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FRESNEL DIFFRACTION IN DEUTERIUM *)

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A B S T R A C T

In Glauber's theory of high energy scattering the wave function inside the nucleus is given by geometrical optics. The theory is extended to lower energies by incorporating the effects due to Fresnel diffraction, and applied to elastic and inelastic scattering off deuterium.

*) This article is dedicated to the memory of my old and irreplaceable friend Amos de-Shalit.

+) On leave from Cornell University.

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I. - INTRODUCTION

Glauber's remarkably successful theory (1) of nuclear diffraction scattering is based on two crucial circumstances that appear to set in at sufficiently high incident energy :

- I - during the collision the target acts as though it were a momentarily stationary¹ assembly of scatterers ;
- II - each scatterer produces a distortion of the wave incident upon it which, over nuclear dimensions, is accurately described by geometrical optics.

Aside from these statements which are clearly associated with high energy phenomena, the theory also assumes that overlap between target nucleons can be treated additively² to the extent that it occurs. The success of the theory is then another indication that the independent particle model is a good zeroth order approximation in nuclear physics.

In the high energy limit assertion I is surely correct, and so is II if the phrase "distortion of the wave" is understood to include the effects due to diffractive production channels. As the energy is lowered, the various members of the theoretical framework come under suspicion at different points on the energy scale, the first being geometrical optics.

The natural dimensionless variable for describing the longitudinal development of a diffracted wave is

$$\xi = \frac{z}{ka^2} , \quad (1)$$

where z is the distance from the scatterer along the incident direction, k the wave number, and a the radius of the scatterer. The three regimes of optics are then characterized by³

$$\zeta \ll 1 \quad : \text{ geometrical optics} \quad (2a)$$

$$\zeta \simeq 1 \quad : \text{ Fresnel diffraction} \quad (2b)$$

$$\zeta \gg 1 \quad : \text{ Fraunhofer diffraction} \quad (2c)$$

In a complex nucleus II will then be valid if ka^2 exceeds the mean-free-path of the projectile in nuclear matter,

$$ka^2 \gg \ell, \quad (3a)$$

while in deuterium, the double scattering amplitude will only be accurately described by geometrical optics when

$$ka^2 \gg \bar{r}, \quad (3b)$$

where $\bar{r}$ is the r.m.s. separation between neutron and proton.

When either of the conditions (3) are violated, the eikonal form of the wave function does not suffice : the collapse of the shadow, and the flaring out of the scattered wave become significant. But these phenomena can still be described by diffraction theory provided only that $ka \gg 1$, and this last inequality can be valid even though $ka^2 \simeq \ell$, because $\ell \gg a$. Now $\ell/a^2 \simeq 1 \text{ GeV/c}$, and $\bar{r}/a^2 \simeq 0.6 \text{ GeV/c}$. There is thus a fairly broad range of momenta ⁴ in the neighbourhood 1 GeV/c where assertion I continues to hold, where a fairly large number of partial waves participate significantly in the basic two-body interaction, but where geometrical optics fails. It is our purpose here to show that the theory can be extended into this regime if II is replaced by :

II' - each scatterer produces a distortion of the wave incident upon it which is accurately described by Fresnel diffraction theory.

In this article we apply the Fresnel theory to scattering by deuterium. Our calculations are of an exploratory nature in that we have only used a Gaussian wave function for the deuteron, and ignored the D state. Nevertheless, they are sufficiently realistic to demonstrate clearly that effects characteristic of Fresnel diffraction are of significance in the momentum range from several hundred MeV/c to about 2 GeV/c.

The contents of this paper are the following. The diffraction pattern of an individual nucleon is described in Section II, and that of two stationary nucleons in Section III ; the latter results are then used in Section IV in an evaluation of various deuterium cross-sections. The extension of the Fresnel theory to complex nuclei is considerably more complicated than the usual Glauber theory, and is not taken up here.

II. - THE DIFFRACTION PATTERN OF A NUCLEON

In Glauber's theory the wave function resulting from one scatterer is described by geometrical optics

$$\begin{aligned}\psi(\vec{b}, z) &= \psi_{inc}(\vec{b}, z) & (z < -a) \\ &= S(b) \psi_{inc}(\vec{b}, z) & (z > a) .\end{aligned}\tag{4}$$

Here $\vec{b}$ is the impact parameter vector, a is the range outside of which the interaction is negligible, and the origin is at the scattering centre. The interaction is described by the elastic S matrix elements $S(b)$, which must satisfy $|S(b)| \leq 1$. As we know from elementary physics, Eq. (4) does not provide an adequate description of the wave as z tends to large positive values. For one thing,

Eq. (4) has no outgoing spherical waves. Furthermore, if $|S(b)| \ll 1$ when $b < a$, Eq. (4) describes a shadow which persists forever, whereas we know that diffraction eventually destroys the shadow. Our first objective must then be a better representation of the wave function.

We begin by quoting a formulation of Huygens' principle due to Fresnel, Kirchhoff and Sommerfeld (3), which provides an exact relationship between a solution ψ of the free wave equations at the point $(\vec{b}, z)$, and the values ψ assumes at points $(\vec{b}', z')$ on an infinite plane Σ perpendicular to the z axis :

$$\psi(\vec{b}, z) = \int_{\Sigma} \mathcal{G}(\vec{b}, z; \vec{b}', z') \psi(\vec{b}', z') d^2 b'. \quad (5)$$

The Green's function is

$$\mathcal{G}(\vec{b}, z; \vec{b}', z') = \int \frac{d^3 p}{(2\pi)^3} \frac{2ip_n}{k^2 - p^2 + i\epsilon} e^{i\vec{p}_\perp \cdot (\vec{b} - \vec{b}')} e^{ip_n(z - z')} \quad (6)$$

$$= \frac{k}{2\pi i} \frac{e^{ik\rho}}{\rho} \left(1 + \frac{i}{k\rho}\right) \cos \Theta, \quad (7)$$

where $\vec{\rho}$ is the vector from the integration point on Σ to the observation point, and Θ is the angle between $\vec{\rho}$ and the z axis, i.e.,

$$\rho = \sqrt{(\vec{b} - \vec{b}')^2 + (z - z')^2},$$

$$\cos \Theta = \frac{z - z'}{\rho}. \quad (8)$$

We shall always use Eq. (5) at separations ρ large compared to the wavelength, and henceforth we drop the factor $(1 + i/k\rho)$ from $\mathcal{G}$.

It is now quite clear what we must do in order to obtain a better rendition of the wave function at lower energies, or equivalently, larger distances. We shall assume that the eikonal form Eq. (4) is valid on a plane Σ just beyond the interaction region, i.e., for $z' > a$, and then use Huygens' construction, Eq. (5), to evaluate ψ elsewhere. However, it turns out that keeping Σ at a slightly arbitrary point $z' > a$ is somewhat awkward, and we wish to rid ourselves of this complication at the outset. Strictly speaking, Eq. (5) is only valid in a half-space free of interaction. But when diffraction concepts are applicable the wave function maintains the geometrical optics form over a distance larger than a [see Eq. (2a), and argument below], and so we can displace Σ to $z' = 0$ even though it then intersects a portion of space where the free wave equation does not hold. This we shall do henceforth, that is, we shall describe the interaction by the statement that at $z = 0$ an incident plane wave e^{ikz} produces the wave function

$$S(b) \equiv 1 - \Gamma(b). \quad (9)$$

When the term "1" of Eq. (9) is inserted into Eq. (5) (with $z' = 0$), it obviously reproduces the incoming wave. The scattered wave for $z > 0$ is therefore

$$\psi_{sc}(\vec{b}z) = - \int \mathcal{G}(\vec{b}z; \vec{b}'0) \Gamma(b') d^2b'. \quad (10)$$

To determine the distance over which ψ_{sc} maintains the eikonal form, we ask under what circumstances $\mathcal{G}$ behaves like $\delta^2(\vec{b} - \vec{b}')$. When this is the case we certainly have $(\vec{b} - \vec{b}')^2 \ll z^2$, and $\mathcal{G}$ in Eq. (10) can be replaced by

$$\frac{k}{2\pi i} \frac{e^{ikz}}{z} e^{ik(\vec{b}-\vec{b}')^2/2z} = e^{ikz} \int \frac{d^2\kappa}{(2\pi)^2} e^{i\vec{\kappa}(\vec{b}-\vec{b}')} e^{-i\kappa^2 z/2k}. \quad (11)$$

As $\Gamma \rightarrow 0$ for $b' \gtrsim a$, $|\vec{b} - \vec{b}'| \lesssim a$, and thus we see that $\mathcal{G}$ behaves effectively like $e^{ikz} \delta^2(\vec{b} - \vec{b}')$ for longitudinal distances satisfying $z \lesssim ka^2$; we have therefore verified Eq. (2a). Whenever diffraction concepts are applicable, $ka \gg 1$, and ka^2 therefore exceeds a . The shifting of Σ carried out before is seen to be legitimate.

Having established the short-distance behaviour of ψ_{sc} , we now turn to the asymptotic form, i.e., the limit $b \rightarrow \infty$, $z \rightarrow \infty$, with $b/z \equiv \tan \theta$ fixed. We immediately find that Eq. (10) yields the scattering amplitude

$$f(kq_{\perp}) = \frac{ik}{2\pi} \cos \theta \int e^{i\vec{q}_{\perp} \cdot \vec{b}} \Gamma(b) d^2b, \quad (12)$$

where $\vec{q}$ is the momentum transfer. Except for the factor $\cos \theta$, which is unimportant when $\theta^2 \ll 1$, this agrees with the familiar formula for the amplitude. It should be noted that only $\vec{q}_{\perp}$, the perpendicular component of $\vec{q}$, enters the integral, and that it is therefore a function only of $q_{\perp} = k \sin \theta$, not of $q = 2k \sin \frac{1}{2} \theta$. Naturally $q_{\perp} \simeq q$ for $\theta^2 \ll 1$.

Our first objective is an evaluation of the wave function in the region that is neither asymptotic nor so close to the scatterer that the eikonal approximation holds. Of course, Eq. (10) is an exact interpolation formula between these two limits, but as it stands it does not lend itself to analytic integration. In the multiple scattering processes that we wish to study the deviations from the eikonal limit are significant, but not overwhelming; that is to say, the transverse spreading of the beam continues to be small in comparison to the distance from the scatterer, and $(b - b')^2 \ll z^2$ is a well-satisfied inequality. Hence, Eq. (11) is an adequate approximation to $\mathcal{G}$ for our purpose:

$$\psi_{sc}(\vec{b}, z) = -\frac{k}{2\pi i} \frac{e^{ikz}}{z} \int e^{ik(\vec{b} - \vec{b}')^2/2z} \Gamma(b') d^2b'. \quad (13)$$

For $\theta^2 \ll 1$ the two-particle cross-sections are well described by a Gaussian in θ^2 , and we therefore assume that $\Gamma(b)$ has the form

$$\Gamma(b) = \Gamma_0 e^{-b^2/2B} \quad (14)$$

If a more refined rendition of the data is desired, this can be achieved by a superposition of Gaussians, and the calculations that follow would be no more difficult, though the book-keeping would be more tedious. As we are not concerned here with detailed fits to experiments, but only with the general features of the problem, we restrict ourselves to Eq. (14), and furthermore assume that the elastic scattering amplitude is pure imaginary, in which case

$$\Gamma_0 = \frac{\sigma}{4\pi B} \quad (15)$$

where σ is the total two-body cross-section. Unitarity requires $\Gamma_0 \leq 1$; in the numerical results quoted below we usually set $\Gamma_0 = 1$.

The integration of Eq. (13) is now elementary, and gives a remarkably simple result :

$$\begin{aligned} \psi(\vec{b}; z) &\equiv e^{ikz} [1 + \varphi(b; z)] , \\ \varphi(b; z) &= \frac{\Gamma_0}{1+i\zeta} \exp\left(-\frac{b^2}{2B} \frac{1}{1+i\zeta}\right). \end{aligned} \quad (16)$$

Here

$$\zeta = \frac{z}{2B} \quad ; \quad (17)$$

that is to say, ζ is just the variable introduced in Eq. (1). We see immediately that φ reduces to the eikonal form for $\zeta \rightarrow 0$. For $\zeta \rightarrow \infty$ φ becomes an outgoing spherical wave with a scattering amplitude that agrees with the small-angle form of Eq. (12) :

$$f(k, \theta) = \frac{i\sigma k}{4\pi} e^{-\frac{1}{2}Bk^2\theta^2} \quad (18)$$

That φ does not lead to the correct form of f for large θ was to be expected in view of the discussion preceding Eq. (13), and also does not matter as we do not need φ for $\zeta \gg 1$.

Intensity profiles computed with Eq. (16) are shown in Fig. 1 for various values of z/k . The transition from the geometrical shadow to the beginnings of a Fraunhofer diffraction pattern are clearly visible. It is important to realize that Fig. 1 shows intensities, i.e., that the deviation of the amplitude from the geometrical optics limit sets in much more rapidly as z/k increases than Fig. 1 indicates. As a consequence processes involving coupled diffractive channels (e.g., Compton scattering) should be more sensitive to Fresnel effects than conventional elastic scattering. In this context one should note that if τ is real $e^{-ikz}\psi$ is also real in the eikonal approximation, whereas in our case this function must (and does) change phase drastically so as to give a purely imaginary scattering amplitude.

With the help of Fig. 1, we can now make a somewhat more educated guess concerning the numerical importance of Fresnel effects than was possible on the basis of the inequalities of Eq. (3). For example, if we are willing to admit an error of, say, 30 % in $|\psi|^2$ over a distance of $3F$ (a typical mean-free-path), Fig. 1 would indicate that the Glauber theory should be satisfactory for $k \gtrsim 3$ GeV/c. In the deuteron, where the wave is only needed over a distance of order $1.5 F$, $k \geq 1.5$ GeV/c should be equally good.

These estimates are actually too pessimistic, and the reason for this can already be glimpsed from Fig. 1. We note that as the shadow begins to fill in, there is a tendency towards a compensating diminution of intensity in the illuminated region. Indeed, if we use the representation Eq. (6) of $\mathcal{G}$ in Eq. (10), we obtain the exact relationship⁵

$$\int \psi_{sc}(\vec{b}, z) d^2b = e^{ikz} \int T(b) d^2b. \quad (19)$$

In words, if the diffracted wave is averaged over all impact parameters, the result is independent of z apart from a trivial phase, and is therefore the same as in the eikonal approximation. In a heavy nucleus, where secondary scatterers are more or less randomly distributed, an average resembling Eq. (19) surely enters, and we must therefore expect a considerable cancellation of Fresnel effects and a corresponding improvement of the Glauber approximation. For essentially the same reason Fresnel effects will be surprisingly small in total deuteron cross-sections, and only the angular distribution of elastic scattering will reveal effects of the order of magnitude that one would expect on looking at Fig. 1.

III. - DIFFRACTION BY TWO NUCLEONS

Consider two nucleons placed symmetrically about the origin at $(-\frac{1}{2}\vec{y}, -\frac{1}{2}z)$ and $(\frac{1}{2}\vec{y}, \frac{1}{2}z)$, where $\vec{y}$ is transverse to the incident direction. The wave that strikes the downstream nucleon at a transverse point $\vec{b}$ is then

$$\psi_2(\vec{b}) = e^{\frac{1}{2}ikz} [1 + \varphi(\vec{b} + \frac{1}{2}\vec{y}; z)]. \quad (20)$$

The total wave at an observation point $(\vec{b}_0, z_0)$ is once more given by Huygens' principle,

$$\Psi(\vec{b}_0, z_0) = \int \mathcal{G}(\vec{b}_0, z_0; \vec{b}, z) \psi_z(\vec{b}) S(\vec{b} - \frac{1}{2}\vec{y}) d^2b.$$

The amplitude for double scattering can be extracted from Ψ 's asymptotic behaviour, with the result

$$f_D(kq_L; \vec{r}) = \frac{ik}{2\pi} \cos \theta e^{\frac{1}{2}i\vec{q} \cdot \vec{r}} \int e^{i\vec{q} \cdot \vec{b}} \varphi(\vec{b} + \frac{1}{2}\vec{y}; z) T(b) d^2b. \quad (21)$$

where $\vec{r} = (\vec{y}, z)$.

With the assumed form of T , Eq. (21) is again a Gaussian integral :

$$f_D(kq_L; \vec{r}) = \frac{\frac{1}{2}ikB\Gamma_0^2}{1 + \frac{1}{2}i\zeta} \cos \theta e^{\frac{1}{2}i q_{||} z} \exp\left(-\frac{1}{4}q_L^2 B \frac{1+i\zeta}{1+\frac{1}{2}i\zeta}\right) \cdot \exp\left(\frac{\frac{1}{4}\vec{q}_L \cdot \vec{y} \zeta}{1 + \frac{1}{2}i\zeta}\right) \exp\left(-\frac{y^2}{4B} \frac{1}{1 + \frac{1}{2}i\zeta}\right), \quad (22)$$

where $q_{||}$ is longitudinal component of $\vec{q}$, i.e., $q_{||} = k - \sqrt{k^2 - q_L^2}$. This formula reduces to Glauber's for high energies ($\zeta \rightarrow 0$) and small angles⁶.

IV. - DEUTERIUM CROSS-SECTIONS

The complete amplitude for scattering by two fixed nucleons is

$$F(kq_L; \vec{r}) = 2f_0(kq_L) \cos \frac{1}{2}\vec{q} \cdot \vec{r} + f_D(kq_L; \vec{r}), \quad (23)$$

where f_0 is the amplitude for scattering off one nucleon. The elastic amplitude, and differential cross-section irrespective of energy loss, are then obtained by averaging F and $|F|^2$, respectively, in the deuteron ground state.

If accurate results are desired one must use a realistic wave function and include the D state. The resulting calculations are then necessarily numerical, and opaque. The general features of the problem are better studied by analytical methods. This we shall do in two ways : by a detailed evaluation of the cross-section with a Gaussian wave function, and for even greater clarity, by considering the limiting case of a very large deuteron.

A. - Limit of large r.m.s. radius

The deuteron is a loosely bound structure, and we therefore begin by exploiting the assumption that its r.m.s. radius $\bar{r}$ greatly exceeds the other lengths that enter the problem, the interaction range $\sqrt{B}$ and the wavelength. When $k \rightarrow \infty$ our results must then reduce to Glauber's well-known formulas involving $\langle r^{-2} \rangle$.

It will be convenient to rewrite the double scattering amplitude as ⁷

$$f_D(\vec{y}, z) = \frac{k}{2\pi i} e^{\frac{1}{2}i\vec{q} \cdot \vec{r}} \int \Delta(\vec{b} - \vec{b}' + \vec{y}; z) e^{i\vec{q} \cdot \vec{b}} \Gamma(b) \Gamma(b') d^2b d^2b', \quad (24)$$

where

$$\Delta(\vec{b}; z) = \int \frac{d^2k}{(2\pi)^2} e^{-i\vec{k} \cdot \vec{b}} e^{-ik^2 z / 2k}. \quad (25)$$

For elastic scattering we need the expectation value of f_D in the ground state χ :

$$\langle f_0 \rangle = 2 \int_0^\infty dz \int d^2y |\chi(\sqrt{y^2+z^2})|^2 f_D(\vec{y}z). \quad (26)$$

The width δy of $f_D(\vec{y}z)$ can be estimated from Eq. (22) to be of order

$$\frac{\delta y}{\bar{r}} \sim \frac{\sqrt{B}}{\bar{r}} + \frac{1}{\sqrt{k\bar{r}}}.$$

As we wish to extract the leading term as $\bar{r} \rightarrow \infty$, we can replace Eq. (26) by

$$\langle f_0 \rangle = 2 \int_0^\infty dz |\chi(z)|^2 \int d^2y f_D(\vec{y}z). \quad (27)$$

With the help of Eq. (25), this becomes

$$\langle f_0 \rangle = \frac{i}{k} \left\langle \frac{e^{i\lambda r}}{r^2} \right\rangle [f(k_\perp^2 q_\perp)]^2 \quad (28)$$

where ⁸

$$\lambda = \frac{1}{2} (q_0 - \frac{q_\perp^2}{4k}) \simeq \frac{q^2}{8k}. \quad (29)$$

Hence Glauber's formula for the elastic double scattering amplitude is modified by Fresnel effects by the factor

$$\frac{\langle r^{-2} e^{i\lambda r} \rangle}{\langle r^{-2} \rangle} = C(k, q_\perp). \quad (30)$$

In this $\bar{r} \rightarrow \infty$ approximation there is no Fresnel correction to the total cross-section. The physical reason for this absence of a Fresnel correction was already discussed in connection with Eq. (19).

Precisely the same modification [Eq. (28)] of Glauber's formula has just been obtained by Blankenbecler and Gunion (4) from an approach that appears to be based on a quite different physical picture. In their work, the Green's function of the static eikonal theory is replaced by an improved version (5) which approximately takes the motion of the target into account. In our approach the target is treated as static during the collision. It is therefore rather surprising - even disturbing - that these two treatments give the same result ⁹.

We now turn to the summed cross-section. By the same reasoning as led from Eqs. (26) to (28) we have

$$\begin{aligned} \lim_{\bar{r} \rightarrow \infty} \langle |f_D|^2 \rangle &= \left(\frac{k}{2\pi} \right)^2 2 \int_0^\infty dz |\chi(z)|^2 \int d^2y |f_D(\vec{y}, z)|^2 \\ &= \frac{k^2}{(2\pi)^3} \langle \frac{1}{r^2} \rangle \int d^2b \dots d^2b' e^{i\vec{q} \cdot (\vec{b} - \vec{b}')} \delta^2(\vec{b} + \vec{b}' - \vec{b}_1 - \vec{b}_1') \\ &\quad \cdot \Gamma(b) \Gamma^*(b') \Gamma(b_1) \Gamma^*(b_1'). \end{aligned}$$

In this formula all z dependent phase factors have cancelled, and this means that the Fresnel effects have too. The same cancellation also occurs in the interference term between single and double scattering, $\langle f_D \cos \frac{1}{2} \vec{q} \cdot \vec{r} \rangle$.

We therefore have a rather astonishing result : in the limit of a large deuteron radius only the elastic angular distribution reveals Fresnel effects. This of course means that there are precise cancellations between elastic and inelastic scattering so that the total cross-section and the differential cross-section summed over all final states of the target are still given by Glauber's formulas even when Fresnel diffraction occurs.

B. - Gaussian wave function

Franco and Glauber (6) have shown that the wave function

$$\chi(r) = (2A\pi)^{-3/4} e^{-r^2/2A} \quad (31)$$

leads to results that agree remarkably well with those obtained with more realistic functions. As this function leads to analytic expression for the amplitudes we adopt it here, with their value of the r.m.s. radius, $\sqrt{A} = 8 \text{ GeV}^{-1}$.

On examining Eq. (22) we observe that the dependence on z (i.e., on ζ) is rather intricate. As a consequence it does not seem possible to evaluate $\langle f_D \rangle$ in closed form for arbitrary angles. Nevertheless the forward amplitude, and therefore the total cross-section can be given in closed form. Let σ_D be the total cross-section in deuterium, σ the same cross-section on one nucleon, and define δ by

$$\sigma_D = 2\sigma(1-\delta).$$

In Glauber's theory, and with our assumptions of a Gaussian wave function and profile $\Gamma(b)$, δ is given by

$$\delta_G = \frac{\frac{1}{2} \Gamma_0 B}{A + 2B}. \quad (32)$$

We, however, find that $\delta \equiv \delta_G c(\kappa)$, where the correction factor for Glauber's eclipse effect is

$$c(\kappa) = \frac{1}{\sqrt{\pi}} e^{\kappa^2} \text{erfc } \kappa, \quad (33)$$

with

$$\kappa = \frac{\sqrt{2}}{\bar{\xi}} \left(1 + \frac{A}{2B}\right), \quad \text{erfc } \kappa = \frac{2}{\sqrt{\pi}} \int_{\kappa}^{\infty} e^{-t^2} dt,$$

and

$$\bar{\xi} = \frac{\sqrt{A}}{kB}. \quad (34)$$

$\bar{\xi}$ is just our longitudinal variable $\xi = z/kB$ evaluated for z equal to the r.m.s. size of the target.

Our result reduces to Glauber's for $\kappa \gg 1$:

$$C(\kappa) \simeq 1 - \frac{1}{2\kappa^2} + \frac{3}{4\kappa^4} - \dots$$

For a typical interaction radius, say $B = 8 \text{ GeV}^{-2}$, $\kappa \simeq 7k$, where k is in GeV/c. Our correction to Glauber's formula for the total cross-section is therefore ridiculously small for all momenta :

$$C \simeq 1 - \frac{1}{100 k^2}. \quad (35)$$

That the total cross-section suffers a small correction was expected from the preceding estimates in the limit $\sqrt{A} \rightarrow \infty$ [cf. Eq. (28) et seq.], and from our earlier discussion [cf. Eq. (19)] of compensation between eclipsed and illuminated regions. Because of this compensation the dimensionless variable that characterizes the momentum dependence of the total cross-section is κ , which is far larger than $1/\bar{\xi}$, as one would have expected from the simple arguments based on the inequality (3b).

The preceding discussion changes considerably if the two-body amplitude has an important real part. The compensation is not nearly so complete then, presumably because there is no well defined shadow. We shall return to this point shortly [see Eq. (39) et seq.]

We now turn to the angular distribution of elastic scattering. In evaluating $\langle f_D \rangle$ the $\vec{y}$ integration is trivial with a Gaussian wave function, but the subsequent z integration is impossible. We again take advantage of the fact that the deuteron is a loose structure, and expand the integrand in terms of the small quantity

$$\frac{2B(1 + \frac{1}{2}i\zeta)}{A} . \quad (36)$$

We again express our result in terms of a function $C(k, q_1)$ that multiplies the usual Glauber double scattering amplitude. To within terms of order B/A - in practice to better than 10 % in the forward hemisphere - we find that

$$C(k, q_1) = \gamma [w(Q) + \sqrt{2} \bar{\gamma} \frac{B}{A} (Qw(Q) - \frac{i}{\sqrt{\kappa}})] , \quad (37)$$

where

$$\begin{aligned} \gamma &= \left(1 + \frac{q_1^2}{4k^2}\right)^{-1/2} , \\ Q &= \frac{\bar{\gamma}}{2\sqrt{2}} (q_1 k - \frac{1}{4} q_1^2) B \approx \frac{\bar{\gamma} q_1^2 B}{8\sqrt{2}} , \\ w(Q) &= e^{-Q^2} \operatorname{erfc}(-iQ) ; \end{aligned} \quad (38)$$

the function $w(Q)$ is tabulated in Ref. (7).

If we compare Eq. (37) with our earlier and more approximate expression Eq. (30), we find that the latter gives $C = w(Q/\gamma)$. This agrees with (37) in the limit $B/A \rightarrow 0$ except ¹⁰ for the rather unimportant factor γ .

The correction factor C is shown in Fig. 2 ; to this order in B/A , $\text{Re}C$ vanishes in the forward direction, as we would expect from our earlier calculation of the total cross-section. $\text{Im}C$ does not vanish at $\theta = 0$:

$$\text{Im} C(k, 0) = -i\sqrt{\frac{2}{\pi}} \frac{1}{k\bar{r}} . \quad (39)$$

Thus there will be a significant modification of the Glauber formula for the total cross-section if the two-body amplitude has an appreciable real part. [Note that this feature is missing in Eq. (30), because there $\bar{r} \rightarrow \infty$.]

The elastic differential cross-section evaluated with Eq. (37) is shown in Fig. 3 for $k = 1 \text{ GeV}/c$, and is compared there to the high energy limit where the Glauber formula becomes the exact result of diffraction theory. There are two modifications of Glauber's result : a tendency to fill the minimum due to the interference of single and double scattering, and a depletion of the cross-section at large angles where double scattering dominates. The first effect is caused by the z dependent phase of the Fresnel wave function, Eq. (16), which is absent in the eikonal wave function. The second feature is due to the lateral spreading of the shadow, as a result of which the double scattering looks as if it were brought about by two objects that are somewhat larger than the nucleon. As one can readily see from Fig. 2, the Fresnel effects are very small for $k \gtrsim 2 \text{ GeV}/c$, and grow quite rapidly as k falls below $1 \text{ GeV}/c$.

Our results cannot be compared directly with the data because we have not included the D state which is known to play a crucial role in the interference minimum, (1), (8). Nevertheless, it is quite clear that in fairly low energy collisions, e.g., the π^- -d experiment at $895 \text{ MeV}/c$ (1), (8), (9), Fresnel effects are of the same order as those due to the D state.

We have also evaluated the differential cross-section summed over final states with the Gaussian wave function by using the expansion that led to Eq. (37). Our results confirm the $\bar{r} \rightarrow \infty$ limit in that the Fresnel corrections are found to be very small. To be precise ¹¹, the correction is of order $q^2 B^2 \xi^{-2}/A$, or $q^2 \xi^{-2}$ if q is in GeV/c ; this is to be compared with the correction to elastic scattering which is of order $q^2 \xi^{-2}$.

V. - CONCLUSIONS

We have shown that effects arising from Fresnel diffraction are important in elastic scattering by deuterium for moderate and large momentum transfers, and for incident momenta from several hundred MeV/c to 2 GeV/c. On the other hand, there are important cancellations between waves diffracted into the eclipsed and illuminated regions, and also between Fresnel corrections to elastic and inelastic scattering. As a consequence Glauber's formulas for the total cross-section, and for the angular distribution irrespective of energy loss, are virtually unaltered by Fresnel effects.

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I am indebted to D.R. Yennie for an exceptionally helpful correspondence.

FOOTNOTES

- 1 - This is not simply due to the high velocity of the projectile, because if it were, assertion I would inevitably fail for a sufficiently large target. This assertion is a consequence of the incident energy being high in comparison to the energies excited in the system ; cf., e.g., Ref. (2).

- 2 - If the interactions are describable by a real potential, additivity is not a further assumption. But in the energy domain where the theory is applied the interactions cannot be so described.

- 3 - The shadow of a nucleus therefore has a length $\sim 5 k A^{2/3} \cdot 10^{-5} \text{ \AA}$, where k is in GeV/c. For a heavy nucleus and $k \simeq 500 \text{ GeV/c}$, this length is of interatomic dimensions. I have not been able to devise an experiment that would take advantage of this astonishing fact. At first sight one might think that one could observe interference of these shadows in a crystal, but the zero-point motion of the lattice produces a calamitous reduction of the effect. I should like to thank C. Michael for bringing the importance of zero-point motion to my attention.

- 4 - Observe that it is always the momentum, and not the energy, that enters the criteria concerning diffraction phenomena. The theory is therefore applicable to considerably lower energies for nucleons than for pions.

- 5 - I owe this observation to D.R. Yennie.

- 6 - The phase factor $\exp(\frac{1}{2} i q_{\parallel} z)$ does not disappear at high energies, only at small angles. When the wave function is given by Eq. (24), this phase contributes a factor $\exp(-k^2 \theta^4 A/32)$, i.e., an angular dependence governed by the deuteron size. Nevertheless, it is negligible compared to the conventional (Glauber) double scattering term $\exp(-\frac{1}{4} q_{\perp}^2 B)$ when $\theta^2 \ll 1$. In the Fresnel regime, on the other hand, this phase factor is comparable to other factors in the amplitude ⁷.

7 - In this very approximate treatment we drop the rather unimportant factor $\cos\theta$.

8 - Here we can see that the phase factor $\exp(\frac{1}{2}i q_{\parallel} z)$ must be kept in the Fresnel regime ⁶. It is also essential in the evaluation of the summed cross-section discussed below.

9 - A possible link between our approaches is the following. In the eikonal approximation the free Green's function is $(iv)^{-1} e^{ikz} \theta(z) \delta^2(\vec{b})$, where v is the velocity. Recall that this function follows from the momentum representation $(E_{\vec{k}} - E_q + i\epsilon)^{-1}$ if one sets $\vec{q} = \vec{k} + \vec{\kappa}$, and keeps only linear terms in $\vec{\kappa}$, which are then parallel to $\vec{k}$. If one also retains the leading contribution of $\vec{\kappa}$ transverse to $\vec{k}$ one obtains the Green's function.

$$\frac{1}{iv} e^{ikz} \theta(z) \frac{1}{2\pi i} \frac{k}{z} e^{ib^2 k/2z},$$

and on comparing with Eq. (11) we see that this describes Fresnel diffraction. Perhaps a somewhat better treatment of the intermediate state energy denominators (4) than in the closure approximation is equivalent to our treatment.

10 - The factor γ is missing from Eq. (30) because the calculation leading to Eq. (37) shows that the limit $\vec{r} \rightarrow \infty$ and the $\vec{y}$ integration cannot be interchanged in passing from Eq. (26) to Eq. (27).

11 - To be even more precise, Glauber's result for $\langle |f_D|^2 \rangle$ is multiplied by the factor γ , and his result for $\langle f_D \cos \frac{1}{2} \vec{q} \cdot \vec{r} \rangle$ by $\frac{1}{2} [1 + (1+q_{\perp}^2/k^2)^{-\frac{1}{2}}]$.

REFERENCES

- 1) R.J. Glauber - "High Energy Physics and Nuclear Structure",
edited by S. Devons, Plenum Press, New York 1970, and
references cited therein.
- 2) L.L. Foldy and J.D. Walecka - Ann.Phys. 54, 447 (1969).
- 3) A. Sommerfeld - "Optics", Academic Press, New York 1954 ;
Section 34 C.
- 4) R. Blankenbecler and J.F. Gunion - private communication.
- 5) R.L. Sugar and R. Blankenbecler - Phys.Rev. 183, 1387 (1969).
- 6) V. Franco and R.J. Glauber - Phys.Rev. 142, 1195 (1966).
- 7) M. Abramowitz and I.A. Stegun - "Handbook of Mathematical
Functions", Dover Publications, New York 1965, p. 325.
- 8) D.R. Harrington - Phys.Rev.Letters 21, 1496 (1968) ;
V. Franco and R.J. Glauber - Phys.Rev.Letters 22, 370 (1968) ;
C. Michael and C. Wilkin - Nuclear Phys. B11, 99 (1969) ;
G. Alberi and L. Bertocchi - Nuovo Cimento 63A, 285 (1969).
- 9) F. Bradamante, S. Conetti, G. Fidecaro, M. Fidecaro, M. Giorgi,
A. Penzo, L. Piemontese, F. Sauli and P. Schiavon - Phys.
Letters 28B, 191 (1968).

FIGURE CAPTIONS

Figure 1 Intensity profiles as a function of distance z beyond the scatterer. The function ψ is given by Eq. (16), with $\tau_0 = 1$, $B = 8 \text{ GeV}^{-2}$. The curves are labelled by the parameter z/k , where z is in F and k in GeV/c. The dashed curve is the shadow according to geometrical optics.

Figure 2 The correction factor $C(k, q_{\perp})$ which multiplies Glauber's formula for the double scattering amplitude. The analytic expression for C is given in Eq. (37). The upper curves are $\text{Re}C$, the lower $\text{Im}C$. They are labelled by k in GeV/c. The curves for $k = 0.5$ are not carried beyond $q_{\perp}^2 = 0.2$, or $\theta \simeq 50^\circ$, because we do not trust the whole theory for such large angles.

Figure 3 The differential cross-section for elastic scattering from deuterium. As before, $B = 8 \text{ GeV}^{-2}$, $A = 64 \text{ GeV}^{-2}$, $\tau_0 = 1$.

FIG. 1

FIG. 2

FIG. 3