed at the Rutherford laboratory for elastic scattering in the 875-1579 MeV/c range³⁵⁰). The number of counters employed was larger than 50. The beam was defined by a coincidence-anticoincidence $b = B_1 B_2 B_3 \bar{A}_1$; timing requirements in the B-counters eliminated the proton contamination; the muon contamination was measured with a gas Čerenkov counter placed before B_3 . A coincidence $a = b T A_2$ indicated an interaction in the liquid hydrogen target, giving at least a charged particle in the T counter. An event was a coincidence $e = a \pi_i p_i$, where π_i and p_i were a pair of kinematically conjugate counters. The water Čerenkov counter, placed behind some of the pion counters, eliminated the protons in the kinematically ambiguous region. The solid angles were defined by the π -counters, while the p-counters were made slightly larger than necessary in order to count all the protons conjugate to elastically scattered pions. The data were stored in a pulse-height analyser. Typical beam intensities of 1100 pions per second produced 3 events/sec. Typical statistical standard deviations of about $\pm 5\%$ were achieved, with a scale error of about $\pm 3\%$. The systematic error arose from beam contamination, beam size, counter efficiencies, light-guide counting, etc.

At still higher energies, the total elastic cross-section decreases, while the inelastic one increases. Because of these facts, one has to measure also momenta, as well as scattering angles, in order to obtain a high signal-to-noise ratio. The elastic scattering is concentrated in the forward direction, and the large-angle cross-section becomes very small. Measurements in the diffraction region were performed by a Brookhaven group employing hodoscopes with hundreds of scintillation counters, controlled by a large computer on-line²⁹¹). The group used two geometries: for scattering in the Coulomb-nuclear interference region ($|t| < 0.1$), both incoming and outgoing pions were measured in angle, location, and momentum to a very high accuracy, by means of counter hodoscopes. For scattering in the $0.1 < |t| < 1.0$ (GeV/c)² region a side hodoscope measured also the scattered proton. For the $|t| < 0.1$ region, Fig. 4.3 applies: pions were selected by two threshold Čerenkov counters CT1 and CT2. Hodoscopes H01 and H02 measured the location and the angle of the incident particle to an accuracy of ± 0.15 mrad. Hodoscopes H1 and H2 determined the location and direction of the scattered pion; hodoscope H4, placed after three bending magnets, measured the pion momentum. The over-all resolutions were ± 0.4 mrad in scattering angle and $\pm 0.4\%$ in momentum. These precisions were relaxed for the measurements in the $0.1 < |t| < 1$ (GeV/c)² region. The whole system was about 120 metres long. Several precautions were taken in trying to determine absolute values to something like $\pm 1\%$. This is an exceptionally small systematic error; in the coincidence measurement the systematic error was much larger. Beam intensities of the order of 10,000 pions per second were used; the data-handling capacity was able to cope with about 10^6 interactions per hour.

At high energies and large angles the smallness of the cross-section introduces severe problems of counting rates and of background. One has to use a high intensity beam and cover as large a solid angle as possible, and both recoil particles have to be momentum analysed. Figure 4.4 shows one of the two layouts employed in another Brookhaven experiment³⁵⁶). The beam, defined by scintillation and Čerenkov counters, impinged on a 2 ft-long liquid hydrogen target. Scattered and recoil particles were momentum analysed by

magnets M_1 and M_2 , and detected by a large number of scintillation counter hodoscopes π_1, π_2, π_3 and P_1, P_2, P_3 . There were ten such telescopes covering 10 angular regions, so that a large fraction of the total solid angle was observed. C was a threshold gas Čerenkov counter, which vetoed forward-going pions. A coincidence between a beam particle, a pion telescope, and a proton telescope caused the optical spark chambers SC1-SC6 to be pulsed. Beam intensities were 2×10^5 pions per pulse, and event rates varied between 0.1 to 30 events per hour depending on angle and energy. The statistical precisions of the measurements were about $\pm 10\%$, with a scale error of $\pm 15\%$.

4.2 Total elastic cross-sections

The total elastic cross-sections are obtained by integration of differential cross-sections, which are usually not available for the whole angular range; in particular, it is difficult to measure the differential distribution for small as well as for very large scattering angles. Therefore, in order to obtain the total elastic cross-section, an extrapolation must be made in the differential data. At energies higher than 1 GeV the elastic scattering is concentrated at small angles, and the extrapolation in this region involves a large part of the cross-section, from 10% to even 50% of the total elastic cross-section; the extrapolation of the cross-section to large angles introduces a small error because of the smallness of the large-angle elastic cross-section. At small angles the presence of Coulomb scattering, which must be removed in order to obtain the total elastic nuclear cross-sections, may further complicate the analysis.

The error on the extrapolations and the contributions of several systematic uncertainties give a large total systematic error for the total elastic cross-section. For this reason, the data scatter considerably from experiment to experiment.

Table 3.1 gives a list of interpolated values for π^-p and π^+p total elastic cross-sections. The errors quoted include systematic as well as statistical errors. The experimental total elastic cross-sections are shown in Figs. 3.2 and 3.3, together with the total cross-sections. Note that for π^+p , the total and total elastic cross-sections coincide for energies up to 170 MeV, because up to this energy only the elastic channel is open. Instead, for π^-p in the same energy region, σ_{el} and σ_{tot} are different because of the presence of the charge exchange channel.

The behaviour of the total elastic cross-sections follows quite closely the behaviour of the total cross-sections.

For energies larger than 5 GeV, the elastic cross-sections have been fitted to formula (3.8) with $b = 1$. The results are tabulated in Table 3.3.

Figure 4.5 shows σ_{el}/σ_{tot} for π^+p as function of the pion laboratory kinetic energy. Above 8 GeV the ratio is either constant or slowly decreasing. The average values of σ_{el}/σ_{tot} and of the opacity, defined as $0 = 2\sigma_{el}/\sigma_{tot}$, are listed in Table 4.1.

Table 4.1
Average values of σ_{el}/σ_{tot} above 8 GeV

	σ_{el}/σ_{tot}	Opacity
π^-p	0.172 ± 0.004	0.344 ± 0.008
π^+p	0.180 ± 0.005	0.360 ± 0.010
pp	0.262 ± 0.004	0.524 ± 0.008

Note that π^-p and π^+p have approximately the same opacity, while the pp opacity is larger.

4.3 Angular distributions -- low energies

Typical angular distributions for different incoming energies are shown in Fig. 4.6 for $\pi^-p \rightarrow \pi^-p$ and in Fig. 4.7 for $\pi^+p \rightarrow \pi^+p$.

A preliminary analysis of the data may be performed by least squares fitting the angular distributions to a polynomial in $\cos \theta$:

$$\frac{d\sigma}{d\Omega} = \sum_{n=0}^{n_{\max}} a_n \cos^n \theta \quad (4.1)$$

or to Legendre polynomials:

$$\frac{d\sigma}{d\Omega} = \sum_{n=0}^{n_{\max}} c_n P_n(\cos \theta) . \quad (4.2)$$

Both expressions are found in the literature as well as expression (4.2) with $c_n = c'_n/k^2$. The Legendre polynomial expansion (4.2) is preferable, since the Legendre polynomials form an orthonormal set; therefore the errors are less correlated than in the case of Eq. (4.1), and the coefficients c_n change little when $n_{\max}$ is changed. For both expansions $n_{\max}$ is found empirically, requiring that the highest coefficient is different from zero, outside the errors.

Figures 4.8 and 4.9 show the behaviour of the $c_n^\pm$ coefficients as function of the energy. By using interpolated values of the $c_n^\pm$ coefficients, one may compute the differential cross-sections at a number of selected energies. Figures 4.10 and 4.11 show interpolated values of the cross-sections.

At low energies the angular distributions are typical of S + P-waves. The forward-backward (F - B) asymmetry is negative for π^+p elastic scattering. At 180 MeV, the P-wave resonances ($\delta_{33} = 90^\circ$); the angular distribution is typical of a large P-wave and is symmetric around 90° ; therefore F - B = 0. Above the 3/2,3/2 resonance, the angular distribution still has S + P dominant waves, but for π^+p the F - B asymmetry is now positive. The change in sign of the asymmetry is a clear indication that the δ_{33} phase has really passed through 90° .

At higher energies other waves start to be important and the angular distribution has a higher complexity. In particular, the differential cross-section for π^-p changes conspicuously when passing the 600 MeV and 900 MeV, $I = \frac{1}{2}$ resonances.

Above 1 GeV a forward diffraction peak becomes prominent, while the large-angle cross-section decreases considerably. A secondary bump becomes evident at angles corresponding approximately to the same t -value of $1.2 (\text{GeV}/c)^2$, independent of energy (see Table 4.2). The cross-section at $\theta = 180^\circ$ has large fluctuations, clearly connected with s -channel resonances (Fig. 4.12).

Perl and Corey⁶³⁾ least squares fitted the experimental differential cross-sections in the 1-4 GeV/c region to the spinless partial wave expansion (2.31) and (2.32), using the a_ℓ 's as parameters to be fitted, with the condition $-1 \leq a_\ell < +1$. The procedure is clearly

Table 4.2
Approximate positions of the secondary maxima in $\pi^\pm p$ elastic and charge exchange angular distributions at different energies

Reaction	Laboratory momentum (GeV/c)	Locations of maxima		Reference
		First $-t(\text{GeV}/c)^2$	Second $-t(\text{GeV}/c)^2$	
$\pi^+ p \rightarrow \pi^+ p$	2.02	1.3	-	378
	2.3	1.3	-	361
	2.5	1.4	3.4	↓
	2.7	1.3	3.6	
	3.0	1.3	4.0	
	3.5	1.2	3.5	
	3.7	1.3	-	
	4.0	1.3	-	382
	8.0	1.4	3.5	p.c.
	12.0	1.4	-	p.c.
$\pi^- p \rightarrow \pi^- p$	2.5	1.4	3.7	361
	3.0	1.4	3.6	361
	3.5	1.2	3.7	361
	5.9	1.25	3.5	356
	8.0	1.3	3.5	p.c.
	12.0	1.4	4	p.c.
$\pi^- p \rightarrow \pi^0 n$	3.07	1.2		409
	3.67	1.1		409
	4.83	1.0		409
	5.9	1.1		409
	10.0	1.0		412
	13.3	1.0		409
	18.2	1.0		409

oversimplified, but it gives an interesting result. In the plot of $(1 - a_\ell)$ (determined as stated above) versus ℓ , one has a smoothly decreasing function, as shown schematically in Fig. 4.13. A simple interpretation of Fig. 4.13 may be obtained semi-classically by associating each a_ℓ with the strength of the interaction at a distance $r \approx \ell/K$. Thus, if with increasing ℓ we have at some value of ℓ_1 a sudden rise in a_ℓ (a drop in $1 - a_\ell$) from nearly 0 to nearly 1, this is interpreted as a sharp drop in the interaction force at a distance $r_1 \approx \ell_1/K$ (curve A of Fig. 4.13). On the other hand, a slow rise of a_ℓ (slow decrease of $1 - a_\ell$) would mean that there is no sharp boundary to the interaction region (curve B of Fig. 4.13). If the interaction could be described by a potential, then the curve of Fig. 4.13 could represent $(1 - a_\ell)$ versus ℓ , as well as $V(r)$ versus r , apart from some modifications. For increasing energies the curves in the $(1 - a_\ell)$ versus ℓ plane move toward higher values of ℓ , while in the $V(r)$ plane they may remain fixed if the interaction region does not change with energy.

The results of Perl and Corey indicate, as expected, that there is no sharp boundary in the pion-nucleon interaction: $1 - a_\ell$ decreases monotonically and continuously to zero, and the pion-nucleon interaction decreases smoothly with distance.

We shall take up the results of phase-shift analyses in Chapter 7.

4.4 Angular distributions -- high energies

The main features of the high-energy angular distributions, as they are shown in Figs. 4.6 and 4.7, are the presence of a large and narrow diffraction peak at small angles, some wiggles at intermediate angles, the smallness of the large-angle cross-section, and the presence of a narrow backward peak.

We shall divide the high-energy angular distributions into four regions:

- a) the very small angle region, $[|t| < 0.1 \text{ (GeV/c)}^2]$, where Coulomb scattering is important and where an interference between Coulomb and nuclear scattering can be observed;
- b) the diffraction region proper (approximately for $0.1 < |t| < 1$);
- c) the large-angle region (for $|t| > 1$);
- d) the backward region, close to 180° .

4.4.1 The diffraction region proper

It is immediately evident from the small-angle, high-energy angular distributions of Figs. 4.6 and 4.7, that the experimental points, expressed as $d\sigma/d|t|$ versus $|t|$, follow an exponential very closely.

It has become customary to fit the differential cross-sections with the functions:

$$\frac{d\sigma}{d|t|} = \exp[a + b|t|] \quad (4.3)$$

and
$$\frac{d\sigma}{d|t|} = \exp[A + B|t| + Ct^2] . \quad (4.4)$$

The first formula represents a purely exponential diffraction pattern. Its use is adequate for $|t| < 0.5 \text{ (GeV/c)}^2$. In a larger t -interval, Eq. (4.4) gives a better fit to the data and is generally used, though it predicts nonsense at very large t -values.

Figure 4.14 shows the results of the fits of the published π^+p data to formula (4.3) for $0.1 < |t| < 0.5$ (GeV/c)²; the fit to Eq. (4.4) over $0.1 < |t| < 1.0$ is shown in Fig. 4.15. It is quite important to analyse different pieces of data for the same $|t|$ intervals because of the approximate validity of both Eqs. (4.3) and (4.4); by using the data over a slightly different $|t|$ range, one may obtain different results.

Figure 4.14 shows that, while for π^+p the parameter b remains constant as function of energy, for pp it increases indicating a narrowing (or shrinking) of the diffraction peak. Phenomenologically this corresponds to an increase of the interaction radius. The π^+p and π^-p slopes are almost equal, the π^-p being slightly steeper.

The deviation of the diffraction pattern from a pure exponential can be represented by the dimensionless quantity C/B^2 , which is the second coefficient in the expansion.

$$\log \left[\frac{d\sigma/dt}{(d\sigma/dt)_{t=0}} \right] = Bt \left[1 + \frac{C}{B^2} (Bt) + \dots \right]. \quad (4.5)$$

The values for C/B^2 given in Fig. 4.16 show that the deviation from a pure exponential is small.

Serber⁵³⁾ writes Eq. (4.4) with dimensionless quantities:

$$\left[\frac{d\sigma/dt}{(d\sigma/dt)_{t=0}} \right] = \exp [B't' + C't'^2] \quad (4.6)$$

$$t' = \left(\frac{d\sigma}{dt} \right)_{t=0}^{\frac{1}{2}} \frac{t}{\sqrt{\pi}}. \quad (4.7)$$

Figure 4.17 shows the B' and C' parameters for some π^+p , K^+p , and p^+p data at high energies. The remarkable fact is that the pseudoscalar mesons appear to have approximately the same values of B' and C' , while $\bar{p}p$ and pp have identical values of B' , but substantially smaller than the pseudoscalar meson case. This difference may be related to the difference in opacity (or in σ_{el}/σ_{tot}) that exists between p^+p , π^+p , and K^+p . In the limit of very high energies, the equality of B' , C' would be converted to a statement of the identity of the elastic scattering of related particles.

4.4.2 The very small angle region

In this region, Coulomb scattering is important and must be properly taken into account in order to derive the pure nuclear scattering effects. Measurements at very small angles yield information on the real part of the forward non-spin-flip scattering amplitude (the spin-flip part contains a $\sin \theta$ term and therefore does not contribute at small angles). There are essentially two methods to measure the real part of the forward elastic scattering amplitude.

i) In the first method the contribution of the real part to the cross-section is obtained as the difference between the extrapolated nuclear cross-section at $t = 0$ and the contribution of the imaginary part obtained by the total cross-section via the optical theorem:

$$|\text{Re } f(s,0)|^2 = \left(\frac{d\sigma}{dt} \right)_{t=0} - \frac{\sigma_{tot}^2}{16\pi}. \quad (4.8)$$

This method leaves the sign of the real part undetermined. Moreover, the extrapolation required to $t = 0$ is usually not very accurate; the real part is obtained as the difference of two independent measurements, which must therefore be known with good accuracy also in an absolute scale. Finally, the data at small angles require a Coulomb correction which is not always reliable.

ii) The second method is based on the Coulomb-nuclear interference effect and it has been used most extensively at high energies. Several experimental problems arise here from the very small scattering angles involved, for the need of good absolute values both for σ_{tot} and $d\sigma/dt$, and because of the high statistics needed. Figure 4.18 shows some angular distributions in the Coulomb-nuclear interference region²⁹¹) [the two amplitudes become approximately equal at $|t| \approx 0.002 \text{ (GeV/c)}^2$]. The interference between Coulomb and nuclear scattering is evident.

The absence of spin-spin terms makes the πN system much more amenable to a phenomenological analysis in the interference region than, say, the NN system.

The Coulomb non-spin-flip amplitude is:

$$g_C(s,t) = \mp \frac{2\sqrt{\pi} \ell^2}{\hbar c \beta_{\pi\text{lab}} t} e^{2i\delta} G_\pi(t) G_p(t) \frac{1}{1 - \frac{|t|}{4m_p^2}} \approx \mp A_C \frac{e^{i\delta}}{t}, \quad (4.9)$$

where the minus (plus) sign is appropriate for π^+p (π^-p) elastic scattering; e is the electron charge; $(\beta_{\pi\text{lab}} c)$ is the velocity of the incident pion in the laboratory system; δ , the phase shift between Coulomb and nuclear scattering, is given by Bethe as^{11,201,227}):

$$\delta \approx \frac{e^2}{\hbar c \beta_{\pi\text{lab}}} \ln \frac{1.06}{pa \Theta}. \quad (4.10)$$

δ is not really known to better than 15%. G_π and G_p are the electric form factors for the pion and the proton. For lack of better knowledge, it is usually assumed $G_\pi = G_p$.

The nuclear non-spin-flip amplitude may be rewritten as:

$$g_N(s,t) = \text{Re } g_N(s,t) + i \text{Im } g_N(s,t) = [\rho(s,t) + i] \text{Im } g_N(s,t), \quad (4.11)$$

where $\rho(s,t)$ is the ratio of the real to imaginary part of the nuclear scattering amplitude. A simplifying assumption is:

$$\rho(s,t) \approx \rho(s,0), \quad (4.12)$$

which is not too unreasonable in view of the small t -interval in which the interference may be studied ($0.001 < |t| < 0.04$). At high energies ρ is small; the amplitude is mainly imaginary and may be approximated as in the previous section:

$$\text{Im } g_N(s,t) \approx \exp \left[\frac{A}{2} + \frac{B}{2} |t| + \frac{C}{2} t^2 \right] = \frac{\sigma_{\text{tot}}}{4\sqrt{\pi}} \exp \left[\frac{B}{2} |t| + Ct^2 \right]. \quad (4.13)$$

By substituting Eqs. (4.9) to (4.13) into Eq. (2.67), one obtains:

$$\frac{d\sigma}{dt}(\pi^{\pm}p) \underset{s \gg t \ll}{\Rightarrow} \underbrace{\frac{A_c^2}{t^2}}_{\text{Coulomb term}} + (1 + \rho_{\pm}^2) \underbrace{\frac{\sigma_{\text{tot}}^2}{16\pi}}_{\text{nuclear term}} e^{B|t|+Ct^2} \mp \underbrace{\frac{2A_c}{|t|} (\rho_{\pm} \cos 2\delta \pm \sin 2\delta) \frac{\sigma_{\text{tot}}}{4\sqrt{\pi}} e^{B|t|+Ct^2}}_{\text{interference term}} + \text{multiple scattering.} \quad (4.14)$$

This is the formula used for computing $\rho^{\pm}$. The multiple scattering term represents an experimental correction, which may be an important one, because of the very small angles involved (much less than 1° in the l.s.).

In Fig. 4.18 two curves have been drawn for each distribution; one curve corresponds to no interference ($\rho = 0$), while the second curve represents the result of a least squares fit according to Eq. (4.14).

The interference is constructive for π^+p ; it is destructive for π^-p . Therefore the real part of the nuclear scattering amplitude gives rise to a repulsive interaction.

Figure 4.19 shows the experimental results for $\rho = (\text{Re } f / \text{Im } f)_{t=0}$ for $\pi^{\pm}p$ at several energies. The curves drawn are the predictions of dispersion relations. The real part of the forward scattering amplitude decreases slowly with increasing energy. At the highest energies it is about 10% of the imaginary part. Therefore all the models which neglect the real part of the scattering amplitude must be considered as first-order approximations in the presently accessible high-energy range.

4.4.3 Scattering at large angles

For $|t| > 1$ the differential cross-section, for both $\pi^{\pm}p$ scattering, continues to decrease, but not as fast as in the diffraction region. Moreover, it exhibits structures at $|t| \approx 1.3$ and 3.5 (GeV/c)^2 (187). Table 4.2 gives the experimental location of the maxima at different energies.

The experimental points shown in Figs. 4.6 and 4.7 indicate that for $|t| > 1 \text{ (GeV/c)}^2$ the bulk of the data around 4.0 GeV/c have considerably larger cross-sections than the data around 8 GeV/c. Therefore the $\pi^{\pm}p$ elastic scattering data for $|t| > 1 \text{ (GeV/c)}^2$ exhibit an s-dependence, while for $|t| < 1$ there is nothing appreciable. The experimental results are not sufficiently accurate to establish more detailed conclusions.

The comparison of the large-angle $\pi^{\pm}p$ elastic scattering data with the pp data shows that the pion cross-sections decrease at large angles in a way not too dissimilar from pp, although the slope may be faster. Therefore, the pion may have a less opaque core than the proton.

Opinions on the interpretation of the large-angle cross-sections are divided. One group of authors interprets both the small-angle and the large-angle cross-sections as diffractive. For this interpretation one needs at least two types of potentials^{57, 54)} in order to obtain a reasonable fit for different t-values; but the large energy dependence remains partly unexplained.

Another group of authors favours a statistical interpretation^{76, 154)}. The elastic channel is in statistical competition with all inelastic channels, and the observed exponential dependence of the integrated large-angle cross-sections on c.m. energy is essentially the Boltzman factor. Nevertheless, the angular distributions are more pronounced than the

limiting angular distributions predicted by a statistical model with angular momentum conservation. Moreover, the predicted statistical fluctuations⁴¹⁾ have not been found experimentally in pp scattering²⁰⁹⁾.

Chou and Yang picture the nucleon as an extended object²²⁴⁾; therefore it is difficult to accelerate its various parts without breaking it, making large momentum transfers improbable.

Hybrid models combine some of the above features. In any case, it is clear that more measurements of large-angle scattering for all types of particles are needed in order to clarify the situation.

4.4.4 Backward scattering

The main features of backward scattering seem to be the following:

- i) In the resonance region the energy variation of the backward cross-section exhibits a series of maxima and minima clearly connected to the pion-nucleon resonances. The structures are superimposed on a background which decreases with increasing energy. These features suggest that two mechanisms are effective, one connected with the resonances and the other with the background.
- ii) At high energies, the value of the differential cross-section at $\theta = 180^\circ$ is smaller by 3-4 orders of magnitude than that at $\theta = 0^\circ$.
- iii) The π^-p backward cross-section is about an order of magnitude smaller than the corresponding π^+p one.
- iv) There are backward peaks, which seem to be narrower than the forward ones. Their dependence on the momentum transfer is roughly exponential:

$$\left(\frac{d\sigma}{dt}\right)_{\text{backward}} = \left(\frac{d\sigma}{dt}\right)_{t=t_{\text{max}}} e^{-b_B(t_{\text{max}}-t)}. \quad (4.15)$$

Several models have been proposed as explanations of the backward elastic peaks in high-energy elastic scattering. These models are reminiscent of the explanations for the forward elastic peaks. In fact, Byers and Yang⁸⁵⁾ proposed that all momentum transfer peaks are due to coherent phenomena. Some of the specific models that have been proposed are:

- i) The diffraction model⁴³⁸⁾ (see Section 8.7 and Fig. 8.1).
- ii) A second model assumes that the backward cross-section is dominated by the nearby singularities in the u-channel. In the pole approximation one has the baryon exchange diagrams of Fig. 4.20 (N and Δ for π^+p , Δ states only for π^-p). A Born approximation yields a prediction which is about two orders of magnitude larger than the observed cross-sections, and a wrong energy dependence. Absorption effects reduce these contributions, but at present seem unlikely to yield peaks as narrow as those observed experimentally.
- iii) In the Regge-pole model, the baryon states that are exchanged are Reggeized. This improves the situation considerably (see Sections 10.5 and 12.1).
- iv) Shadow of inelastic processes through the unitarity of the S-matrix¹⁰⁵⁾.

5. CHARGE EXCHANGE SCATTERING

The experimental results on the charge exchange process are considerably less precise than the results on elastic scattering because of the difficulties connected with the detection of the neutral particles, neutron and π^0 , produced in the scattering process. At high energies, the detection of the neutral particles becomes easier, but the charge exchange cross-section becomes very small, and the background from multi- π^0 events becomes important.

Below 400 MeV there are several relatively old synchro-cyclotron experiments, where the two γ 's from the π^0 decay and/or the neutrons were detected with relatively low efficiencies by means of large scintillation counters and/or lead-glass Čerenkov counters. At higher energies the experiments used heavy optical spark chambers, which surrounded the liquid-hydrogen target almost completely. The γ -rays are detected in the chambers with efficiencies approaching 100%, through the characteristic electromagnetic showers.

5.1 Experimental methods

We shall discuss the experiment of Bulos et al.⁴⁰²⁾, which concerned the measurement of the charge exchange differential distributions in the energy range 545-1151 MeV at the Brookhaven cosmotron. The experimental layout is shown in Fig. 5.1. The incoming π^- was counted as $\pi_1 = 12\bar{3}4$, where $\bar{3}$ was a scintillation counter with a hole. The trigger for the spark chambers was a neutral trigger: $t = \pi_1 \bar{3} \bar{5}$, i.e. a charged particle arrived and none emerged. Counters $\bar{3}$ and $\bar{5}$ were scintillation counters in anticoincidence, which surrounded the small liquid-hydrogen target as much as possible (5 cm in diameter and 4 cm long). The γ -rays were converted in the four large heavy-plate spark chambers that surrounded the target. The detection efficiency for γ -rays, with energy above 20 MeV, was practically 100%. The three small thin-plate spark chambers TF1,2,3 were used to supplement the electronics information, by requiring a good incident track in the first two chambers and none in the third one. They reduced the number of spurious triggers by a factor of about four.

Approximately 90% of the two γ events were examples of reactions

$$\pi^- p \rightarrow \pi^0 n \quad (5.1)$$

$$\rightarrow \eta^0 n . \quad (5.2)$$

The remaining 10% were coming from higher multiplicity events, such as $2\pi^0$, $3\pi^0$, $N\pi^0$ production in which, for geometrical reasons, only two γ -rays were seen, since the detection solid angle was less than 4π . A Monte Carlo calculation was used to determine the correction for these events. In order to analyse the two γ events and to subtract the background, it is convenient to transform the measured quantities in the c.m. frame of the πp system. In this system the π^0 's are monochromatic and give a theoretical opening angle distribution for the two γ -rays peaked toward the minimum allowed angle. Figure 5.2 shows the experimental opening angle distribution after background subtraction of 619 MeV. The π^0 and η^0 positions are clearly visible and well separated.

In this experiment only the directions of the γ -rays were measured, not their energies. This leaves a non-serious kinematical ambiguity. In practice, the authors have taken the bisector of the two γ 's as the π^0 direction; the resulting distribution was later corrected.

5.2 Analysis of the charge exchange data

The analysis of the charge exchange data may proceed in the same form as the analysis of elastic scattering. Figure 5.3 shows some of the experimental angular distributions. The typical statistical standard deviation is 5-10%, and the systematic error is of about the same order.

The total charge exchange cross-section may be obtained by integration of the differential distribution. At low energies, when multi- π^0 production is not important, the total charge exchange cross-section may be obtained directly with a neutral trigger as discussed in the previous section.

Table 3.1 gives a list of interpolated values for the total charge exchange cross-section. The experimental total charge exchange cross-sections are shown in Fig. 3.2.

In the resonance region the behaviour of the charge exchange cross-section follows closely the behaviour of the other cross-sections, while in the high-energy region it decreases monotonically toward zero. Table 3.3 shows the result of fitting the high-energy cross-section to formula (3.8), with $b = 1$.

As for elastic scattering, a preliminary analysis of the charge exchange data in the resonance may be performed by best fitting the angular distributions to the functions (4.1) or (4.2). Figure 5.4 gives the c_n^0 coefficients of the Legendre polynomial expansion²²¹.

The main characteristics of the high-energy charge exchange angular distributions are the following:

- i) The forward cross-section is smaller than the elastic cross-section by about a factor of 30 at 3 GeV/c and about 200 at 18 GeV/c. The decrease is directly connected to the decrease of the difference $\sigma_{\text{tot}}(\pi^-p) - \sigma_{\text{tot}}(\pi^+p)$ as the energy increases. In fact the optical theorem and charge independence yield:

$$\left(\frac{d\sigma}{dt}\right)_{\text{ce}}(t=0) = \frac{1}{32\pi} [\sigma_{\text{tot}}(\pi^-p) - \sigma_{\text{tot}}(\pi^+p)]^2 + \frac{\pi}{K^2} |\text{Re } G_{\text{ce}}(t=0)|^2. \quad (5.3)$$

- ii) From the experimental knowledge of the forward charge exchange cross-section and the total cross-section difference, Eq. (5.3) predicts that the real part of the forward scattering amplitude is about equal to the imaginary part.
- iii) The angular distribution is peaked forward and has a diffraction-type pattern. Since the particles involved in the scattering process change their nature, a real diffraction cannot occur. One may instead say that the reaction is peripheral, in the sense that non-central collisions between the π^- and the proton are preferred, and therefore the angular distribution is peaked toward the small momentum transfers. In other words, the incoming pion interacts mainly with the meson cloud surrounding the nucleon. In terms of a Feynman diagram, one may think of the interaction as proceeding via ρ -meson exchange (Fig. 5.5). In practice, the calculation of the diagram of Fig. 5.5 leads nowhere, unless one Reggeizes the ρ meson.
- iv) The angular distribution has probably three dips: at $|t| = 0$, 0.6, and perhaps around 2.0 (GeV/c)². The dip at $t = 0$ may arise from a large spin-flip amplitude, which

contains a $\sin \theta$ dependence and therefore vanishes at $\theta = 0$. The dip at $|t| = 0.6$ has a logical explanation in the Regge-pole model.

- v) In the t -interval $0.1 < |t| < 0.5$, the angular distribution is essentially exponential. A fit of the distribution in this t -interval to Eq. (4.3) shows that b_0 is larger than $b^\pm$ and probably that it increases with increasing energy (Fig. 4.14) (shrinking of the forward peak).

6. POLARIZATION MEASUREMENTS

The polarization measurements performed until now concern the P-parameter only; there are no measurements of the A- and R-parameters, though an experiment to measure them is in progress at CERN. Until a few years ago it was possible to perform only measurements of the type shown in Fig. 2.5a, that is a measurement of P by scattering the recoil proton off carbon nuclei. The advent of polarized targets greatly improved the experimental situation. Nevertheless, polarization measurements remain difficult experiments: the amount of available data is limited and is of limited accuracy.

6.1 Experimental methods in elastic scattering

We shall discuss only measurements with polarized targets. The polarized targets at present in use are made of a $[\text{La}_2 \text{Mg}_3(\text{NO}_3)_{12} \cdot 24 \text{H}_2\text{O}]$ crystal, having a typical volume of about 10 cm^3 . The density of the material is about 2 g cm^{-3} , with a free hydrogen density approximately equal to the liquid hydrogen density (0.07 g cm^{-3}); this corresponds to only about 3% free hydrogen in weight. The crystal is cooled to 1.1°K , placed in a strong magnetic field, and polarized by microwaves with a frequency of 69,000 MHz. The polarization of the free hydrogen is about 70%; the polarization direction may be reversed by changing the microwave frequency by 160 Hz.

Polarized targets using ethyl-alcohol are now being developed. With these targets the density of free hydrogen is doubled and the density of the material reduced by about a factor of two, resulting in a much larger hydrogen percentage (about a factor of four). Unfortunately, up to the present, the polarization of the free hydrogen is only 30%, so that there is no net gain in the measurement precision.

A typical problem of an experiment with a polarized proton target is the separation of good events from background, coming from inelastic events and from elastic events from bound nucleons. The separation is achieved by requiring that the two outgoing particles be coplanar and have the correct angular correlation. In the experiments now in progress, stricter requirements are required at higher energies: Čerenkov counters are added, in order to detect with certainty the scattered pions, and a magnetic analysis of the recoil proton is performed.

Figure 6.1 shows the experimental layout used at CERN to measure the P-parameter in elastic $\pi^\pm p$ scattering from 6 GeV/c to 12 GeV/c ^{474, 477}). The incoming pion is counted as $\pi_1 = AK\bar{H}$ where A and C are scintillation counters in coincidence, $\bar{H}$ is an anticoincidence scintillation counter with a hole, and K is a gas differential Čerenkov counter. The outgoing pion is counted by a scintillation counter combination as $\pi_0 = S_1S_2\bar{V}$, where $\bar{V}$ is again an anticoincidence counter with a hole. The recoil proton is counted as $p = PR$. Every time a coincidence $(\pi_1\pi_0p)$ occurs, the information from the counter hodoscopes H_1 ,

H₂, and R are written in a magnetic tape and later analysed by a computer. The R hodoscope is made of 10 long counters having photomultipliers at both extremes. The location of the proton in one direction is directly given by the counter number; in the other direction the position is determined, to a precision of a few centimetres, by measuring the time difference of the light reaching the two photomultipliers at the two extremes of each counter. With this arrangement the experimenters were able, in most cases, to reduce the background to a few per cent of the signal from free hydrogen.

6.2 Experimental methods in charge exchange scattering

Polarization measurements for the charge exchange reaction are even more difficult, because the difficulties of measuring the charge exchange reaction are now combined with the difficulties of polarization measurements.

Figure 6.2 shows the experimental layout of the charge exchange polarization experiment of Bonamy et al. at CERN in the 6 GeV/c to 11 GeV/c momentum region⁴⁸⁵). The beam is counted in approximately the same way as in the experimental layout of Fig. 6.1. Many large scintillation counters are used to detect the recoil neutron (NC-L and NC-R), while the forward-going γ -rays are detected in an optical, heavy-plate spark chamber, which has a thickness of 12 radiation lengths.

6.3 Elastic scattering polarization data

Figure 6.3 shows some typical polarization data in elastic scattering. The statistical standard deviations are usually larger than 10%, and the systematic error is of about the same order; a considerable contribution to the systematic error usually arises from the uncertainty in the target polarization and target composition.

6.3.1 Low energies

At low energies the polarization parameter P changes considerably with angle and with energy, as shown in Fig. 6.3a. The polarization is very sensitive to the spin-parity of the interference between the large waves, and therefore polarization measurements in this energy region are very useful for resolving phase-shift ambiguities and allow the determination of the parity of the resonances.

A preliminary analysis of the data may be performed by least squares fitting the product of the differential elastic cross-section times the polarization P to a polynomial in $\cos \theta$:

$$\left(\frac{d\sigma}{d\Omega}\right) P = \sin \theta \sum_{n=1}^{n_{\max}} a'_n \cos^n \theta \quad (6.1)$$

or in terms of associate Legendre polynomials:

$$\left(\frac{d\sigma}{d\Omega}\right) P = \sum_{n=1}^{n_{\max}} d_n P'_n(\cos \theta) = \sin \theta \sum_{n=1}^{n_{\max}} d_n \frac{dP_n(\cos \theta)}{d(\cos \theta)}. \quad (6.2)$$

These expressions are analogous to the expressions (4.1) and (4.2) for elastic scattering. Again formula (6.2) is preferable for the same reason that formula (4.2) is preferred to (4.1). Figure 6.4 gives the d_n coefficients as function of the energy.

As an example of the usefulness of a polarization measurement in determining the resonance parity, we shall mention the d_5^- coefficient, which in the energy region between the $N(1688)$ and $\Delta(1920)$ resonances has a sign and an energy dependence characteristic of $F_{5/2} - F_{7/2}$ interference; this established the parity of these resonances.

6.3.2 High energies

The high-energy behaviour of the polarization in the diffraction region is shown in Fig. 6.3b. The π^+p and π^-p polarizations have opposite sign, a maximum magnitude of about 20%, and probably become zero at $|t| \approx 0.6$ (GeV/c)². The magnitude of the polarization seems to decrease with increasing energy, but is still different from zero. Therefore spin effects cannot be neglected even at the highest energies.

It has been noted that the polarization in elastic scattering looks like a logarithmic derivative of the angular distribution. In particular this implies that the polarization vanishes at the minima of $(d\sigma/d\Omega)$.

The rapid variation of $P(\theta)$ with θ around the minimum of $(d\sigma/d\Omega)_{el}$ at $|t| \sim 0.6$ (GeV/c)² may be interpreted in different ways:

- 1) in terms of a diffraction mechanism; 2) as a simple reflection of a minimum in $(d\sigma/d\Omega)_{el}$ without particular reference to the mechanism that produces the minimum;
- 3) in terms of Regge poles, as the location at which the ρ -trajectory crosses $\alpha = 0$.

Predazzi and Soliani¹⁵⁶⁾ considered the following quantity:

$$\Pi(E, \theta) = \frac{P}{2 \sin \theta} \frac{d\sigma}{d\Omega}, \quad (6.3)$$

which is plotted in Fig. 6.5 for both π^+p and π^-p elastic scatterings at high energies. Clearly, the behaviour of $\pi^\pm$ in the small-angle region is of exponential form:

$$\Pi^\pm = e^{\alpha_\pm + \beta_\pm |t|}, \quad (6.4)$$

which is similar to the elastic scattering behaviour. The β_+ and β_- slopes are approximately equal; they are about twice as large as the slopes of elastic scattering ($\beta_+ \approx \beta_- \approx 2b_+ \approx 2b_-$).

In elastic scattering at forward angles one may neglect the spin-flip amplitude h , since it contains $\sin \theta$, and the real part of the non-spin-flip amplitude g , as discussed in the chapter on elastic scattering. Therefore:

$$\frac{d\sigma}{dt} \approx e^{a+b|t|} \approx |\text{Im } g|^2 \quad (6.5)$$

$$\text{and:} \quad \Pi = \text{Im} (g h_R^*) \approx \left[e^{(a/2)+(b/2)|t|} \right] \text{Im } h_R^*, \quad (6.6)$$

where $h_R^* = h/\sin \theta$ is defined as the residual spin-flip amplitude. Figure 6.5 and formula (6.6) imply that at high energies, the residual spin-flip amplitude h_R^* behaves very much as the non-spin-flip amplitude g , in the sense that it has a marked forward peak but with a much larger slope). In a more complete treatment, one has to consider the effect of the real parts (they will contribute an effect of 10-20% compared to the imaginary part).

6.4 Polarization data in charge exchange scattering

There are very few measurements of the P polarization parameter in the charge exchange scattering, all with very large statistical and systematic uncertainties (10-20%).

Figure 6.6 shows all the experimental results above 2 GeV/c ^{485, 486}). The data indicate that at low energies the polarization is influenced by the presence of the pion-nucleon resonances. At high energies (5.9 - 11.2 GeV/c) the polarization is still different from zero.

7. PHASE-SHIFT ANALYSIS

The determination of the phase-shifts, on the one side from the experimental data and on the other from theoretical models, has been the traditional task of low-energy scattering. At present there are no models that are able to make reliable and complete predictions. More recently the phase-shift analyses have become a method for discovering new resonant states and/or for determining their quantum numbers. We shall discuss in particular this new approach.

In Section 2.5 we have expressed the observable quantities (total and partial cross-sections, angular distributions, and polarizations) in terms of phase-shifts, via complicated non-linear equations. A phase-shift analysis consists in solving these relations with respect to the phases and the absorption parameters, and selecting the right solution from among the many possible ones.

For the energies that have been considered until now (about 2000 MeV), it is satisfactory to stop the phase-shift expansions at $\ell = 4$, with the exception of the highest energies where $\ell = 5$ has been used. For each ℓ -value different from zero, one has eight parameters, (four for $\ell = 0$), four for the phases ($I = \frac{1}{2}, J = \ell + \frac{1}{2}$), ($\frac{1}{2}, \ell - \frac{1}{2}$) ($3/2, \ell + \frac{1}{2}$) ($3/2, \ell - \frac{1}{2}$), and four for the corresponding absorption coefficients. Therefore a phase-shift analysis with S + P + D + F + G + H waves at a single energy contains 44 parameters. The number of experimental data points to be analysed is about 4000, at more than 60 different energies. The problem becomes mathematically very complicated and the analyses are now restricted to a limited number of superspecialists, the "phase-shifters", who use elaborate and lengthy computer programs. The physical intuition is now of little help, and non-specialists can only check *a posteriori* if the different "phase-shifters" agree in their results.

There is essentially no purely experimental phase-shift analysis, since much theoretical information and many hypotheses have to be used at different stages. Most of the analyses differ on these grounds. For instance, the functional form of the phases with energy is not known, and different authors use different forms.

The basic assumption used in the hunting of resonances in the phase-shift analysis is that every circle in the Argand diagram of a certain partial wave corresponds to a resonance in that same partial wave. This assumption would be correct if there were no other known simple mechanisms (other than Breit-Wigner type formulae) that could produce circles in the Argand plots. Schmidt¹⁸⁶) recently pointed out that one may obtain Argand circles in the s-channel partial-wave analysis from the t-channel exchange of a Regge trajectory. Therefore it would appear that resonance interpretation of a circle in the Argand plot is no longer unique. In effect, the scattering amplitude may be described either as a

sum of s-channel resonances (eventually with some background) or as a sum of t-channel Regge poles or other t-channel singularities. Both descriptions represent a complete parametrization of the physical scattering amplitude (duality). This is exactly correct in the Veneziano model¹⁸⁶). Therefore the utilization of one or the other representation is a question of practical convenience. At low energies, the interpretation in terms of the s-channel resonances is simpler, and therefore we shall use it.

7.1 The methods

There are basically two methods of phase-shift analyses: energy-dependent and energy-independent.

In the energy-independent method, a phase-shift analysis is performed at each energy. The number of possible solutions depends on the quality and quantity of the data available at each energy. The energy behaviour of the different sets of phases is then investigated.

In the energy-dependent approach, one assumes certain energy dependences for the phases and absorption parameters, and all the data are analysed at one time^{50,52}).

Phase-shift analyses of data at a single energy have for a long time been incapable of giving a unique set of phases that had a smooth energy behaviour. Therefore, energy-dependent analyses were tried, by using as inputs and as guiding lines the single-energy analyses performed at the energies where the best data were available. Present analyses use sophisticated procedures, which may really be classified as combinations of the two basic methods.

The three most recent analyses that cover the data up to about 2100 MeV, total c.m. energy are: Saclay¹⁷⁵), CERN¹⁶⁵), and Berkeley²⁰⁷).

The first step of these analyses is a random search for solutions at the energies where most data are available. The solutions are defined as valleys in the χ^2 hypersurface. Usually one has many of such solutions.

The second step consists in selecting the solution (or solutions) that is continuous with energy. At Saclay and at CERN this was done by eye; that is by plotting and looking at the solutions. At Berkeley it was done automatically by the computer.

At this stage one may conclude that the dominant features of the found solution are probably unique. But the graphs of some of the phases or absorption parameters versus energy are quite bumpy, and it may not be clear which bumps are significant or not. Systematic errors among the different experiments contribute significantly to the bumpiness and to the poor χ^2 's obtained.

The third stage is a smoothing operation, at a cost in χ^2 , but without destroying the fit. The Saclay group did it by eye to the small phases, and then repeated the fit to see what happened to the large phases. The Berkeley analysis is not finished in this respect. They added some parameters to normalize the different types of experiments, thus reducing effects arising from systematic errors. CERN has the most elaborate procedure. They assumed that the energy dependence of the phases and absorption parameters are represented by partial wave dispersion relations. One has a series of functions, each of which has the correct analytic properties in the energy and is multiplied by an adjustable parameter. The experimental values of a particular partial wave are fitted to this sum of functions.

These functions represent a refinement over previous polynomial dependences; in particular, the data are now smoothed not only for continuity but also for analyticity. If, for instance, there is an inelastic threshold, η will decrease as illustrated in Fig. 2.14a. Analyticity then requires that the phase δ has a woolly cusp. Similarly, if η has a valley, corresponding to an inelastic resonance (Fig. 2.14b), then the phase δ will have the behaviour shown in Fig. 2.14b.

The last step is an iteration procedure: the results of the fits are put back into the phase-shift analysis at a single energy, and so on. With this procedure one may obtain smooth energy behaviours, but one certainly has a bias against narrow resonances, which may be missed altogether.

It is not completely clear by how much the solution could be biased by imposing theoretical behaviours, such as those from dispersion relations. Lovelace says that their "experimental" solution contains something like 25% theory²⁰³). They also use dispersion relation predictions for the small phases as "theoretical data" to supplement the experimental information.

The analyses differ not only in the methods, but also in the selection of the data used. This is partly deliberate, since one may be suspicious of some experimental data and therefore not include it.

All groups claim to have satisfactory χ^2 , which is somewhat surprising in view of the large systematic uncertainties that affect the data. In reality the errors do not have a simple meaning. In fact the errors on the different partial waves are strongly correlated. This means that the smoothing procedure on separate partial waves yields a situation with poor χ 's. It is therefore necessary to redo the fit, starting with the smoothed values, but with errors usually enlarged by a large and essentially arbitrary factor.

7.2 The results

The results are shown in the form of Argand diagrams in Fig. 7.1 for the CERN and Berkeley analyses, and as plots of the phases and absorption coefficients versus energy in Figs. 7.2 and 7.3 for the CERN analysis only^{203,163}).

The dominant features are the same in all three analyses. In particular, all previously established resonances are present. All three groups agree that there are more of them, but disagree on how many and at which energies. The main disagreements concern P_{11} and S_{11} . The CERN group presents nine new candidates; another one is presented by the other two groups. All the new possible resonances are very inelastic, and are therefore difficult to see. Their corresponding Argand diagrams are rather distorted and usually do not show completely closed figures. Some of the inelastic resonances are best seen as minima in the absorption coefficients.

Table 7.1 gives the resonance parameters. Most of the parameters come from an Argand diagram, i.e. the resonant energy corresponds to the top of the circle and the half-width points to the rightmost and leftmost positions of the circle. For some inelastic resonances the parameters are obtained from the absorption coefficients.

The resonances are divided into well-established, probable, and unconfirmed.

Table 7.1

Status of nucleon resonances as at June 30 1968.
 E_R and Γ are the resonant energy and full width at half-maximum in MeV. $x = \Gamma_{el}/\Gamma_{tot}$ is the elasticity. Errors on E_R and Γ are always larger than ± 10 MeV, with the exception of the $\Delta(1236)$ (± 1 MeV) and $N(1520)$ (± 5 MeV).

N					Δ					Experimental status
E_R	Γ	J^P	$L_{2I,2J}$	x	E_R	Γ	J^P	$L_{2I,2J}$	x	
939	-	$1/2^+$	-	-	1237	123	$3/2^+$	P_{33}	1.0	Well established
1460	260	$1/2^+$	P_{11}	0.57	1630	160	$1/2^-$	S_{31}	0.27	
1515	115	$3/2^-$	D_{13}	0.52	1940	210	$7/2^+$	F_{37}	0.42	
1525	80	$1/2^-$	S_{11}	0.34	1880	250	$5/2^+$	F_{35}	0.18	
1675	145	$5/2^-$	D_{15}	0.43	1905	300	$1/2^+$	P_{31}	0.25	
1690	125	$5/2^+$	F_{15}	0.61						
1710	280	$1/2^-$	S_{11}	0.66						
1785	405	$1/2^+$	P_{11}	0.34	1690	280	$3/2^+$	P_{33}	0.10	Very probable (seen reasonably well in two phase-shift analyses)
1855	335	$3/2^+$	P_{13}	0.27	1670	225	$3/2^-$	D_{33}	0.13	
1983	225	$7/2^+$	F_{17}	0.13	1954	310	$5/2^-$	D_{35}	0.15	Unconfirmed (seen in only one phase-shift analysis or not so well in two)
2030	290	$3/2^-$	D_{13}	0.26	2420	340	$11/2^+$	H_{311}	0.18	
2190	300	$7/2^-$	G_{17}	0.35						
2450	350	$9/2^+$	H_{19}	0.4						
2650	300	?	?	?	2850	300	$15/2^+$	?	?	Unconfirmed (seen only in one total cross-section measurement)

The only purely elastic -- one could say the only classical -- resonance, is the famous $3/2, 3/2$ resonance at 1236 MeV. The corresponding Argand diagram makes an almost complete circle of radius 0.5, while the phase-shift is purely real and crosses 90° . The $P_{33}(1236)$ resonance is a dominant feature in both formation and production experiments. In particular, it causes the first, very large bump in the total cross-sections. The parameters of this state measured by using different definitions are shown in Table 2.3.

Other well-established resonances are $P_{11}(1470)$, $D_{13}(1520)$, $D_{15}(1680)$, $F_{15}(1690)$, $S_{11}(1710)$, $G_{17}(2200)$ with $I = \frac{1}{2}$, and $F_{37}(1950)$, with $I = 3/2$. There is little doubt that S_{31} resonates at about 1640 MeV, although the behaviour of this amplitude is rather strange.

The second class of resonances includes resonances seen reasonably well in at least two of the three phase-shift analyses; there is a good chance that these resonances are real. The first $S_{11}(1535)$ resonance seems to be present in the Saclay and Berkeley results, but is missing in the case of CERN, though a little discontinuity is observed there^{*)}. It is very likely that the CERN smoothing procedure is responsible for eliminating this narrow candidate. The CERN and Berkeley analyses show common resonance-like features for the $P_{33}(1688)$, $D_{33}(1691)$, $F_{35}(1913)$, $P_{31}(1934)$, and probably for $D_{13}(2057)$.

*) The $S_{11}(1535)$ is now there also in the latest CERN analysis (1968).

It takes some imagination to see resonance-like behaviour in the Argand plots for $P_{13}(1863)$ and $D_{35}(1954)$.

The $P_{11}(1751)$ is seen only in the CERN analysis, where it seems very clear. $F_{17}(1983)$, $H_{311}(2420)$, and $H_{19}(2450)$ are much less clear.

Even if all of these resonances were to survive, it seems likely that the quoted resonance parameters may well undergo significant changes as more information becomes available and as the analyses become more refined.

The indication from phase-shift analyses of so many new states with large masses, low spin-values and low elasticities has complicated the physical picture. In analogy (a somewhat doubtful one) with atomic and nuclear spectroscopy, one could say that while the bumps in the total cross-sections individuated the dominant, strong lines, the recent phase-shift analyses discovered weak lines and/or found splittings of some of the strong ones. In other words, the phase-shift analysis behaves as a highly sensitive spectroscope with high resolution (and some instability). Clearly this "instrument" should be put to further use for clearing up the situation.

Among the many resonance classification schemes that have been proposed, we shall discuss the Regge-pole one in Section 10.2. The proceedings of some recent conferences offer good reviews of the other schemes¹⁹⁹⁾.

8. OPTICAL MODELS

8.1 Comparison with classical optics

In this section we shall review some of the optical models used in high-energy physics. The optical model is the geometrical limit of wave phenomena when the wavelength of the incoming particle is short compared to the region of interaction. The conditions under which the diffraction theory provides a good approximation of the scattering of a wave are²⁰⁴⁾:

1) Short wavelength condition^{*)}:

$$\lambda \ll R, \quad (8.1)$$

which implies that the scattering will be concentrated at small angles and that a large number of partial waves contribute coherently to the scattering process.

2) Large distance condition: the distance from the source of the wave to the scattering system and from the scattering system to the wave detector should be large compared to the region of interaction. If d is the smaller of the two distances,

$$R \ll d. \quad (8.2)$$

In optics one distinguishes two cases:

$$\text{Fraunhofer diffraction} \quad \frac{R}{\lambda} \cdot \frac{R}{d} \ll 1 \quad \text{or } d \rightarrow \infty \quad (8.3a)$$

---

*) For $p_{\pi \text{ lab}} = 10 \text{ GeV/c}$, it is $\lambda_{\pi \text{ lab}} = h/p = 0.124 \text{ fermi}$ and $\lambda_{\pi \text{ c.m.}} = 0.59 \text{ fermi}$; $\lambda_{\pi \text{ lab}}$ and $\lambda_{\pi \text{ c.m.}}$ are the wavelengths of the incident pion in the lab. and c.m. frames, respectively.

Fresnel diffraction

$$\frac{R}{\lambda} \cdot \frac{R}{d} \gtrsim 1 . \quad (8.3b)$$

In particle physics one is always in the Fraunhofer diffraction case.

A simple physical picture of the optical model is obtained when there exist simple shape parameters for the scattering system.

The basic equations we shall use in particle physics are the impact parameter expansions of the scattering amplitude (2.44a,b); the different optical models are obtained by using different expressions for the $f(E,b)$'s. In the limit of very high energies, as a first approximation usually one assumes: a) a spin-independent interaction, so that $f(E,b^+) = f(E,b^-)$ and therefore $h(E,\theta) = 0$; b) a purely imaginary non-spin-flip amplitude. In this case, Eqs. (2.44a,b) reduce to

$$g(E,\theta) \approx i K \int_0^\infty b[1 - a(E,b)] J_0(Kb \theta) db \quad (8.4a)$$

$$h(E,\theta) \approx 0 . \quad (8.4b)$$

It may be worth while recalling that Eqs. (2.44a,b) and therefore all equations in this chapter are valid for large values of s and not too large values of the scattering angle.

In optics, for light-rays travelling in different media, the spatial part of the wave function may be written in the form:

$$\psi(\vec{r}) = \exp [i X(\vec{r})] . \quad (8.5)$$

In the case in which the wavelength of the light is small compared to the distances that characterize the rate of change of the refraction index, χ obeys the eikonal equation:

$$\left(\frac{\partial \chi}{\partial x}\right)^2 + \left(\frac{\partial \chi}{\partial y}\right)^2 + \left(\frac{\partial \chi}{\partial z}\right)^2 = K^2 n^2(\vec{r}) = K'^2(\vec{r}) . \quad (8.6)$$

$\vec{K}$ and $\vec{K}'(\vec{r})$ are the wave vectors in vacuum and inside matter, respectively; $n(\vec{r})$ is the index of refraction. A formal solution of Eq. (8.6) is:

$$X = \int_{\infty}^z K'(\vec{r}') dr' . \quad (8.7)$$

The use of this formalism in particle physics results in a connection between typical wave quantities and particle quantities. The wave vectors $\vec{K}$ and $\vec{K}'$ become the momenta of the incoming and outgoing particles. If the interaction between the particles could be described by a complex potential

$$V(\vec{r}) = U(\vec{r}) + i W(\vec{r}) , \quad (8.8)$$

the relation between the wave vectors is given by energy conservation

$$\sqrt{K'^2(\vec{r}) + m_\pi^2} + U(\vec{r}) + i W(\vec{r}) = \sqrt{K^2 + m_\pi^2} , \quad (8.9a)$$

which at high energies, neglecting the pion mass, becomes

$$K'(\vec{r}) \approx K - [U(\vec{r}) + i W(\vec{r})] . \quad (8.9b)$$

Therefore:

$$n(\vec{r}) = \frac{K'(\vec{r})}{K} \approx 1 - \frac{U(\vec{r}) + i W(\vec{r})}{K} . \quad (8.10)$$

Equation (8.10) gives the relation between an optical quantity (the index of refraction) and a particle quantity (the potential). The wave function (8.5) may now be written explicitly in terms of the potential via Eqs. (8.7) and (8.10).

By using the same physical picture discussed in Section 2.5, we may expand the incident plane wave as a superposition of ingoing and outgoing partial spherical waves. The ingoing partial wave is not affected by the scattering process, while the phase of the outgoing partial wave is modified by a factor:

$$(1 - a_1) = \exp [2i(\delta_1 + i \epsilon_1)] . \quad (8.11)$$

In the description of the wave function in terms of a potential, the phase of the outgoing wave is modified by

$$\exp \left\{ -i \int_{-\infty}^z [U(\vec{r}') + i W(\vec{r}')] dz' \right\} . \quad (8.12)$$

A comparison of Eq. (8.12) with Eq. (8.11) yields the desired connection between the phase shift and the potential:

$$2(\delta_1 + i \epsilon_1) = - \int_{-\infty}^z [U(\vec{r}') + i W(\vec{r}')] dz' . \quad (8.13)$$

In practice, considerations on the potential are of limited use since the form for the potential is not simpler than the form for a_ℓ .

8.2 Optical model with constant opacity

All optical models with a single potential and a purely imaginary scattering amplitude have two parameters, the opacity and the radius of interaction. The average opacity is determined by the ratio σ_{el}/σ_{tot} , while the radius is determined by the slope of the diffraction pattern.

Let us assume that the target particle is an opaque sphere of radius R and constant opacity. Semiclassically, only partial waves with $\ell \gtrsim KR$ can be affected by the scattering and have to be considered. For each of these waves we have the same constant opacity $a_\ell = a$; a is real and smaller than unity. For all waves with $\ell > KR$, it is $a_\ell = 0$. Therefore Eq. (8.4a) reduces to

$$\begin{aligned} g(E, \theta) &\approx i K(1 - a) \int_0^R b J_0(Kb \theta) db \\ &= \frac{i}{K\theta^2} (1 - a) \int_0^R (Kb \theta) J_0(Kb \theta) d(Kb \theta) \end{aligned} \quad (8.14a)$$

$$= i(1 - a) KR^2 \left[\frac{J_1(KR \theta)}{KR \theta} \right] . \quad (8.14b)$$

It follows that:

$$\frac{d\sigma}{d\Omega}(E, \theta) \approx (1 - a)^2 K^2 R^4 \left[\frac{J_1(KR \theta)}{KR \theta} \right]^2. \quad (8.15a)$$

$$f(E, \theta) \approx \frac{i}{2} (1 - a) K R^2 \quad (8.15b)$$

$$\sigma_{el}(E) \approx \pi R^2 (1 - a)^2 \quad (8.15c)$$

$$\sigma_{in}(E) \approx \pi R^2 (1 - a^2) \quad (8.15d)$$

$$\sigma_{tot}(E) \approx 2\pi R^2 (1 - a) \quad (8.15e)$$

$$\sigma_{el}/\sigma_{tot} \approx \frac{1}{2} (1 - a). \quad (8.15f)$$

For complete absorption (black sphere) $a = 0$, and it is easy to obtain the relevant formulae from Eqs. (8.15a,f). It is interesting to note that in this case:

$$\sigma_{el} = \sigma_{in} = \pi R^2 = \frac{1}{2} \sigma_{tot}, \quad (8.16)$$

i.e. the total cross-section is twice the geometrical cross-section.

The differential cross-section [Eq. (8.15a)] may be rewritten in terms of the square of the momentum transfer t ; it assumes a particularly simple form for large energies and small scattering angles, where $|t| \approx K^2 \theta^2$:

$$\frac{d\sigma}{d|t|} \approx (1 - a)^2 \frac{\pi R^4}{4} \exp[-R^2 |t|/4] \quad (8.17)$$

For small t -values, the Bessel function J_1 may be replaced by an exponential. In fact, the power series expansions for the Bessel and exponential functions are:

$$\frac{J_1(R\sqrt{|t|})}{\sqrt{|t|}} = \left\{ \frac{R}{2} \left[1 - \frac{R^2 |t|}{2^2 1! 2!} + \frac{R^4 |t|^2}{2^4 2! 3!} - \frac{R^6 |t|^3}{2^6 3! 4!} + \dots \right] \right\}^2 \quad (8.18a)$$

$$\exp(-R^2 |t|/8) = 1 - \frac{R^2 |t|}{(2^3) 1!} + \frac{R^4 |t|^2}{(2^3)^2 2!} - \frac{R^6 |t|^3}{(2^3)^3 3!} + \dots \quad (8.18b)$$

The first two terms of the expansions are equal; therefore for small $|t|$ values formula (8.17) reduces to:

$$\frac{d\sigma}{d|t|} \underset{t \ll \dots}{\approx} (1 - a)^2 \frac{\pi R^4}{4} \exp[-R^2 |t|/4]. \quad (8.19)$$

The equality of formulae (8.17) and (8.19) is limited to rather small t -values, as seen in Fig. 8.1. At such values of t , the results of the calculation from different optical models may all be approximated by exponential distributions, different in the values of the interaction radius and of the opacity.

The optical model with constant opacity is not expected to provide a good description of the experimental data because a constant opacity over a spherical region is difficult to imagine. The sharp boundary gives rise to diffraction minima in the differential cross-section at the points where $J_1 = 0$ (for $R\sqrt{|t|} = 3.8, 7.0, 10.2, \dots$).

8.3 Optical model with Gaussian opacity

Let us assume the opacity of the nucleon to be Gaussian

$$1 - a(E, b) = \alpha \exp \left[-\frac{5b^2}{2R^2} \right], \quad (8.20)$$

where α is a real constant, b is the impact parameter, and R the root mean square radius of the distribution. By substituting in formula (8.4a):

$$f(E, \theta) \simeq i K \alpha \int_0^\infty b \exp \left[-\frac{5b^2}{2R^2} \right] J_0(Kb \theta) db \quad (8.21)$$

the integration yields:

$$f(E, \theta) \simeq i \frac{K \alpha R^2}{5} \exp \left[-\frac{2}{5} (KR \theta/2)^2 \right] \quad (8.22)$$

$$\frac{d\sigma}{d\Omega}(E, \theta) \simeq \frac{K^2 \alpha^2 R^4}{25} \exp \left[-\frac{K^2 R^2 \theta^2}{5} \right] \quad (8.23a)$$

For small scattering angles $|t| \simeq K^2 \theta^2$; therefore the following formulae are valid [within (1-2)% for $|t| < 1$ (GeV/c)² and $T_{\pi \text{lab}} \gtrsim 10$ GeV]:

$$\frac{d\sigma}{dt} \simeq \frac{\pi \alpha^2 R^4}{25} \exp \left[-\frac{R^2}{5} |t| \right] \quad (8.23b)$$

$$\left(\frac{d\sigma}{dt} \right)_{t=0} \simeq \frac{\pi \alpha^2 R^4}{25} \quad (8.23c)$$

$$\sigma_{el}(E) \simeq \frac{\pi}{K^2} \int_0^\infty 2\pi b |1 - a|^2 K db = \pi \int_0^\infty \alpha^2 \exp \left[-\frac{5b^2}{R^2} \right] db^2 = \frac{\pi \alpha^2 R^2}{5} \quad (8.23d)$$

$$\sigma_{tot}(E) \simeq \frac{4\pi \alpha R^2}{5} \quad (8.23e)$$

$$\sigma_{el}/\sigma_{tot} \simeq \frac{\alpha}{4}. \quad (8.23f)$$

Several variations of this model are found in the literature. Cottingham and Peierls use the unitarity condition to express the high-energy value of $\text{Im } f_b$ in terms of a new function $\rho(b)$ to which a Gaussian expression is attributed. The differential elastic cross-section is then predicted to fall off exponentially in $|t|$ for small t -values; it flattens off until it is exponential in $\sqrt{|t|}$ for large scattering angles. In the case of total absorption, the model predicts a maximum ratio $(\sigma_{el}/\sigma_{tot})_{\text{max}} = 0.1845$. The experimental $(\sigma_{el}/\sigma_{tot})$ ratio is smaller than this value for πN scattering, while it is larger for pp . Therefore one must introduce another parameter for pp scattering, which could be the real part of the scattering amplitude or a spin-spin term.

8.4 Optical models with more than one potential

The need for more than one potential was stressed by many authors.

Serber^{36,57)}, starting from Eq. (8.13), shows that it is possible to find a combination of two purely imaginary potentials, which can fit the gross features of the pp data both for large and small angles (small and large impact parameters). For small impact parameters, the potential is of the Yukawa type, while for large ones it is of the Gaussian type. The model fails to explain all the energy dependence of the data.

Krisch⁵⁴⁾ assumes that the π meson, the K meson and the baryon-antibaryons are the quanta of three types of interactions, each having a different Gaussian region of interaction. Therefore the nucleon has an onion-type structure with pion phenomena in the outside, kaon processes in the intermediate region, and baryon-antibaryon processes on the inner region. The elastic scattering pp data, plotted as function of the square of the transverse momentum, agree with this picture over 10 orders of magnitude.

The natural extension of these pictures is a superposition of an infinite number of potentials. Predazzi writes:

$$\frac{d\sigma}{dt} = c \sum_{n=0}^{\infty} \alpha_n e^{-\beta_n x^2}, \quad (8.24)$$

where x is the transverse momentum, and c, α_n, β_n are constants to be determined. The first three terms are made to coincide with the results of Krisch. In this way the agreement with the pp data is extended for another order of magnitude. Moreover, Eq. (8.24) satisfies the bound of Cerulus and Martin, while this is often violated by other composite models.

8.5 The radius of interaction

In the study of the electromagnetic structure of nucleons and of nuclei, one usually defines the root mean square radius of the charge distribution as

$$\langle R^2 \rangle = \frac{\int_0^{\infty} r^2 \rho(r) 4\pi r^2 dr}{\int_0^{\infty} \rho(r) 4\pi r^2 dr}, \quad (8.25)$$

where $\rho(r)$ is the electric charge density. The infinitesimal volume $dv = 4\pi r^2 dr$ is the volume of an element of sphere. All the points inside this elementary volume have the same charge density.

In the case of optical models, one may define the root mean square radius of the opaque zone as:

$$\langle R^2 \rangle = \frac{\int_0^{\infty} (1 - a_b) b^2 4\pi b \sqrt{r^2 - b^2} db}{\int_0^{\infty} (1 - a_b) 4\pi b \sqrt{r^2 - b^2} db}. \quad (8.26)$$

The infinitesimal volume $dv = 4\pi b \sqrt{r^2 - b^2} db$ is the volume of a cylinder where all the incoming particles have the same impact parameter and they experience the same opacity

(Fig. 8.2). The constant $5/2$ which appears in the exponent of Eq. (8.20) is the correct constant in order that R has the meaning of root mean square radius according to Eq. (8.26).

The optical radius defined by formula (8.26) would be the radius of the target particle if the incident particle had no dimensions. In practice, both particles have finite dimensions (of the order of one fermi for pions and protons) and the radius defined by formula (8.26) is a combination of the radii of the two particles. The impact parameter b is the transverse distance between the lines of flight of the centre of the two particles. Moreover, the finite range of the nuclear forces comes in. We may consider that the nuclear radius of each particle is equal to the radius of the nuclear matter (probably equal to the charge radius) plus the range of nuclear forces. Finally, relativistic effects (Lorentz contraction) prevent a better definition.

We shall use the term "interaction radius" to denote the radius defined mathematically by Eq. (8.26); such a radius will depend on the size of both particles as well as on the finite range of the forces.

It may be worth adding that, though it is more intuitive to discuss the optical model in the laboratory system, it is more appropriate to use it in the c.m. frame. The two systems of reference coincide in the case of light rays, and, with good approximation, in the case of light particles against nuclei.

The behaviour of the cross-sections at small t -values may yield information on the interaction radius. Figure 8.1 shows that in the small t -region the various models yield exponential diffraction patterns. Nevertheless, the radii of interactions differ by a constant factor. The absolute value of the radius may therefore have a systematic error, while the change with energy is significant. We shall consider the radius for the model with constant opacity. In terms of the exponential (4.3), one has

$$R = 2 \sqrt{-b} . \quad (8.27)$$

In order to apply Eq. (8.27) to the experiments, we must choose limits on $|t|$. On the low- t side we want to be far away from the Coulomb interference region; on the large- t side, we have to limit ourselves to values of t such that the C -term of Eq. (4.4) is not significantly different from zero, and the momentum transfers involved are not too large. By choosing the interval $0.1 < |t| < 0.5 \text{ (GeV/c)}^2$, one computes the radius at an average value of $|t| \approx 0.3 \text{ (GeV/c)}^2$, i.e. to momentum transfers of about 550 MeV/c corresponding to a distance of about 0.37 f, which may be a kind of compromise (on the smaller side) on outer pion cloud and inner core region.

The results of the analysis are shown in Fig. 4.14. While the $\pi^{\pm}p$ interaction radius remains constant, the pp radius increases with increasing energy.

The average values of the $\pi^{\pm}p$ and $\pi^{\mp}p$ interaction radii are slightly different (Table 8.1). It is not clear if the difference is significant.

Table 8.1
Average values of the interaction radii at high energies

Energy interval (GeV)	$\langle r_{\pi^- p} \rangle$ (fermi)	$\langle r_{\pi^+ p} \rangle$ (fermi)
$3 < T_{\pi_{lab}} < 8$	(1.080 ± 0.009)	(1.056 ± 0.011)
$8 < T_{\pi_{lab}} < 20$	(1.099 ± 0.005)	(1.070 ± 0.005)

8.6 The opacity

In the various optical models the opacity of the target particle changes with the impact parameter (with the exception of the model with constant opacity). Mathematically this is expressed by giving different forms to $[1 - a(b)]$. In practice, as we discussed for the interaction radius, we have a collision between two different particles of finite dimensions; therefore the opacity we may measure is in reality a combination of the structures of the two particles.

The ratio $(\sigma_{el}/\sigma_{tot})$ is a measure of the average opacity, which we may define as $0 = 2 \sigma_{el}/\sigma_{tot}$. In this way the opacity of a black disc is equal to one.

Figure 4.5 shows the opacity for $\pi^\pm p$. Table 4.1 gives the $\pi^\pm p$ and pp average opacities for energies larger than 8 GeV. The $\pi^+ p$ and $\pi^- p$ opacities are approximately equal and somewhat smaller than the pp opacity.

8.7 The backward peak

In the model of Simmons⁶⁴⁾ for scattering at low energies, all waves with $l < L$ are assumed to be equally absorbed. In this case one may sum the partial wave expansion Eq. (2.31) and obtain

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\theta=0} \left[\frac{P'_L(\cos \theta) + P'_{L+1}(\cos \theta)}{(L+1)^2} \right]^2, \quad (8.28)$$

which predicts a narrow backward peak (see Fig. 8.3). This feature could be easily destroyed by varying the relative amount of absorption among the waves interested in the scattering process, as happens at low energies (see Fig. 4.12).

All the formulae used in the previous paragraphs of this section are valid at high energies and not too large scattering angles. One may treat the backward region by an optical model, by applying to the small values of the angle

$$\theta' = 180^\circ - \theta \quad (8.29)$$

the same approximations used at forward angles^{202, 438)}. In this way one obtains a situation that is completely analogous to the forward one. In particular, one predicts a backward peak whose angular dependence is given, at first order in the optical model with constant opacity, by:

$$\frac{d\sigma}{d\Omega} \underset{\theta' \ll \frac{\pi}{2}}{\approx} (1-a)^2 R^2 B(KR\theta') = (1-a)^2 \frac{R_2}{2} \left[1 - \left(\frac{KR\theta'}{2} \right)^2 + \dots \right]. \quad (8.30)$$

The backward peak is predicted to have a width equal or narrower than the forward one, and a cross-section ratio (backward/forward) $(1/K^2 R^2) = 1/L^2$, where L is the maximum value of the orbital angular momentum. These predictions are qualitatively correct, but the difference between π^+p and π^-p elastic backward scattering remains unexplained together with the energy dependence of the total elastic backward cross-section which is predicted to be only $1/K^2$, that is $\sigma_{el}(\text{backward}) \approx 0.78 (1 - a)^2 \pi/K^2$.

8.8 Some considerations on the validity of the optical models

The two-parameter optical models with an imaginary non-spin-flip scattering amplitude are able to fit the existing experimental data in the multi-GeV region for a limited t -range (t small). When data at lower energies or at larger momentum transfers are included, the models find some difficulties, such as in explaining a) the difference at low t -values in the energy dependence for pp compared to π^+p (the pp peak shrinks, while the π^+p does not); b) the detailed large-angle angular distributions; c) the energy dependence of the large-angle and backward π^+p cross-sections; d) the difference between the π^+p and π^-p large-angle and backward cross-sections. Nevertheless, the general appearance of the differential cross-section is quite consistent with a diffraction mechanism. The high-energy distributions of Figs. 4.6 and 4.7 contain the forward diffraction peak, indications of secondary diffraction maxima, and a backward peak; all these features are predicted by an optical model. It is the quantitative explanation of all these features that are difficult to reconcile within the model.

It is clear that the model must be refined in order to explain such detailed features. In particular the implications of the presence of a real part of the scattering amplitude (about 10% of the imaginary part at $\theta = 0$) with an unknown angular dependence, together with the possibility that also the spin-flip amplitude is almost certainly different from zero, have not been fully analysed. Cottingham and Peierls⁶⁵⁾ as well as Durand and Greider³⁵⁾ were able to fit the π^+p and pp elastic scattering data over a considerable angular range, making some assumptions on the behaviour of the real part. Feld¹⁵⁵⁾ emphasized the importance of the spin-flip amplitude (see Fig. 8.4).

Another way out of some of the difficulties is to use more than one potential. It is nevertheless quite possible that a part or all of the large-angle cross-section arises from a different mechanism.

A purely imaginary scattering amplitude predicts a shrinking of the diffraction pattern if the total cross-section remains constant with increasing energy, while the elastic cross-section decreases. Semiclassically, this would imply that the radius of interaction increases with increasing energy, while the opacity decreases. Within the large errors with which the opacities and the real parts are known, a situation of this sort could qualitatively explain the pp scattering data.

Byers and Yang⁸⁵⁾ observe that all the forward and backward elastic peaks in π^+p , K^+p , p^+p , as well as all charge-exchange forward peaks, may be expressed, over a limited region, as exponentials in the momentum transfer variable [Eqs. (4.3) and (4.15)]. The existence of these large peaks leads to the conclusion that whereas large momentum transfers are difficult to realize, it is instead relatively easy to transfer, in a coherent way, quantum numbers such as charge, strangeness, nucleon number, and so on. Moreover, the universality

of the form of the peaks suggests that the mechanisms of these phenomena may be related to one another. They therefore suggest an impact parameter model both for elastic and charge exchange scattering, with the relations:

$$\alpha_{ce}(E,b) - 1 = K_{nf}(E) a(E,b) \log a(E,b) \quad (8.31a)$$

$$\beta_{ce}(E,b) - 1 = K_f(E) a(E,b) b^2 \log a(E,b) \quad (8.31b)$$

where α_{ce} and β_{ce} are the non-spin-flip and spin-flip partial charge exchange amplitudes; $K_{nf}(E)$ and $K_f(E)$ are complex quantities and $a(E,b)$ is the ordinary partial amplitude. Deriving $a(E,b)$ from elastic scattering data, it is possible to predict the small- t charge-exchange cross-section in terms of the only parameter K_f/K_{nf} , which turns out to be of the order of 1.

9. THE STATISTICAL MODEL

9.1 The classical statistical model

The high-energy large-angle cross-section is characterized by a slower t -dependence and a faster s -dependence than at smaller angles (the 90° c.m. cross-section shows an exponential decrease with energy). These features may suggest a statistical mechanism. Following the compound model picture for nuclear reactions, one may visualize the statistical process in this way: for central high-energy collisions, the two colliding particles form a compound system. The total energy available is concentrated in a certain volume in which a kind of thermal equilibrium is reached. The volume expands, and at a certain moment the equilibrium ceases and the compound state decays into the many open channels, one of which is just the initial channel. The large-angle elastic cross-section due to the statistical process will therefore be proportional to the probability of formation of the compound state and to the branching ratio for decay of the compound state into the elastic channel. The analogy with nuclear physics does not really go very far, because of the very short lifetimes involved in elementary particle physics, which means that a statistical equilibrium cannot be reached.

Because of the relativistic contraction, the central collision occurs between two flat discs, as illustrated in Fig. 9.1. In the classical statistical picture, the resulting volume will not remain in shape, but will quickly expand in all directions and will cease to be in thermal equilibrium when it becomes approximately spherical, with a radius $r \approx 1/m_\pi$.

The cross-section for the formation of the compound state σ_c will be proportional to the total cross-section σ_{tot} minus the diffraction cross-section σ_d ; at high energies this difference is practically given by the inelastic cross-section σ_{in} . Every channel has a small cross-section; there is almost no direct elastic scattering, but only as shadow of inelastic processes. Therefore:

$$\sigma_c = (\sigma_t - \sigma_d) \alpha \approx \sigma_{in} \alpha, \quad (9.1)$$

where α gives the fraction of the inelastic cross-section which is due to central processes. α is expected to decrease slowly and monotonically with increasing energy, because at high

energies peripheral processes become more important. A plausible first approximation of Fast et al.^{33,34}) is:

$$\alpha \approx 1/E, \quad (9.2)$$

where E is the total c.m. energy.

$\sigma_c/\sigma_{\text{tot}}$ is roughly 1-10%. The division of the collisions into central and non-central is somewhat analogous to the division of nuclear interactions into compound nucleus formation and direct reactions.

For the pion-nucleon system, the old statistical model of Hagedorn^{33,34,76}) predicts for the branching ratio of the compound state into the elastic channel the value:

$$P = \frac{P_{\pi N}}{P_{\pi N} + P_{2\pi N} + P_{3\pi N} + \dots} = \exp [-3.17(E - 1.40)], \quad (9.3)$$

where E is in GeV; $P_{\pi N}$ is the probability for decaying into the πN state, $P_{2\pi N}$ into $2\pi N$, etc. From Eqs. (9.1) to (9.3) the "compound elastic" cross-section $\sigma_{c,el}$, i.e. the elastic cross-section due to a statistical process, is given by:

$$\sigma_{c,el} = \sigma_c P \approx \frac{\sigma_{in}}{E} \exp [-3.17(E - 1.40)]. \quad (9.4)$$

Equations (9.3) and (9.4) are relative to elastic plus elastic charge exchange scatterings.

Figure 9.1 shows a naive picture of the collision between two particles of finite size. The time of contact $t = d/c$ between the particles should be greater than $t_0 = b/c$, $\sim R/c \sim 3 \times 10^{-24}$ sec, if the interaction has to spread out (at the speed c) over the whole volume, and if the system has to reach a kind of thermodynamical equilibrium during the time of contact. In practice, the interaction spreads out over the whole particle only for central collisions. In peripheral collisions only "part of the particle" is involved in the statistical process. For identical particles of radius R and mass m the Lorentz contraction factor is given by:

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}} = \frac{E}{2m}. \quad (9.5)$$

A sort of equilibrium could result only in the region for which

$$b/c \leq t = \frac{2R}{\gamma c} = \frac{4Rm}{cE}. \quad (9.6)$$

The equality is obtained for a certain value $b = b_{\text{max}}$, to which semiclassically corresponds L, the maximum orbital angular momentum possible to achieve equilibrium:

$$L \approx b_{\text{max}} K \approx \frac{2R}{\gamma} \frac{E}{2} = 2 Rm \approx 10 \text{ for pp}, \quad (9.7)$$

i.e. only a partial region and only a limited number of ℓ -values (≤ 10 for pp) contribute to the real statistical process; this is independent of incoming energy, provided the energy is high enough. For $\ell > L$ it is not possible to achieve an equilibrium condition.

The branching ratio given by Eq. (9.3) is essentially $1/N$, where N is the number of states into which the compound nucleus may decay. The total number of states is related to the entropy of the system $S(E)$ by the relation^{62,77}):

$$N = e^S. \quad (9.8)$$

The correlation between entropy, total c.m. energy, and temperature T is given by

$$\frac{dS}{dE} = \frac{1}{T}. \quad (9.9)$$

The dependence of S on E may be assumed to be of the form:

$$S = aE^n, \quad (9.10)$$

where a is a constant and n characterizes the "gas" of the compound system: $n = \frac{1}{2}$ corresponds to a Fermi gas, i.e. to the case where in the constant volume of the compound system only one kind of particle is present, whose temperature increases when E increases; $n = 1$ corresponds to the case of a system in which the temperature remains essentially constant and the number of particles increases with increasing c.m. energy. The model of Fast and Hagedorn³³) produces an essentially constant temperature of $T = 0.3 \text{ GeV} = 2.18 m_\pi$.

9.2 Limitations of the statistical interpretation

Some of the main limitations of this statistical interpretation are:

- i) the fraction of collisions to be considered central (α) is not known precisely;
- ii) in a practical experiment the compound elastic contribution may be completely hidden under the tail of the diffraction peak;
- iii) the statistical model in this form is not able to answer detailed questions such as: What is the angular distribution? Including angular momentum conservation effects, one gets an angular distribution with an angular dependence not stronger than a limiting distribution $1/\sin \theta$ ⁵⁵). The experimental data at large angles seem to indicate an angular distribution that is more pronounced than this limiting one. This is evidence that some other mechanism is also present.

Ericson⁴¹) pointed out that if the large-angle scattering is of a statistical nature, the angular distribution should show peaks and valleys, with separations of the order of a few degrees and peak-to-valley ratios as high as 10. These fluctuations should appear when the compound system lifetime is comparable to or smaller than the interaction time of the incident wave packet. Precise pp elastic scattering data at 19 GeV/c did not find any evidence for these fluctuations²⁰⁹).

In conclusion, there is no clear evidence for the validity of the statistical model in high-energy elastic scattering.

9.3 The inclusion of collective motions

Hagedorn and Ranft¹⁵⁴) have recently improved their statistical model by assuming that, inside the interaction region, equilibrium is never reached, but that there are strong collective motions reflecting the situation just before the collision. In this way kinematics comes in immediately through the collective motion.

A volume element of the particle is considered in its rest frame (of the element): here thermodynamics governs alone. The particles produced in this element are Lorentz transformed to the c.m. frame of the collision. This is done for a continuum of volume elements whose velocity distribution is described by weight functions. Inside the volume element there are, potentially, all sorts of particles and resonances. All these hadronic states are treated on an equal footing, and the name fireball is used to denote them.

The basic postulate of the thermodynamics of this model is:

1) a fireball is a statistical equilibrium of an undetermined number of all kinds of fireballs; each of these fireballs is, in turn, a statistical equilibrium of an undetermined number of fireballs, and so on *ad infinitum*. The immediate consequence of this postulate is the existence of a highest temperature $T_0 \approx 160$ MeV. This is needed in order to explain the experimental fact that in inelastic high-energy collisions the transverse momentum distribution of secondaries is of a form similar to a Boltzmann distribution, with a constant average transverse momentum, independent of energy. One can make an analogy with a liquid at boiling point: at boiling-point particles are emitted from the liquid into the vapour and thereby must overcome a large potential barrier; in this process so much energy is used that although energy flows constantly in, the temperature stays constant. Now, in the fireball gas, an infinity of different kinds of particles may be created. With increasing total energy, more states are created and the temperature remains constant at T_0 .

Two other postulates are needed to describe the collective motion and to correlate it to the thermodynamics:

2) collective motions (other than thermodynamics) have only longitudinal components with respect to the collision axis.
3) any loss of kinetic energy density during the collision is adiabatically transformed into heat.

The collective motion is really added from the outside; this means that in practice one arrives at a mixed model, which is a combination of two models. The predictions of the model are in fair agreement with the experiment.

10. REGGE POLES

As far as we are concerned, the applications of the Regge-pole model in elementary particles appear in two different, although closely connected, contexts:

a) In resonance physics the Regge-pole model correlates, by means of a Regge trajectory, particles and resonances having the same internal quantum numbers (such as baryon number, isospin, parity, G-parity, and strangeness), but with spins differing by units of two. As an example, the $N(938)$ and $N(1680)$ with $I = \frac{1}{2}$, $J = \frac{1}{2}^+$ and $5/2^+$, respectively, are believed to lie in the same trajectory. In the J versus E^2 plane, the Regge trajectory is a straight line connecting these two states (Chew-Frautschi plot). Therefore, a Regge trajectory may be thought of as interpolating the angular momentum between physical values.

b) In the high-energy field the two-body or quasi-two-body collisions are predicted to be dominated by the exchange of a few of these trajectories. For instance, the $\pi^- p \rightarrow \pi^0 n$ charge exchange reaction is predicted to be dominated by the exchange of the ρ trajectory, i.e. by the meson trajectory which passes through the ρ meson.

In this way, in elementary particle physics, the Regge-pole model gives a connection between low-energy phenomena (resonances) and high-energy ones (cross-sections at high energies). The Regge-pole model makes full use of the analyticity properties.

In the following we shall recall the main results of the Regge-pole theory, starting from potential scattering. The reader is referred to one of the numerous lecture notes¹⁹⁶⁾ for a more thorough analysis.

10.1 Regge poles in potential scattering

Regge¹³⁾ considered the Schrödinger equation in the presence of a superposition of Yukawa potentials and explored the solutions for values of ℓ other than non-negative integer ones, in fact for arbitrary complex ℓ -values. His main conclusions were as follows:

- i) The radial Schrödinger equation has solutions for complex values of ℓ , provided $\text{Re } \ell \geq -\frac{1}{2}$.
- ii) The partial wave scattering amplitude $f_\ell(E)$, which we now write as $f(E, \ell)$, may be defined for complex ℓ -values, and is a meromorphic function of ℓ for $\text{Re } \ell \geq -\frac{1}{2}$ (analytic except for a finite number of poles).
- iii) The poles of $f(E, \ell)$ in the complex ℓ -plane are called Regge poles. They occur: on the real ℓ -axis if E is below the scattering threshold, E_{th} ; in the upper half plane, with $\text{Im } \ell > 0$, if $E > E_{\text{th}}$. A Regge pole moves in the complex ℓ -plane if the energy varies. The trajectory of the pole, that is $\alpha = \alpha(E)$, is called the Regge trajectory.

Figure 10.1 illustrates the analyticity domain of $f(E, \ell)$ and the characteristics of a Regge trajectory: starting with $E < E_{\text{th}}$, the pole lies to the left of $\text{Re } \ell = -\frac{1}{2}$; with increasing energy the pole moves to the right, enters the meromorphic domain and may, depending on the potential, cross integer values of $\text{Re } \ell$ on the real axis. It may eventually acquire a positive imaginary part for $E > E_{\text{th}}$. Eventually the trajectory turns over and returns to the left of $\text{Re } \ell = -\frac{1}{2}$. We now want to understand what happens when the trajectory crosses positive integers of $\text{Re } \ell$. Let us consider the partial wave amplitude and the angular momentum in the neighbourhood of a non-negative integer. If the pole is a simple pole at $\alpha = \ell$, the partial wave scattering amplitude in the vicinity of the pole may be written as:

$$f(E, \ell) = \frac{R(E)}{\ell - \alpha(E)} + \dots, \quad (10.1)$$

where $R(E)$ is the residue. For $E < E_{\text{th}}$, $\alpha(E)$ is purely real; let us expand $\alpha(E)$ in a Taylor series about the integer value n , corresponding to the energy E_n :

$$\alpha(E) = \alpha(E_n) + (E - E_n) \left(\frac{d\alpha}{dE} \right)_{E_n} + \dots \approx n + (E - E_n) \alpha'(E). \quad (10.2)$$

By substituting Eq. (10.2) in Eq. (10.1):

$$f(E, \ell) \approx \frac{R(E)}{\ell - n - (E - E_n) \alpha'(E_n)} = \frac{R(E)/\alpha'(E_n)}{[\ell - n/\alpha'(E_n)] - (E - E_n)} \quad (10.3)$$

Equation (10.3) says that if the partial amplitude $f(E, \ell)$ has a pole in the angular momentum

plane for $\ell = n$, then it also has a pole in the energy plane at $E = E_n$. This means that we have a bound state at $E = E_n$ with $\ell = n$. In Fig. 10.1 we have two bound states, with $\text{Re } \ell = 0$ and 1.

For $E > E_{th}$, $\alpha(E)$ has a real as well as an imaginary part. Let us again expand $\alpha(E)$ in a Taylor series about the integer $\text{Re } \alpha(E) = n$:

$$\alpha(E) = \text{Re } \alpha(E) + i \text{Im } \alpha(E) \approx n + (E - E_n) \left(\frac{d \text{Re } \alpha(E)}{dE} \right)_{E_n} + i \text{Im } \alpha(E_n). \quad (10.4)$$

Notice that in Eq. (10.4) we put $\text{Im } \alpha(E) \approx \text{Im } \alpha(E_n)$: this implies that the imaginary part is small compared to the real part. By letting

$$a = \left[\frac{d}{dE} \text{Re } \alpha(E) \right]_{E=E_n} \quad (10.5a)$$

$$\frac{\Gamma}{2} = \frac{\text{Im } \alpha(E_n)}{a}, \quad (10.5b)$$

and by substituting Eqs. (10.4) and (10.5a,b) into Eq. (10.1), one has:

$$f(E, \ell) \approx - \frac{R(E)/a}{E - E_n + i \frac{\Gamma}{2} - \frac{1-n}{a}}. \quad (10.6)$$

At $\ell = n$, Eq. (10.6) gives a Breit-Wigner formula; therefore for $E > E_{th}$ one has a resonant state when the Regge trajectory crosses a positive integer of $[\text{Re } \alpha(E)]$. Notice that Eq. (10.4) is valid if the imaginary part of $\alpha(E)$ is small; Γ is positive if a is positive: therefore, in order to have a resonance, the Regge trajectory must increase through the value $\text{Re } \alpha = n$ as the energy increases. In Figure 10.1 one has a resonance for $\text{Re } \ell = 2$, with $E = E_2$; there are no resonant states when the trajectory bends over and decreases through the values $\text{Re } \alpha = 2, 1, 0$.

We shall now see what are the consequences of these properties of $f(E, \ell)$ on the total amplitude $f(E, \theta)$. Sommerfeld and Watson have shown that the partial wave expansion of the amplitude may be written as the sum of residues of an analytic function, which then by Cauchy's integral theorem may be transformed into an integral over the contour C of Fig. 10.2. Regge deformed the contour by including C' ; the explicit contribution of the Regge poles comes in by another application of Cauchy's theorem over the contour $C + C'$. The final result is the Regge-Sommerfeld-Watson (RSW) representation of the scattering amplitude:

$$f(E, \theta) = \sum_i \frac{\beta_i(E) P_{\alpha_i}(-\cos \theta)}{\sin \pi \alpha_i} + (\text{background integral}), \quad (10.7)$$

where the summation is over all Regge poles; $\beta_i(E)$ is a known function of $\alpha_i(E)$ and of the residues of the partial scattering amplitude, $R_i(E)$. The background integral is an integral over the line $\text{Re } \alpha = -\frac{1}{2}$; it is hoped that it may be neglected. There are two other integrals over the contour C' , which, when C' is enlarged to infinity, do not contribute if $f(E, \ell)$ is well behaved, as is the case for Yukawa potentials. The comparison of

Eq. (10.7) with the partial wave expansion shows that one Regge pole contributes in general to all partial waves.

10.2 Regge trajectories for the nucleon resonances

It has been conjectured that the Regge pole ideas may be taken over from potential scattering to all strong interactions²⁰). This is a conjecture, which cannot be proved in the frame of the existing theories. It can only be checked *a posteriori* by comparing its predictions with experiments.

The RSW formula (10.7) with all its hypotheses and implications is now taken over to all types of strong interactions. It follows that a Regge pole gives bound states and resonances as in potential theory. Now each partial wave scattering amplitude, and therefore each Regge trajectory, carries quantum numbers, such as baryon number, isospin, strangeness, etc. Particles can lie on the same trajectory only if they have the same quantum numbers, except spin.

The amplitude (10.7) is a non-spin-flip amplitude and does not take into account the possibility of exchange potentials or exchange interactions. Since such interactions are known to occur, we have to modify the expansion (10.7) to accommodate them. The presence of a space-exchange potential modifies the wave function by a factor $(-1)^\ell$. This factor causes difficulties when one studies solutions for ℓ -complex. The difficulty is circumvented by treating odd and even values of ℓ separately. Formally, this may be done by considering the two combinations $[P_\ell(\cos \theta) \pm P_\ell(-\cos \theta)]$ which are even (odd) in ℓ . The corresponding partial wave amplitudes $f^{(\pm)}(E, \ell)$ have different analytic continuations in ℓ and different Regge poles. One says that a Regge pole of $f^{(+)}(E, \ell)$ has a positive signature, $\tau = +$, while a pole of $f^{(-)}(E, \ell)$ has a negative signature, $\tau = -$. With the signature concept one may write Eq. (10.7) as:

$$f(E, \theta) = \sum_i \frac{\beta_i(E)}{2 \sin \pi \alpha_i(E)} [P_{\alpha_i(E)}(-\cos \theta) + \tau_i P_{\alpha_i(E)}(\cos \theta)] + \int \dots$$

The consequence of the signature concept is very important: a positive signature Regge trajectory can only give particles and resonances of even orbital angular momentum, a negative signature trajectory only one of odd orbital angular momentum. In other words, particles and resonances are spaced by $\Delta \ell = 2$ in a given Regge trajectory.

In pion-nucleon scattering for each ℓ -value there are two partial wave amplitudes $f(E, \ell^\pm)$, with $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$, respectively. Because of the possible existence of a spin-orbit potential, the partial wave amplitudes $f(E, \ell^\pm)$ may be determined by different potentials, and therefore have different Regge trajectories. Therefore, for nucleon resonances, there are four possible types of trajectories for each isospin state (remember that the intrinsic parity of the pion is negative).

Table 10.1
The four types of Regge trajectories for nucleon resonances
with a definite isotopic spin

τ	ℓ	j	States
+	even	$\ell + \frac{1}{2}$	$1/2^-, 5/2^-, 9/2^-, \dots$
+	even	$\ell - \frac{1}{2}$	$3/2^-, 7/2^-, 11/2^-, \dots$
-	odd	$\ell + \frac{1}{2}$	$3/2^+, 7/2^+, 11/2^+, \dots$
-	odd	$\ell - \frac{1}{2}$	$1/2^+, 5/2^+, 9/2^+, \dots$

From very general assumptions, MacDowell¹⁴⁾ was able to show

$$f(E, +, \ell) = f(-E, -, \ell + 1). \quad (10.9)$$

Therefore the two partial waves with the same j , opposite parity and opposite sign of the energy are equal. The MacDowell symmetry implies that if $f(E, +, \ell)$ has an energy or angular momentum pole at a positive energy E , there is a similar pole for $f(-E, -, \ell + 1)$ at a negative energy. For a Regge trajectory for which $\alpha(E) = a + bE^2$, the sign change of the energy is unimportant. Consequently, in the Chew-Frautschi plot one expects the fermion resonances to occur in parity doublets ($\frac{1}{2}^+, \frac{1}{2}^-$), ($5/2^+, 5/2^-$), etc., the two particles in a doublet having the same mass if the trajectories are really straight lines. The absence of an expected doublet partner could be explained by a vanishing residue function at the energy where it should appear.

Figure 10.3 shows the Regge trajectory plot, or Chew-Frautschi plot, for the nucleon isobars. $I = \frac{1}{2}$ resonances are shown as black circles if well-established, as open circles if not firmly established. Solid lines connect the $I = \frac{1}{2}$ states. Black and open squares together with dotted lines are used for the $I = 3/2$ states. Vertical lines indicate states for which the spin is not yet known.

Three trajectories may be individuated, two with $I = \frac{1}{2}$ and one with $I = 3/2$. They are roughly straight parallel lines, with slopes of about one unit of spin per GeV^2 . All the new resonances found in the recent phase-shift analyses have low spin-values, and at the moment do not have recurrences. Their presence complicates the physical picture considerably.

A comparison of these trajectories with those computed in potential scattering from a Gaussian potential may be instructive (Fig. 10.4). The finite number of states in potential theory is due to the finite range of the potential. In fact the trajectories turn over so quickly that there are no recurrences on the trajectories unless the strength of the potential becomes very large. The relativistic interaction must somehow have a capacity for making the trajectories continue to rise for a substantial distance above threshold, though they are expected eventually to turn over.

There may be some indications for parity doublets, as predicted by the MacDowell symmetry; in particular, the $N_{5/2}^-(1680)$ and $N_{5/2}^+(1690)$ and $\Delta_{5/2}^+(1913)$ and $\Delta_{5/2}^-(1954)$, the $\Delta_{3/2}^+(1688)$ and $\Delta_{3/2}^-(1691)$, and so on.

There may also be some indications that the resonance spacing in a trajectory is not two units of spin, but one unit with alternance in parity. Another way of expressing the same thing is to say that two trajectories with opposite signature, but otherwise identical quantum numbers, may be approximately coincident. This would indicate that exchange forces play a minor role.

Figure 10.5 shows the elasticities of the nucleon resonances, using the same notations of Fig. 10.3. The elasticities decrease with increasing resonant masses, with roughly an exponential dependence.

One may also individuate a kind of alternance in isotopic spin for resonances with large masses; alternatively, resonances of large masses tend again to come in pairs.

10.3 The high-energy Regge scattering amplitude

The high-energy approximation of the scattering amplitude in the Regge-pole model for pion-nucleon elastic scattering is obtained in three steps:

i) First one starts with the RSW formula of the amplitude for the t -channel reaction $N\bar{N} \rightarrow \pi\pi$. The RSW formula is written in the relativistic equivalent of the form (10.7), that is (neglecting exchange forces):

$$G_i(s_t, t_t) = \sum_i \frac{\beta_i(t_t) P_{\alpha_i(t_t)}(-\cos \theta_t)}{\sin \pi \alpha_i(s_t)} + \int \dots, \quad (10.10)$$

where the subscript t indicates t -channel quantities; $\beta_i(t_t)$ is renormalized so as to take care of the proper factors needed to go from Eq. (10.7) to Eq. (10.10), according to the prescription of Eq. (2.18). The cosine of the c.m. scattering angle for the t -channel reaction is given by:

$$\cos \theta_t = - \frac{2t_t + s_t - 2m_\pi^2 - 2m_N^2}{\sqrt{4m_\pi^2 - s_t} \sqrt{4m_N^2 - s_t}}. \quad (10.11)$$

ii) The second step is to invoke crossing symmetry to conclude that the same analytic expression for the scattering amplitude is valid in the s -channel reaction $\pi N \rightarrow \pi N$. By replacing $s_t \Rightarrow t_s$ and $t_t \Rightarrow s_s$ one has:

$$G_s(t_s, s_s) = \sum_i \frac{\beta_i(s_s) P_{\alpha_i(s_s)}(-\cos \theta_t)}{\sin \pi \alpha_i(t_s)} + \int \dots, \quad (10.12)$$

$$\cos \theta_t = - \frac{2s_s + t_s - 2m_\pi^2 - 2m_N^2}{\sqrt{4m_\pi^2 - t_s} \sqrt{4m_N^2 - t_s}}. \quad (10.13)$$

iii) The third step is to consider the high-energy limit as $s_s \rightarrow \infty$ in order to get particularly simple formulae. One has:

$$\cos \theta_t \xrightarrow{s \rightarrow \infty} \frac{-2s_s}{\sqrt{4m_\pi^2 - t_s} \sqrt{4m_N^2 - t_s}} \approx -s_s \quad (10.14)$$

$$P_\alpha(-\cos \theta_t) \xrightarrow{s \rightarrow \infty} (\cos \theta_t)^{\alpha(t)} \Rightarrow K(t) \left(\frac{s}{s_0} \right)^{\alpha(t)}. \quad (10.15)$$

s_0 is an arbitrary scale factor introduced for convenience in order to have a dimensionless quantity, s/s_0 . From Eqs. (10.14), (10.15), and (10.12) (dropping from now on the subscripts):

$$G(s,t) \underset{s \gg}{\approx} \sum_i \frac{\gamma_i(t)}{\sin \pi \alpha_i(t)} \left(\frac{s}{s_0} \right)^{\alpha_i(t)} + O(s^{-1/2}), \quad (10.16)$$

where $\gamma_i(t)$ includes the previous $\beta_i(t)$ and $k(t)$ terms; the background integral is written as $O(s^{-1/2})$, since theoretical considerations suggest that its energy dependence is in fact of the $s^{-1/2}$ type.

The inclusion of exchange forces, according to the recipe of Eq. (10.8), yields the final Regge-pole model expression for the high-energy non-spin-flip scattering amplitude, in the s -channel:

$$G(s,t) \underset{s \gg}{\approx} \sum_i \gamma_i(t) \xi_i(t) \left(\frac{s}{s_0} \right)^{\alpha_i(t)}, \quad (10.17a)$$

where the signature factor $\xi_i(t)$ has the forms:

$$\xi(t) = \frac{\exp\{-i[\pi \alpha(t)/2]\}}{\sin[\pi \alpha(t)/2]} \quad \text{for } \tau = + \quad (10.18a)$$

$$\xi(t) = \frac{\exp\{-i[\pi \alpha(t)/2]\}}{\cos[\pi \alpha(t)/2]} \quad \text{for } \tau = - . \quad (10.18b)$$

$\alpha(t)$ is the exchanged Regge trajectory considered in the t -channel unphysical region (see Fig. 10.6) The residue function $\gamma(t)$ is real if $\alpha(t)$ is real; $\alpha(t)$ is real for boson trajectories, and is in general complex for baryon trajectories.

By using the same procedure as was used to derive $G(s,t)$, one obtains the formula for the relativistic spin-flip scattering amplitude:

$$H(s,t) \underset{s \gg}{\approx} \sum_i \delta_i(t) \xi(t) \sqrt{-t} \alpha(t) \left(\frac{s}{s_0} \right)^{\alpha(t)}, \quad (10.17b)$$

which is analogous to $G(s,t)$, apart from the extra factors $\alpha(t)\sqrt{-t}$. The factor $\sqrt{-t}$ arises from the presence of the $(\sin \theta_t)$ term, which at large s -values yields:

$$\sin \theta_t = \sqrt{1 - \cos^2 \theta_t} \approx \cos \theta_t \approx s. \quad (10.19)$$

The factor $\alpha(t)$ arises from the presence of the derivatives of the Legendre polynomials, which at large s -values yield:

$$\frac{d P_\alpha(-\cos \theta_t)}{d(\cos \theta_t)} \underset{s \gg}{\approx} \frac{d}{ds} s^\alpha = \alpha s^{\alpha-1}. \quad (10.20)$$

From Eq. (10.17) it is evident that the terms with the largest $\text{Re } \alpha$ contribute most. Therefore the terms in the summation could be arranged in decreasing order of importance, and the formula represents a kind of an asymptotic series in powers of s . In fact, one hopes that at high energies only one term, or few terms, will be important.

10.4 High-energy total cross-sections

The optical theorem and formulae (10.17) give:

$$\sigma_{\text{tot}}(s) = \frac{1}{s} \text{Im } G(s, t=0) \underset{s \gg s_0}{\approx} \frac{1}{s} \sum_i \text{Im} \left[\gamma_i(0) \xi_i(0) \left(\frac{s}{s_0} \right)^{\alpha_i(0)} \right]. \quad (10.21)$$

In order to obtain constant high-energy cross-sections, one must clearly have $\alpha_1(0) = 1$. The Regge trajectory with $\alpha(0) = 1$ and all the quantum numbers of the vacuum ($B = S = I = 0$, $\tau = G = +$), except the spin, is called the Pomeron trajectory (P). All other trajectories have $\alpha_i(0) < 1$, and therefore in the limit of very high energies become negligible.

Let us consider sufficiently high energies, so that the P-trajectory is by itself adequate enough to reproduce the total cross-sections; one has [$\xi(0) = -i$ for $\tau = +$]:

$$\sigma_{\text{tot}}(s) \underset{s \gg s_0}{\approx} - \frac{\gamma(0)}{s_0}. \quad (10.22)$$

At the energies available at present, the π^+p and π^-p total cross-sections are slightly different and seem to converge to the asymptotic limit from "above" (Figs. 3.2 and 3.3). Therefore other trajectories must be present. The difference between π^-p and π^+p total cross-sections may be explained by the presence of the ρ trajectory, which has a negative signature and therefore contributes with opposite sign to the π^-p and π^+p scattering. Another trajectory is required, because both total cross-sections seem to approach the asymptotic limit from above; it must be a trajectory with a positive signature: for this purpose one invents a second vacuum trajectory, the P'. Therefore the high-energy $\pi^\pm p$ total cross-sections are explained with the presence of three trajectories¹⁷⁰):

$$\sigma_{\text{tot}}(\pi^\pm p) = P + P' \pm \rho. \quad (10.23)$$

The total cross-sections depend on the parameters of the trajectories only through the intercept at $t = 0$. For the P, P', and ρ one finds¹⁷⁰): $\alpha_P(0) = 1$, $\alpha_{P'}(0) \approx 0.7$, $\alpha_\rho(0) \approx 0.6$.

10.5 High-energy differential cross-sections

We shall assume that the high-energy $\pi^\pm p \rightarrow \pi^\pm p$ elastic scattering processes are dominated by the presence of the P, P', and ρ poles. Charge exchange scattering is dominated by the presence of a single pole, the ρ , and is therefore much simpler to analyse. In fact, because of isospin conservation:

*) The new measurements in the 20-65 GeV/c region cast some doubts on these simple conclusions.

$$F_{ce}(s,t) = \frac{1}{\sqrt{2}} [F_{\pi^+p} - F_{\pi^-p}] = \frac{1}{\sqrt{2}} [P+P'+\rho - P-P'+\rho] = \sqrt{2} \rho . \quad (10.24)$$

Consequently:

$$\left(\frac{d\sigma}{dt}\right)_{ce} \underset{s \gg}{\approx} \frac{1}{16 \pi s^2} |F_{ce}|^2 = \frac{\sqrt{2}}{16 \pi} |\xi_p(t)|^2 \left\{ |\gamma_p(t)|^2 + (-t) \alpha_p^2(t) |\delta_p(t)|^2 \right\} \left(\frac{s}{s_0}\right)^{2\alpha_p(t)-2} \quad (10.25)$$

The s -dependence is concentrated in the last factor. If one assumes a ρ trajectory which is linear in t (Fig. 10.6)

$$\alpha_p(t) = \alpha_p(0) + t\alpha'_p , \quad (10.26)$$

then follows:

$$\left(\frac{s}{s_0}\right)^{2\alpha_p(t)-2} = \left(\frac{s}{s_0}\right)^{2\alpha_p(0)-2} e^{2t \alpha'_p \ln(s/s_0)} . \quad (10.27)$$

The argument of the exponential in the differential cross-section has a mixed s - and t -dependence. By neglecting other terms that are slowly varying, at fixed s it implies an exponential function of $|t|$; the slope of this exponential is: $[2\alpha'_p \ln(s/s_0)]$, i.e. it is an increasing function of s . This is the famous shrinking of the diffraction peak implied by the presence of a single Regge pole. The shrinking could be masked by the presence of the other t -dependent terms.

The presently available charge exchange data are consistent with a shrinking of the forward peak. In practice, one uses the above equations in the form:

$$\ln X = \ln [(d\sigma/dt)/(d\sigma/dt)]_{t=0} \underset{s \gg}{\approx} \ln D(t) + [2\alpha_p(t) - 2] \ln \frac{s}{s_0} , \quad (10.28)$$

and one tries to determine $\alpha_p(t)$ from the experimentally measured cross-sections. Figure 10.7 shows a fit of the charge exchange data to a slightly modified form of Eq. (10.28)⁸²⁾. A good parametrization of these data is given by^{82,99)}

$$\alpha_p(t) = 0.57 + 0.96 t . \quad (10.29)$$

We shall now try to explain the other main characteristics of the charge exchange cross-sections at high energies, which were reviewed in Section 5.2, by using Eqs. (10.24) to (10.28).

The simplest explanation of the dips at $|t| = 0$ and 0.6 (GeV/c)^2 is obtained by assuming the presence of a large spin-flip contribution, which becomes zero both at $t = 0$, because of the $\sqrt{-t}$ factor, and at $-t \approx 0.6$ because $\alpha_p(-t \approx 0.6) = 0$.

The almost exponential behaviour in the $0.1 < |t| < 0.5$ range is explained directly in terms of formula (10.28).

As soon as several trajectories contribute to a cross-section, as in πN elastic scattering, the predictions of the Regge-pole model are no longer clear-cut. For instance, the shrinkage of the diffraction peak may be washed out by compensations between the different trajectories.

The parametrization (10.28) offers a test for the shrinking or non-shrinking of the diffraction peak, which is more sensitive than the radius of interaction, because one can use data over a larger $|t|$ interval.

Figures 10.8 to 10.10 show the ratio X versus $\log_{10}s$ at the fixed values of $|t| = 0.1, 0.3, 0.5, 0.7$ and 0.9 $(\text{GeV}/c)^2$. The cross-sections were computed from the best fits of the diffraction cross-sections as $\exp[Bt + Ct^2]$ over the range $0.1 < |t| < 1$ $(\text{GeV}/c)^2$. The situation would be only slightly different if one had used as normalization $\sigma^2(s)/16\pi$ instead of $(d\sigma/dt)_{t=0}$ from the best fits. Because of error correlations, the errors have been computed by using the error of the independent experimental points which contribute to each cross-section for a fixed t -value.

The π^-p and π^+p cross-sections at fixed t show very little s -dependence, while the pp cross-sections show a strong s -dependence. The global experimental information indicates that for $|t| < 1$ $(\text{GeV}/c)^2$, the pp and K^+p diffraction peaks shrink, the $\bar{p}p$ expands, and the π^+p , K^-p remain unchanged. These facts seem to indicate that the P trajectory has a slope considerably smaller than 1 $(\text{GeV}/c)^2$.

The backward peak in high-energy elastic scattering is explained as being due to the exchange of baryon Regge trajectories, corresponding to the reggeization of the diagram in Fig. 4.20. The reggeization improves the situation considerably over a simple Born approximation, but at present the parameters of the model have still to be found phenomenologically.

10.6 High-energy polarization

In the presence of a single Regge trajectory, one has an unambiguous prediction for the polarization at high energies. If $\alpha(t)$ is real, the residue functions $\gamma(t)$ and $\delta(t)$ are both real and the phases of G and H are the same. The polarization parameter P , which is proportional to $\text{Im}(GH^*)$, is then predicted to be zero. Therefore there should be no polarization in high-energy charge exchange scattering. Experimentally, the polarization is found to be about 15% at 11 GeV/c (Fig. 6.6), in disagreement with the prediction.

Several suggestions have been put forward as explanations of the observed charge exchange polarization, among which are the following:

- a) $\text{Im } \alpha_\rho(t) \neq 0$, that is a complex ρ trajectory;
- b) contribution from a cut in the angular momentum plane;
- c) presence of a second ρ trajectory;
- d) influence of direct channel resonances;
- e) inclusion of absorption corrections²²³).

At present it is not possible to choose with certainty which is the correct solution of the problem.

For elastic scattering the presence of three trajectories means the presence of too many unknown parameters. Nevertheless, one can make some guesses. It is believed that the P and P' trajectories give a large contribution to the non-spin-flip amplitude and a small contribution to the spin-flip one. From the analysis of the charge exchange data, one knows that the ρ trajectory has a large spin-flip contribution. Therefore it is believed that

the largest contribution to the polarization in πN elastic scattering is given by the interference of the $P + P'$ non-spin-flip amplitudes with the ρ spin-flip one:

$$P_{el} \approx \text{Im} [(G_P + G_{P'}) H_\rho^*] . \quad (10.30)$$

Since the ρ contributes with opposite signs to the $\pi^+ p$ and $\pi^- p$, while the P and P' contributions stay the same, the model predicts opposite polarizations for $\pi^+ p$ and $\pi^- p$ elastic scattering:

$$P_{el}(\pi^+ p) = -P_{el}(\pi^- p) \quad (10.31)$$

Moreover, H_ρ changes sign at $-t = 0.6 \text{ GeV}^2/c$, and therefore the polarization should change sign at this value. Both predictions seem to be verified by the experiments, as shown in Fig. 6.3.

In backward scattering, a baryon Regge pole is expected to contribute. Since for $t \leq 0$ the baryon Regge pole is in general complex, G and H do not have any more the same phase, and one may have a non-vanishing polarization even if only one Regge pole contributes. There are no data in this angular region.

10.7 Further considerations on Regge poles

Several theoretical problems are still open in the Regge-pole model. We shall list some of the known properties together with some open problems. The proceedings of some recent conferences should be consulted for further details and further references¹⁹⁸.

10.7.1 The factorization principle

This principle says that the residue function $\gamma(t)$ may be written as:

$$\gamma(t) = (\text{known kinematical factors}) \gamma_{aRc}(t) \gamma_{bRd}(t) , \quad (10.32)$$

where, as shown in Fig. 10.10, $\gamma_{aRc}(t)$ refers to the coupling of the Regge pole R to the t -channel initial state $(a\bar{c})$, while γ_{bRd} refers to the coupling of R to $(\bar{b}d)$.

As a consequence of the factorization principle, one has relations between different cross-sections. Consider, for example, the πp , pp , and $\pi\pi$ elastic scatterings. At very high energy, these scattering processes are thought to be governed by the exchange of the Pomeron trajectory only. Their elastic amplitude for large s are proportional to the following residues:

$$\begin{aligned} F_{\pi p} &\underset{s \gg}{\sim} \gamma_{\pi P \pi} \gamma_{p P p} \\ F_{pp} &\underset{s \gg}{\sim} \gamma_{p P p} \gamma_{p P p} \\ F_{\pi\pi} &\underset{s \gg}{\sim} \gamma_{\pi P \pi} \gamma_{\pi P \pi} , \end{aligned} \quad (10.33)$$

which imply:

$$F_{pp} F_{\pi\pi} \underset{s \gg}{\sim} F_{\pi p}^2 \quad (10.34)$$

and, by using the optical theorem:

$$\sigma_{\text{tot}}(pp) \sigma_{\text{tot}}(\pi\pi) \underset{s \gg}{\sim} \sigma_{\text{tot}}^2(\pi p) . \quad (10.35)$$

The factorization property is really useful only when a single Regge pole is exchanged; otherwise it becomes very complicated.

10.7.2 *SU(3) symmetry for residues*

In the context of the Regge-pole model, SU(3) symmetry may be applied to the residue $\gamma(t)$ together with the factorization property. SU(3) can be used at the vertices of Fig. 10.10 to relate the contributions of a given Regge pole in different processes. One obtains sum rules among different cross-sections, which are particularly simple for the total cross-sections and the charge exchange reactions, as for instance:

$$[\sigma_{\text{tot}}(K^-p) - \sigma_{\text{tot}}(K^-n)] - [\sigma_{\text{tot}}(K^+p) - \sigma_{\text{tot}}(K^+n)] = [\sigma_{\text{tot}}(\pi^-p) - \sigma_{\text{tot}}(\pi^+p)] \quad (10.36)$$

$$\frac{d\sigma}{dt}(K^-p \rightarrow \bar{K}^0n) + \frac{d\sigma}{dt}(K^+n \rightarrow K^0p) = \frac{d\sigma}{dt}(\pi^-p \rightarrow \pi^0n) + \frac{d\sigma}{dt}(\pi^-p \rightarrow \eta n). \quad (10.37)$$

10.7.3 *Presence of cuts*

If a Regge pole may be exchanged in a certain strong interaction, it is almost certain that two Regge poles may be exchanged together. This implies a cut in the angular momentum plane as indicated schematically in Fig. 10.1. For instance, in the charge exchange reaction, a cut should be connected with the joint exchange of P and ρ . The dominating contribution from a cut is of the type:

$$\frac{\gamma_c(t) (s/s_0)^{\alpha_c(t)}}{(\ln s/s_0)_v} \quad (10.38)$$

where c means cut and v is a positive constant.

10.7.4 *Ghosts*

The signature factor $\xi(t)$ has a pole at $\alpha(t) = 0$ for a positive signature trajectory. One may get rid of this undesired pole (or ghost) by assuming:

$$\gamma(t) = \alpha(t) [\alpha(t) + 1] \tilde{\gamma}(t) \quad (10.39)$$

The $\tilde{\gamma}$ may have to replace the γ for the factorization principle.

10.7.5 *Daughter trajectories*

The problem of the daughter trajectories starts when one considers two-body reactions involving particles which are not equal pairwise, as in charge exchange. In this case, $\cos \theta_u$, the cosine of the scattering angle in the u -channel, does not grow with s , but in some interval remains bound. Therefore one cannot derive in a simple way the characteristic $(s/s_0)^{\alpha(u)}$ behaviour, which one expects in any case from the requirement of analyticity of the scattering amplitude. The behaviour may be still obtained if one thinks that the pole is accompanied by another pole, the daughter trajectory, which has the same internal quantum numbers, opposite signature, and angular momentum differing by one unit.

10.7.6 *Gribov phenomenon*

In the $t < 0$ region, if a fermion trajectory $\alpha_f(t)$ contributes to a particular scattering process, then also the trajectory $\alpha_f^*(t)$ contributes.

The problem of the multiplication of the trajectories required to satisfy analyticity and crossing in each channel, or some other basic principle, starts to be serious: one speaks of evasion, conspiracy, etc. Their phenomenological implications are for the moment of some importance for photoproduction and np charge exchange scattering.

11. ELASTIC SCATTERING AS THE SHADOW OF INELASTIC PROCESSES

At high energies the inelastic scattering amplitude is believed to become purely imaginary; then elastic scattering results only as the shadow of inelastic processes, and is completely determined by the knowledge of the inelastic collisions. The presence of a real part of the scattering amplitude modifies this picture, which nevertheless is a good first approximation.

The above statement on the elastic scattering, being the shadow of inelastic processes, follows from the unitarity relation (2.73). The intermediate states $|k_n\rangle$ may be either the elastic state or some inelastic state. The two cases may be mathematically separated and the inelastic part may be expressed in terms of an "overlap function". At high energies, forward and backward elastic scattering peaks arise from the fact that inelastic processes have the following characteristics:

- a) In the c.m.s. the secondary particles are concentrated into two narrow cones around the forward and backward directions.
- b) The average transverse momentum of the secondary is small, about 0.4 GeV/c, independent of incoming energy. This implies a similarity in the widths of the forward and backward peaks.
- c) About 80% of the secondaries consists of pions, whose angular distribution is symmetric in the c.m.s.
- d) A large fraction of the energy is taken by a few energetic particles, whose angular distribution is strongly asymmetric in the c.m.s. This symmetry has the effect of depressing the backward compared to the forward cross-section.

In a naïve approach, we shall consider situations where only one inelastic process is dominant. This clearly represents an oversimplification, which nevertheless gives a qualitative insight into the problem.

11.1 Intermediate energies

For pion-nucleon collisions at intermediate energies (1-2 GeV), the main inelastic channel is the production of one pion, i.e. a three-body final state. The production goes mainly via the one-pion exchange mechanism (OPE) illustrated in the diagram of Fig. (11.1a). If one assumes this process to be really the dominant inelastic process, one has a simple picture for the diffraction scattering. Its shadow is shown in Fig. (11.1b). The shadow goes via the exchange of two pions. The computation of the inelastic process of Fig. 11.1a has been performed in several ways, with and without the inclusion of the nucleon spin. The inclusion of the spin into the analysis allows a computation of $(d\sigma/dt)$ on an absolute basis²⁷⁾. In the 1 GeV to 2 GeV region, the agreement between the predictions of the model and the experiments is reasonable as far as the shape of the diffraction pattern is concerned, while the predicted absolute cross-section is much smaller than the measured one. This fact may simply indicate the inadequacy of assuming that there are no other inelastic scattering

processes besides those of the type illustrated in Fig. 11.1a; moreover, the real part of the scattering amplitude is certainly different from zero in this energy region.

11.2 Very high energies

For very high energies, Amati et al.^{24,42)} proposed an extension of the peripheral model, i.e. the multiperipheral model, which is consistent with the large increase in multiplicity and with the near constancy of the total cross-sections. Figure 11.2a shows the basic Feynman diagram involved for inelastic processes, while Fig. 11.2b shows their shadow.

Amati et al. were able to sum the whole series of multiperipheral graphs by means of a linear integral equation, whose kernel depends on low-energy parameters. The part of the elastic amplitude arising from intermediate inelastic states in the unitarity condition, shows the Regge pole behaviour, while the part due to elastic intermediate states behaves as a continuous superposition of powers of s , i.e. a cut in the complex angular momentum plane. At high energy the pole represents the main contribution in the forward direction, but as soon as t increases, the contribution of the cuts starts to dominate.

Some of the consequences of this model for the diffraction peak are that

- i) at very high energies there should be shrinking as in the Regge pole model, but the rate of shrinking of the diffraction peak

$$\frac{d[\ln(d\sigma/dt)]}{d(\ln s)} \quad (11.1)$$

decreases with increasing energy (at 30 GeV it would be ≈ 1.5 smaller than at 10 GeV for $|t| = 1$ (GeV/c)²;

- ii) the shrinking would be smaller for the processes with smaller cross-sections, i.e. smaller for πN than for NN .

Also this model has several limitations: for instance, it neglects the exchange of heavier particles or systems of particles at some stage of the chains of Figs. (11.2a), and it neglects the presence of effective two-body collisions.

Rough calculations by Bertocchi and Capella¹⁰⁵⁾, using the unitarity relation (2.73) and the statements (a) to (d) made at the beginning of this section seem to be qualitatively in agreement with the experimental results of the π^+p backward peak at 8 GeV/c. In this model the difference between the π^+p and π^-p backward peaks may possibly be explained by the charge asymmetry in multipion final states: in the π^+p case, both the forward and backward cones (excluding the energetic particles) have a net positive charge, while in the π^-p the forward cone has a net negative charge and the backward one has a net positive charge. Therefore π^-p reactions have less coherence. This feature should reduce the height of the elastic backward peak.

12. MIXED MODELS

The interpretation of the scattering data in the 2 GeV to 8 GeV region is difficult, because the energy is already too large for a phase-shift analysis, and it is too small to use profitably a high-energy model. The interpretation is therefore often made in terms of a combination of a low-energy and a high-energy model, such as resonance formation in the direct channel, and Regge poles (interference model), resonances, and diffraction, etc.

Similar problems may arise also at higher energies for scattering regions where it is likely that two or more mechanisms are at work simultaneously. For instance, the elastic scattering at large angles may be interpreted in terms of the diffraction and the statistical model or diffraction plus Regge poles. Even the statistical model with collective motions of Hagedorn and Ranft¹⁵⁴) (see Section 7.3) is a kind of mixed model.

If f_1 and f_2 are the scattering amplitudes derived from models 1 and 2, the combined effect of the two mechanisms is usually written in terms of a total scattering amplitude

$$f = f_1 + f_2 . \quad (12.1)$$

There is no real complete justification for the form (12.1). In fact a common doubt about these models is that they may violate unitarity.

12.1 Interference model

Since the Regge-pole model seems to provide a reasonable description of some scattering processes at high energies, a logical objective is to try to extend the model to the intermediate-energy region. To this end an interference model has been proposed, which takes into account the probability of resonance formation in the direct channel as well as the Regge exchange mechanism¹⁶²). The model has been applied successfully to a) $\pi^- p \rightarrow \pi^0 n$ forward scattering; b) the total cross-section difference $[\sigma_{\text{tot}}(\pi^- p) - \sigma_{\text{tot}}(\pi^+ p)]$; c) the π^- elastic scattering in the backward direction. When the resonance contribution is large and there is no cancellation between neighbouring resonances, the predictions of the model are less reliable¹⁶⁶). According to the duality picture of Schmidt some "double counting" is probably unavoidable in this model.

12.1.1 Charge exchange scattering and total cross-section differences

The amplitude for the charge exchange reaction is written as:

$$f_{ce} = f_\rho + f_R , \quad (12.2)$$

where f_ρ is the Regge amplitude, which, in the case of the charge exchange reaction, is determined by the exchange of the ρ trajectory and is known from high-energy scattering; f_R is the resonance amplitude, which is computed as a sum of Breit-Wigner formulae arising from the known πN resonances. Figure 12.1 shows the results of the analysis of Barger applied to the total cross-section differences¹⁶²). The dotted curve is the ρ -exchange contribution alone. The solid curve includes both the resonance and Regge amplitudes. The $I = 3/2$ resonances interfere destructively and the $I = 1/2$ resonances constructively with the ρ -exchange amplitude.

12.1.2 Backward $\pi^- p$ elastic scattering

Figure 12.2 shows diagrammatically the relevant amplitudes chosen to explain the $\pi^- p$ backward scattering: nucleon resonances in the direct channel, and exchange of $I = 3/2$ nucleon trajectories in the crossed channel. At 180° , the contribution of each Breit-Wigner contains a factor $(-1)^\ell$, where ℓ is the orbital angular momentum of the resonance. Because of this factor, the nature of the interference is sensitive to the parities of the resonances in the direct channel: $P = (-1)^{\ell+1}$. Figure 4.12 shows the 180° $\pi^- p$ cross-sections, together

with the results of the interference model calculations¹⁶²⁾: the agreement is good. The Regge amplitude alone accounts for most of the cross-sections above 5 GeV. The two-decade valley near 2.1 GeV/c results from destructive interference between the resonance and Regge amplitudes. The interference patterns depend heavily on the parities of the individual resonances; in this respect the 180° cross-section measurement plus the interference model may give indirect evidence for the correct spin-parity assignment of some resonances.

12.2 Diffraction and resonant scattering

In the 1-5 GeV region, the elastic scattering cross-sections clearly exhibit a diffraction peak at forward angles and considerable structure at large angles, most of which arises from the presence of nucleon resonances. There is therefore some justification for expressing the elastic scattering amplitude as the sum of a diffraction term plus several resonant terms¹⁵¹⁾:

$$g(E, \theta) = g_D(E, \theta) + g_R(E, \theta) \quad (12.2a)$$

$$h(E, \theta) = h_D(E, \theta) + h_R(E, \theta) , \quad (12.2b)$$

where, as usual, g and h denote the non-spin-flip and spin-flip amplitudes, respectively; the subscript D means diffraction and R means resonant. In the applications of the model, further simplifications have been made: a) all amplitudes are taken as pure imaginary; b) $g_D = G \exp(b|t|)$; c) $h_D = 0$; d) for g_R and h_R one uses the partial wave expansions (2.36a,b), retaining only those terms corresponding to the nearby resonances; e) for these terms the partial wave amplitudes are assumed to have the Breit-Wigner form (2.62). Refinements, such as the addition of a real part in g_D with a certain angular dependence, are being done.

The model has had some success; it was used to determine from the elastic scattering data the parameters (mass, width, elasticity) of the nearby resonances, as well as the forward elastic cross-section, the ratio of the real to imaginary part at 0°, and the slope of the diffraction pattern.

12.3 Hybrid model for scattering at high energies

This model is a combination of the optical and Regge models¹⁷¹⁾. This hybrid combination gives enough parameters to understand the s - and t -dependences of elastic scattering at both small and large angles. In particular, when applied to pp elastic scattering, it explains:

- a) the shrinking of the diffraction peak and the stronger s -dependence at large angles;
- b) the structures in the angular distributions at $|t| \approx 1.2$ and 5.5 (GeV/c)².

The model could be reinterpreted within the context of the Regge theory, by noticing that it corresponds to specific forms for the Pomeron trajectory and for the Regge cuts.

13. DISPERSION RELATIONS FOR THE FORWARD SCATTERING AMPLITUDE

For our purposes, a dispersion relation (d.r.) is an equation which expresses the real part of a function of a complex variable in terms of an integral over the imaginary part. In the case of the forward scattering amplitude, the imaginary part is computed, via the

optical theorem, from the total cross-section; therefore the forward dispersion relation (f.d.r.) gives the real part of the forward scattering amplitude in terms of an integral over total cross-sections.

In the forward direction, the spin-flip contribution is zero and we need discuss only the non-spin-flip amplitude. The formulae we shall derive are valid in the laboratory system.

13.1 The basic formulae for πN scattering

Let us define the symmetric and antisymmetric combinations:

$$f^{(\pm)} = \frac{1}{2}(f_- \pm f_+) , \quad (13.1)$$

where f_- , f_+ , and f_0 are the amplitudes for $\pi^- p \rightarrow \pi^- p$, $\pi^+ p \rightarrow \pi^+ p$, and $\pi^- p \rightarrow \pi^0 n$ scatterings, respectively. σ_- and σ_+ will denote the $\pi^- p$ and $\pi^+ p$ total cross-sections. The combinations (13.1) are the right ones for expressing spin-dependence and crossing: the crossing relations at a fixed angle $\theta_{\pi_{lab}}$ read:

$$f^{(\pm)*}(E_{\pi_{lab}}, \theta_{\pi_{lab}}) = \pm f^{(\pm)}(-E_{\pi_{lab}}, \theta_{\pi_{lab}}) . \quad (13.2)$$

Let us recall the basic steps needed for deriving the f.d.r.:

1. Microcausality ensures that the $f^{(\pm)}$ combinations of Eq. (13.1) are analytic in the upper half-plane of the complex energy (Fig. 13.1). Therefore Cauchy's theorem over the closed contour $C_1 + C_2$ of Fig. 13.1 yields:

$$\oint \frac{g^{(\pm)}(\omega')}{E_{\pi_{lab}} - \omega'} d\omega' = 0 . \quad (13.3)$$

where ω' is the complex laboratory energy, and $E_{\pi_{lab}}$ the physical laboratory energy at which we want to compute the real part of the amplitude.

2. We assume that the $f^{(\pm)}$'s behave properly at infinity, so that the integral along C_2 becomes zero when the contour is expanded to infinity.
3. The integration along C_1 in the limit of $r \rightarrow 0$, where r is the radius of the semicircle around $E_{\pi_{lab}}$, yields the principal part of the integration along the real axis plus the discontinuity at $\omega = E_{\pi_{lab}}$, which equals $[-i \pi g^{(\pm)}(E_{\pi_{lab}})]$.
4. By taking the real part of the expansion, one has the desired form, i.e. the real parts of $g^{(\pm)}$ is given in terms of an integral over the imaginary parts.
5. The integral along the real axis is decomposed into two:

$$\int_{-\infty}^{+\infty} = \int_{-\infty}^0 + \int_0^{+\infty} . \quad (13.4)$$

The crossing relation (13.2) allows change of the integral from negative to positive energies.

6. The integral over positive energies is again decomposed:

$$\int_0^{+\infty} = \int_0^{m_\pi} + \int_{m_\pi}^{+\infty} . \quad (13.5)$$

The second integral involves physical energies, where the imaginary part may be expressed in terms of total cross-sections. The integral in the interval $(0, m_\pi)$ involves an intermediate single nucleon state (as in Fig. 2.9a) at a non-physical laboratory total pion energy $m_\pi^2/2m_N$. The computation involves the unrenormalized pion-nucleon coupling constant f^2 .

The final result is

$$\text{Re } g^{(+)}(E_{\pi_{1ab}}, 0) = \frac{1}{2}(\text{Re } g_- + \text{Re } g_+) =$$

$$-\frac{2f^2(m_\pi^2/2m_N)^2}{E_{\pi_{1ab}}^2 - (m_\pi^2/2m_N)^2} + \frac{1}{4\pi^2} P \int_{m_\pi}^{\infty} \frac{E'_{\pi_{1ab}} [\sigma_-(E'_{\pi_{1ab}}) + \sigma_+(E'_{\pi_{1ab}})]}{E'^2_{\pi_{1ab}} - E_{\pi_{1ab}}^2} K'_{\pi_{1ab}} dE'_{\pi_{1ab}} \quad (13.6a)$$

$$\text{Re } g^{(-)}(E_{\pi_{1ab}}, 0) = \frac{1}{2}(\text{Re } g_- - \text{Re } g_+) =$$

$$= \frac{2f^2 E_{\pi_{1ab}}}{E_{\pi_{1ab}}^2 - (m_\pi^2/2m_N)^2} + \frac{E_{\pi_{1ab}}}{4\pi^2} P \int_{m_\pi}^{+\infty} \frac{\sigma_-(E'_{\pi_{1ab}}) - \sigma_+(E'_{\pi_{1ab}})}{E'^2_{\pi_{1ab}} - E_{\pi_{1ab}}^2} K'_{\pi_{1ab}} dE'_{\pi_{1ab}} \quad (13.6b)$$

These are usually referred to as unsubtracted f.d.r. A close examination of the behaviour at infinity of the total cross-section shows that in the case of $g^{(+)}$, the integral over the semicircle cannot be neglected. Therefore only the relation (13.6b) is usable.

N-times subtracted d.r. may be obtained by using Cauchy's theorem for the function:

$$\frac{g(E_{\pi_{1ab}}, 0)}{(\omega - E_{\pi_{1ab}})(\omega - \omega_1) \dots (\omega - \omega_n)} \quad (13.7)$$

instead of the function $f(E_{\pi_{1ab}})/(\omega - E_{\pi_{1ab}})$. $\omega_1, \dots, \omega_n$ are the subtraction points. Once-subtracted dispersion relations with $\text{Re } \omega_1 = m_\pi$ are the following:

$$\text{Re } g^{(+)}(E_{\pi_{1ab}}, 0) = \text{Re } g^{(+)}(m_\pi, 0) + \frac{2f^2 K_{\pi_{1ab}}^2 (m_\pi^2/2m_N)}{[m_\pi^2 - (m_\pi^2/2m_N)^2][E_{\pi_{1ab}}^2 - (m_\pi^2/2m_N)^2]} +$$

$$\frac{K_{\pi_{1ab}}^2}{4\pi^2} P \int_{m_\pi}^{+\infty} \frac{E'_{\pi_{1ab}}}{K'_{\pi_{1ab}}} \frac{\sigma_-(E'_{\pi_{1ab}}) + \sigma_+(E'_{\pi_{1ab}})}{E'^2_{\pi_{1ab}} - E_{\pi_{1ab}}^2} dE'_{\pi_{1ab}} \quad (13.8a)$$

$$\frac{\text{Re } g^{(-)}(E_{\pi_{1ab}}, 0)}{E_{\pi_{1ab}}} = \frac{\text{Re } g^{(-)}(m_\pi, 0)}{m_\pi} - \frac{2f^2 K_{\pi_{1ab}}^2}{[m_\pi^2 - (m_\pi^2/2m_N)^2][E_{\pi_{1ab}}^2 - (m_\pi^2/2m_N)^2]} +$$

$$\frac{K_{\pi_{1ab}}^2}{4\pi^2} P \int_{m_\pi}^{+\infty} \frac{1}{K'_{\pi_{1ab}}} \frac{\sigma_-(E'_{\pi_{1ab}}) - \sigma_+(E'_{\pi_{1ab}})}{E'^2_{\pi_{1ab}} - E_{\pi_{1ab}}^2} dE'_{\pi_{1ab}}, \quad (13.8b)$$

where the real parts at $E_{\pi_{1ab}} = m_\pi$ are given in terms of the $I = 3/2$, $J = 3/2$, and $I = \frac{1}{2}$, $J = \frac{1}{2}$ scattering lengths a_3 and a_1 :

$$\text{Re } g^{(+)}(m_\pi, 0) = \frac{1}{3}(a_1 + 2a_3)A \quad (13.9a)$$

$$\text{Re } g^{(-)}(m_\pi, 0) = \frac{1}{3}(a_1 - a_3)A \quad (13.9b)$$

Therefore:

$$\text{Re } g_+(m_\pi, 0) = a_3 A \quad (13.10a)$$

$$\text{Re } g_-(m_\pi, 0) = \left(\frac{2}{3} a_1 + \frac{1}{3} a_3 \right) A ; \quad (13.10b)$$

$A = 1$ for c.m. values; $A = (m_\pi + m_N)/m_N$ for lab. values.

By solving Eq. (13.8a,b) with respect to $\text{Re } g_+$, $\text{Re } g_-$, one obtains:

$$\begin{aligned} \text{Re } g_\pm(E_{\pi_{lab}}, 0) &= \frac{1}{2} \left(1 + \frac{E_{\pi_{lab}}}{m_\pi} \right) \text{Re } \rho_\pm(m_\pi, 0) + \\ &\quad \frac{1}{2} \left(1 - \frac{E_{\pi_{lab}}}{m_\pi} \right) \text{Re } \rho_\pm(m_\pi, 0) \pm \frac{2f^2 K_{\pi_{lab}}^2}{\left[m_\pi^2 - (m_\pi^2/2m_N)^2 \right] \left[E_{\pi_{lab}} \mp m_\pi^2/2m_N \right]} \\ &\quad + \frac{K_{\pi_{lab}}^2}{4\pi^2} P \int_{m_\pi}^{\infty} \frac{dE'_{\pi_{lab}}}{K'_{\pi_{lab}}} \left[\frac{\sigma_\pm(E'_{\pi_{lab}})}{(E'_{\pi_{lab}} - E_{\pi_{lab}})} + \frac{\sigma_\mp(E'_{\pi_{lab}})}{(E'_{\pi_{lab}} + E_{\pi_{lab}})} \right]. \end{aligned} \quad (13.11)$$

By computing the relation (13.6b) at $E_{\pi_{lab}} = m_\pi$ one obtains:

$$a_1 - a_3 = \frac{2f^2}{m_\pi^2 - (m_\pi^2/2m_N)^2} + \frac{2m_\pi m_N}{2\pi^2(m_\pi + m_N)} P \int_{m_\pi}^{\infty} \frac{\sigma_-(E') - \sigma_+(E')}{K'} dE'. \quad (13.12)$$

This is a sum rule, which connects the scattering length combination $(a_1 - a_3)$ to the coupling constant f^2 .

In practice, one uses only the relations (13.6b), (13.11), and (13.12). Relation (13.12) allows the computation of f^2 once $(a_1 - a_3)$ is known from other sources (or vice versa). The subtracted f.d.r. (13.6b) allows the computation of $\text{Re } g_+(E_{\pi_{lab}}, 0)$, $\text{Re } g_-(E_{\pi_{lab}}, 0)$. The unsubtracted f.d.r. (13.11) yields via charge independence the real part of the forward charge exchange cross-section from total cross-section differences:

$$\text{Re } g_0(E_{\pi_{lab}}, 0) = - \frac{1}{\sqrt{2}} \text{Re } g^{(-)}(E_{\pi_{lab}}, 0). \quad (13.13)$$

By performing a subtraction at infinity, one may obtain a so-called inverse dispersion relation¹⁸⁰:

$$\sigma^{(+)}(\infty) = \sigma^{(+)}(T_{\pi_{lab}}) + \frac{8\pi f^2 T_0}{(T_0^2 - T_{\pi_{lab}}^2) \sqrt{m_\pi^2 - T_0^2}} - 8P \int \frac{\text{Re } g^{(+)}(T) T dT}{\sqrt{T^2 - m_\pi^2} (T_{\pi_{lab}} - T^2)}, \quad (13.14)$$

which can be used to "compute" the cross-section at infinity, by assuming a Regge energy

dependence with P , P' , and ρ trajectories. Grecchi et al.¹⁸⁰⁾ computed a value $\sigma_{\text{tot}}(\pi N)_{\infty} \approx 20.0$ mb, which is somewhat low compared with the results of Tables 3.2 and 3.3.

The integration over the total cross-sections should be performed from m_{π} to ∞ . In practice, the integration is split into two parts: from 0 to about 30 GeV, where one uses interpolations of the measured cross-sections, and from 30 GeV to infinity, where one uses empirical or theoretical continuations of the cross-sections.

In the pion-nucleon forward d.r., the unphysical region is small and contributes only with the term in f^2 . In contrast, the NN, KN, and $\bar{K}N$ have wider unphysical regions, resulting in a considerable number of uncertainties. Hence the πN f.d.r. are the most exact and suitable for a critical experimental test.

The forward dispersion relations have been applied to the pion-nucleon scattering data in a number of ways for a variety of purposes: a) to compute low-energy parameters, such as f^2 , from the sum rule (13.12); b) to compute the real parts of the forward scattering amplitudes for elastic²²⁸⁾ and charge exchange scattering⁴⁹⁾ via Eqs. (13.11) and (13.6b), (13.13); c) as a test of some fundamental principles such as microcausality²⁹¹⁾ and/or charge independence⁴⁹⁾ by comparing the computed forward real parts with experimentally determined ones (Figs. 13.2 and 4.19); d) to estimate the asymptotic value of the total cross-sections at infinity^{178, 180)}, and so on.

Figure 4.19 shows the ratio $\rho = \text{Re } f(E,0)/\text{Im } f(E,0)$ for $\pi^{\pm}p$ elastic scattering as computed by Carter²²⁸⁾. $\text{Im } f(E,0)$ was computed via the optical theorem from the total cross-sections; $\text{Re } f(E,0)$ was computed via the dispersion relations (13.11) using $a_1 = 0.259 f$, $a_3 = 0.154 f$, and $f^2 = 0.0815$. A Regge-type behaviour was used for the high-energy cross-sections. Figures 13.2a and 13.2b show the real part of the forward elastic scattering cross-sections versus the imaginary parts for the $I = \frac{1}{2}$ and $I = 3/2$ states. (In reality: the complex diagram of $K f_{\frac{1}{2}}$ and $K f_{3/2}$.) The solid lines were computed by Hhler and co-workers (private communication), while the dashed lines are the predictions of the latest CERN phase-shift analysis. The agreement is reasonable for the $I = 3/2$ state, while it is poor for the $I = \frac{1}{2}$ state. There is now the possibility of using the dispersion relation computation of the real parts as "theoretical input data" for the next phase-shift analysis.

Figure 13.3 shows the real part of the forward charge-exchange amplitude versus laboratory pion momentum. Shown are the predictions of Hhler et al.⁸²⁾ using the unsubtracted dispersion relation (13.6b) and those of Hamilton⁸⁸⁾, who used subtracted dispersion relations. The large difference between the two predictions is due to the different relations used and to the different experimental data used as input. This is another example of how cautiously one should take as valid both the experimental data and the use of certain theoretical tools. The experimental points shown in Fig. 13.3 are obtained from the square root of the difference between the measured forward differential cross-sections and the imaginary part squared, computed via the optical theorem. This determination does not give the sign of the real part, which is taken equal to the sign predicted by Hhler, whose results agree better with the experiments.

Figure 13.4 shows the complex diagram of $K f^{(-)}$ ⁸²⁾.

13.2 The parameters of low-energy pion-nucleon physics

The parameters of low-energy pion-nucleon physics are the coupling constant f^2 , the S-wave scattering lengths a_1 , a_3 , and the P-wave scattering lengths a_{11} , a_{13} , a_{31} , a_{33} . Two basic methods have been used until now for the computations of these parameters.

a) One method is based on the results of phase-shift analyses below 100 MeV. Only S- and P-waves are present here. Because of the small energies involved, simple energy dependences for the phase shifts may be used. The Rochester group^{317-319,273}) used non-relativistic c.m. momentum dependences for the nuclear part of the phase shifts:

$$\delta = a \left(\frac{K}{m_\pi} \right)^{21+1}, \quad (13.15)$$

where a is the scattering length. The coupling constant f^2 may be obtained by assuming a form suggested by the static Chew-Low theory^{210,318}):

$$\text{tg } \delta_{33} = \frac{4}{3} \left(\frac{K}{m_\pi} \right)^3 \frac{f^2}{T[1 - (T/T_0)]}, \quad (13.16)$$

where T and T_0 are c.m. pion kinetic energies; T_0 is the energy at which the $3/2, 3/2$ phase-shift resonates. The results of this simple approach are shown in Table 13.1.

More complicated dependences have been assumed; for instance, a six-parameter polynomial for δ_{33} and two-parameter formulae for the other phases. None of these parametrizations are completely satisfactory.

The S-wave parameters that enter directly in the low-energy scattering experiments are:

$$\frac{d\sigma}{d\Omega} (\pi^+ p \rightarrow \pi^+ p) \approx a_3^2 \quad (13.17a)$$

$$\frac{d\sigma}{d\Omega} (\pi^- p \rightarrow \pi^- p) \approx \frac{1}{9} (2a_1 + a_3)^2 \quad (13.17b)$$

$$\frac{d\sigma}{d\Omega} (\pi^- p \rightarrow \pi^0 n) \approx \frac{2}{9} (a_1 - a_3)^2. \quad (13.17c)$$

b) The second method for determining the low-energy parameters is based on forward dispersion relations, which are manipulated so as to yield the most sensitive equations. For instance, Hamilton and Woolcock³⁸) used the f.d.r. for the B amplitude, where B is defined as:

$$f(T_{\pi_{lab}}, 0) = \frac{m_p}{4\pi E} [A(T_{\pi_{lab}}, 0) + T_{\pi_{lab}} B(T_{\pi_{lab}}, 0)]. \quad (13.18)$$

For B one may use an unsubtracted f.d.r., while for A one needs at least one subtraction.

The best results are probably obtained by a combination of the two methods, as originally performed by Hamilton and Woolcock³⁸). In this case the procedure becomes rather involved, since a best fit over several determinations is performed.

Table 13.1

Values of the low-energy pion-nucleon parameters obtained by different groups. The scattering lengths are in units of m_π . Method 1 means obtaining the parameters from phase-shift analyses, using formulae of the type (13.15-16); method 2 by means of dispersion relations; method 1 + 2 a combination of methods 1 and 2. Method 3 is from threshold photoproduction and the Panofsky ratio.

Method	1	1	1 + 2	2	3	2	1	2
Author and reference	Rochester (317-319, 275)	Donald (463, 464, 110)	Hamilton, Woolcock (38)	Samaranyake, Woolcock (78)	Hamilton, Woolcock (15)	Höhler et al. (103)	Hamilton (88)	Lovelace, Heidelberg Conf. (203)
Year	1963	1966	1963	1965	1960	1966	1966	1967
f^2	0.088 ± 0.002		0.081 ± 0.002	0.0822 ± 0.0018		0.081 0.076 ± 0.002		
a_1	0.17 ± 0.03		0.171 ± 0.005				-0.091 ± 0.005	
a_3	-0.107 ± 0.003		-0.088 ± 0.004	-0.109 ± 0.008			0.270 ± 0.008	
$2a_1 + a_3$		0.255 ± 0.022	0.254 ± 0.011	-0.035 ± 0.012			0.271 ± 0.007	
$a_1 - a_3$	0.272 ± 0.025	0.282 ± 0.015		0.292 ± 0.020	0.278 ± 0.026	0.290 0.271		0.069 0.263
a_{11}	0.05 ± 0.33		-0.101 ± 0.007					0.195 ± 0.0111
a_{13}	-0.14 ± 0.07		-0.029 ± 0.005					-0.070 ± 0.005
a_{31}	-0.041 ± 0.04		-0.038 ± 0.005					
a_{33}			0.215 ± 0.005					

The results presented in Table 13.1 indicate that there are some discrepancies among the various authors. Also the consistency with the determinations from the threshold pion photoproduction and the Panofsky ratio is not particularly satisfactory. The errors quoted in Table 13.1 are statistical only. They do not include either the large systematic errors from the measurements or the theoretical uncertainties that may be intrinsic in the method used.

In the above discussion it is implicitly assumed that the $(\pi^- p \pi^-)$ and $(\pi^+ p \pi^+)$ coupling constants are equal. It may be worth while to investigate thoroughly if this is really the case.

In conclusion, there are still doubts about the low-energy parameters and it is difficult to really ascribe an error to them. More low-energy, high-precision data, together with a thorough analysis, are really needed in order to improve the situation significantly.

Some of the low-energy parameters offer significant tests for various theories¹⁸¹⁾. For instance, the determination of the S-wave scattering lengths via the algebra of currents predicts¹⁵⁰⁾: $a_1 \approx 0.20$ and $a_3 \approx -0.10$. These values are in reasonable agreement with the experiments, within the estimated 15% error expected from the validity of the theory¹⁵⁰⁾.

14. PION-NUCLEON SCATTERING AND CONSERVATION LAWS

Several fundamental theoretical hypotheses have been made in the analysis of the pion-nucleon scattering data. For instance, we made extensive use of the charge independence hypothesis. Pion-nucleon scattering experiments offer the possibility of testing some of these hypotheses, though not with good precision.

14.1 Charge independence

All checks of charge independence are based on the relation (2.49), which connects the amplitudes for the three measurable cross-sections at the same angle and energy.

a) The direct application of the formula leads to a triangular relation illustrated in Fig. 14.1. This presents difficulties in interpretation, since it leads to inequalities in measurable cross-sections, as for instance:

$$\sqrt{2} \sqrt{\left(\frac{d\sigma}{d\Omega}\right)_0} - \sqrt{\left(\frac{d\sigma}{d\Omega}\right)_-} \leq \sqrt{2} \sqrt{\left(\frac{d\sigma}{d\Omega}\right)_+} + \sqrt{\left(\frac{d\sigma}{d\Omega}\right)_-} . \quad (14.1)$$

Törnqvist^{91, 149, 161)} has found an ingenious way of representing the triangular relation diagrammatically. He plots the three quantities y_+ , y_- , and $1.5 y_0$ in a triangle. The y_i are defined as:

$$y_i = \frac{(\frac{d\sigma}{d\Omega})_i}{(\frac{d\sigma}{d\Omega})_+ + (\frac{d\sigma}{d\Omega})_- + (\frac{d\sigma}{d\Omega})_0} \quad (14.2)$$

with the differential cross-sections at the same angle θ . Charge independence requires that the point individuated by y_+ , y_- , and $1.5 y_0$ be inside the circle inscribed in the triangle. Figure 14.2 shows that this is barely the case for backward scattering. For the points in the circumference the triangle inequality becomes an equality.

b) Low-energy analysis. The S-wave nuclear contribution to the three measurable differential cross-sections is given by Eqs. (13.17a,b,c). The situation is overdetermined; for instance one may obtain a_3 and $a_3 + 2a_1$ from elastic scattering, compute from these values the combination $(a_1 - a_3)$, and then check it with the value measured from charge exchange data. The comparison offers a rough test of charge independence, because there are very few accurate low-energy experiments (see Table 13.1).

c) Test of the validity of a forward dispersion relation and/or charge independence.

The charge-independent relation (2.49) may be written, by using the optical theorem, as:

$$f_0(E,0) = -\frac{1}{\sqrt{2}} \{ [\text{Re } f_-(E,0) - \text{Re } f_+(E,0) + i \frac{K}{4\pi} [\sigma_-(E) - \sigma_+(E)]] \}. \quad (14.3)$$

The total cross-sections are measured; $(\text{Re } f_- - \text{Re } f_+)$ may be obtained via the unsubtracted dispersion relation (13.6b); $f_0(E,0)$ is obtained as the square root of the measured forward differential charge exchange cross-section. If the purpose of the test is to compare the real parts, the best energies at which this can be done are those for which $(\sigma_- - \sigma_+)$ is small, i.e. at about 0-120, 350-550 MeV and for energies larger than 1100 MeV. Figures 13.3 and 13.4 show that the agreement between right and left side of Eq. (14.3) is reasonable within the large statistical and systematic errors (always larger than 10%)^{49,103}.

d) Equation (14.3) may be rewritten in terms of measurable quantities, specifically forward differential cross-sections and total cross-sections only. By squaring Eq. (14.3), one has:

$$2 \left[\frac{d\sigma}{d\Omega} (E,0) \right]_0 = [\text{Re } f_-(E,0) - \text{Re } f_+(E,0)]^2 + \frac{K^2}{16\pi} (\sigma_- - \sigma_+)^2. \quad (14.4)$$

The real parts may be expressed in terms of the forward cross-sections and the imaginary parts, which may again be computed from the cross-sections:

$$\text{Re } f_{\pm}(E,0) = \sqrt{\left[\frac{d\sigma}{d\Omega} (E,0) \right]_{\pm} - \frac{K^2}{16\pi^2} \sigma_{\pm}^2}. \quad (14.5)$$

A comparison of the left and right side of Eq. (14.4) at many energies leads to the conclusion that charge independence is verified to something like 10%^{10,221}. The main difficulties of the test are connected with the fact that the forward cross-sections have total errors not smaller than 5 to 10%. Coulomb corrections bring in an extra uncertainty.

The conclusion of this analysis is that the whole body of information and analysis of the pion-nucleon scattering is barely consistent with the charge independence hypothesis.

14.2 Microcausality

A BNL group tested the validity of the forward dispersions (13.6b) and (13.8a) at high energies. They assumed that f^2 and $\text{Re}^{(+)}(m_{\pi})$ were determined by low-energy analyses (at high energy one is not very sensitive to these values). High-precision total cross-sections²⁹² and forward real parts measured by Coulomb-nuclear interference²⁹¹ were used in the analysis. The agreement between the right and left sides of relations (13.6b) and (13.8a) is good, within the relatively large errors ($\approx 10\%$).

This may be considered a test of microcausality. If a small acausal region or "fundamental length" ℓ_0 existed, the forward dispersion relations would break down at an energy $\sim 1/\ell_0$. By applying this consideration to this test, one obtains $\ell_0 < 10^{15}$ cm.

15. CONCLUSIONS

It is clear that although the number of theoretical and experimental works on pion-nucleon scattering is impressive, many problems are still open. The experiments that are required in order to reach a further understanding are in general of the systematic type, to be carried out over a wide range of energies and angles with good precision and high statistics. Considerable technical improvements are still required in order to obtain measurements of all the polarization parameters for elastic and, in particular, for charge exchange scattering. The theoretical insight provided by the many existing models is modest. The situation is reminiscent of that existing some time ago with all the different models in nuclear physics. These models are at first rough and are born independently from one another. One needs to refine them and to find connections among them. It is in fact clear that each model explains or emphasizes only one aspect of the physical situation. On the other hand, a more fundamental approach to the whole problem of scattering should be found. In a certain sense the pioneering stage is over, but this does not mean that unexpected things will not come out. On the contrary, it happened quite often that more systematic, often routine and tedious works brought the real understanding. The phase-shift analysis may be an example of this.

We shall close this review on pion-nucleon scattering by listing a number of unsettled problems:

1. There still are doubts and discrepancies with respect to the low-energy parameters, and it is in fact difficult to assign an error to them. The situation could be improved by systematic precision measurements at energies below 80 MeV. Some improvements in the theoretical analysis may be needed to make use of the improved precision in the experiment. This holds true also for the methods of predicting these parameters.
2. The discrepancies among the existing phase-shift analyses should be removed, so that one may really be sure of how many resonances do exist. The solution to the problem lies in more precise and systematic experimental data, particularly on charge exchange scattering and charge exchange polarization.
3. Phase-shift analyses should be carried over to higher energies. At these energies, the number of parameters is so large that one really has to use all the possible experimental information. The analysis will very likely lead to the discovery of new higher mass states, with small elasticities, typically not larger than 20%. It would therefore be particularly helpful to study experimentally the inelastic channels, for instance with a bubble chamber, at many closely spaced energies. At the moment, there is hardly any data. At the same time one should find good theoretical means of relating inelastic channels to the elastic one.
4. Which mechanism produces the structures at $|t| \approx 1.3$ and 3.5 (GeV/c)² ? They might be secondary diffraction maxima; then they should appear also in other two-body elastic scatterings. But they could also be connected in some way with an oscillatory character, as seems to be the case for p-p scattering.

5. Which mechanism gives rise to high-energy, large-angle scattering? What kind of core, if any, do the particles involved in the scattering have? The experiments required to solve the problem have to cope with very small cross-sections and therefore low counting rates.
6. What is the t -dependence of the real part of the scattering amplitude? More refined experiments in the Coulomb interference region may lead to an answer, but presumably over a limited t -range only. New theoretical methods have to be devised.
7. The polarization in both elastic and charge exchange scatterings is still quite sizeable even at 12 GeV/c. More data are required to give the energy dependence of the polarization.
8. What is the production mechanism of the high-energy backward bump? A detailed determination of the angular distribution near 180° in c.m. and of the energy dependence is required.
9. Finally: to what precision is the charge independence hypothesis valid?

Acknowledgements

This report is an outgrowth of the CERN 66-18 report, written in collaboration with Dr. M.N. Focacci. It is based on the lectures given to the Scuola di Perfezionamento di Fisica dell'Università di Bologna. It is a pleasure to acknowledge the help of Drs. M.N. Focacci, P. Pini and S. Stagni in the preparation of the review.

I gratefully acknowledge the fruitful discussions I have had with my colleagues, and the assistance they have given in clarifying certain points; also their helpfulness in letting me have data before publication. In particular, I would like to thank Drs. S. Bergia, L. Bertocchi, J. Chahoud, L. Di Lella, E. Ferrari, G. Höhler, W.A. Love, C. Lovelace, A. Minguzzi, G. Puppi, R. Rubinstein, P. Santacrose, P. Serra, J.J. Thresher, A.M. Wetherell and W.S. Woolcock.

Finally, I would like to thank Professors R.L. Cool and E. Clementel for their encouragement and advice.

REFERENCES

An extensive list of references on pion-nucleon scattering is given. The papers containing experimental data have been divided by subject, in the order -T, +T, -E, +E, OC, -D, +D, OD, -P, +P, OP. (The codes are explained below). Within one subject, the references have been grouped by year in ascending order. Besides the experimental papers, a list of theoretical, phenomenological, and review papers is added.

Each reference is written onto two IBM cards: the first card contains the journal, the authors and the technique, data type, and data range, in a code form; the second card contains the title of the paper.

Figure captions

- Fig. 2.1 : Momentum configuration in c.m. and lab. systems before and after collision.
- Fig. 2.2 : Illustration of the s- and t-channels.
- Fig. 2.3 : Domains of variation for s, t, u.
- Fig. 2.4 : Illustration of the coordinate system used for the wave function (2.7).
- Fig. 2.5 : Illustration of the various types of polarization measurements:
- a) measurement of the P-parameter with a second scattering;
  - b) measurement of the P-parameter with a polarized target;
  - c) measurement of the R-parameter with a polarized target and a second scattering;
  - d) measurement of the A-parameter with a polarized target and a second scattering.
- Fig. 2.6 : Illustration of the singularities of the scattering amplitude in the complex energy plane.
- Fig. 2.7 : The two Feynman diagrams contributing poles to the partial wave scattering amplitude.
- Fig. 2.8 : Definition of the impact parameter b.
- Fig. 2.9 : Illustration of the exchange amplitudes:
- a) direct channel;
  - b) exchanged channel;
  - c) cross-exchanged channel.
- Fig. 2.10 : Illustration of the singularities of the partial wave scattering amplitude in the complex energy plane.
- Fig. 2.11 : Resonant amplitudes in the Argand diagrams for three different elasticities: $x = 1, \frac{1}{2}, \frac{1}{4}$ ¹⁹⁴).
- Fig. 2.12 : The dependence of δ and η on energy for resonances of elasticities $x = 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}$. Although the elasticity is independent of energy, the absorption parameter η varies rapidly in the resonance region ¹⁹⁴).
- Fig. 2.13 : Resonant trajectory of the partial wave amplitude f_l in the presence of
- a) an attractive non-resonant background of same spin and parity;
  - b) a repulsive non-resonant background of same spin and parity.

- Fig. 2.14 : Illustration of the analyticity relationships between δ and η .
- Fig. 2.15 : Illustration of meson-nucleon interaction, as pictured in the independent quark model. M and M' represent meson states and B, B' baryon states. The f 's represent form factors.
- Fig. 3.1 : The geometry of a total cross-section measurement, using the absorption technique^{221, 287}). The gas Cerenkov counter in the beam separates pions from protons and, with less efficiency, from protons and, with less efficiency, from electrons and muons. S_1-S_{12} are scintillation counters. S_1-S_3 count the incident beam; S_4-S_{12} are the transmission counters. The various targets are indicated as H_2 , D_2 , and Dummy.
- Fig. 3.2 a) π^-p total, elastic, and
b) charge exchange cross-sections as function of pion laboratory kinetic energy.
- Fig. 3.3 : π^+p total and elastic cross-sections.
- Fig. 3.4 : $I = \frac{1}{2}$ total cross-section.
- Fig. 3.5 : $(\pi^-p) - (\pi^+p)$ total cross-section differences.
- Fig. 4.1 : The geometry for a typical elastic scattering low-energy Rochester measurement³¹⁷). It is a tight geometry, with no-wall scattering. Scintillation counters 1-2 detect the incoming pion, 3-4 the scattered pion, $\bar{5}$ is an anti-coincidence. Counter 3 is pulse-height analysed.
- Fig. 4.2 : Counter arrangement for a Nimrod measurement³⁵⁰).
- Fig. 4.3 : A simplified sketch of the Brookhaven counter hodoscope elastic scattering experiment in the Coulomb-Nuclear interference region²⁹). Hodoscopes HO_1 and HO_2 measure the location and direction of the incoming pion; hodoscopes H_1 and H_2 determine its location and direction after scattering; hodoscope H_4 , placed after the magnets, determines its momentum.
- Fig. 4.4 : The experimental arrangement for the 135° - 165° scattering experiment at Brookhaven³⁵⁶). Both scattered particles are momentum analysed.
- Fig. 4.5 : σ_{el}/σ_{tot} for π^+p scattering. Errors include also systematic uncertainties.
- Fig. 4.6 : A collection of $\pi^-p \rightarrow \pi^-p$ differential cross-sections.
- Fig. 4.7 : A collection of $\pi^+p \rightarrow \pi^+p$ differential cross-sections.

- Fig. 4.8 : Results of the best fit analysis according to formula (4.2) for π^-p .
- Fig. 4.9 : Results of the best fit analysis according to formula (4.2) for π^+p .
- Fig. 4.10 : Interpolated $\pi^-p \rightarrow \pi^-p$ differential cross-sections.
- Fig. 4.11 : Interpolated $\pi^+p \rightarrow \pi^+p$ differential cross-sections.
- Fig. 4.12 : a) The π^-p elastic differential cross-section at $\theta = 180^\circ$ versus energy³⁵⁹.
The solid line is a free-hand fit to the data.
b) The π^+p 180° differential cross-section.
- Fig. 4.13 : Illustration of $(1 - a_\ell)$ versus ℓ , or $V(r)$ versus r in the analysis of Perl and Corey.
- Fig. 4.14 : Results of the best fit analysis of $\pi^\pm p \rightarrow \pi^\pm p$, $pp \rightarrow pp$ scattering data for $0.1 < |t| < 0.5$ according to formula (4). The radius of interaction is defined as $R = 2\sqrt{-b}$.
- Fig. 4.15 : Results of the best fit analysis of $\pi^\pm p$ elastic scattering data in the diffraction region, for $0.1 < |t| < 1.0$ (GeV/c)², to formula (4.4).
- Fig. 4.16 : C/B^2 for $\pi^\pm p$. The experimental data in the interval $0.1 < |t| < 1.0$ (GeV/c)² have been least squared fitted to Eq. (4.4).
- Fig. 4.17 : Results of the best fit analysis with dimensionless formulae (4.6) and (4.7) of the elastic scattering data in the diffraction region for $\pi^\pm p$, $K^\pm p$, and $p^\pm p$ (approximately for $0.1 < |t| < 1.0$).
- Fig. 4.18 : Differential $\pi^\pm p$ elastic cross-sections in the Coulomb interference region. The 9.86 GeV/c π^+p data are displaced down one decade. The solid curves are the results of best fits where ρ was constrained to be zero. The number of degrees of freedom (d.f.) of each fit is shown.
- Fig. 4.19 : The ratio $\rho = \text{Re } f(E,0)/\text{Im } f(E,0)$ of the forward scattering amplitude for $\pi^\pm p$ ²²⁸.
- Fig. 4.20 : Virtual baryon exchange mechanisms for backward pion-nucleon scattering.
- Fig. 5.1 : Schematic view of the experimental arrangement of Bulos et al.⁴⁰² for charge exchange scattering. The trigger is given by the counters 12345; TF1, TF2, and TF3 are thin-plate spark chambers; S1, S2, S3, and S4 are heavy-plate spark chambers.
- Fig. 5.2 : Experimental opening angle distribution at 619 MeV, after background subtraction⁴⁰².

- Fig. 5.3 : Typical charge exchange angular distributions:
a) low energies⁴¹⁰);
b) high energies²¹⁶).
- Fig. 5.4 : The c_n^0 coefficients of the Legendre polynomial expansion for charge exchange angular distributions.
- Fig. 5.5 : ρ -meson exchange in $\pi^-p \rightarrow \pi^0n$.
- Fig. 5.6 : The forward charge-exchange cross-section versus laboratory pion momentum¹⁰³). The solid line is the dispersion-relation prediction [Eq. (13.6b)]; the dashed line is the contribution of the imaginary part. The data in Fig. 5.6a are from Refs. 398, 400, 402, 411, and in Fig. 5.6b from 403, 404, 409.
- Fig. 6.1 : Schematic drawing of a polarization measurement in elastic scattering^{474,477}). The cryostat and the magnet are required for the polarized target.
- Fig. 6.2 : Schematic view of a polarization measurement in charge exchange scattering⁴⁸⁵). The magnet and the cryostat of the polarized target are not shown. The π^- beam enters through a hole in the magnet yoke.
- Fig. 6.3 : The P polarization parameter in $\pi^\pm p$ elastic scattering:
a) low energy⁴⁷⁶);
b) high energy^{205,474,477}).
The curves represent theoretical predictions of Chiu et al.¹¹⁴).
- Fig. 6.4 : The coefficients d_n resulting from least squares fitting the product differential cross-section) $\times$ (polarization) to the formula
- $$P \frac{d\sigma}{d\Omega} = \frac{1}{K^2} \sum_n D_n P'_n(\cos \theta).$$
- Fig. 6.5 : Plots of $\pi^\pm$, defined by Eq. (6.3), as function of t for $\pi^\pm p \rightarrow \pi^\pm p$ ¹⁵⁶).
- Fig. 6.6 : The P polarization parameter in $\pi^-p \rightarrow \pi^0n$ scattering at all incident momenta above 2 GeV/c^{485,486}).
- Fig. 7.1 : Argand diagrams from πN phase-shift analyses²⁰³). Solid lines are the results of the CERN group; dashed lines are the results of the Berkeley group.
- Fig. 7.2 : The larger phases and the corresponding absorption coefficients according to the CERN analysis²⁰³). The solid lines refer to $I = \frac{1}{2}$ quantities, while the dotted ones refer to $I = \frac{3}{2}$.

- Fig. 7.3 : The smaller phases and the corresponding absorption coefficients according to the CERN analysis²⁰³). Solid lines for $I = \frac{1}{2}$, dashed ones for $I = 3/2$.
- Fig. 8.1 : Illustration of the predictions of various optical models at 12.4 GeV/c: optical model with constant opacity with $(1 - a) = 0.32$, $R = 1.16$ f, using Eq. (8.17) (curve A); using Eq. (8.18) (curve B). Curve C is for a model with Gaussian opacity. The data are from Ref. 346.
- Fig. 8.2 : Definition of the root mean square radius.
- Fig. 8.3 : Fit of the 2 GeV/c π^-p elastic scattering data according to formula (8.28)⁶⁴).
- Fig. 8.4 : The differential cross-sections obtained by assuming different spin-flip to non-spin-flip ratios in the optical model of Feld²²²) with a modified Gaussian opacity. The non-spin-flip contribution is also shown.
- Fig. 9.1 : A naive picture of peripheral (a) and central (b) collisions, with particles represented as contracted spheres.
- Fig. 10.1 : An example of the motion in potential scattering of a Regge pole in the complex l -plane as the energy E varies from large negative values through the scattering threshold E_{th} to large positive values¹⁹⁶). A cut is also indicated.
- Fig. 10.2 : The contour C used in the Sommerfeld-Watson transformation and the deformation $C + C'$ used to pick up the contribution from the Regge poles.
- Fig. 10.3 : Regge trajectories for nucleon resonances. $I = \frac{1}{2}$ states are shown as circles, black if they correspond to well-established states, open if the states are uncertain. The $I = \frac{1}{2}$ Regge trajectories are shown as solid lines. For the $I = 3/2$ states squares and dotted lines are used.
- Fig. 10.4 : Regge trajectories for two Yukawa potentials of different strengths ($G = 20$ and $G = 9$) in the $\text{Re } \alpha(E)$ versus E plane¹⁹⁷).
- Fig. 10.5 : Elasticities of nucleon resonances.
- Fig. 10.6 : Regge trajectory of the ρ meson; its exchange in the t -channel unphysical region gives rise to the high-energy scattering formula (10.16).

- Fig. 10.7 : $\log (d\sigma/dt)_{ce}$ as a function of $\log E_{lab} = \log w^{82})$. The solid lines are a best fit to $d\sigma/dt$ assuming $\alpha(t) = 0.57 + 0.91 t$.
- Fig. 10.8 : $X = (d\sigma/dt)/(d\sigma/dt)$ versus energy for π^-p elastic scattering.
- Fig. 10.9 : " " " " " π^+p " "
- Fig. 10.10 : " " " " " pp " "
- Fig. 10.11 : Illustration of the factorization principle. R is the exchanged Regge trajectory.
- Fig. 11.1 : a) One-pion exchange (OPE) inelastic process and
b) its shadow.
- Fig. 11.2 : a) Inelastic processes in the multiperipheral model and
b) their shadow.
- Fig. 12.1 : Interference model fit to the total cross-section difference $[\sigma_{tot}(\pi^-p) - \sigma_{tot}(\pi^+p)]$. The dashed curve represents the contribution of the ρ -exchange amplitude.
- Fig. 12.2 : Illustration of the interference model approximation for π^-p backward elastic scattering.
- Fig. 13.1 : Illustration of the integration path in the complex plane of the laboratory pion total energy.
- Fig. 13.2 : The real part of the forward scattering amplitudes versus the imaginary parts for $I = 1/2$ and $I = 3/2$ (complex diagram of $K f_{1/2}$ and $K f_{3/2}$). The solid curve is obtained computing the imaginary part via the optical theorem and the real part via dispersion relations. The dotted lines are the 1968 CERN phase-shift analysis predictions (Höhler, private communication).
- Fig. 13.3 : The behaviour of the real part of the forward charge-exchange cross-section. The solid line is the prediction of Höhler et al.¹⁰³⁾, the dashed line the prediction of Hamilton).
- Fig. 13.4 : The complex diagram of $Kf^{(-)}$ ¹⁰³⁾. The solid line is the dispersion relation prediction.
- Fig. 14.1 : Illustration of the charge independence triangle. σ_+ , σ_- , and σ_0 are the π^+p , π^-p elastic and π^-p charge-exchange differential cross-sections. S is the area of the triangle.
- Fig. 14.2 : The triangle graph of Törnqvist at $\cos \theta = -0.85$ obtained from the phase-shift analyses and from direct measurements¹⁶¹⁾.

Fig. 2.1

Fig. 2.2

Fig. 2.3

Fig. 2.4

Fig. 2.5

Fig. 2.6

Fig. 2.7

Fig. 2.8

Fig. 2.9

Fig. 2.10

Fig. 2.11

Fig. 2.12

Fig. 2.13

Fig. 2.14

Fig. 2.15

Fig. 3.1

a

b

Fig. 3.2

Fig. 3.3

Fig. 3.4

Fig. 3.5

Fig. 4.2

Fig. 4.3

Fig. 4.4

Fig. 4.5

Fig. 4.6

Fig. 4.7

Fig. 4.8

Fig. 4.9

Fig. 4.10

Fig. 4.11

Fig. 4.12

Fig. 4.13

Fig. 4.14

Fig. 4.15

Fig. 4.16

Fig. 4.17

Fig. 4.18

Fig. 4.19

Fig. 4.20

Fig. 5.1

Fig. 5.2

a

b

Fig. 5.3

Fig. 5.4

Fig. 5.5

Fig. 5.6

Fig. 6.1

Fig. 6.2

Fig. 6.3

Fig. 6.4

Fig. 6.5

Fig. 6.6

Fig. 7.1

Fig. 7.2

Fig. 7.3

Fig. 7.3

Fig. 8.1

Fig. 8.2

Fig. 8.3

Fig. 8.4

Fig. 9.1

Fig. 10.1

Fig. 10.2

Fig. 10.3

Fig. 10.4

Fig. 10.5

Fig. 10.6

Fig. 10.7

Fig. 10.8

Fig. 10.9

Fig. 10.10

Fig. 10.11

Fig. 11.1

Fig. 11.2

Fig. 12.1

Fig. 12.2

Fig. 13.1

Fig. 13.2

Fig. 13.3

Fig. 13.4

Fig. 14.1

Fig. 14.2