CM-P00058731

Archive

Ref.TH.1341-CERN

ONE-LOOP AMPLITUDES IN THE MODEL OF NEVEU AND SCHWARZP. Goddard ^{*)} and R.E. Waltz ^{**)}

CERN - Geneva

A B S T R A C T

Dual one-loop amplitudes for external pions are constructed from the Born term of Neveu and Schwarz. Non-orientable diagrams are needed, although the model has $SU(2)$ symmetry. The Pommeranchukon-like singularity now has intercept greater than one and there is also a negative G parity isoscalar singularity with zero intercept generated by the same mechanism. The divergences can be removed by an extension of previous procedures.

^{*)} On leave of absence from Trinity College, Cambridge.

^{**)} NSF Postdoctoral Fellow.

INTRODUCTION

Recently, Neveu and Schwarz ^{1),2),3)} have proposed a new dual resonance model Born term which seems to be a substantial improvement on the original Veneziano model, while not sacrificing any of the latter's benefits. The new model has the advantage of a more realistic spectrum ; each of the low mass states can be associated with some physical particle with the same quantum numbers, though the masses are not always correct (e.g., the pion is a tachyon) and some particles are missing. It maintains the essential property of factorization and also has analogues of the Virasoro ⁴⁾ gauge conditions which make it probable that the model is ghost free ⁵⁾. Further Halpern and Thorn ⁶⁾ have shown that adding a fifth dimension permits the varying of the pion mass, thus giving a dual resonance model with no tachyons, no exotics and, quite possibly, no ghosts (but at the cost of an unphysical spectrum).

On another front, much attention ^{7),8)} has been devoted, in the last year, to multiloop amplitudes, in the original dual resonance model, and their divergences. While some serious difficulties remain, considerable progress has been made. The ultimate usefulness of such studies depends on their general features being the same for a class of Born terms, including some of more physical interest. Studying loops in a new model, such as that of Neveu and Schwarz, is a way of testing this. Conversely, the features of such loops act as a constraint on possible Born terms, if one is ambitious enough to require that they should generate a formally unitary perturbation expansion.

One constraint produced in this way is from orientable non-planar diagrams (i.e., those with an even, non-zero number of twists) which possess a singularity ^{9),10)}, having vacuum quantum numbers in the usual model, besides those put there by construction. Either this singularity must be regarded as a deficiency of the sewing

procedure, or, more optimistically, one hopes that in a better model it will become the Pomeranchukon. At present, this singularity is a t plane cut, rather than a pole, so violating unitarity and leaving the constraint unsatisfied. We will show that it, and in particular its intercept, is rather more sensitive to changes in the model than might have been supposed.

In section 2, we discuss the diagrams which are needed at the one-loop level, showing that non-orientable diagrams are present in the Neveu-Schwarz model. Then, we calculate the expression for a planar loop in Section 3, eliminating spurious states by adapting Ref. 5), and removing the divergence after the fashion of Ref. 9). In section 4, we examine orientable non-planar loops and investigate the new singularities generated. Section 5 deals with non-orientable loops and Section 6 summarizes the conclusions. An appendix lists various properties of functions used in the text.

2. DIAGRAMS REQUIRED BY UNITARITY

In the usual dual resonance model, in order to obtain unitarity^{9),11)}, three types of loops have to be considered : planar, and both orientable and non-orientable non-planar loops. Shapiro¹²⁾ has shown that when isospin [or $SU(3)$ symmetry] is put into this model via the Chan-Paton¹³⁾ procedure, non-orientable loops should not be added. The difference arises because in the former case the intermediate states which are odd under twisting (negative signature) cancel when the various Born terms corresponding to different permutations of external lines are added. Thus the usual propagator has too many states and can be replaced by a signaturized propagator. When isospin is added, the situation changes ; the spectrum now consists of the product of the whole of the spectrum of the original propagator with the isospin space. The states odd under twisting are now just negative charge conjugation states, and every state couples to some number of external pions. Thus, there is no "signaturizing" to do. Every state is degenerate with one of the opposite G parity and different isospin : the $\rho^- f^0$ trajectory is degenerate with the $\omega - A_2$.

In the Neveu-Schwarz model, this is no longer true. The $\varrho - f^0$ trajectory is split by half a unit from the $\omega - A_2$ ^{*)}; the space of physical states is not the direct product of an isospin space with an oscillator space. Certain states are prevented from coupling to any number of external pions by the anti-commutation properties of the b operators. In fact, the effect of twisting a state built up from r external pions is $(-)^{r+1} \Omega$ where Ω is the twisting operator given in Ref. 1). In this new model, this may be written entirely in terms of operators $\hat{\Omega} = G\Omega(-)^I$, where G is the G parity operator ¹⁾. States which are odd under $\hat{\Omega}$ will not couple to any number of pions (after the various Born terms have been added) and should be excluded by using signaturized propagators formed with $\frac{1}{2}(1+\hat{\Omega})$. Thus Shapiro's result is model dependent. [Note that in both the original model without isospin and this model we could construct a theory in which the states odd under Ω ($\hat{\Omega}$) are physical, being pair-produced from pions. Then they would occur in loops and non-orientable diagrams would be excluded. This would result in making the spectrum of the present model much less physical !] That the model does not reproduce the loop structure suggested by quark diagrams is not surprising. This is a direct consequence of its not having degenerate triplets and singlets which an $SU(2)$ quark structure would imply.

3. PLANAR LOOPS

Two formulations of the Neveu-Schwarz model have been given, both in the space of states generated by the operators a_r^+ , $r = 1, 2, \dots, b_n^+$, $n = \frac{1}{2}, \frac{3}{2}, \dots$, which satisfy

$$[a_r, a_s^{*+}] = g^{rs} \delta_{rs}, \quad \{b_m, b_n^{*+}\} = g^{mn} \delta_{mn} \quad (3.1)$$

*) Since there is no strangeness in the Neveu-Schwarz model, no real distinction can be made between the ω and φ or the η and η' .

In each case the vertex operator is given by

$$V(p) = \left(\phi \cdot \sum_{m=-\infty}^{\infty} b_m \right) \exp \left\{ \sqrt{2} p \sum_{r=1}^{\infty} \frac{a_r^+}{\sqrt{r}} \right\} \exp \left\{ -\sqrt{2} p \sum_{r=1}^{\infty} \frac{a_r}{\sqrt{r}} \right\} \quad (3.2)$$

(We use $b_m = b_{-m}^+$, $a_r = a_{-r}^+$). In the first formulation ¹⁾, in which the space is denoted by $\mathcal{H}_1$, the propagator is $(L_0 - 1)^{-1}$ and in the second formulation ²⁾, $\mathcal{H}_2$, it is $(L_0 - \frac{1}{2})^{-1}$ where

$$L_0 = \pi^2 + \sum_{r=1}^{\infty} r a_r^+ a_r + \sum_{m=\frac{1}{2}}^{\infty} m b_m^+ b_m \equiv \pi^2 + \mathcal{R}_a + \mathcal{R}_b \quad (3.3)$$

The second formulation, in which we work, is simpler because the gauge conditions that on shell real states $|\Phi\rangle$ satisfy, are ²⁾

$$G_m |\Phi\rangle = 0, \quad m = \frac{1}{2}, \frac{3}{2}, \dots \quad (3.4)$$

where

$$G_m = \sqrt{2} \pi b_m^+ \sum_{r \neq 0} \sqrt{r} a_r b_{m-r}, \quad m = \pm \frac{1}{2}, \pm \frac{3}{2}, \dots$$

and we have the algebra

$$[L_m, G_n] = \left(\frac{m}{2} - n \right) G_{m+n} \quad (3.5)$$

$$\{G_m, G_n\} = 2 L_{m+n} + \frac{D}{2} (m^2 - \frac{1}{4}) \delta_{m,-n} \quad (3.6)$$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{8} m(m^2-1) \delta_{m,-n} \quad (3.7)$$

[D is the number of values μ runs over being 4 in Ref. 1) and 5 in Ref. 6).]

The $2N$ point planar one-loop amplitude is given by the usual sewing procedure ¹⁴⁾. We introduce a function $P^+(w)$ to correct for the removal of spurious states which we will calculate later

$$M = \int d^4k \int \prod_{i=1}^{2N} x_i^{-\alpha(\pi_i^2)-1} dx_i P^+(w) \text{Tr}\{x_1^R V(p_1) \dots x_{2N}^R V(p_{2N})\} \quad (3.8)$$

where $\alpha(\pi^2) = \frac{1}{2} - \pi^2$, $\pi_i = k + p_1 + \dots + p_{i-1}$, $w = x_1 x_2 \dots x_{2N}$. The loop momentum integration can be performed as before ⁹⁾ and the trace is a product of a trace of a operators.

$$M = 4\pi^2 \int \prod_{i=1}^{2N} dx_i \frac{w^{-3/2} [f(w)]^{-D}}{w^{2N}} P^+(w) \prod_{1 \leq i < j \leq 2N} [\psi(x_{ij})]^{2p_i p_j} \cdot \text{Tr}\{x_1^R V_b(p_1) \dots x_{2N}^R V_b(p_{2N})\} \quad (3.9)$$

where $x_{ij} = x_i x_{i+1} \dots x_{j-1}$ and the vertex V has been written as $V_a V_b$ as suggested by (2.2). The functions f and ψ are defined in Ref. 9). The trace of b operators can be expressed as a sum over possible divisions of the $2N$ momenta into pairs (no attention being paid to the order of the pairs or within the pairs; there will be $[(2N)! 2^{-N} (N!)^{-1}]$ terms). To each pairing is attached a sign,

$\epsilon_{(i_1 i_2) \dots (i_{2N-1} i_{2N})}$, which comes from reordering the Fermi operators. This sign can be calculated by putting the numbers 1 to $2N$ in order round a circle, joining paired points and taking

$\epsilon = +1(-1)$ as the number of intersections is even (odd). In fact,

$$\begin{aligned} & \text{Tr} \{x_1^R V_b(p_1) \dots x_{2N}^R V_b(p_{2N})\} \\ &= \sum_{\text{pairings}} \epsilon_{(i_1 i_2) \dots (i_{2N-1} i_{2N})} \prod_{j=1}^N [p_{i_{2j-1}} \cdot p_{i_{2j}} \chi^+(x_{i_{2j-1} i_{2j}})] \cdot [\Phi_+(w)]^D \quad (3.10) \end{aligned}$$

where

$$\chi^{\pm}(x) \equiv \chi^{\pm}(x, w) = \sum_{n=1}^{\infty} \frac{x^{n-\frac{1}{2}} \pm (w/x)^{n-\frac{1}{2}}}{1 \pm w^{n-\frac{1}{2}}} \quad (3.11)$$

and

$$\Phi^{\pm}(w) = \prod_{n=1}^{\infty} (1 \pm w^{n-\frac{1}{2}}) \quad (3.12)$$

The new functions χ, Φ that we are led to introduce are expressed in terms of Jacobi θ functions, and their behaviour as $w \rightarrow 1$ given in the Appendix. The χ functions are immediately related to the Jacobi cn and sn functions. In particular, Φ^+ diverges like $q^{-1/24}$ (i.e., half as fast as $[f(w)]^{-1}$ where $q = \exp(2\pi^2/\ln w)$). As $w \rightarrow 1$,

$$\chi^+(x, w) \sim -\frac{\pi}{\ln w} \left[\sin\left(\frac{\pi \ln x}{\ln w}\right) \right]^{-1} (1 + O(q)) \quad (3.13)$$

exactly like ψ^{-1} . Ignoring P^+ , the integrand, therefore, diverges like $q^{-D/8}$ as $w \rightarrow 1$. This divergence can be removed ^{15), 9)} by subtracting a counter term $\tilde{M}$ given by replacing each f, ψ, χ, Φ function in (3.9) by its leading behaviour in powers of q as $w \rightarrow 1$. Also, the integrand can be multiplied by an arbitrary $G(w)$ ⁸⁾ satisfying $G = 1 + O(q)$ as $w \rightarrow 1$. If we change to the usual variables θ, w , where $\theta_{ij} = 2\pi \ln x_{ij} / \ln w$, the counter term factorizes again into θ and w integrals. Thus the total arbitrariness introduced by this counter term amounts to one free parameter. The θ integrand is again just the integrand of the original Born term in the Koba Nielsen form. Thus factorization and duality properties and Scherk's arguments that the divergent part of the mass matrix is diagonal to order g^2 should carry over.

We can calculate $P_+(w)$ under similar assumptions to those made by Brower and Thorn ⁵⁾. The spurious states of the form $|\psi\rangle = \sum \alpha_m G_{-m} |\Phi_m\rangle$ satisfying $R|\psi\rangle = n|\psi\rangle$ can be written as linear combinations of the states

$$|\{\epsilon, \lambda\}, \mu, m\rangle = G_{-\frac{1}{2}}^{\epsilon_1} G_{-\frac{3}{2}}^{\epsilon_2} \dots G_{-a+\frac{1}{2}}^{\epsilon_a} L_{-1}^{\lambda_1} L_{-2}^{\lambda_2} \dots L_{-b}^{\lambda_b} |\mu, n-m\rangle \quad (3.14)$$

where $\epsilon_i = 0$ or 1 , $\sum_{i=1}^a (i-\frac{1}{2}) \epsilon_i + \sum_{j=1}^b j \lambda_j = m$, and $|\mu, n-m\rangle$ is a basis for the space of non-spurious states satisfying $R|\mu, n-m\rangle = (n-m)|\mu, n-m\rangle$. As before these states can be removed recursively from the trace, when we consider the residue at a particular pole, provided that we use certain identities, which really only hold for the integrated (infinite) expression, and also note that amplitudes with missing propagators vanish ²⁾. This would be sufficient if none of the spurious states in (3.14) have zero norm. But we can see that the states $G_{-\frac{1}{2}} |\mu, n-1\rangle$ have zero norm on mass shell ($L_0 = \frac{1}{2}$). Further, as far as we know ¹⁾ they are the only such states and making a parallel assumption to that in Ref. 5), we can correct for these states. We find that P^+ is defined by

$$P^\pm(w) = (1 \pm w^{\frac{1}{2}}) f(w) \{\Phi^\pm(w)\}^{-1} \quad (3.15)$$

The isospin factor for the loop is just twice the trace of the Pauli matrices τ associated with the external particles.

4. ORIENTABLE NON-PLANAR LOOPS

We now consider a $2N$ particle loop with two twists and M particles on one boundary, $2N-M$ on the other. We explained in Section 1 that the appropriate twisting operator is $\hat{\Omega} = G\Omega(-)^I$, where in the $\mathfrak{F}_1$ space ¹⁾,

$$\hat{\mathcal{Q}}_{(1)} = (-)^{R - \frac{1}{4}(1-G)} e^{-L-1} (-)^I \quad (4.1)$$

Using the relationship between $\mathcal{F}_1$ and $\mathcal{F}_2$ given in Ref. 2), we find that in $\mathcal{F}_2$

$$\hat{\mathcal{Q}}_{(2)} = G_{\frac{1}{2}} G_{-\frac{1}{2}} G (-)^{R + \frac{1}{4}(1+G)} e^{-L-1} (-)^I \quad (4.2)$$

where the G parity G , is $-(-)^{2R}$ in $\mathcal{F}_2$. As $G_{\frac{1}{2}} G_{-\frac{1}{2}}$ has no effect on a physical $\mathcal{F}_2$ state ²⁾ on mass shell, we shall henceforth omit it in the definition of $\hat{\mathcal{Q}}$. Thus, the trace we consider can be written

$$\text{Tr} \{ x_1^R \hat{\mathcal{Q}} V(p_1) x_2^R V(p_2) \dots x_M^R V(p_M) x_{M+1}^R \hat{\mathcal{Q}} V(p_{M+1}) \dots x_{2N}^R V(p_{2N}) \} \quad (4.3)$$

The factors of e^{-L-1} can be removed ^{5),9)} and the elimination of spurious states effected by incorporating $P^\pm(w)$, defined by Eq. (3.15). Here and in the following, we take the upper (lower) sign as M is even or odd. After momentum integrations, we have

$$M = 4\pi^2 \int \prod_{i=1}^{2N} dx_i \frac{w^{-\frac{3}{2}} (1 \pm w^{\frac{1}{2}})}{m^2 w} \frac{[f(w)]^{1-D}}{\Phi^\pm(w)} \prod_{1 \leq i < j \leq 2N} [\psi_{(\mp)}(x_{ij})]^{2p_i p_j}$$

$$\cdot \left[\pm (-)^{\frac{1}{2}} \text{Tr} \{ (-)^{\frac{1}{4}(1 \pm G)} (-)^{R_b} x_1^{R_b} V_b(p_1) \dots V_b(p_M) (-)^{R_b} x_{M+1}^{R_b} V_b(p_{M+1}) \dots x_{2N}^{R_b} V_b(p_{2N}) \} \right] \quad (4.4)$$

In Eq. (4.4), $\psi_{(\mp)}(x_{ij})$ is to be read as $\psi(x_{ij})$ if p_i and p_j have an even number of twists between them and $\psi_{\bar{T}}(x_{ij})$ ⁹⁾ otherwise.

A similar convention will be adopted for $\mathcal{X}_{(\mathbb{T})}^{\pm}$ below. For the trace of b operators in Eq. (4.4), we obtain

$$\begin{aligned} \pm (-)^{\frac{1}{2}} \text{Tr} \{ \dots \} &= (\pm)^{-\frac{1}{2}} \sum_{\text{pairings}} \varepsilon_{(i_1 i_2) \dots (i_{2N-1} i_{2N})} \\ &\cdot \prod_{j=1}^N \left[\phi_{i_{2j-1}} \cdot \phi_{i_{2j}} \cdot \mathcal{X}_{(\mathbb{T})}^{\pm}(x_{i_{2j-1} i_{2j}}) \right] (\Phi_{\pm}(w))^D \end{aligned} \quad (4.5)$$

The new functions $\mathcal{X}_{\mathbb{T}}^{\pm}(x)$ appearing in Eq. (3.5) are defined by

$$\mathcal{X}_{\mathbb{T}}^{\pm}(x) \equiv \mathcal{X}_{\mathbb{T}}^{\pm}(x, w) = \sum_{n=1}^{\infty} \frac{(-x)^{n-\frac{1}{2}} \mp (-w/x)^{n-\frac{1}{2}}}{1 \pm w^{n-\frac{1}{2}}} \quad (4.6)$$

Because $\mathcal{X}_{\mathbb{T}}^{+}(x)$ and $\mathcal{X}_{\mathbb{T}}^{-}(x)$ are both antisymmetric under $x \rightarrow w/x$, we have to choose their arguments $x_{i_{2j-1} i_{2j}}$ so that, in the first case, we pass through a fixed twisted line (say x_1) in going cyclically from $p_{i_{2j-1}}$ to $p_{i_{2j}}$ and, in the second case, we pass through no twisted lines.

When we calculate the isospin factor for this loop, taking account of the $(-)^I$ in Eq. (4.2), we obtain the product of the traces of the $\mathcal{T}$ matrices for the two boundaries, showing that the channel whose invariant is $t = -(p_1 + p_2 + \dots + p_M)^2$ is indeed isoscalar only.

We now consider the behaviour of the integrand as $w \rightarrow 1$ and the associated singularities in the t channel. First consider M even, when it has vacuum quantum numbers. From Eq. (3.13), we note that $\mathcal{X}^{+}(x, w)$ involves no power of q as $w \rightarrow 1$ and further, from the Appendix,

$$\mathcal{X}_{\mathbb{T}}^{+}(x, w) \sim -i \frac{4\pi}{\ln w} \cos\left(\frac{\pi \ln x}{\ln w}\right) q^{\frac{1}{2}} (1 + O(q)) \text{ as } w \rightarrow 1 \quad (4.7)$$

Thus, if we replace each term in Eq. (4.4) by its leading behaviour, the various terms behave like different powers of q . The largest

term is of the order of $q^{-(D-1)/8-t/2}$, so that Eq. (4.4) converges for $t < -(D-1)/4$, having a scalar singularity at this point, and further singularities at $t = -(D-1)/4 + 2n$. However, smaller terms may give rise to singularities lying on higher trajectories. To determine the trajectory height, we must analyse the spin of these singularities, taking into account the momentum factors $p_{i_{2j-1}} \cdot p_{i_{2j}}$. For definiteness, take the special case of $N = M$. A priori, the terms with the highest spin in (4.5) are those in which all pairs consist of momenta on different boundaries. The total of such terms has a leading behaviour like

$$q^{N/2 - (D-1)/8 - t/2} \det \{ p_i \cdot p_{N+j} \cos \left(\frac{\theta_{i,N+j}}{2} \right) \} \quad (4.8)$$

the determinant being an $N \times N$ one. The highest spin that (3.8) could contain is spin N . If there, such a singularity would lie on a singularity of slope $1/2$ and intercept $(D-1)/8 + N/2$, $N=2,4,6\dots$ resulting in a catastrophic series of ancestors. To decide which spins are in fact there, we analyse the expression, first reducing to tensors of a definite symmetry type, which are irreducible with respect to the general linear group, and then further with respect to the Lorentz group. Because Eq. (4.8) can be shown to be zero when either $p_1 = p_2 = p_3$ or both $p_1 = p_2$ and $p_3 = p_4$, only the components corresponding to totally antisymmetric tensors, or those antisymmetric on all but one index, survive. Thus, we only get contributions of spin 0 or spin 2 and this can be shown to be true for all the terms in the leading behaviour. Further for $N = 2$, we have checked explicitly that the spin 2 contribution does not vanish. So the highest trajectory has intercept $1 + (D-1)/8$ and no singularities at negative t but the next trajectory has intercept $(D-1)/8$ with a spin zero tachyon cut.

Actually, we have not fully proved that the catastrophe is totally avoided, or that $1 + (D-1)/8$ is the highest intercept, because we have considered only the leading behaviour of each term. A complete discussion would involve a similar tensor analysis of the lower orders.

Now consider M odd; the t channel has quantum numbers $I = 0$, $G = -1$. From formulae in the Appendix, we see that if $N = M$ (4.8) is replaced by

$$q^{-t/2} \det \{ p_i \cdot p_{N+j} \} \quad (4.9)$$

which is always a zero spin object. This gives a negative parity negative G parity positive signature isoscalar trajectory of zero intercept. It might be expected that this singularity would cancel when the permutations of external momenta are summed over. This is not the case because, for e.g., $N = 3$, the isospin factor is anti-symmetric and so is (4.9). (The rest of the leading terms are symmetric and vanish).

5. NON-ORIENTABLE LOOPS

Consider a $2N$ particle loop with one twist. The corresponding trace is

$$\text{Tr} \{ x_1^R \hat{Q} V(p_1) x_2^R V(p_2) \dots x_{2N}^R V(p_{2N}) \} \quad (5.1)$$

When we evaluate the trace of b operators, we are forced to write

$$G(-)^{\frac{1}{4}(1+G)} = \frac{1}{\sqrt{2}} \{ e^{i\pi/4} G - e^{-i\pi/4} \} \quad (5.2)$$

and we obtain for the loop amplitude

$$M = 4\pi^2 \int \prod_{i=1}^{2N} dx_i \frac{w^{-3/2} [f(w)]^{1-D}}{w^2 w} \prod_{1 \leq i < j \leq 2N} [\psi_{N(\pi)}(x_{ij})]^{2p_i p_j} \{ B_+ + B_- \} \quad (5.3)$$

where

$$B_{\pm} = -e^{\mp i\pi/4} (1 \pm (-w)^{\frac{1}{2}}) [\Phi_{\pm}(e^{i\pi w})]^{D-1} \sum_{\text{pairings}} \varepsilon(i_1 i_2) \dots (i_{2N-1} i_{2N}) \quad (5.4)$$

$$\cdot \left[\prod_{j=1}^N (\phi_{i_{2j-1}} \phi_{i_{2j}}) \chi^{\pm}(x_{i_{2j-1} i_{2j}}, e^{i\pi w}) \right]$$

$x_{i_{2j-1} i_{2j}}$ is chosen not to include the twisted line (x_1) and we note that $B_+ = B_-^*$. The formulae in the Appendix show that the loops can be renormalized as before.

6. CONCLUSION

The theory of dual loops that has been developed ⁷⁾ depends on the connection between the loop integrands and Θ functions, or, more generally, automorphic functions. The loop integrands in the Neveu Schwarz model naturally contain more structure. We have found that this can also be expressed in terms of Θ functions at the one loop level, and in such a way that the work on the Veneziano model is easily extended.

One result which is, perhaps, surprising, is that the intercept of the Pomeron-like singularity is no longer simply proportional to the number of effective sets of oscillators, D' , but is in fact always greater than 1, being $1 + D'/8$. Also the same mechanism which produces this singularity produces a negative G parity object with zero intercept. Both these singularities are, in fact, combinations of logarithmic and square root branch points, hence violating unitarity.

ACKNOWLEDGEMENTS

We wish to thank D. Amati, D.B. Fairlie and A. Neveu for helpful suggestions and the CERN Theoretical Study Division for hospitality.

APPENDIX

We give expressions for the various χ and Φ functions introduced in the text and their behaviour as $w \rightarrow 1$ (these expressions are based on formulae in Vol. 2 of the Bateman Manuscript Project ¹⁷).

Define

$$v = \frac{\ln x}{2\pi i}, \quad \tau = \frac{\ln w}{2\pi i}, \quad \theta = \frac{2\pi \ln x}{\ln w} \quad (A1)$$

$$q = \exp \left\{ \frac{2\pi^2}{\ln w} \right\}, \quad q_N = -i \exp \left\{ \frac{\pi^2}{2 \ln w} \right\} \quad (A2)$$

a) Orientable loops

$$\chi^+(x, w) = \frac{i}{2} \theta_2(0|\tau) \theta_4(0|\tau) \frac{\theta_3(v|\tau)}{\theta_1(v|\tau)} \sim -\frac{\pi}{\ln w} \left(\sin \frac{\theta}{2} \right)^{-1} [1 + O(q)] \quad (A3)$$

$$\chi_T^+(x, w) = \frac{i}{2} \theta_2(0|\tau) \theta_4(0|\tau) \frac{\theta_4(v|\tau)}{\theta_2(v|\tau)} \sim -\frac{4\pi i}{\ln w} \cos \frac{\theta}{2} q^{\frac{1}{2}} [1 + O(q)] \quad (A4)$$

$$\chi^-(x, w) = \frac{i}{2} \theta_3(0|\tau) \theta_2(0|\tau) \frac{\theta_4(v|\tau)}{\theta_1(v|\tau)} \sim -\frac{\pi}{\ln w} \cot \frac{\theta}{2} [1 + O(q)] \quad (A5)$$

$$\chi_T^-(x, w) = \frac{i}{2} \theta_3(0|\tau) \theta_2(0|\tau) \frac{\theta_3(v|\tau)}{\theta_2(v|\tau)} \sim -\frac{\pi i}{\ln w} [1 + O(q)] \quad (A6)$$

$$\Phi^+(w) = [\theta_3(0|\tau)/f(w)]^{\frac{1}{2}} \sim w^{1/48} q^{-1/24} [1 + O(q)] \quad (A7)$$

$$\Phi^-(w) = [\theta_4(0|\tau)/f(w)]^{\frac{1}{2}} \sim w^{1/48} q^{1/12} [1 + O(q)] \quad (A8)$$

b) Non-orientable loops

$$\chi^{\pm}(x, e^{i\pi}w) \sim -\frac{\pi}{\ln w} \left(\sin \frac{\theta}{4} \right)^{-1} [1 + O(q_N)] \quad (\text{A9})$$

$$\varphi^{\pm}(e^{i\pi}w) \sim |q_N|^{-1/24} w^{1/48} e^{\pm i\pi/8} [1 + O(q_N)] \quad (\text{A10})$$

REFERENCES

- 1) A. Neveu and J.H. Schwarz, Princeton University preprint (1971),
to be published in Nuclear Phys.B.
- 2) A. Neveu, J.H. Schwarz and C.B. Thorn, Princeton University
preprint (1971), to be published in Phys.Letters.
- 3) A. Neveu and J.H. Schwarz, Princeton University preprint (1971).
- 4) M. Virasoro, Phys.Rev. D1, 2933 (1970).
- 5) R.C. Brower and C.B. Thorn, CERN preprint, TH 1293 (1971).
- 6) M.B. Halpern and C.B. Thorn, Berkeley preprint (1971).
- 7) C. Lovelace, Phys.Letters 32B, 703 (1970) ;
V. Alessandrini, Nuovo Cimento 2A, 321 (1970) ;
V. Alessandrini and D. Amati, CERN preprint TH.1278 (1971).
- 8) J. Scherk, Orsay preprints LPTHE 70/45 (1970) and LPTHE 71/7 (1971).
- 9) D.J. Gross, A. Neveu, J. Scherk and J.H. Schwarz, Phys.Rev. D2,
697 (1970).
- 10) G. Frye and L. Susskind, Phys.Letters 31B, 589 (1970).
- 11) P.H. Frampton, P. Goddard, D.A. Wray, CERN preprint TH.1270
(1970), to be published in Il Nuovo Cimento.
- 12) J.A. Shapiro, University of Maryland preprint 71-096 (1971).
- 13) Chan Hong-Mo and J. Paton, Nuclear Phys. B10 (1969).
- 14) K. Bardakçi, M.B. Halpern and J.A. Shapiro, Phys.Rev. 185,
1910 (1969) ;
D. Amati, C. Bouchiat and J.L. Gervais, Nuovo Cimento Letters 2,
399 (1969).
- 15) A. Neveu and J. Scherk, Phys.Rev. D1, 2355 (1970).
- 16) M. Hamermesh, Group Theory (1962).
- 17) The Bateman Manuscript Project, Vol. 2 (New York, 1953).