Explicitly Covariant Light-Front Dynamics and Relativistic Few-Body Systems

J. Carbonell\(^1\), B. Desplanques\(^2\),
Institut des Sciences Nucléaires\(^a\),
53 avenue des Martyrs, F-38026 Grenoble Cedex, France

V.A. Karmanov\(^3\),
Lebedev Physical Institute, Leninsky Prospekt 53, 117924 Moscow, Russia

J.-F. Mathiot\(^4\)
Laboratoire de Physique Corpusculaire\(^b\),
24 avenue des Landais, F-63177 Aubière Cedex, France

June 6, 2005

PACS numbers: 03.65.Pm, 21.45.+v, 25.30.-c
Keywords: Relativity, Light-Front Dynamics, Few-Body systems

\(^1\)e-mail: carbonel@isnhp4.in2p3.fr
\(^2\)e-mail: desplanq@isn.in2p3.fr
\(^3\)e-mail: karmanov@sci.lpi.msk.su
\(^4\)e-mail: mathiot@in2p3.fr
\(^a\)Unité mixte de recherche CNRS-Université Joseph Fourier
\(^b\)Unité mixte de recherche CNRS-Université Blaise Pascal
Abstract

The wave function of a composite system is defined in relativity on a space-time surface. In the explicitly covariant light-front dynamics, reviewed in the present article, the wave functions are defined on the plane $\omega \cdot x = 0$, where $\omega$ is an arbitrary four-vector with $\omega^2 = 0$. The standard non-covariant approach is recovered as a particular case for $\omega = (1, 0, 0, -1)$. Using the light-front plane is of crucial importance, while the explicit covariance gives strong advantages emphasized through all the review.

The properties of the relativistic few-body wave functions are discussed in detail and are illustrated by examples in a solvable model. The three-dimensional graph technique for the calculation of amplitudes in the covariant light-front perturbation theory is presented.

The structure of the electromagnetic amplitudes is studied. We investigate the ambiguities which arise in any approximate light-front calculations, and which lead to a non-physical dependence of the electromagnetic amplitude on the orientation of the light-front plane. The elastic and transition form factors free from these ambiguities are found for spin 0, 1/2 and 1 systems.

The formalism is applied to the calculation of the relativistic wave functions of two-nucleon systems (deuteron, scattering state), with particular attention to the role of their new components in the deuteron elastic and electrodisintegration form factors and to their connection with meson exchange currents. Straightforward applications to the pion and nucleon form factors and the $\rho - \pi$ transition are also made.
Contents

1 Introduction
   1.1 The relevance of relativity in few-body systems
   1.2 Why light-front dynamics?

2 Covariant formulation of light-front dynamics
   2.1 Transformation properties of the state vector
      2.1.1 Kinematical transformations
      2.1.2 Dynamical transformations
      2.1.3 Role of the angular condition
   2.2 Covariant light-front graph technique
      2.2.1 General derivation
      2.2.2 Spin 0 system
      2.2.3 Spin 1/2 system
      2.2.4 Spin 1 and coupling with derivatives
      2.2.5 Ultraviolet and infrared behavior
   2.3 Simple examples
      2.3.1 Exchange in $t$-channel
      2.3.2 Compton scattering
      2.3.3 Self-energy contributions

3 The two-body wave function
   3.1 General properties of the wave function
      3.1.1 Transformation properties
      3.1.2 Parametrization in the spinless case
      3.1.3 Normalization
      3.1.4 New representation
   3.2 Equation for the wave function
   3.3 Relation with the Bethe-Salpeter function
   3.4 Application to the Wick-Cutkosky model
      3.4.1 Solution in the covariant formulation of LFD
      3.4.2 Solution in the Bethe-Salpeter approach
   3.5 Angular momentum and angular condition
4 The nucleon-nucleon potential
   4.1 Mesonic degrees of freedom in nuclei .................................................. 53
   4.2 The non-relativistic NN potential .......................................................... 55
      4.2.1 Spin structure of the non-relativistic NN potential ......................... 55
      4.2.2 Physical inputs ................................................................................. 57
      4.2.3 Choice of the parametrization .......................................................... 58
   4.3 Meson exchange interaction on the light front ........................................ 60
      4.3.1 OBEP on the light front ..................................................................... 61
      4.3.2 Beyond OBEP on the light front ....................................................... 63

5 Applications to the nucleon-nucleon system ................................................ 67
   5.1 The deuteron wave function ...................................................................... 67
      5.1.1 Spin structure ..................................................................................... 67
      5.1.2 Two-body contribution to the deuteron normalization ...................... 70
      5.1.3 Equation for the wave function ......................................................... 71
   5.2 Connection with the Bethe-Salpeter amplitude ........................................ 71
   5.3 Two nucleon wave function in the $J^\pi = 0^+$ scattering state ............ 73
      5.3.1 Non-relativistic wave function .......................................................... 73
      5.3.2 Spin structure of the relativistic wave function in LFD ....................... 74
      5.3.3 Equation for the wave function .......................................................... 75
   5.4 Numerical results ....................................................................................... 76

6 The electromagnetic amplitude ......................................................................... 82
   6.1 Factorization of the electromagnetic amplitude ........................................ 82
   6.2 Extracting the physical form factors ......................................................... 84
   6.3 Light-front electromagnetic vertex ............................................................ 86
      6.3.1 Spin 0 system ..................................................................................... 86
      6.3.2 Spin 1/2 system .................................................................................. 88
      6.3.3 Spin 1 system ..................................................................................... 93
      6.3.4 The transitions $1^+ - 0^+$ and $1^- - 0^-$ ........................................... 100
      6.3.5 The transitions $0^- - 1^+$ and $0^+ - 1^-$ .......................................... 102

7 Electromagnetic observables ............................................................................ 104
   7.1 Electromagnetic form factors in the Wick- Cutkosky model ....................... 104
      7.1.1 Spin 0 ............................................................................................... 104
      7.1.2 Spin 1 ............................................................................................... 106
   7.2 Applications to the quark model ................................................................. 109
      7.2.1 The pion form factor ......................................................................... 109
      7.2.2 The nucleon form factors ................................................................. 114
   7.3 Application to the nucleon-nucleon systems ............................................. 117
      7.3.1 Light-front dynamics and meson-exchange currents ......................... 117
      7.3.2 The elastic deuteron form factors ...................................................... 125

8 Concluding remarks ......................................................................................... 131
A  Notations  133

B  Relation to other techniques  135
   B.1 Relation to the Weinberg rules  135
   B.2 Relation between the Feynman amplitudes and the Weinberg rules  137
   B.3 Relation between the Feynman amplitudes and the covariant light-front
      graph technique  140
         B.3.1 Spin 0  140
         B.3.2 Spins 1/2 and 1  144

C  Relation between the deuteron components  146

D  Two-body kinematical relations  148
   D.1 One loop diagram  148
   D.2 Two-loop diagram  149

E  Three-body kinematical relations  152
Chapter 1

Introduction

1.1 The relevance of relativity in few-body systems

The relevance of a coherent relativistic description of few-body systems, both for bound and scattering states, is now well recognized in nuclear as well as in particle physics.

This is already clear in particle physics for the understanding of the wave functions of the valence quarks in the nucleon or in the pion, as revealed for instance in exclusive reactions at very high momentum transfer. The typical example of such a reaction is elastic scattering, and the extraction of the electromagnetic form factors, with their well known behavior at very high momentum transfer, the so-called scaling laws expected from QCD.

The need for a coherent relativistic description of few-body systems has also become clear in nuclear physics in order for instance to check the validity of the standard description of the microscopic structure of nuclei in terms of mesons exchanged between nucleons. In this case, electromagnetic interactions play also a central role in “seeing” meson exchanges in nuclei. The forthcoming experiments at Thomas Jefferson National Accelerator Facility (former CEBAF) at momentum transfer of a few (GeV/c)^2 are here of particular importance.

In both domains, it is obligatory to have, first, a relativistic description of the bound and scattering states. It is also necessary to have a consistent description of the electromagnetic current operator needed to probe the system. This is mandatory in order to have meaningful predictions for the various cross-sections.

A few relativistic approaches have been developed in the past ten years in order to meet these goals. Among them, two have received particular attention in the last few years.

The first one is based on the Bethe-Salpeter formalism [1] or its three-dimensional reductions. The Bethe-Salpeter formalism is four-dimensional and explicitly Lorentz covariant. The calculational technique to evaluate electromagnetic amplitudes is based on Feynman diagrams and associated rules. Three-dimensional reductions result in equations of the quasi-potential type.

The second one is Light-Front Dynamics (LFD) [2]. In this case, the state vector
describing the system is expanded in Fock components with increasing number of particles. The state vector is defined on a surface in four-dimensional space-time which should be indicated explicitly. The Fock components – the relativistic wave functions in this formalism – are the direct generalization of the non-relativistic wave functions.

In the non-relativistic limit, \((c \to \infty)\), the wave function is defined at time \(t = 0\), and the time evolution is governed by the Schrödinger equation, once the Hamiltonian of the system is known. This is the “instant” form of dynamics. Physical processes are thus calculated according to old fashioned (time ordered) perturbation theory. This form of dynamics is however not very well suited for relativistic systems, since the interaction of a probe (the electron, for instance) with the constituents of the system is not separated from its interaction with the vacuum fluctuations. Moreover, the plane \(t = cte\) is not conserved by a Lorentz boost.

In the standard version of LFD, the wave function is defined in the plane \(t + \frac{z}{c} = cte\) (\(c = 1\)). It is also equivalent to the usual equal time formalism in the infinite momentum frame. From a qualitative point of view, all the physical processes become as slow as possible because of time dilation in this system of reference. This greatly simplifies the description of the system. The investigation of the wave function is equivalent to make a snapshot of a system not spoiled by vacuum fluctuations. It is thus very natural in the description of high energy experiments like deep inelastic scattering. The calculational technique is here based on the Weinberg rules \([3]\).

This formulation has however a serious drawback since the equation of the plane \(t + \frac{z}{c} = cte\) (we take here and in the following \(c = 1\)) is still not invariant by an arbitrary rotation in space and time (Lorentz boost). This plane breaks rotational invariance. As we shall see later on, this fact has many important consequences as far as the construction of bound states (or scattering states) of definite angular momentum is concerned, or in the calculation of electromagnetic amplitudes.

### 1.2 Why light-front dynamics?

We shall present in this review a covariant formulation of LFD which provides a simple, practical and very powerful tool to describe relativistic few-body systems, their bound and scattering states, as well as their physical electromagnetic amplitudes. In this formulation, the state vector is defined on the plane characterized by the invariant equation \(\omega \cdot x = 0\), where \(\omega\) is an arbitrary light-like four vector \(\omega = (\omega_0, \vec{\omega})\), with \(\omega^2 = 0\) \([4, 5]\). With the particular choice \(\omega = (1, 0, 0, -1)\), we recover the standard LFD defined on the plane \(t + z = 0\). The covariance of our approach is realized by the invariance of the light-front plane \(\omega \cdot x = 0\) under any Lorentz transformation of both \(\omega\) and \(x\). This implies in particular that \(\omega\) cannot be kept the same in any system of reference, as it is the case in the usual formulation of LFD with \(\omega = (1, 0, 0, -1)\).

There is of course equivalence, in principle, between these approaches. Any exact calculation of a given electromagnetic process should give the same physical cross-section, regardless of the relativistic formalism which is used. In practical calculations however,
one can hardly hope to carry out an exact calculation of physical observables starting from “first principles”, although some simple theoretical systems can of course be used as test cases (the Wick-Cutkosky model for instance).

From a practical point of view, one may thus be led to choose a particular formalism depending on the system one is interested in. This choice is of course dependent on various criterions, as well as function of personal taste. Among these criterions, let us mention those we think are the most important:

i) The formalism should enable us to have physical insights into the various processes under consideration, at each step of the calculation.

ii) It should have a direct and transparent non-relativistic limit in order to gain more physical intuition from our present knowledge in the non-relativistic domain. This is particularly important in nuclear physics where a lot is already known from the non-relativistic description of the microscopic structure of nuclei and their electromagnetic interactions at the scale of 1 GeV or less [6].

iii) It should provide a as simple as possible calculational procedure to evaluate physical processes and compare them with experimental results.

We would like to show in this review that the covariant formulation of LFD is, according to these criterions, of particular interest. As we shall see extensively in the following, it has definite advantages as compared to both the Bethe-Salpeter formalism, and the usual formulation of LFD. Let us recall the most important ones below.

i) The calculational formalism - time ordered graph technique (in the light-front time) in contrast to old fashioned perturbation theory - does not involve vacuum fluctuation contributions. This strongly simplifies a priori the physical picture and calculations.

ii) In this approach, the wave functions – the Fock components of the state vector – satisfy a three-dimensional equation, and have the same physical meaning (probability amplitudes) as the non-relativistic wave functions. Their non-relativistic limit is thus explicit from the general structure of the wave function. This enables a transparent link with non-relativistic approaches in first $1/m$ order and in particular with the contribution of the dominant meson exchange current (the so-called pair term) in the two-nucleon systems.

iii) The explicit covariance of LFD is very important in practical applications: it allows to construct states with definite total angular momentum to separate contributions of relativistic origin from non-relativistic ones, and simplifies very much the calculations in the framework of a special graph technique. In these respects, the advantages of the covariant formulation in comparison to ordinary LFD are the same as the advantages of the Feynman graph technique in comparison to old fashioned perturbation theory.

iv) A very important property of relativistic wave functions and off-shell amplitudes is their dependence on the orientation of the light-front plane. It takes place both in non-covariant and covariant approaches. In the covariant approach this dependence is parametrized explicitly in terms of the four-vector $\omega$. On the contrary, exact on-shell physical amplitudes should not depend on the orientation of light-front plane. However, in practice, this dependence survives due to approximations. The covariant representation of the electromagnetic amplitudes allows to separate the physical form factors from the
unphysical contributions.

As already mentioned, the present approach differs from the standard ones by the parametrization of the current choice of the \( z \) direction by an arbitrary one \( \vec{n} \). A second major difference is the reference to a particular field theory inspired dynamics, which implies that our approach describes states with an unfixed number of particles. This is a source of many problems, especially related to the mass dependence of the interaction. These problems, which will receive a particular attention, have a strong relationship with those encountered in nuclear physics when using an energy dependent nucleon-nucleon interaction (Bonn-E potential [7]), or with the contribution of recoil and norm corrections to meson-exchange currents [8].

Our aim in this review is to present the various facets of the covariant formulation of LFD, and compared them to the standard formulation. For completeness, we shall also make contacts with the BS formalism. The relevance of Light-Front Quantization in quantum field theory is the subject of an intense present activity. Among others, the non-perturbative problems of how the condensates present in many theories appear on the light-front, or how renormalization should be implemented, are not yet completely solved [9]. However, directions of research are clearly identified (zero modes in LFD in particular). These problems are outside the scope of this review, although the explicit covariance of our formalism may be of particular interest in solving these problems.

We believe that the real advantages of the covariant formulation of LFD can be apprehended in practice, i.e., in applications to relativistic nuclear and particle physics and field theoretical problems. The first applications to relativistic nuclear and particle physics are reviewed in the present paper, with particular attention to two-nucleon systems for which many experimental data exist at present or are expected in the near future.

Many extensive or review papers have been devoted to the description of few-body systems in relativistic approaches. Some of them are based on different versions of the quasi-potential equations [10, 11]. Many applications were made using the three-dimensional Gross equation [12, 13, 14, 15, 16]. Two- and three-nucleon systems were investigated in the framework of the Bethe-Salpeter equation in refs. [17, 18, 19, 20] (see for a review [21]). The solution of the Bethe-Salpeter equation was also found and applied to deep inelastic scattering on deuteron and to the deuteron electrodisintegration in refs. [22, 23, 24].

The standard version of the LFD and its applications to few-body systems was reviewed in refs. [25, 26, 27, 28, 29, 30, 31]. The first results on the covariant formulation of LFD are given in ref. [3]. This approach has been investigated and developed also in refs. [32].

The content of this review is the following. We develop in chapter 2 the general properties of the covariant formulation of LFD, and derive the graph technique associated to it. Chapter 3 is devoted to the properties of the two-body wave function: spin structure, equation for the wave function, as well as its connection to the Bethe-Salpeter amplitude. We pay particular attention to the construction of the angular momentum operator. We apply our formalism to the two-nucleon system in chapter 4 by first deriving the nucleon-nucleon potential in this formalism. The two-nucleon wave function
(deuteron and $^1S_0$ wave function) is then constructed in chapter 3. The general structure of the electromagnetic amplitude is detailed in chapter 6, where we show how to extract the physical form factors for spin 0, 1/2 and 1 systems, as well as transition form factors. We discuss in chapter 7 the electromagnetic form factor of the simplest states with $J = 0$ and 1 of the Wick-Cutkosky model, as well as a few hadronic systems (pion, nucleon, ...). We apply our formalism to the two-nucleon system (deuteron form factors and electrodisintegration cross-section). We also discuss, in a $1/m$ expansion, the relationship between our formalism and first order relativistic corrections taken into account as meson exchange currents (the so-called pair term) in non-relativistic approaches.

The chapters 2, 3 and 6 are quite general and apply to any system. Chapters 4, 5 and 7 are direct applications to the two-nucleon systems, as well as three quark and quark-antiquark systems.
Chapter 2

Covariant formulation of light-front dynamics

We detail in this chapter the general properties of the covariant formulation of LFD. In contrast to the standard approach, the transformations of the coordinate system and of the light-front plane can be done independently from each other. These two types of transformations entail the corresponding transformation properties of the state vector and its Fock components. Particular attention is paid to the angular momentum operator. This formal field-theoretical introduction (sect. 2.1) can be omitted by a reader not interested in these details.

We then present the graph technique associated to this formulation, for particles of spin 0, 1/2 and 1. We illustrate peculiarities of this graph technique by a few simple examples.

2.1 Transformation properties of the state vector

The state vector is defined in general on a hypersurface in space-time and therefore depends dynamically on its position. For example, the non-relativistic wave function \( \psi(t) \) depends dynamically on translations of the plane \( t = \text{cte} \), i.e. on time \( t \) (by the trivial phase factor \( \exp(-iEt) \) for a bound state, where the binding energy \( E \) is determined by dynamics). In ordinary LFD, the wave function is defined on the plane \( t + z = t^+ = \text{cte} \). It depends dynamically on any translation of this plane. However, some Lorentz transformations and rotations change the orientation of this plane \( t^+ \). These transformations are thus also dynamical ones. That means lack of explicit covariance (i.e., the impossibility to transform the state vector from one reference system to another one without knowledge of the dynamics). This does not mean the absence of Lorentz covariance at all, since there exist anyhow a closed system of generators of the Poincaré group, and the observable amplitudes calculated exactly would be covariant. In practical approximate calculations, however, the covariance is lost.

In the covariant formulation of LFD, the wave function is defined on the general plane \( \omega \cdot x = \sigma \), where \( \omega \) is an arbitrary four vector restricted to the condition \( \omega^2 = 0 \), and
\( \sigma \) is the "light-front time". When there is no need to refer to the \( \sigma \) evolution, we shall take \( \sigma = 0 \). The kinematical transformations of the system of reference are thus separated from the dynamical transformations of the plane \( \omega \cdot x = \sigma \). The dynamical dependence of the wave function on the light-front plane results in that case in their dependence on \( \omega \).

This separation of kinematical and dynamical transformations has a definite advantage in the sense that it provides a definite prescription for constructing bound and scattering states of definite angular momentum. Since the total angular momentum of a composite system is determined by the transformation properties of its wave function under rotation of the coordinate system, the construction of systems with definite \( J \) is now purely kinematical, in contrast to the formulation on the plane \( t + z = t^+ \). The dynamical part of this problem, resulting from the presence of the interaction in the generators of the Poincaré group which change the position of the light-front plane, is separated out and replaced by the so-called angular condition.

### 2.1.1 Kinematical transformations

Let us first specify the transformation properties of the state vector with respect to transformations of the coordinate system. We will use for this purpose a field-theoretical language. The operators associated to the four-momentum and four-dimensional angular momentum are expressed in terms of integrals of the energy-momentum \( T^{\mu \nu} \) and the angular momentum \( M^{\rho}_{\mu \nu} \) tensors over the light-front plane \( \omega \cdot x = \sigma \), according to:

\[
\hat{P}_\mu = \int T^{\mu \nu} \omega^\nu \delta(\omega \cdot x - \sigma) d^4x = \hat{P}_\mu^0 + \hat{P}_\mu^{int},
\]

\[
\hat{J}_{\mu \nu} = \int M^{\rho}_{\mu \nu} \omega^\rho \delta(\omega \cdot x - \sigma) d^4x = \hat{J}_{\mu \nu}^0 + \hat{J}_{\mu \nu}^{int},
\]

where the \( 0 \) and \( int \) superscripts indicate the free and interacting parts of the operators respectively. For generality, we consider here the light-front time \( \sigma \neq 0 \).

The description of the evolution along the light-front time \( \sigma \) implies a fixed value of the length of \( \vec{\omega} \), or, equivalently, of \( \omega_0 \). This is necessary in order to have a scale of \( \sigma \). However, the most important properties of the physical amplitudes following from covariance do not require to fix the scale of \( \omega \) and will be invariant relative to its change.

We work in the interaction representation in which the operators are expressed in terms of free fields. For example, for a scalar field \( \varphi(x) \), the free operators \( \hat{P}_\mu^0 \) have the form:

\[
\hat{P}_\mu^0 = \int a^\dagger(\vec{k}) a(\vec{k}) k_\mu d^3k,
\]

\[
\hat{J}_{\mu \nu}^0 = \int a^\dagger(\vec{k}) a(\vec{k}) i \left( k_\mu \frac{\partial}{\partial k_\nu} - k_\nu \frac{\partial}{\partial k_\mu} \right) d^3k,
\]

where \( a^\dagger \) and \( a \) are the usual creation and destruction operators, with \([a(\vec{k}), a^\dagger(\vec{k}')] = \delta^3(\vec{k} - \vec{k}')\). The operators \( \hat{P}^{int} \) and \( \hat{J}^{int} \) contain the interaction Hamiltonian \( H^{int}(x) \):

\[
\hat{P}_\mu^{int} = \omega_\mu \int H^{int}(x) \delta(\omega \cdot x - \sigma) d^4x,
\]
\[ j^{int}_{\mu\nu} = \int H^{int}(x)(x_\mu \omega_\nu - x_\nu \omega_\mu) \delta(\omega \cdot x - \sigma) \, dx . \] (2.6)

For the particular applications considered in this review, we do not need to develop the field theory in its full form. We therefore do not pay attention here to the fact that the field-theoretical Hamiltonian \( H^{int}(x) \) is usually singular and requires a regularization. The regularization of amplitudes in our formulation will be illustrated by the example of a typical self-energy contribution at the end of this chapter.

Eq. (2.3) is consistent with the expectation that only the component \( P_\mu \) along the direction \( \omega \) of the light-front "time" has a dynamical character in the light-front formalism \[2\]. Under translation \( x \to x' = x + a \) of the coordinate system \( A \to A' \), the equation \( \omega \cdot x = \sigma \) takes the form \( \omega \cdot x' = \sigma' \), where \( \sigma' = \sigma + \omega \cdot a \). The state vector is transformed in accordance with the law:

\[ \phi_\omega(\sigma) \to \phi'_\omega(\sigma') = U_{P^0}(a)\phi_\omega(\sigma) , \] (2.7)

where \( U_{P^0}(a) \) contains only the operator of the four-momentum (2.3) of the free field:

\[ U_{P^0}(a) = \exp(i\hat{P}^0 \cdot a) . \] (2.8)

The "prime" at \( \phi'(\sigma) \) indicates that \( \phi'(\sigma) \) is defined in the system \( A' \) on the plane \( \omega \cdot x' = \sigma' \) in contrast to \( \phi(\sigma) \) defined in the system \( A \) on the plane \( \omega \cdot x = \sigma \) (the value of \( \sigma \) being the same). The state vector \( \phi'(\sigma') \) is defined in \( A' \) on the plane \( \omega \cdot x' = \sigma' \), which coincides with \( \omega \cdot x = \sigma \). Therefore no dynamics enters into the transformation (2.7). This is rather natural, since under translation of the coordinate system the plane \( \omega \cdot x = \sigma \) occupies the same position in space while it occupies a new position with respect to the axes of the new coordinate system, as indicated in fig. 1. The formal proof of (2.7), (2.8) can be found in \[3\].

![Fig.1. Translation of the reference system along the light-front time.](image)
In the case of infinitesimal four-dimensional rotations \( x_\mu \rightarrow x'_\mu = gx_\mu = x_\mu + \epsilon_{\nu\mu}x^\nu \),
the result is similar \([33]\):
\[
\phi_\omega(\sigma) \rightarrow \phi'_{\omega'}(\sigma) = U_{\mathcal{J}^\mu}(g)\phi_\omega(\sigma),
\]
(2.9)
where \( \omega'_\mu = \omega_\mu + \epsilon_{\nu\mu}x^\nu \) and
\[
U_{\mathcal{J}^\mu}(g) = 1 + \frac{1}{2} \hat{J}^0_{\mu\nu} \epsilon^{\mu\nu}.
\]
(2.10)
The operator \( \hat{J}^0_{\mu\nu} \) is given by \((2.4)\).

### 2.1.2 Dynamical transformations

The properties of the state vector under transformations of the hypersurface are determined by the dynamics and follow from the Tomonaga-Schwinger equation \([34]\):
\[
i\delta\phi / \delta\sigma(x) = H^{\text{int}}(x) \phi.
\]
(2.11)
From the definition of the variational derivative in \((2.11)\) we obtain:
\[
i\delta\phi = H^{\text{int}}(x) \phi \delta V(x),
\]
where \( \delta V(x) \) is the volume between the initial surface and the surface obtained from the original one by the variation \( \delta\sigma(x) \) around the point \( x \).

Under the translation \( \sigma \rightarrow \sigma + \delta\sigma \) of the plane, the total increment of the state vector is obtained through the increment at each point of the surface:
\[
i\delta\phi = \int H^{\text{int}}(x) \delta(\omega \cdot x - \sigma) d^4 x \phi \delta\sigma.
\]
(2.12)
This relation gives the Schrödinger equation. In the interaction representation in the light-front time, we have:
\[
i\partial\phi / \partial\sigma = H(\sigma)\phi(\sigma),
\]
(2.13)
where:
\[
H(\sigma) = \int H^{\text{int}}(x) \delta(\omega \cdot x - \sigma) d^4 x,
\]
(2.14)
and \( H^{\text{int}}(x) \) may differ from \( H^{\text{int}}(x) \) because of singularities of the field commutators on the light cone. This point is explained below in sect. \([2.2.1]\).

Similarly, in the case of a rotation of the light-front plane, \( \omega_\mu \rightarrow \omega'_\mu = \omega_\mu + \delta\omega_\mu, \delta\omega_\mu = \epsilon_{\nu\mu}x^\nu \), we find:
\[
\phi_\omega(\sigma) \rightarrow \phi_{\omega + \delta\omega}(\sigma) = \phi_\omega + \delta\phi_\omega, \delta\phi_\omega = \frac{1}{2} \epsilon_{\mu\nu} \left( \omega^\mu \frac{\partial}{\partial\omega_\nu} - \omega^\nu \frac{\partial}{\partial\omega_\mu} \right) \phi_\omega(\sigma).
\]
(2.15)
The increment of the volume over the point \( x \) is:
\[
\delta V = \epsilon_{\mu\nu} x^\mu \omega^\nu \delta(\omega \cdot x - \sigma) d^4 x,
\]
(2.16)
and it follows from (2.12) that 

\[ \hat{J}_{\mu \nu}^{int} \phi_\omega(\sigma) = \hat{L}_{\mu \nu}(\omega) \phi_\omega(\sigma) , \tag{2.17} \]

where:

\[ \hat{L}_{\mu \nu}(\omega) = i \left( \omega_\mu \frac{\partial}{\partial \omega_\nu} - \omega_\nu \frac{\partial}{\partial \omega_\mu} \right) , \tag{2.18} \]

and \( \hat{J}_{\mu \nu}^{int} \) is given by (2.6).

Equation (2.17) is called the angular condition. It plays an important role in the construction of relativistic bound states, as we shall explain below.

The transformation of the coordinate system and the simultaneous transformation of the light-front plane, which is rigidly related to the coordinate axes, correspond to the successive application of the two types of transformations considered above (kinematical and dynamical). Thus, under the infinitesimal translation \( x \rightarrow x' = x + a \) of the coordinate system, \( A \rightarrow A' \), and of the plane, we have:

\[ \phi_\omega(\sigma) \rightarrow \phi'_\omega(\sigma) = (1 + i \hat{P} \cdot a) \phi_\omega(\sigma) . \tag{2.19} \]

Note that for the state with definite total four-momentum \( p \) (i.e., for an eigenstate of the four-momentum operator), the equations (2.7) and (2.19) give:

\[ \exp(i \hat{P}_0 \cdot a) \phi(\sigma) = \exp(ip \cdot a) \phi(\sigma + \omega \cdot a) . \tag{2.20} \]

This equation determines the conservation law for the four-momenta of the constituents, given in chapter 3.

### 2.1.3 Role of the angular condition

We are interested here in the state vector of a bound system. It corresponds to a definite mass \( M \), four-momentum \( p \), total angular momentum \( J \) with projection \( \lambda \) onto the \( z \) axis in the rest frame, i.e., the state vector forms a representation of the Poincaré group. This means that it satisfies the following eigenvalue equations:

\[ \hat{P}_\mu \phi^{J\lambda}(p) = p_\mu \phi^{J\lambda}(p) , \tag{2.21} \]
\[ \hat{P}^2 \phi^{J\lambda}(p) = M^2 \phi^{J\lambda}(p) , \tag{2.22} \]
\[ \hat{S}^2 \phi^{J\lambda}(p) = -M^2 J(J + 1) \phi^{J\lambda}(p) , \tag{2.23} \]
\[ \hat{S}_3 \phi^{J\lambda}(p) = M \lambda \phi^{J\lambda}(p) , \tag{2.24} \]

where \( \hat{S}_\mu \) is the Pauli-Lubanski vector:

\[ \hat{S}_\mu = \frac{1}{2} \epsilon_{\mu \nu \rho \gamma} \hat{P}^\nu \hat{J}^{\rho \gamma} . \tag{2.25} \]

The state vector \( |p, \lambda\rangle_\omega \) for a given \( J \) is normalized as follows:

\[ \omega \langle \lambda', p' | p, \lambda \rangle_\omega = 2p_0 \delta^{(3)}(\vec{p} - \vec{p}') \delta^{J\lambda} . \tag{2.26} \]
For convenience we introduce here another notation for the same state vector:
\[
|p, \lambda\rangle_\omega \equiv \phi^J_{\omega}(p).
\]

We omit in \(\phi^J_{\omega}(p)\) the argument \(\sigma\), but show explicitly the argument \(p\). For simplicity, we have left out the \(\omega\) underscript when not absolutely necessary.

We can now use the angular condition \((2.17)\) and replace the operator \(\hat{J}^{int}_{\mu\nu}\), which is contained in \(\hat{J}_{\mu\nu}\) in \((2.25)\), by \(\hat{L}_{\mu\nu}(\omega)\). Introducing the notations:
\[
\hat{M}_{\mu\nu} = \hat{J}^{0}_{\mu\nu} + \hat{L}_{\mu\nu}(\omega),
\]
\[
\hat{W}_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\gamma} \hat{P}_{\nu} \hat{M}_{\rho\gamma},
\]
we obtain instead of eqs.\((2.23)\) and \((2.24)\):
\[
\hat{W}_2 \phi^J_{\omega}(p) = -M^2 J(J+1) \phi^J_{\omega}(p),
\]
\[
\hat{W}_3 \phi^J_{\omega}(p) = M \lambda \phi^J_{\omega}(p).
\]

*These equations do not contain the interaction Hamiltonian, once \(\phi\) satisfies \((2.21)\) and \((2.22)\). The construction of states with definite angular momentum becomes therefore a purely kinematical problem.*

At the same time, the state vector must satisfy the dynamical equation \((2.17)\). In terms of the operators \(\hat{J}_{\mu\nu}\) and \(\hat{M}_{\mu\nu}\), the angular condition \((2.17)\) can be rewritten as:
\[
\hat{M}_{\mu\nu}(\omega) \phi_\omega(\sigma) = \hat{J}_{\mu\nu} \phi_\omega(\sigma).
\]

The commutation relations between the operators \(\hat{P}_\mu, \hat{J}_{\mu\nu}, \hat{M}_{\mu\nu}\) have the form:
\[
[\hat{P}_\mu, \hat{P}_\nu] = 0, \tag{2.32}
\]
\[
\frac{1}{i} [\hat{P}_\mu, \hat{J}_{\kappa\rho}] = g_{\mu\kappa} \hat{P}_\rho - g_{\mu\rho} \hat{P}_\kappa, \tag{2.33}
\]
\[
\frac{1}{i} [\hat{J}_{\mu\nu}, \hat{J}_{\rho\gamma}] = g_{\mu\rho} \hat{J}_{\nu\gamma} - g_{\nu\rho} \hat{J}_{\mu\gamma} + g_{\nu\gamma} \hat{J}_{\mu\rho} - g_{\mu\gamma} \hat{J}_{\nu\rho}, \tag{2.34}
\]
\[
\frac{1}{i} [\hat{P}_\mu, \hat{M}_{\kappa\rho}] = g_{\mu\kappa} \hat{M}_\rho - g_{\mu\rho} \hat{M}_\kappa, \tag{2.35}
\]
\[
\frac{1}{i} [\hat{M}_{\mu\nu}, \hat{M}_{\rho\gamma}] = g_{\mu\rho} \hat{M}_{\nu\gamma} - g_{\nu\rho} \hat{M}_{\mu\gamma} + g_{\nu\gamma} \hat{M}_{\mu\rho} - g_{\mu\gamma} \hat{M}_{\nu\rho}, \tag{2.36}
\]
\[
\frac{1}{i} [\hat{J}_{\mu\nu}, \hat{M}_{\rho\gamma}] = g_{\mu\rho} \hat{J}_{\nu\gamma} - g_{\nu\rho} \hat{J}_{\mu\gamma} + g_{\nu\gamma} \hat{J}_{\mu\rho} - g_{\mu\gamma} \hat{J}_{\nu\rho}. \tag{2.37}
\]

Equations \((2.32)\), \((2.33)\) and \((2.34)\) are the standard commutation relations of the Poincaré group. The equations \((2.35)\), \((2.36)\) and \((2.37)\) simply reflect the tensor nature of the operators. This set also corresponds to the Poincaré group transformations. The derivation of these equations is explained in ref. [5].

We have thus shown that under the condition \((2.17)\) the problem of constructing states with definite angular momentum can be formulated in terms of the kinematical
operators $\hat{M}_{\mu\nu}$ and $\hat{W}_\mu$ in exactly the same way as in terms of the operators $\hat{J}_{\mu\nu}$ and $S_\mu$ which depend on the interaction. This naturally reflects the fact that the angular momentum of a system determines the kinematical properties of the wave function relative to transformations of the coordinate system. The dynamics is involved in the composition of the angular momentum of the system from the spins of its constituents.

As already mentioned, the angular momentum generators in ordinary LFD contain the interaction. To avoid any misunderstanding, note that the interaction Hamiltonian does not disappear in our approach. It is moved into the angular condition (2.31). However, it is very convenient to first construct on purely kinematical grounds the general form of the light-front wave function for a given angular momentum. One can then find from dynamics the coefficients in front of all the spin structures. This way of separating dynamical from kinematical transformation properties is of particular interest since the interaction Hamiltonian is often approximate.

We emphasize that by introducing the operator $\hat{L}_{\mu\nu}(\omega)$, eq.(2.18), containing the derivatives over $\omega$, we enlarge the Hilbert space where the state vector is defined. It is now the direct sum of the Hilbert space where the “normal” field-theoretical operators act and of the Hilbert space, where the operator $\hat{L}_{\mu\nu}(\omega)$ acts. Hence, in this enlarged space the scalar product and, correspondingly, the normalization condition contains integration over $\omega$ with the appropriate measure $\omega d\mu_\omega$:

$$\int_\omega \langle \lambda', p'|p, \lambda \rangle_\omega d\mu_\omega = 2p_0 \delta(3)(\vec{p} - \vec{p}') \delta^{\lambda\lambda'} .$$

(2.38)

The particular form of the measure $d\mu_\omega$ corresponds to integration over the directions of $\vec{\omega}$ in a particular system of reference. The angular condition (2.17), (2.31) just ensures the equivalence of the approach developed in the enlarged Hilbert space to the ordinary approach. In particular, the orthogonality condition (2.38), which contains integration over $\omega$, has to be equal to the orthogonality condition (2.26), where $\omega$ is a fixed parameter. This means, of course, that the product $\omega \langle \lambda', p'|p, \lambda \rangle_\omega$ does not depend on $\omega$ at all, (though for an arbitrary vector $|\cdots\rangle_\omega$ from the enlarged Hilbert space, the scalar product $\omega \langle \cdots | \cdots \rangle_\omega$ depends on $\omega$). However, any separate contribution of the Fock component to $\omega \langle \lambda', p'|p, \lambda \rangle_\omega$ depends on $\omega$, while the $\omega$-dependence disappears in the sum over all Fock components.

In the enlarged Hilbert space, the angular momentum operator is given in terms of derivatives on the momenta and $\omega$ and do not contain the interaction. The construction of states with definite angular momentum becomes therefore very simple. In practice, it is convenient first to solve the kinematical part of the problem – the construction of states with definite angular momentum – and then satisfy the angular condition (2.17).

The transformation properties of a state with definite angular momentum are discussed in chapter 3. We will see that they completely determine the structure of the wave function, and, in particular, the number of its spin components (six in the case of the deuteron). These components will be found below by solving the light-front generalization of the Schrödinger equation. At first glance, this procedure is unambiguous and nothing remains to be determined by the angular condition. What is then the role of this condition?
Without solving the angular condition, there is an ambiguity in finding states with definite angular momentum. To show that, let us construct the operator:

\[ \hat{A} = (\hat{W} \cdot \omega)^2, \]  

(2.39)

where \( \hat{W}_\mu \) is the kinematical Pauli-Lubansky vector (2.28). It is readily verified that this operator commutes with \( \hat{P}_\mu, \hat{M}_{\mu\nu}, \hat{W}_{\mu\nu} \) and also with the parity operator. It seems therefore that the state vector must be characterized not only by its mass, momentum and angular momentum, but also by the eigenvalue \( \alpha \) of the operator \( \hat{A} \):

\[ \hat{A} \phi_\alpha = \alpha \phi_\alpha. \]  

(2.40)

One can show that, for instance, for the total angular momentum \( J = 1 \), there are only two states with \( \alpha = 0 \) and \( \alpha = 1 \). However, these states \( \phi_\alpha \) are degenerate. Since the commutators of \( \hat{M}_{\mu\nu} \) and of \( \hat{J}_{\mu\nu} \) with \( \hat{P}_\mu, \hat{J}_{\mu\nu} \) are equal to each other (see eqs.(2.32)-(2.37)), the operator

\[ \Delta \hat{J}_{\mu\nu} = \hat{M}_{\mu\nu} - \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu}(\omega) - \hat{J}^{\text{int}}_{\mu\nu} \]  

(2.41)

commutes with \( \hat{P}_\mu \) and with \( \hat{J}_{\mu\nu} \), but \([\Delta \hat{J}_{\mu\nu}, \hat{A}] \neq 0\). The state \( \phi' = \Delta \hat{J}_{\mu\nu} \phi_\alpha \) is therefore not an eigenvector of the operator \( \hat{A} \), i.e., it can be represented in the form \( \phi' = \sum \beta_\alpha \phi_\alpha \). But it corresponds to the same mass as \( \phi_\alpha \). We would thus always get for the deuteron and for any state with \( J = 1 \) two degenerate states with \( \alpha = 0 \) and 1, in evident contradiction with reality. The angular condition (2.17) is just distinguishing a definite superposition of states \( \phi_\alpha \),

\[ \phi = \sum \alpha c_\alpha \phi_\alpha, \]  

(2.42)

which is such that \( \Delta \hat{J}_{\mu\nu} \phi = 0 \). This equation eliminates the problem of the “spurious” states of relativistic composite systems. This procedure will be illustrated in section 3.5 by a simple example.

### 2.2 Covariant light-front graph technique

The light-front graph technique is a method for calculating the \( S \)-matrix. In the framework of perturbation theory, the on-shell amplitude given by this graph technique coincides with the one given by the Feynman graph technique. However, the methods to calculate them drastically differ from each other, as we shall see in this section. The most important difference lies in the fact that in the LFD all four-momenta are always on the mass shell. This three-dimensional form of the theory has enormous advantages in solving several problems. In applications to relativistic few-body systems, it provides a direct and close connection between the relativistic wave functions and the non-relativistic ones, as we shall detail in the next chapters. Moreover, the diagrams corresponding to vacuum fluctuations are absent. This simplifies very much the theory as compared to other three-dimensional approaches (like the old fashioned time ordered perturbation theory). Another important advantage results from the explicitly covariant formulation.
of the light-front plane defined by $\omega \cdot x = 0$ as compared to the standard formulation on the plane $t + z = 0$.

We shall derive in this section the rules pertinent to the covariant light-front graph technique by transforming the standard expression for the $S$-matrix. We explicitly show in appendix B the relations between the covariant light-front amplitudes and the amplitudes given by the Weinberg and Feynman rules.

### 2.2.1 General derivation

The graph technique described below was developed by Kadyshevsky [35] (see ref. [36] for a review) and applied to LFD in ref. [4]. It is manifestly covariant, like the Feynman graph technique, and retains all the positive features of the old fashioned perturbation theory developed by Weinberg [3]. Following ref. [35], we start from the standard expression for the $S$-matrix:

$$
S = T \exp \left[ -i \int H^{\text{int}}(x) d^4x \right] = 1 + 
\sum_n \int (-i)^n H^{\text{int}}(x_1) \theta(t_1 - t_2) H^{\text{int}}(x_2) \ldots \theta(t_{n-1} - t_n) H^{\text{int}}(x_n) d^4x_1 \ldots d^4x_n ,
$$

(2.43)

where $H^{\text{int}}(x)$ is the interaction Hamiltonian. The sign of the $T$-product (and $1/n!$) are omitted, since the time ordering is made explicit by means of the $\theta$-functions. The expression (2.43) is then represented in terms of the light-front time $\sigma = \omega \cdot x$:

$$
S = 1 + 
\sum_n \int (-i)^n H^{\text{int}}_\omega(x_1) \theta(\omega \cdot (x_1 - x_2)) H^{\text{int}}_\omega(x_2) \ldots \theta(\omega \cdot (x_{n-1} - x_n)) H^{\text{int}}_\omega(x_n)
\times d^4x_1 \ldots d^4x_n .
$$

(2.44)

The index $\omega$ at $H^{\text{int}}_\omega$ indicates that $H^{\text{int}}$ and $H^{\text{int}}_\omega$ may differ from each other in order to provide the equivalence between (2.43) and (2.44). The region in which this can happen is a line on the light cone. Indeed, if $(x_1 - x_2)^2 > 0$, the signs of $\omega \cdot (x_1 - x_2)$ and $t_1 - t_2$ are the same and hence $H^{\text{int}}_\omega = H^{\text{int}}$. If $(x_1 - x_2)^2 < 0$, the operators commute:

$$
[H^{\text{int}}(x_1), H^{\text{int}}(x_2)] = 0,
$$

(2.45)

and their relative order has no significance. On the light cone, i.e. if $(x_1 - x_2)^2 = 0$, $\omega \cdot (x_1 - x_2)$ can be equal to zero while $t_1 - t_2$ may be different from zero. If the integrand has no singularity at $(x_1 - x_2)^2 = 0$, this line does not contribute to the integral over the volume $d^4x$. However, if the integrand is singular, some care is needed. To eliminate the influence of this region on the $S$-matrix, we have introduced in (2.44) a new Hamiltonian $H^{\text{int}}_\omega$, such that expressions (2.43) and (2.44) be equal to each other. The form of $H^{\text{int}}_\omega$, which provides the equivalence between (2.43) and (2.44), depends on the singularity of the commutator (2.45) at $(x_1 - x_2)^2 = 0$. For the scalar fields, the singularity is weak
enough, and the expressions (2.43) and (2.44) are the same, so that \( H^{\text{int}}_\omega = H^{\text{int}} \). For fields with spins 1/2 and 1 or with derivative couplings, the equivalence is obtained with \( H^{\text{int}}_\omega \) differing from \( H^{\text{int}} \) by an additional contribution (counter term) leading to contact terms in the propagators (or so called instantaneous interaction). We shall come back to this point later on in this section.

Introducing the Fourier transform of the Hamiltonian:

\[
\tilde{H}_\omega(p) = \int H^{\text{int}}_\omega(x) \exp(-ip\cdot x) d^4x ,
\]  

and using the integral representation for the \( \theta \) function:

\[
\theta(\omega \cdot (x_1 - x_2)) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\exp(i\tau \omega \cdot (x_1 - x_2))}{\tau - i\epsilon} d\tau ,
\]

we can transform the expression (2.44) to the form:

\[
S = 1 + R(0)
= 1 - i\tilde{H}_\omega(0)
+ \sum_{n \geq 2} (-i)^n \int \tilde{H}_\omega(-\omega \tau_1) \frac{d\tau_1}{2\pi i(\tau_1 - i\epsilon)} \tilde{H}_\omega(\omega \tau_1 - \omega \tau_2) \ldots \frac{d\tau_{n-1}}{2\pi i(\tau_{n-1} - i\epsilon)} \tilde{H}_\omega(\omega \tau_{n-1}) .
\]

The \( \tau \) variable appears here as an auxiliary variable, as defined in eq.(2.47); \( \omega \tau \) has the dimension of a momentum.

The \( S \)-matrix (2.48) gives the state vector \( \phi(\sigma) = S(\sigma) \phi_0 \) for asymptotic states, i.e. for an infinite value of the light-front time \( \sigma = \infty \). It determines the on-energy-shell amplitude. The off-energy shell amplitude is determined by \( S(\sigma) \) at finite \( \sigma \). Similarly, introducing another \( \theta \)-function \( \theta(\sigma - \omega \cdot x_1) \) in (2.44), one can find easily that the \( S \)-matrix on a finite light-front plane \( \sigma \) is represented in the form:

\[
S(\sigma) = 1 + \int_{-\infty}^{\infty} \frac{\exp(i\tau \sigma)}{2\pi i(\tau - i\epsilon)} R(\omega \tau) d\tau ,
\]

where

\[
R(\omega \tau) = \sum_{n} (-i)^n \int \tilde{H}_\omega(\omega \tau - \omega \tau_1) \frac{d\tau_1}{2\pi i(\tau_1 - i\epsilon)} \tilde{H}_\omega(\omega \tau_1 - \omega \tau_2) \ldots \frac{d\tau_{n-1}}{2\pi i(\tau_{n-1} - i\epsilon)} \tilde{H}_\omega(\omega \tau_{n-1}) .
\]

These formulae give the iterative solution of the equation (2.13). The matrix elements of the operator \( R(\omega \tau) \) for \( \tau \neq 0 \) correspond to the off-energy shell amplitudes, whereas at \( \tau = 0 \) they give the on-energy shell amplitude. We shall precise the difference between off-energy shell and off-mass-shell amplitudes below in sect. 2.3.1. At \( \sigma \to \infty \), the only residue in (2.49) for \( \tau = i\epsilon \to 0 \) survives, and we recover eq.(2.48).

We emphasize that despite the presence of the four-vector \( \omega \) in eq.(2.48), the \( S \)-matrix and any physical amplitude do not depend on \( \omega \), since eq.(2.48) gives the same
S-matrix, as the initial one given by eq. (2.43). Similarly, the off-shell matrix \( R(\omega \tau) \) in eq. (2.50), depends on \( \omega \) and off-shell light-front amplitude does not coincide with the Feynman one. This is natural since \( R(\omega \tau) \) determines the \( S \)-matrix at a given light-front plane \( \omega \cdot x = \sigma \) in the interaction region. As we shall see in chapter 3, the same is also true for light-front wave functions. The latters depend on \( \omega \) since they are always off-shell objects and are defined not as asymptotic states, but at any finite time (at any given light-front plane in our case).

### 2.2.2 Spin 0 system

The covariant light-front graph technique arises when, as usual, one represents the expression (2.48) in normal form. Let us consider for example the simple case of an interaction Hamiltonian of the form \( H = -g\varphi^3(x) \), where \( \varphi \) is a scalar field. We introduce the Fourier transform \( \tilde{\varphi}(k) \) of the field \( \varphi(x) \) given by:

\[
\varphi(x) \equiv \frac{1}{(2\pi)^{3/2}} \int \tilde{\varphi}(k) \exp(ik \cdot x) d^4k
\]

\[
= \frac{1}{(2\pi)^{3/2}} \int \left[ a(\vec{k}) \exp(-ik \cdot x) + a^\dagger(\vec{k}) \exp(ik \cdot x) \right] \frac{d^3k}{\sqrt{2 \varepsilon_k}} . \tag{2.51}
\]

We thus have:

\[
\tilde{\varphi}(k) = [a(-\vec{k})\theta(-k_0) + a^\dagger(\vec{k})\theta(k_0)]\sqrt{2 \varepsilon_k} \delta(k^2 - m^2) . \tag{2.52}
\]

When the \( S \)-matrix (2.48) is reduced to normal form, we obtain the contractions:

\[
\tilde{\varphi}(k)\tilde{\varphi}(p) = \tilde{\varphi}(k)\tilde{\varphi}(p) - :\tilde{\varphi}(k)\tilde{\varphi}(p): = \theta(p_0)\delta(p^2 - m^2)\delta^{(4)}(p + k) . \tag{2.53}
\]

It is convenient to replace in the following \( \theta(p_0) \) in the propagator (2.53) by \( \theta(\omega \cdot p) \). This is always possible, since \( p^2 = m^2 > 0 \).

We would like to emphasize at this point that the propagator (2.53) contains the delta-function \( \delta(p^2 - m^2) \), and therefore all particles are always on their mass shells. The reason of this property, which drastically differs from the Feynman approach, is the absence of the T-product operator in (2.43).

In this graph technique, the four-vectors \( \omega \tau_j \) in (2.48) are associated with a fictitious particle – called spurion – and the factors \( 1/(\tau_j - i\epsilon) \) are interpreted as the propagator of the spurions responsible for taking the intermediate states off the energy shell. This spurion should be interpreted as a convenient tool in order to take into account off-energy shell effects in the covariant formulation of LFD (in the absence of off-mass shell effects), and not as a physical particle. It is absent, by definition, in all asymptotic, on-energy shell, states. We shall show below on simple examples how the spurion should be used in practical calculations.
The general invariant amplitude $M_{nm}$ of a transition $m \rightarrow n$ is related to the $S$-matrix by:

$$S_{nm} = 1 + i(2\pi)^{4}\delta^{(4)} \frac{\left( \sum_{i=1}^{m} k_i - \sum_{i=1}^{n} k'_i \right)}{(2\pi)^{3}2\varepsilon_{k'_1} \cdots (2\pi)^{3}2\varepsilon_{k'_n} (2\pi)^{3}2\varepsilon_{k_1} \cdots (2\pi)^{3}2\varepsilon_{k_m})^{1/2}},$$

(2.54)

where, e.g., $\varepsilon_{k_1} = \sqrt{m_1^2 + k_1^2}$. The cross-section of the process $1 + 2 \rightarrow 3 + \ldots + n$ is thus expressed as:

$$d\sigma = \frac{(2\pi)^{4}}{4j\varepsilon_{k_1}\varepsilon_{k_2}} |M|^2 \frac{d^3k_3}{(2\pi)^{3}2\varepsilon_{k_3}} \cdots \frac{d^3k_n}{(2\pi)^{3}2\varepsilon_{k_n}} \delta^{(4)}(k_1 + k_2 - k_3 - \ldots - k_n),$$

(2.55)

where $j$ is the flux density of the incident particles:

$$j\varepsilon_{k_1}\varepsilon_{k_2} = \frac{1}{2} [s - (m_1 + m_2)^2]^{1/2} [s - (m_1 - m_2)^2]^{1/2}, \quad s = (k_1 + k_2)^2.$$

Fig.2. The vacuum vertices.

To find the matrix element $M$ of order $n$ one must proceed as follows: \[35, 36, 37, 4, 5\]

1. Arbitrary label by a number the vertices in the Feynman graph of order $n$. Orientate continuous lines (the lines of physical particles) in the direction from the smaller to the larger number. Initial particles are oriented as incoming into a graph, and final particles as outgoing. Connect by a directed dashed line (the spurion line) the vertices in the order of decreasing numbers. Diagrams in which there are vertices with all incoming or outgoing particle lines (vacuum vertices, as indicated in fig. 2) can be omitted. Associate with each continuous line a corresponding four-momentum, and with each $j$-th spurion line a four-momentum $\omega_{\tau_j}$.

---

1In order to stick to conventional notations, the normalization of the amplitude given by the standard formulae (2.54), (2.55) and some factors in the rules of the graph technique (mainly, the degrees of $2\pi$) differ from the ones previously used in ref. \[33, 34, 35, 37\]. Besides, the numbering of vertices here and in \[33, 34, 35, 37\] has opposite order.
2. To each internal continuous line with four-momentum \( k \), associate the propagator \( \theta(\omega \cdot k) \delta(k^2 - m^2) \), and to each internal dashed line with four-momentum \( \omega \tau_j \) the factor \( 1/(\tau_j - i\epsilon) \).

3. Associate with each vertex the coupling constant \( g \). All the four-momenta at the vertex, including the spurion momenta, satisfy the conservation law, i.e., the sum of incoming momenta is equal to the sum of outgoing momenta.

4. Integrate (with \( d^4k/(2\pi)^3 \)) over those four-momenta of the internal particles which remain unfixed after taking into account the conservation laws, and over all \( \tau_j \) for the spurion lines from \(-\infty\) to \( \infty \).

5. Repeat the procedure described in 1-4 for all \( n! \) possible numberings of the vertices.

We omit here the factorial factors that arise from the identity of the particles and depend on the particular theory.

The vacuum vertices indicated in fig. 2 disappear for a trivial reason: it is impossible to satisfy the four-momentum conservation law for them. Indeed, the conservation law for the vertex of fig. 2 has the form \( k_1 + k_2 + k_3 = \omega(\tau_1 - \tau_2) \). Since the four-momenta are on the mass shell: \( k_{1-3}^2 = m^2 > 0 \), so that the left-hand side is always strictly positive: \( (k_1 + k_2 + k_3)^2 \geq 8m^2 \), whereas the right-hand side is zero since \( \omega^2 = 0 \). However, it will be seen that the vacuum contributions that vanish in the light-front approach leave their track in a different way, making in the cases discussed below the light-front interaction \( H_\omega(x) \) in eq.(2.44) different from the usual interaction \( H(x) \) in (2.43).

The light-front diagrams can be interpreted as time-ordered graphs. As soon as the vertices are labelled by numbers, any deformation of a diagram changing the relative position of the vertex projections on the “time direction” from left to right does not change the topology of the diagram and the corresponding amplitude. Therefore it is often convenient to deform the diagram so that the vertices with successively increasing numbers are disposed from left to right. This just corresponds to time ordered graphs.

The light-front amplitudes can be also obtained by direct transformation of a given Feynman amplitude. This transformation is given in section 3.3 of appendix B.

### 2.2.3 Spin 1/2 system

The rules of the graph technique for spin 1/2 particles are similar to those given above except for the fact that one has to worry about contact interactions we already mentioned in section (2.2.1). To see this from a more practical point of view, let us first consider the diagrams of figs. 3(a) and (b) for scalar particles and consider for a moment the space-like plane \( \lambda \cdot x = \sigma \) with \( \lambda^2 = 1 \), as developed by Kadyshevsky [35].

In accordance with the rules given above, the amplitude for the diagram of fig. 3(a) gets the form:

\[
M_a = g^2 \int \delta \left[ (k + p + \lambda \tau_1)^2 - m^2 \right] \theta [\lambda(k + p) + \tau_1] \frac{d\tau_1}{\tau_1 - i\epsilon}.
\] (2.56)
Fig. 3. Exchange of a particle in the s channel: (a) contribution from intermediate state containing one particle; (b) contribution from antiparticle intermediate state; (c) same as (b) but rewritten as a time ordered diagram.
Integrating over $d\tau_1$ by means of the $\delta$-function, we get:

$$M_a = \frac{g^2}{2\tau_1 \sqrt{[\lambda \cdot (k + p)]^2 + m^2 - (k + p)^2}},$$

$$\tau_1 = -\lambda \cdot (k + p) + \sqrt{[\lambda \cdot (k + p)]^2 + m^2 - (k + p)^2}.$$ (2.57)

Similarly, we obtain the following expression for the amplitude of fig. 3(b):

$$M_b = \frac{g^2}{2\tau_2 \sqrt{[\lambda \cdot (k + p)]^2 + m^2 - (k + p)^2}},$$

$$\tau_2 = \lambda \cdot (k + p) + \sqrt{[\lambda \cdot (k + p)]^2 + m^2 - (k + p)^2}.$$ (2.58)

The sum of (2.57) and (2.58) gives the usual Feynman amplitude in the $s$ channel:

$$M_a + M_b = \frac{g^2}{m^2 - (k + p)^2}.$$ (2.59)

In the case where $\lambda = (1, \vec{0})$, we recover the contribution of the usual time ordered graphs in the old fashioned perturbation theory:

$$M_a = \frac{g^2}{2\varepsilon_{\vec{k} + \vec{p}} \varepsilon_{\vec{k} + \vec{p}} - \varepsilon_{\vec{k}} - \varepsilon_{\vec{p}}}$$
$$M_b = \frac{g^2}{2\varepsilon_{\vec{k} + \vec{p}} \varepsilon_{\vec{k} + \vec{p}} + \varepsilon_{\vec{k}} + \varepsilon_{\vec{p}}}.$$ (2.59)

The diagram of fig. 3(b) and the amplitude $M_b$ correspond to a vacuum contribution. This is clear if we draw it as a time ordered diagram, as it is done in fig. 3(c).

The light-front case can be obtained by introducing first the four vector $\omega$ with $\omega^2 = \delta^2$, replacing in the above formulae $\lambda$ by $\omega / \delta$ and then taking the limit $\delta \to 0$, i.e. $\omega^2 \to 0$. This limit is however a delicate issue. For the scalar case under consideration above, it can be checked that the amplitude $M_b$ disappears. In this limit, the system of reference where $\lambda = (1, \vec{0})$ moves with a velocity close to $c$. The graph technique for $\omega^2 = 0$ is therefore naturally related to the old fashioned time ordered perturbation theory in the infinite momentum frame, corresponding to the particular case $\omega = (1, 0, 0, -1)$, that defines the “standard” light front $t + z = 0$.

The full consideration of spin 1/2 particles shows that the above result has to be completed. In such a case, indeed, one should associate to each propagator the factor $(\hat{k} + m)$ for a fermion and $(m - \hat{k})$ for an antifermion, where $\hat{k} = k_\mu \gamma^\mu$ (see for instance section B.3.2 in appendix B).

The amplitude $M_a$, in eq. (2.57), is thus multiplied by $(\hat{p}_1 + m) = (\hat{p} + \hat{k} + \lambda \tau_1 + m)$ and $M_b$ in eq. (2.58), is multiplied by $(m - \hat{p}_2) = -(\hat{\lambda} \tau_2 - \hat{p} - \hat{k} - m)$ (for shortness we
omit the initial and final spinors). Replacing again $\lambda$ by $\omega/\delta$ with $\omega^2 = \delta^2$, we get in the limit $\delta \to 0$:

$$
M_a = g^2 \frac{\hat{k} + \hat{p} + m}{m^2 - (k + p)^2} + g^2 \frac{\hat{\omega}}{2\omega \cdot (k + p)}.
$$

$$
M_b = -g^2 \frac{\hat{\omega}}{2\omega \cdot (k + p)}.
$$

The amplitude (2.61) is the contact term we already mentioned in section 2.2.1. It is just the track of the disappeared vacuum diagrams. The sum of the amplitudes $F_a$ and $F_b$ gives the Feynman amplitude for spin 1/2 particles. As we shall see later on, contact terms are indeed essential in getting fully covariant results (i.e., independent of $\omega$). Other examples will be considered in the following sections.

![Fig.4. Contact interaction contributing to the exchange of a spin 1/2 particle in the s channel](image)

To incorporate the contact term from the very beginning, one should not consider the diagram of fig. 3(b), but add to the diagram of fig. 3(a) the diagram indicated in fig. 4, which is obtained from fig. 3 by deleting the spurion line and marking the internal line by a cross. This contribution can be also derived from a counter term added to $H^{int}$ in order to get $H^{int}$, as it was done for spin 1/2 in ref. [38]. Associating the crossed line to a fermion carrying the momentum $l$, we assign to this line the factor $-\hat{\omega} \theta(\omega \cdot l)/(2\omega \cdot l)$. For the diagram of fig. 4 this rule gives eq.(2.61) back.

We can thus formulate the rules of the covariant light-front graph technique for the case of spin 1/2 particles.

1. Transform the Feynman graph of a given order to the set of the light-front graphs in the same way as described in point 1 of the rules for scalar particles. Without making change in the orientation, replace the ordinary lines corresponding to antifermions by double lines, so that the number of fermions minus the number of antifermions is conserved. Both initial fermions and antifermions are shown by lines incoming to a diagram, and final fermions and antifermions by outgoing lines.

2. Consider the diagrams with internal fermion and antifermion lines labelled at their extremities by two successive numbers (i.e., the ends of the lines are directly connected by the spurion line, see, e.g., fig. 3). In addition to all the diagrams we
already have, we create from the latter diagrams another set of diagrams, by deleting those spurion lines, which connect the ends of the fermion (and antifermion) lines. Put on the corresponding fermion and antifermion lines a cross (see, e.g., fig. [4]). Draw any diagram with incoming lines at the left and outgoing lines at the right.

3. The analytical expression for the amplitude is written from the left to the right. The factors in this expression are written in the order where they are met when one goes through a diagram, starting at the right, from an outgoing fermion line, and continuing in the direction opposite to the orientation of the fermion lines. For an incoming antifermion line, one starts at the left, and continues in the direction of orientation of the (double) antifermion lines.

4. To each internal continuous line with four-momentum \( k \), associate the propagator \((\hat{k} + m)\theta(\omega \cdot k)\delta(k^2 - m^2)\) for a fermion, and the propagator \((m - \hat{k})\theta(\omega \cdot k)\delta(k^2 - m^2)\) for an antifermion, the factor \(-\hat{\omega}\theta(\omega \cdot k)/(2\omega \cdot k)\) for each crossed fermion line, the factor \(\hat{\omega}\theta(\omega \cdot k)/(2\omega \cdot k)\) for each crossed antifermion line, and the factor \(1/(\tau_j - i\epsilon)\) for each internal dashed line with four-momentum \(\omega \tau_j\).

5. Associate with each vertex the factor \(gV\), where \(V\) depends on the type of coupling \((1, i\gamma_5, \gamma_\mu, \text{etc.})\) in the Hamiltonian \(H = -g\bar{\psi}V\psi\cdots\). All the four-momenta at each vertex, including the spurion momenta, satisfy the conservation law, i.e., the sum of incoming momenta is equal to the sum of outgoing momenta (incoming and outgoing momenta always correspond to the incoming and outgoing lines).

6. With any outgoing fermion [antifermion] line with momentum \(p\), associate the spinor \(\pi(p)\) \([v(p)]\) and with any incoming fermion [antifermion] line associate the spinor \(u(p)\) \([\pi(p)]\).

7. Integrate \((d^4k/(2\pi)^3)\) over those four-momenta of the internal particles which remain unfixed after taking into account the conservation laws, and over all \(\tau_j\) for the spurion lines from \(-\infty\) to \(\infty\).

8. Repeat the procedure described in 1-7 for all \(n!\) possible numberings of the vertices. One should also take into account the standard sign factors appearing from permutation of fermion fields.

In the case of a diagram containing boson lines connected to fermion loops, one should go through the outgoing boson line, and then, having reached the fermion line, pass it as indicated in the rules, i.e., in the direction opposite to the orientation of the fermion line and along the orientation of the antifermion line. Both lines in the loop, single and double, are oriented in the same direction, but one makes a cycle when passing through them. We emphasize that the fermion and antifermion are distinguished by the type of the continuous lines (single or double). Both the fermion and antifermion lines are directed
from the smaller to the larger number (in the time ordered graph they both propagate from left to right, i.e., in the direction of time increase in time ordered graph). In contrast to the standard versions of the Feynman rules, any incoming line, both for antifermion and fermion, corresponds to the initial state, and outgoing lines – to the final state. The fermion and antifermion lines are followed in opposite directions (opposite to the orientation for a fermion line and along the orientation for an antifermion line). The fermion and antifermion propagators differ from each other.

In many practical applications, the contact term (or so called instantaneous interaction) can be conveniently incorporated for spin 1/2 particles by replacing the spin part of the propagator \( m \pm \hat{k} \) (for those lines where the contact term contributes at all) by \( [m \pm (k - \hat{\omega} \tau)] \) (however, the delta-function in the propagator \( \delta(k^2 - m^2) \) still depends on the argument \( (k^2 - m^2) \)). Together with the fact that the contact term originates from the lines which ends are labelled by two successive numbers, this rule coincides with the rule given in refs. \[39, 40\]. The contact terms modify only the propagators corresponding to the “lines extending over a single time interval”. Two successive numbers just determine a single time interval.

The above replacement to incorporate systematically the contact term can always be made in perturbation theory. However, it cannot be made in more complicated cases. In particular, the graph for the wave function, for instance, indicated in fig. [10] below, does not contain any external crossed lines. Hence, the contribution of the contact terms to the deuteron electromagnetic form factors or electrodisintegration amplitude, in the impulse approximation for example, cannot be incorporated by the above substitution in the propagators, since it would create an object with an external crossed line. The contact terms in the deuteron electromagnetic form factors and electrodisintegration will be properly taken into account in section \[7.2.1\].

If a diagram contains two fermions (but not fermion and antifermion) turning into a boson, like in the case of the deuteron form factor (the vertex \( NNd \)), one should pass through one fermion line from the right to the left, and then start again from the same final vertex and pass through the second line. Both lines are followed in the direction opposite to their orientation. Such example is given below in section \[5.3.3\].

### 2.2.4 Spin 1 and coupling with derivatives

For particles with spin 1 the rules are similar to the case of particles with spin 1/2. Like in Feynman rules, external lines for spin 1 particles are associated with polarization vectors. However, in the case of spin 1 particles, there is no difference between propagators of particle and antiparticle. The propagator has the form

\[
D_{\mu\nu}(k, m) = \left(-g_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{m^2}\right) \theta(\omega \cdot k) \delta(k^2 - m^2).
\]

The contact term is given by:

\[
\Delta_{\mu\nu} = -\frac{k_\mu \omega_\nu + k_\nu \omega_\mu}{2(\omega \cdot k)m^2} - \frac{\omega_\mu \omega_\nu}{4(\omega \cdot k)^2m^2}(k^2 - m^2).
\]
Like in the case of spin 1/2, the contact term contributes only in the lines extending over a single time interval. It can be incorporated simply by replacing the spin part of the corresponding propagators \((g_{\mu\nu} - k_\mu k_\nu/m^2)\) by \([g_{\mu\nu} - (k - \omega \tau)_\mu (k - \omega \tau)_\nu/m^2]\).

In the case of massless vector boson (the photon for instance) the form of the propagator, as usual, depends on the gauge. For a general gauge, the momentum dependence of the gauge term in the propagator induces also a corresponding contact term. We do not investigate here the general case and give the photon propagator for the Feynman gauge:

\[
D_{\mu\nu}(k, m = 0) = -g_{\mu\nu}\theta(\omega \cdot k)\delta(k^2).
\] (2.64)

There is no contact term in this gauge.

It is sometimes more convenient to use in this formulation of LFD the light-cone gauge defined by \(\omega \cdot A = 0\). The spin-1 propagator for a massless particle in this gauge is [40]:

\[
D_{\mu\nu}(k, m) = \left(-g_{\mu\nu} + \frac{\omega_\mu k_\nu + \omega_\nu k_\mu}{\omega \cdot k}\right)\theta(\omega \cdot k)\delta(k^2 - m^2)
\] (2.65)

and the corresponding contact term has the form:

\[
\Delta_{\mu\nu} = -\frac{\omega_\mu \omega_\nu}{(\omega \cdot k)^2}.
\] (2.66)

The contact term (2.66) can be again incorporated by the replacement \(k \rightarrow k - \omega \tau\) in the spin part of eq.(2.65).

Consider finally the coupling with a derivative, e.g., when the interaction Lagrangian contains a term like \(\partial_\mu \varphi(x)\). Since all the lines are oriented, any line is either incoming in the vertex, or outgoing from it. Let this line be associated with the four-momentum \(k\). The derivative results in the multiplication of a vertex by the extra factor \(ik_\mu\) for an outgoing line, and by the factor \(-ik_\mu\) for an incoming line. The contact term contributes to the lines involving the derivative of a field and having the same topology, as in the case of spins 1/2 and 1. That means that the ends of these lines should be connected by the spurion line. Like in the case of spin 1 propagator, the contact term can be incorporated by the replacement \(k_\mu \rightarrow k_\mu - \omega_\mu \tau\), where \(\omega \tau\) is the spurion momentum, and this spurion line should connect the ends of the particle line, corresponding to the field with derivative coupling.

2.2.5 Ultraviolet and infrared behavior

A peculiarity of the covariant light-front amplitudes is that they have no any ultraviolet divergences for finite values of all the spurion four-momenta. All the ultraviolet divergences in light-front diagrams appear after integrations over \(\tau_j\) with infinite limits [35]. Indeed, the energy-momentum conservation (including the spurion four-momentum) is valid at any vertex. Since all the four-momenta are on the corresponding mass shells, we have at each vertex a real physical process as far as the kinematics is concerned. For finite initial particle energies and for finite incoming spurion energy, the energies of the particles in the
intermediate states are thus also finite. Hence, the integrations over the particle momenta for fixed spurion momenta are constrained by a kinematically allowed finite domain. It is the same reason that provides finite imaginary part of a Feynman diagram found by replacing the Feynman propagators \( \frac{1}{(k^2 - m^2 + i\epsilon)} \) by the delta-functions \( -i\pi\delta(k^2 - m^2) \).

In both cases the internal particle lines are associated with the delta-functions.

The only source of the ultraviolet divergences in the light-front amplitudes is the infinite intermediate spurion energies, i.e., infinite \( \tau_j \). This is the reason why divergences may appear at the upper limit of integration over \( \tau_j \). Since \( \tau_j \) are scalar quantities, one can introduce an invariant cutoff in terms of these variables. This way of regularizing the divergent diagrams is another advantage of the covariant formulation of LFD. We illustrate this property of the light-front amplitudes in sect. 2.3.3 for the example of the self-energy.

For massless particles, light-front amplitudes may have infrared divergences, like in the case of Feynman diagrams.

Another peculiarity of LFD is the appearance of “zero modes”. For constituents of zero mass, for instance, the state vector may contain components with \( \omega \cdot k = 0 \) for non-zero four-momentum \( k \). In the standard approach, this corresponds to the finite light-front energy \( k^- = k^2_\perp / k_+ \) for both \( k_+ = 0 \) and \( k^2_\perp = 0 \). Zero modes can also appear in theories with spontaneously broken symmetry. They make the equivalence between LFD and the instant form of quantization in which nontrivial vacuum structures (condensates) appear.

The detailed discussion of the above problems is beyond the scope of the present review.

\section{2.3 Simple examples}

\subsection{2.3.1 Exchange in \textit{t}-channel}

![Exchange by a particle in \textit{t}-channel.](image)

Fig.5. Exchange by a particle in \textit{t}-channel.
Consider the diagrams shown in fig. 5. It corresponds to the exchange of a scalar particle of mass $\mu$ between two scalar particles, in the $t$ channel. These diagrams determine, in the ladder approximation, the kernel of the equation for the calculation of the light-front wave function. The external spurion lines indicate that the amplitude is off-energy shell. According to the light-front graph technique for spinless particles, the amplitude has the form:

$$K = g^2 \int \theta(\omega(k_1 - k'_1)) \delta \left( (k_1 - k'_1 + \omega \tau_1 - \omega \tau) - \mu^2 \right) \frac{d\tau_1}{\tau_1 - i\epsilon} + g^2 \int \theta(\omega(k'_1 - k_1)) \delta \left( (k'_1 - k_1 + \omega \tau_1 - \omega \tau') - \mu^2 \right) \frac{d\tau_1}{\tau_1 - i\epsilon}$$

$$= \frac{g^2 \theta(\omega(k_1 - k'_1))}{\mu^2 - (k_1 - k'_1)^2 + 2\tau \omega(k_1 - k'_1) - i\epsilon} + \frac{g^2 \theta(\omega(k'_1 - k_1))}{\mu^2 - (k'_1 - k_1)^2 + 2\tau \omega(k'_1 - k_1) - i\epsilon}.$$

The two items in (2.67) correspond to the two diagrams of fig. 5. They cannot be non-zero simultaneously. On the energy shell, i.e. for both $\tau = \tau' = 0$, the expression for the kernel is identical to the Feynman amplitude:

$$K(\tau = \tau' = 0) = \frac{g^2}{\mu^2 - (k_1 - k'_1)^2 - i\epsilon}.$$  (2.68)

Note that the off-shell amplitude (2.67) depends on $\omega$. This agrees with the fact that it is related by eqs. (2.49), (2.50) to the $S$-matrix on a finite light-front plane in the interaction region. It depends therefore on this plane. In the case of a scalar amplitude, this dependence is given in terms of extra scalar variables (in addition to Mandelstam variables $s$ and $t$). These variables are the scalar products of $\omega$ with the four-momenta, as seen from eq. (2.67). Hence, the amplitude has extra singularities as a function of this new variables, which can be found similarly to singularities of the Feynman amplitudes [43].

On the energy shell, corresponding to $\tau = \tau' = 0$ and to the light-front plane shifted to infinity, out of the interaction region, the dependence of the amplitude on $\omega$ disappears. In more complicated cases, when a Feynman diagram corresponds to the sum a few light-front diagrams (like in the case of the box diagrams considered in appendix D), the amplitude for a particular light-front diagram may depend on $\omega$ even on the energy shell. This dependence disappears in the sum of all amplitudes at a given order. In this case the singularities of different amplitudes cancel each other in the sum.

The off-energy shell amplitude corresponds to a diagram with external spurion lines. This term – off-energy shell – is borrowed from the old fashioned perturbation theory. As mentioned above, the latter is obtained, if one introduces in eq.(2.44), instead of $\omega$, the four-vector $\lambda$ with $\lambda = (1, 0, 0, 0)$. The difference of initial and final four-momenta differs from zero by $\lambda \tau$. Due to $\lambda \tau = 0$ the sums of initial and final spatial components of momenta is still equal to each other, but for $\tau \neq 0$ initial and final energies are not equal to each other (their difference is equal to $\tau$). This just characterizes the “off-energy shell”
amplitude, which can be a part of a bigger diagram. In the light-front amplitude for arbitrary \( \omega \), all the four-vector components do not satisfy the conservation law. Their difference is proportional to \( \omega \tau \). Hence the term ”off-energy shell” may be inappropriate for the light-front amplitude, since not only energies, but also three-momenta are not conserved. We emphasize again that the four-momenta are always on the mass shells, whereas the Feynman amplitude depends on the four-momenta which may be off-mass shell, but the conservation law is always valid.

### 2.3.2 Compton scattering

![Diagram](image)

Fig.6. (a) The direct diagram for electron Compton scattering with an electron in the intermediate state. (b) The contact term corresponding to the diagram (a).

The Compton scattering amplitude from an electron (through \( s \)-channel) is indicated in fig. 6(a), and, for the contact term, in fig. 6(b). According to the above rules of graph technique, we associate with fig. 6(a) the following expression:

\[
M_a = g^2 \int \bar{u}'(p') \hat{e}^{\ast s} (m + \hat{p}_1) \hat{e} u(p) \theta (\omega \cdot p_1) \delta (p_1^2 - m^2) \frac{d\tau}{\tau - i\epsilon} \delta^{(4)}(p + k + \omega \tau - p_1) d^4 p_1
\]

\[
= g^2 \bar{u}'(p') \hat{e}^{\ast s} \frac{m + \hat{p} + \hat{k}}{m^2 - (p + k)^2} \hat{e} u(p) + g^2 \bar{u}'(p') \hat{e}^{\ast s} \frac{2 \omega \cdot (p + k)}{2 \omega \cdot (p + k)} \hat{e} u(p),
\]

where \( \hat{e} = \gamma_\mu e_\mu(k) \) and similarly for \( \hat{e}^{\ast s} \). The polarization vector of the photon is denoted by \( e_\mu(k) \). The order of factors in (2.69) corresponds to follow the fermion line from the right to the left, in the direction opposite to its orientation.

The amplitude for the contact term contribution, indicated in fig. 6(b), is:

\[
M_b = -g^2 \bar{u}'(p') \hat{e}^{\ast s} \frac{\hat{\omega}}{2 \omega \cdot (p + k)} \hat{e} u(p).
\]

We omit here the theta-function \( \theta (\omega \cdot (p + k)) \), which is always 1.

The expressions (2.69), (2.70) reproduce the amplitudes (2.60), (2.61). In the sum, \( M_a + M_b \), the omega-dependent items cancel each other.
Fig. 7. (a) The cross diagram for electron Compton scattering with an electron in the intermediate state. (b) The contact term corresponding to the diagram (a).

Fig. 8. (a) The cross diagram for electron Compton scattering with a positron in the intermediate state. (b) The contact term corresponding to the diagram (a).
Now consider the crossed amplitude, indicated in figs. [4] and [5]. The amplitude corresponding to fig. [4](a) has the form:

\[
M_a = g^2 \int \bar{u}'(p') \hat{e} (m + \hat{p}_1) \hat{e}^{s*} u(p) \theta(\omega p_1) \delta(p_1^2 - m^2) \frac{d\tau}{\tau - i\epsilon} \delta^{(4)}(p - k' + \omega \tau - p_1) d^4 p_1
\]

\[
= g^2 \bar{u}'(p') \hat{e} \frac{m + \hat{p} - \hat{k}'}{m^2 - (p - k')^2} \hat{e}^{s*} u(p) \theta(\omega (p - k'))
\]

\[
+ g^2 \bar{u}'(p') \hat{\omega} \frac{2\omega}{(p - k')} \hat{e}^{s*} u(p) \theta(\omega (p - k')).
\]

(2.71)

The contact term contribution in fig. [5](b) writes:

\[
M_b = -g^2 \bar{u}'(p') \hat{e} \hat{\omega} \frac{\omega}{2\omega (p - k')} \hat{e}^{s*} u(p) \theta(\omega (p - k')).
\]

(2.72)

It has the form of the second term in (2.71) and cancel it in the sum \(M_a + M_b\).

The amplitude of fig. [5](a) differs from fig. [4](a) by the time ordering of the vertices. If we draw the vertices in the order of increasing numbers from the left to the right, this amplitude appears as a time ordered diagram. It contains the creation of a \(e^+ e^-\)-pair by the photon, i.e., includes a positron in the intermediate state, shown by the double line, and its subsequent annihilation. The corresponding amplitude has the form:

\[
M_a = g^2 \int \bar{u}'(p') \hat{e} (m - \hat{p}_1) \hat{e}^{s*} u(p) \theta(\omega p_1) \delta(p_1^2 - m^2) \frac{d\tau}{\tau - i\epsilon} \delta^{(4)}(k' - p + \omega \tau - p_1) d^4 p_1
\]

\[
= g^2 \bar{u}'(p') \hat{e} \frac{m + \hat{p} - \hat{k}'}{m^2 - (p - k')^2} \hat{e}^{s*} u(p) \theta(\omega (k' - p))
\]

\[
- g^2 \bar{u}'(p') \hat{\omega} \frac{2\omega (k' - p)}{2\omega (k' - p)} \hat{e}^{s*} u(p) \theta(\omega (k' - p)),
\]

(2.73)

and for the contact term indicated in fig. [5](b):

\[
M_b = g^2 \bar{u}'(p') \hat{\omega} \frac{\omega}{2\omega (k' - p)} \hat{e}^{s*} u(p) \theta(\omega (k' - p)).
\]

(2.74)

The first item in eq.(2.71) differs from zero in the region \(\omega (p - k') > 0\), whereas in eq.(2.73) this occurs in the region \(\omega (p - k') < 0\). The sum of these two contributions reproduce the Feynman amplitude in the \(u\)-channel, in the whole kinematical range.

In order to show how to deal with antifermion states, consider now the Compton scattering amplitude from an antifermion. The \(s\)-channel diagram is similar to fig. [4] with the single lines replaced by the double ones. The corresponding amplitude has the form:

\[
M_a = g^2 \bar{v}(p) \hat{e} \frac{m}{m^2 - (p + k)^2} \hat{e}^{s*} v'(p') - g^2 \bar{v}(p) \hat{\omega} \frac{\omega}{2\omega (p + k)} \hat{e}^{s*} v'(p'),
\]

(2.75)

while the contact term gives:

\[
M_b = g^2 \bar{v}(p) \hat{\omega} \frac{\omega}{2\omega (p + k)} \hat{e}^{s*} v'(p').
\]

(2.76)
The order of factors in (2.73) and (2.76) corresponds to follow the antifermion line from the left to the right, in the direction along its orientation. Note that in eq.(2.76) the sign of the contact term originating from the antifermion line is opposite to the fermion contact term, eq.(2.70). The corresponding \( \omega \)-dependent contribution in (2.75) has also an opposite sign and thus disappears in the sum \( M_a + M_b \).

The amplitude for the cross diagram with an antifermion intermediate state is also similar to fig. 4 and is given by:

\[
M_a = g^2 \int \tilde{v}(p) \bar{\epsilon'}^s(m - \hat{p}_1) \hat{e}v'(p') \theta(\omega p_1) \delta(p_1^2 - m^2) \frac{d\tau}{\tau - i\epsilon} \delta^{(4)}(p' - \omega \tau - p_1) d^4 p_1
\]

\[
= g^2 \tilde{v}(p) \bar{\epsilon'}^s \frac{m - (\hat{p} - \hat{k}')}{m^2 - (p - k')^2} \hat{e}v'(p') \theta(\omega' (p - k'))
\]

\[
- g^2 \tilde{v}(p) \bar{\epsilon'}^s \frac{\hat{\omega}}{2 \omega'(p - k')} \hat{e}v'(p') \theta(\omega' (p - k')). \tag{2.77}
\]

The corresponding contact term is:

\[
M_b = g^2 \tilde{v}(p) \bar{\epsilon'}^s \frac{\hat{\omega}}{2 \omega'(p - k')} \hat{e}v'(p') \theta(\omega' (p - k')). \tag{2.78}
\]

The kinematics of the Compton scattering from fermion and antifermion, i.e., the expression of the intermediate momentum \( p_1 \) through the external momenta, as given in eq.(2.77), is here identical. Besides the spinors \( u \) and \( v \), the difference is in the signs of the spin parts of the propagators and of the contacts terms.

Finally we give the amplitude corresponding to the scattering on an antifermion with a fermion intermediate state (same as fig. 4, but with single lines replaced by the double ones and vice versa):

\[
M_a = g^2 \int \tilde{v}(p) \bar{\epsilon'}^s(m + \hat{p}_1) \hat{e}v'(p') \theta(\omega p_1) \delta(p_1^2 - m^2) \frac{d\tau}{\tau - i\epsilon} \delta^{(4)}(k' - p + \omega \tau - p_1) d^4 p_1
\]

\[
= g^2 \tilde{v}(p) \bar{\epsilon'}^s \frac{m - (\hat{p} - \hat{k}')}{m^2 - (p - k')^2} \hat{e}v'(p') \theta(\omega' (k' - p))
\]

\[
+ g^2 \tilde{v}(p) \bar{\epsilon'}^s \frac{\hat{\omega}}{2 \omega'(k' - p)} \hat{e}v'(p') \theta(\omega' (k' - p)). \tag{2.79}
\]

and the contact term:

\[
M_b = - g^2 \tilde{v}(p) \bar{\epsilon'}^s \frac{\hat{\omega}}{2 \omega'(k' - p)} \hat{e}v'(p') \theta(\omega' (k' - p)). \tag{2.80}
\]

The difference between (2.77), (2.78) and (2.79), (2.80) is in the sign of the argument of the corresponding theta-functions. The sum of both contributions gives the Feynman amplitude for Compton scattering on an antifermion.

As mentioned above, the contact term can be incorporated everywhere by the following replacement in the spin matrix of the propagator: \( \hat{p}_1 \to \hat{p}_1 - \hat{\omega} \tau \).
2.3.3 Self-energy contributions

Another simple example is the self-energy diagram shown in fig. 9. The corresponding amplitude (equal to the self-energy up to a factor) has the form:

\[
\Sigma(p') = g^2 \int \theta(\omega \cdot k) \delta(k^2 - m^2) \theta(\omega \cdot (p_1 + \omega \tau_1 - k)) \delta \left((p_1 + \omega \tau_1 - k)^2 - m^2\right) \frac{d^4k}{(2\pi)^3} \frac{d\tau_1}{\tau_1 - i\epsilon},
\]

with \( p_1 = p' - \omega \tau' \).

Let \( q = p_1 + \omega \tau_1 \). The integral over \( d^4k \) is thus reduced to the well known calculation of the imaginary part of the Feynman amplitude, when all the propagators are replaced by the delta-functions:

\[
\int \delta(k^2 - m^2) \delta \left((q - k)^2 - m^2\right) d^4k = \frac{\pi}{2\sqrt{q^2}} \sqrt{q^2 - 4m^2}.
\]

Inserted in (2.81), it gives:

\[
\Sigma(p_1) = \frac{g^2}{16\pi^2} \int_{4m^2 - p_1^2}^{\infty} \frac{\sqrt{p_1^2 - 4m^2 + \tau_1}}{\sqrt{p_1^2 + \tau_1}} \frac{d\tau_1}{\tau_1 - i\epsilon}.
\]

The logarithmic divergence is at the upper limit of the integration over \( \tau_1 \). This is a particular manifestation of a general property of the light-front amplitudes, discussed above in sect. 2.2.5. One can introduce the invariant cutoff in terms of \( \tau_1 \). In this way, after renormalization, the standard expression for the self-energy amplitude is obtained.
Chapter 3

The two-body wave function

Since the wave functions of a composite system are the Fock components of the state vector, their transformation properties under four dimensional rotations are determined by the corresponding properties of the state vector investigated in the previous chapter. This allows one to construct their angular momentum explicitly. The angular condition, necessary to construct the angular momentum operator, is here of particular relevance. We pay also particular attention to the normalization of the wave function. The comparison with the BS wave function is made, and a simple example (Wick-Cutkosky model) is developed.

3.1 General properties of the wave function

The wave functions we consider are the Fock components of the state vector defined on the light-front plane \( \omega \cdot x = \sigma \). This means that they are coefficients in an expansion of the state vector \( \phi^{J \lambda}(p) \) with respect to the basis of free fields:

\[
|p, \lambda\rangle_\omega \equiv \phi^{J \lambda}(p) = (2\pi)^{3/2} \int \Phi_{j_{1j}j_{2j}j_{3j}}^{J \lambda}(k_1, k_2, p, \omega \tau) a_{\sigma_1}^\dagger(\vec{k}_1) a_{\sigma_2}^\dagger(\vec{k}_2) |0\rangle \\
\times \delta^4(k_1 + k_2 - p - \omega \tau) \exp(i\tau \sigma) 2(\omega \cdot p) d\tau \\
\times \frac{d^3k_1}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_1}}} \frac{d^3k_2}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_2}}} \\
+ (2\pi)^{3/2} \int \Phi_{j_{1j}j_{2j}j_{3j}j_{3j}}^{J \lambda}(k_1, k_2, k_3, p, \omega \tau) a_{\sigma_1}^\dagger(\vec{k}_1) a_{\sigma_2}^\dagger(\vec{k}_2) a_{\sigma_3}^\dagger(\vec{k}_3) |0\rangle \\
\times \delta^4(k_1 + k_2 + k_3 - p - \omega \tau) \exp(i\tau \sigma) 2(\omega \cdot p) d\tau \\
\times \frac{d^3k_1}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_1}}} \frac{d^3k_2}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_2}}} \frac{d^3k_3}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_3}}} + \cdots 
\]

Here \( \lambda \) in \( \phi^{J \lambda}(p) \) is the projection of the total angular momentum of the system on the \( z \)-axis in the rest frame, where \( \vec{p} = 0 \) and \( \sigma_1, \sigma_2, \sigma_3 \) are the spin projections of the particles 1, 2 and 3 in the corresponding rest systems.

We emphasize in (3.1) the presence of the delta-function \( \delta^4(k_1 + \cdots + k_n - p - \omega \tau) \).
\[ \omega \tau)2(\omega p)d\tau. \] Formally it can be obtained from the relation (2.20):

\[ \exp(i\hat{P}_0 a)\phi_\omega(\sigma) = \exp(ip a)\phi_\omega(\sigma + \omega a) \] (3.2)

with

\[ \hat{P}_0^\mu = \int \sum_\sigma a_\sigma^\dagger(\vec{k})a_\sigma(\vec{k})k_\mu d^3k. \]

Indeed, after substituting (3.1) in (3.2), the action of the operator \( \exp(i\hat{P}_0 a) \) on the \( n \)-body sector gives the factor \( \exp\left[ i(k_1 + \cdots + k_n)a \right] \) in the integrand of the l.h.s. of (3.2), whereas the factor \( \exp[i(p + \omega \tau)a] \) appears in the r.h.s. The delta function in (3.1) ensures the equality of these factors and, hence, the relation (3.2).

In the particular case where \( \omega = (1, 0, 0, -1) \), the delta-function \( \delta^{(4)}(k_1 + k_2 - p - \omega \tau) \) gives, after integration over \( \tau \), the standard conservation laws for the \( (\perp, +) \)-components of the momenta, but does not constrain the minus-components.

From (3.1) one can see that the wave function depends on the orientation of the light front, from its argument \( \omega \tau \). This important property of any Fock component is very natural. As explained in the previous chapter, any off-energy shell amplitude is related to the \( S \)-matrix defined on the finite light-front plane in the interaction region and therefore depends on its orientation (see eq.(2.49)). The bound state wave function is always an off-shell object (\( \tau \neq 0 \) due to binding energy). Therefore it also depends on the orientation of the light-front plane. This property is not a peculiarity of the covariant approach. It allows however to parametrize this dependence explicitly. We will investigate below this dependence for a few systems.

### 3.1.1 Transformation properties

We derive in this section the transformation properties of the relativistic wave functions under four dimensional rotations. To do this, we write eq.(2.9) in a more explicit form:

\[ \phi^J_\omega(p) \rightarrow \phi'^J_\omega(gp) = U_{J0}(g)\phi^J_\omega(p) , \] (3.3)

where \( U_{J0}(g) \) is given for infinitesimal transformations by eq.(2.10) and \( g \) is a Lorentz transformation and/or rotation. Here \( \lambda \) is the projection of the angular momentum operator on the \( z \)-axis in the system at rest, \( \vec{p} = 0 \). After transformation, the new state \( \phi' \) does not correspond to a definite projection of \( J \). We then expand \( \phi' \) with respect to the states \( \phi \):

\[ \phi'^J_\omega(gp) = \sum_\lambda D^{J\lambda}_{\lambda\omega}(R(g,p))\phi^J_\omega(gp) . \] (3.4)

Here \( D^{J\lambda}_{\lambda\omega}\{R(g,p)\} \) is the matrix of the rotational group and \( R(g,p) \) is the rotation operator:

\[ R(g,p) = L^{-1}(gp)gL(p) , \] (3.5)

where \( L(p) \) is the Lorentz transformation, corresponding to the velocity \( \vec{v} = \vec{p}/p_0 \), i.e., for example,

\[ L(K)(m,0,0,0) = (mK_0/\sqrt{K^2},m\vec{K}/\sqrt{K^2}). \]
The Euler angles that determine the rotation $R(g,p)$ can be expressed in terms of the momentum $p$ and the parameters of the transformation $g$. The explicit expression of the Euler angles in terms of $g$ and $p$ will not be needed.

To obtain the transformation properties of the state vector, we first substitute $\phi'$ from (3.4) in (3.3), and represent $\phi$ in the form of the expansion (3.1). Since $a_\sigma^\dagger |0\rangle$ in (3.1) is the state vector of a free particle with spin $j$, the operator $a_\sigma^\dagger$ transforms in accordance with the law

$$U_{J0}(g)a_\sigma^\dagger(k)U_{-1}^{-1}(g) = \sum_{\sigma'} D^{(j)}(J,\sigma) D^{(j)}(J,\sigma') \times \Phi_{J_{\lambda} \lambda}(j_1, j_2, \sigma_1, \sigma_2, (g, k)) \times \Phi_{J_{\lambda} \lambda}(j_1, j_2, \sigma_1, \sigma_2, (g, k))$$

Comparing the left- and right-hand sides of the resulting equation, we thus obtain for the two-body wave function:

$$\Phi_{J_{\lambda} \lambda}(j_1, j_2, \sigma_1, \sigma_2, (g, k)) = \sum_{\lambda' \sigma_1' \sigma_2'} D^{(j)}(J, \lambda') D^{(j)}(J, \sigma_1') \times \Phi_{J_{\lambda} \lambda}(j_1, j_2, \sigma_1', \sigma_2', (g, k))$$

and similarly for any $n$-body Fock component. Here $\lambda, \sigma_1, \sigma_2$ are the projections of the spins on the $z$ axis in the rest frame of each of the particles. We emphasize that the transformation (3.7) is a purely kinematical one. As mentioned above, in the usual formulation of LFD, the transformation of the system of reference changes the position of the light-front plane. Indeed, under the transformation $x \rightarrow x' = gx$, the state vector is transformed from the plane $t + z = 0$ in a system $A$ to the plane $t' + z' = 0$ in a system $A'$. Introducing $\omega(0) = (1, 0, 0, -1)$ we indeed obtain different planes $\omega(0) \cdot x = 0$ and $\omega(0) \cdot x' = (g^{-1} \omega(0)) \cdot x = 0$. In the covariant formulation of LFD, the state vector remains to be defined on one and the same plane $\omega \cdot x = \omega' \cdot x' = 0$ in both systems $A$ and $A'$. This property ensures the kinematical transformation law (3.7). The dependence on the surface is then given by the dynamical dependence of the wave function on $\omega$.

### 3.1.2 Parametrization in the spinless case

We will mainly concentrate on the two-body wave function. Generalization to the $n$-body case is straightforward.

Due to the conservation law

$$k_1 + k_2 = p + \omega \tau ,$$

the light-front wave function can be shown graphically like a two-body scattering amplitude as indicated in fig. 10. The broken line corresponds to the fictitious spurion. We emphasize again that although we assign a momentum $\omega \tau$ to the spurion, there is no any fictitious particle in the physical state vector. The basis in eq.(3.1) contains the particle states only. Due to this analogy, the decomposition of the wave function in independent spin structures and their parametrization is analogous to the expansion of a two-body amplitude in terms of invariant amplitudes. We will use this analogy below.
Fig. 10. Graphical representation of the two-body wave function on the light front. The broken line corresponds to the spurion (see text).

Let us consider a system of two spinless particles in a state with zero angular momentum. According to (3.7), its wave function \( \psi = \Phi_{j_1=j_2=j=0} \) is scalar and, hence, depends on invariant variables. We can first use the Mandelstam variables:

\[
s = (k_1 + k_2)^2 = (p + \omega \tau)^2, \quad t = (p - k_1)^2, \quad u = (p - k_2)^2,
\]

with

\[
s + t + u = M^2 + 2m^2. \tag{3.10}
\]

The wave function depends therefore on two independent scalar variables: \( \psi = \psi(s, t) \). It will be convenient in the following to introduce another pair of variables, the relativistic momentum \( \vec{k} \) which corresponds, in the c.m.-system where \( \vec{k}_1 + \vec{k}_2 = 0 \), to the usual relative momentum between the two particles. Note that this choice of variable does not assume, however, that we restrict ourselves to this particular reference frame. We denote by \( \vec{n} \) the unit vector in the direction of \( \vec{\omega} \) in this system. Note that due to the conservation law (3.8), the total momentum \( \vec{p} \neq 0 \) of the system in this reference frame is not zero. In terms of these variables, the wave function takes a form close to the non-relativistic case. Making the appropriate Lorentz transformation, we get:

\[
\vec{k} = L^{-1}(\mathcal{P})\vec{k}_1 = \vec{k}_1 - \frac{\vec{P}}{\sqrt{\mathcal{P}^2}}[k_{10} - \frac{\vec{k}_1 \cdot \vec{P}}{\sqrt{\mathcal{P}^2} + \mathcal{P}_0}], \tag{3.11}
\]

\[
\vec{n} = L^{-1}(\mathcal{P})\vec{\omega}/|L^{-1}(\mathcal{P})\vec{\omega}| = \frac{1}{\sqrt{\mathcal{P}^2}}L^{-1}(\mathcal{P})\vec{\omega}/\omega \cdot \vec{p}, \tag{3.12}
\]

where

\[
\mathcal{P} = p + \omega \tau, \tag{3.13}
\]

and \( L^{-1}(\mathcal{P}) \) is the Lorentz boost. The unit vector \( \vec{n} \) is reminiscent of the unit vector \( \vec{p}/|\vec{p}| \) which appears in the infinite momentum frame.

From these definitions, it follows that under a rotation and a Lorentz transformation \( g \) of the four-vectors from which \( \vec{k} \) and \( \vec{n} \) are formed, the vectors \( \vec{k} \) and \( \vec{n} \) undergo only rotations:

\[
\vec{k}' = R(g, \mathcal{P}) \vec{k}, \quad \vec{n}' = R(g, \mathcal{P}) \vec{n}, \tag{3.14}
\]
where $R$ is the rotation operator (3.3). Therefore $\vec{k}^2$ and $\vec{n} \cdot \vec{k}$ are invariants and can be expressed in terms of $s$ and $t$. For the wave function with zero angular momentum we thus obtain (3.14):

$$\psi = \psi(\vec{k}, \vec{n}) \equiv \psi(\vec{k}^2, \vec{n} \cdot \vec{k}) .$$

(3.15)

It is seen from (3.15) that the relativistic light-front wave function depends not only on the relative momentum $\vec{k}$ but on another variable – the unit vector $\vec{n}$.

Finally, we introduce the third set of variables in which the wave function can be parametrized, in analogy to the equal-time wave function in the infinite momentum frame. We define the variables:

$$x = \frac{\omega \cdot k_1}{\omega \cdot p} , \quad R_1 = k_1 - xp ,$$

(3.16)

and represent the spatial part of $R$ as $\vec{R} = \vec{R}_|| + \vec{R}_\perp$, where $\vec{R}_||$ is parallel to $\vec{\omega}$ and $\vec{R}_\perp$ is orthogonal to $\vec{\omega}$. Since $R_\omega = R_0 \omega_0 - \vec{R}_|| \cdot \vec{\omega} = 0$ by definition of $R$, it follows that $R_0 = |\vec{R}_||$, and, hence, $\vec{R}_\perp^2 = -R^2$ is invariant. Therefore, the two scalars on which the wave function should depend can be $\vec{R}_\perp^2$ and $x$:

$$\psi = \psi(\vec{R}_\perp^2, x) .$$

(3.17)

Using the definitions of the variables $\vec{R}_\perp^2$ and $x$, we can readily relate them to $\vec{k}$ and $\vec{n} \cdot \vec{k}$:

$$\vec{R}_\perp^2 = \vec{k}^2 - (\vec{n} \cdot \vec{k})^2 , \quad x = \frac{1}{2} \left( 1 - \frac{\vec{n} \cdot \vec{k}}{\varepsilon_k} \right) .$$

(3.18)

The inverse relations are

$$\vec{k}^2 = \frac{\vec{R}_\perp^2 + m^2}{4x(1-x)} - m^2 , \quad \vec{n} \cdot \vec{k} = \left[ \frac{\vec{R}_\perp^2 + m^2}{x(1-x)} \right]^{1/2} \left( \frac{1}{2} - x \right) .$$

(3.19)

The variables introduced above can be easily generalized to the case of different masses and an arbitrary number of particles [5]. The corresponding variables $\vec{q}_i, \vec{n}$ are still constructed according to eqs. (3.11), (3.12) and the variables $\vec{R}_{i\perp}, x_i$ according to (3.16) i.e., $x_i = \omega \cdot k_i / \omega \cdot p$ and $R_i = k_i - x_i p$.

### 3.1.3 Normalization

The state vector is normalized by eq.(2.26):

$$\omega \langle p', \lambda' | p, \lambda \rangle_\omega = 2p_0 \delta^{(3)}(\vec{p} - \vec{p}') \delta^{X\lambda} .$$

(3.20)

The Fock components are normalized so as to provide the condition (3.20). Substituting the state vector (3.1) in the left-hand side of eq.(3.20), we get:

$$\omega \langle p', \lambda' | p, \lambda \rangle_\omega = 2p_0 \delta^{(3)}(\vec{p} - \vec{p}') N^{\lambda X} ,$$

(3.21)
with
\[ N^\lambda_n \equiv \sum_n N_n^\lambda = \delta^\lambda_n, \]  
(3.22)

where \( N_n^\lambda \) is the contribution to the normalization integral from the \( n \)-body Fock component. We represent it in terms of all three sets of variables introduced above:

\[
N_n^\lambda = (2\pi)^3 \int \sum_{\sigma_1 \cdots \sigma_n} \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J \delta^{(4)}(k_1 + \cdots + k_n - p - \omega \tau) \\
\times \prod_{i=1}^n \frac{d^3 k_i}{(2\pi)^3 2\varepsilon_{k_i}} 2(\omega \cdot p) d\tau \\
= (2\pi)^3 \int \sum_{\sigma_1 \cdots \sigma_n} \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J \frac{d^3 q_i}{(2\pi)^3 2\varepsilon_{q_i}} \\
= 2(\pi)^3 \int \sum_{\sigma_1 \cdots \sigma_n} \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J 2(\sum_{i=1}^n \varepsilon_{q_i}) 2(\sum_{i=1}^n \varepsilon_{q_i}) \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2\varepsilon_{q_i}} \\
\int \sum_{\sigma_1 \cdots \sigma_n} \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J \Phi_{j_1 \sigma_1 \cdots j_n \sigma_n}^J (\sum_{i=1}^n \varepsilon_{q_i}) 2(\sum_{i=1}^n \varepsilon_{q_i}) \prod_{i=1}^n \frac{d^3 q_i}{(2\pi)^3 2\varepsilon_{q_i}}.
\]
(3.23)

For the state with zero total angular momentum the normalization condition has the form:
\[ \sum_n N_n = 1. \]  
(3.24)

In this case, the two-body contribution to the normalization integral reads, for constituents of equal masses:

\[
N_2 = \frac{1}{(2\pi)^3} \int \psi^2(k_1, k_2, p, \omega \tau) \delta^{(4)}(k_1 + k_2 - p - \omega \tau) \frac{d^3 k_1}{2\varepsilon_{k_1}} \frac{d^3 k_2}{2\varepsilon_{k_2}} 2(\omega \cdot p) d\tau \\
= \frac{1}{(2\pi)^3} \int \psi^2(\vec{k}, \vec{\tau}) \frac{d^3 k}{\varepsilon_k} = \frac{1}{(2\pi)^3} \int \psi^2(\vec{R}, x) \frac{d^2 \vec{R}}{2\pi x(1-x)},
\]  
(3.25)

where \( \psi = \Phi_{j_1 \sigma_1 j_2 \sigma_2}^J (k_1, k_2, p, \omega \tau)|_{J, \lambda=0} \) and we imply in \( (3.27) \) summation over the spin indices of the constituents. To obtain eqs.\( (3.24) \) we used the fact that the first integral is the two-body phase volume:

\[
\int (\cdots) \delta^{(4)}(k_1 + k_2 - p - \omega \tau) \frac{d^3 k_1}{2\varepsilon_{k_1}} \frac{d^3 k_2}{2\varepsilon_{k_2}} 2(\omega \cdot p) d\tau = \int (\cdots) \frac{k d\Omega_k}{8\varepsilon_k} 2(\omega \cdot p) d\tau,
\]
and then use the equality \( M^2 + 2(\omega \cdot p) \tau = (k_1 + k_2)^2 = 4\varepsilon_k^2 \), which gives: \( 2(\omega \cdot p) d\tau = 8kd\tau \).

The change of variables in \( (3.23) \) is made similarly. For the state with \( J = 0 \) the integral \( (3.25) \) and any \( N_n \) do not depend on \( \omega \).

For the wave function of a system with total angular momentum \( J = 1/2 \), the integral \( (3.23) \) does not depend on \( \omega \) as well, since it is impossible to construct any \( \omega \)-dependent terms. At first glance, one could construct the following \( \omega \) dependent structure: \( \tilde{u}'(p) \tilde{\omega} u(p) / \omega \cdot p \) (\( \tilde{u}' \) and \( u \) may correspond to different spin projections). However, since
\[
\tilde{u}'(p) \gamma_\mu u(p) = \frac{p_\mu}{m} \tilde{u}'(p) u(p),
\]
(3.26)
we get: \( \tilde{u}^\lambda \tilde{\omega}^\lambda/\omega \cdot p = \tilde{u}^\lambda u^\lambda/m = 2\delta^\lambda \lambda \), i.e., this structure does not depend on \( \omega \). Hence, any \( N^\lambda \lambda \) is proportional to \( \delta^\lambda \lambda \). For the sum of them we get \( N^\lambda \lambda = A \delta^\lambda \lambda \), and the wave function for a system with \( J = 1/2 \) is normalized by the condition \( A = 1 \). As an example, we give for \( J = 1/2 \) the normalization integral for a \( n \)-body system:

\[
N_n = (2\pi)^3 \int \frac{1}{2} \sum_{\sigma,\sigma_1\ldots\sigma_n} |\Phi^e_n| \delta(4)(k_1 + \cdots + k_n - p - \omega \tau) \prod_{i=1}^n \frac{d^3k_i}{(2\pi)^3 2\varepsilon_k} 2(\omega p) d\tau,
\]

which equals to 1 in the case where the system consists of \( n \) particles only.

We emphasize that the condition (3.20), independent of the light-front plane, is a dynamical property of the state vector provided by the Poincaré group. When \( J \) is different from 0, 1/2, the integral \( N^\lambda \lambda \) depends in general on \( \omega \), though this dependence disappears in the sum (3.22) calculated with all Fock components. For an exact state vector one should still write: \( N^\lambda \lambda = A \delta^\lambda \lambda \), and the normalization condition is reduced to \( A = 1 \), like in the spin 1/2 case. However, for spins higher than 1/2 the \( \omega \)-dependence of \( N^\lambda \lambda \) does not disappear automatically if the state vector is approximated by a limited number of Fock components i.e. if a finite sum is retained in eq. (3.22).

Consider, for example, a system with \( J = 1 \) (the deuteron for instance). Extracting from the Fock components the polarization vector \( e^{(\lambda)}_\mu(p) \) of spin 1 system, we represent \( N^\lambda \lambda \) in the form:

\[
N_n^\lambda \lambda = e^{(\lambda)}_\mu(p) I_n^\mu \nu e^{(\lambda)}_\nu(p).
\]

The general structure of the tensor \( I_n^\mu \nu \) is the following:

\[
I_n^\mu \nu = -A_n g^{\mu \nu} + B_n p^\mu p^\nu + C_n (p^\mu \omega^\nu + p^\nu \omega^\mu) + D_n (p^\mu \omega^\nu - p^\nu \omega^\mu) + E_n \left( \frac{\omega^\mu \omega^\nu}{(\omega \cdot p)^2} + \frac{g_{\mu \nu}}{3 M^2} \right),
\]

where \( A_n, B_n, C_n, D_n, E_n \) are constants, \( M \) is the mass of the composite system. For example, eq. (3.28) has the form in the rest system:

\[
N_n^\lambda \lambda = A_n \delta^\lambda \lambda + E_n M^2 \left( n^\lambda n^\lambda - \frac{1}{3} \delta^\lambda \lambda \right),
\]

where \( n^\lambda \) is a unit vector in the direction of \( \vec{\omega} \). After integration of eq. (3.30) over \( n \), according to (2.38), the irreducible structure \( (n^\lambda n^\lambda - \delta^\lambda \lambda / 3) \) gives zero, and we get the relation \( A_n = N_n \). Using eq. (3.24), we get \( \sum A_n = \sum N_n = 1 \). As it was mentioned above in sect. 2.1.3, the same condition has to be valid without integration over the directions of \( \omega \). This means that the exact Poincaré covariant state vector should provide zero values for the sum of the constants \( C_n, D_n, E_n \) (\( \sum C_n = \sum D_n = \sum E_n = 0 \)), although each of them, or a partial sum of them may not be. While it is not required to do so, the \( \omega \)-dependent structure in eq. (3.30) disappears after averaging over \( n \) (or \( \vec{\omega} \)).

The general normalization condition \( A = 1 \) thus writes, with (3.29):

\[
A = \frac{1}{3} \delta^\lambda \lambda N^\lambda \lambda = -I^\mu \nu \frac{1}{3} \left( g_{\mu \nu} - \frac{p_\mu p_\nu}{M^2} \right) = 1,
\]

(3.31)
where
\[ I^{\mu\nu} = \sum_n I_n^{\mu\nu} \] (3.32)

We emphasize that the normalization (3.31), according to (3.22) and (3.23), contains the sum over the \( n \)-body Fock components for all \( n \). It cannot be generally fulfilled for a two-body component, if the contribution of other components is not negligible. The explicit expression for \( A_2 \) in terms of the deuteron wave function is found in sect.5.1.2.

### 3.1.4 New representation

One can see from (3.7) that the relativistic wave function, in contrast to the non-relativistic one, is transformed in each index by different rotation matrices. It is therefore convenient to use a representation in which the wave function is transformed in each index by one and the same rotation operator \( R(g, \mathcal{P}) \), rotating, according to (3.14), the variables \( \vec{k} \) and \( \vec{n} \). We define the wave function in this new representation as follows:

\[
\psi_{j_1, \sigma_1, j_2, \sigma_2}^{J, \Lambda}(k_1, k_2, p, \omega, \tau) \equiv \sum_{\lambda', \sigma_1', \sigma_2'} D_{\lambda\lambda'}^{(j_1)} \{ R(L^{-1}(\mathcal{P}), p) \} D_{\sigma_1 \sigma_1'}^{(j_1)} \{ R(L^{-1}(\mathcal{P}), k_1) \} \\
\times D_{\sigma_2 \sigma_2'}^{(j_2)} \{ R(L^{-1}(\mathcal{P}), k_2) \} \psi_{j_1, \sigma_1, j_2, \sigma_2}^{J, \Lambda}(k_1, k_2, p, \omega, \tau) ,
\] (3.33)

where, e.g., \( R(L^{-1}(\mathcal{P}), p) \) is given by (3.5) with \( g = L^{-1}(\mathcal{P}) \).

Under the transformation \( g \), the operator \( R(L^{-1}(\mathcal{P}), k_1) \) is factorized as follows [44]:

\[
R \left( L^{-1}(g \mathcal{P}), g k_1 \right) = R(g, \mathcal{P})R(L^{-1}(\mathcal{P}), k_1)R^{-1}(g, k_1) .
\] (3.34)

Using also the property \( D\{R_1R_2\} = D\{R_1\}D\{R_2\} \), we find that the wave function (3.33) is indeed transformed like eq.(3.7), in which the arguments of all the \( D \)-functions are replaced by:

\[
R(g, \mathcal{P}) = L^{-1}(g \mathcal{P})gL(\mathcal{P}) .
\] (3.35)

We thus have:

\[
\psi_{\sigma_1, \sigma_2}^{\lambda}(gk_1, gk_2, gp, g\omega) = \sum_{\lambda', \sigma_1', \sigma_2'} D_{\lambda\lambda'}^{(j_1)} \{ R(g, \mathcal{P}) \} D_{\sigma_1 \sigma_1'}^{(j_1)} \{ R(g, \mathcal{P}) \} \\
\times D_{\sigma_2 \sigma_2'}^{(j_2)} \{ R(g, \mathcal{P}) \} \psi_{\sigma_1', \sigma_2'}^{\lambda}(k_1, k_2, p, \omega, \tau) .
\] (3.36)

The equation (3.36) together with (3.14) shows that in this new representation and in the variables \( k, \vec{n} \) the relativistic wave function transforms exactly as a non-relativistic wave function under a rotation \( R \). This strongly simplifies the spin structure of the relativistic wave function, making it as close as possible to the non-relativistic one. The only difference is the dependence of the wave function on the extra variable \( \vec{n} \). We shall illustrate this dependence in the case of the Wick-Cutkosky model in this chapter, and in the case of the deuteron in chapter 5.
In the particular case of spin 1/2, the matrix $D^{(\frac{1}{2})}_{\sigma\sigma'}$ coming in the transformation (3.33), takes the form (45):

$$D^{\frac{1}{2}}\{R(L^{-1}(\mathcal{P}),k_1}\} = \frac{(k_{10} + m)(\mathcal{P}_0 + \sqrt{\mathcal{P}^2}) - \vec{\sigma} \cdot \vec{P}}{[2(k_{10} + m)(\mathcal{P}_0 + \sqrt{\mathcal{P}^2})(k_{10}\mathcal{P}_0 - \vec{k}_1 \cdot \vec{P} + m\sqrt{\mathcal{P}^2})]^{1/2}}. \quad (3.37)$$

The transformation to the representation (3.33) is similar to the Melosh transformation [16]. We will come back to this point in sect. 3.5.

### 3.2 Equation for the wave function

The equation for the wave function is obtained from the equation for the vertex part shown graphically in fig. 11.

\[\Gamma(k_1, k_2, p, \omega \tau) = \int \Gamma(k_1', k_2', p, \omega \tau') \theta(\omega k_1') \delta(k_1'^2 - m^2) \theta(\omega k_2') \delta(k_2'^2 - m^2) \times \delta^4(k_1' + k_2' - p - \omega \tau') d^4k_1' K(k_1', k_2', \omega \tau'; k_1, k_2, \omega \tau) \frac{d\tau'}{\tau'} - i\epsilon (2\pi)^3. \quad (3.38)\]

The delta-function $\delta^4(\cdots) d^4k_1'$ is kept for convenience. The kernel $K$ is an irreducible block which is calculated directly by the graph technique once the underlying dynamics is known. We should then express the vertex $\Gamma$ through the two-body wave function.

It is the analogue, for a bound state, of the Lippmann-Schwinger equation for a scattering state. Let us first explain its derivation for the case of spinless particles. In accordance with the rules given in sect. 2.2.2, we associate with the diagram of fig. 11 the following analytical expression:

\[\Gamma(k_1, k_2, p, \omega \tau) = \int \Gamma(k_1', k_2', p, \omega \tau') \theta(\omega k_1') \delta(k_1'^2 - m^2) \theta(\omega k_2') \delta(k_2'^2 - m^2) \times \delta^4(k_1' + k_2' - p - \omega \tau') d^4k_1' K(k_1', k_2', \omega \tau'; k_1, k_2, \omega \tau) \frac{d\tau'}{\tau'} - i\epsilon (2\pi)^3. \quad (3.38)\]

The delta-function $\delta^4(\cdots) d^4k_1'$ is kept for convenience. The kernel $K$ is an irreducible block which is calculated directly by the graph technique once the underlying dynamics is known. We should then express the vertex $\Gamma$ through the two-body wave function. This can be done by comparing, for example, two ways of calculating the amplitude for the breakup of a bound state by some perturbation: 1) by means of the graph technique (the result contains $\Gamma$); 2) by calculating the matrix element of the perturbation operator
between the bound state and the free states of \( n \) particles (the result contains \( \Phi \)). We thus get:

\[
\Phi(k_1, k_2, p, \omega \tau) = \frac{\Gamma(k_1, k_2, p, \omega \tau)}{s - M^2},
\]

(3.39)

where \( s = (k_1 + k_2)^2 = (p + \omega \tau)^2 \). The corresponding relation for the \( n \)-body case has the same form. In any practical calculation of the amplitude, we associate \( \Gamma \) with the vertex shown in fig. 10 and then express \( \Gamma \) in terms of \( \phi \) by eq. (3.39). Taking into account that \( s - M^2 = 2(\omega - p) \tau \), and keeping all the spin indices, the equation for the wave function \( \Phi^\lambda_{\sigma_1 \sigma_2} \) has the form [17]:

\[
\left[(k_1 + k_2)^2 - M^2\right] \Phi^\lambda_{\sigma_1 \sigma_2}(k_1, k_2, p, \omega \tau) = -\frac{m^2}{2\pi^3} \sum_{\sigma_1', \sigma_2'} \int \Phi^\lambda_{\sigma_1' \sigma_2'}(k_1', k_2', p, \omega \tau') \times V_{\sigma_1' \sigma_2'}(k_1', k_2', \omega \tau'; k_1, k_2, \omega \tau) \delta^4(k_1' + k_2' - p - \omega \tau') \frac{d^3k_1'}{2\varepsilon_{k_1'}} \frac{d^3k_2'}{2\varepsilon_{k_2'}} 2(\omega - p) d\tau',
\]

(3.40)

where we have defined \( V = -\mathcal{K}/4m^2 \). This equation can be extended in a similar way for the many-body component of the state vector.

For many practical applications, it may be useful to express this equation in terms of the variables \( \vec{k} \) and \( \vec{n} \), i.e. express the wave function in terms of \( \Psi^\lambda_{\sigma_1 \sigma_2}(k_1, k_2, p, \omega \tau) \equiv \Psi^\lambda_{\sigma_1 \sigma_2} (\vec{k}, \vec{n}) \). With the transformation (3.33), one thus gets:

\[
\left[4(\vec{k}^2 + m^2) - M^2\right] \Psi^\lambda_{\sigma_1 \sigma_2}(\vec{k}, \vec{n}) = -\frac{m^2}{2\pi^3} \sum_{\sigma_1', \sigma_2'} \int \Psi^\lambda_{\sigma_1' \sigma_2'}(\vec{k}', \vec{n}) V_{\sigma_1' \sigma_2'}(\vec{k}', \vec{k}, \vec{n}, M^2) \frac{d^3k'}{\varepsilon_{k'}}.
\]

(3.41)

The integration variable \( \vec{k}' \) in eq. (3.41) is defined analogously to \( \vec{k} \) in eq. (3.11):

\[
\vec{k}' = L^{-1}(\mathcal{P}') \vec{k}_1,
\]

(3.42)

where \( \mathcal{P}' = k_1' + k_2' \). The equation (3.41) can also be obtained from (3.40) simply by transcribing it in the system of reference where \( \vec{k}_1 + \vec{k}_2 = 0 \). In this system, the rotation operators in (3.33) are the unit operators, the wave function \( \Phi^\lambda_{\sigma_1 \sigma_2} \) turns into \( \Psi^\lambda_{\sigma_1 \sigma_2} \). However, \( \Psi^\lambda_{\sigma_1 \sigma_2} \) in the integrand differs from \( \Psi^\lambda_{\sigma_1 \sigma_2} \). The wave function \( \Psi^\lambda(\vec{k}', \vec{n}) \) obtains the same form as \( \Psi(\vec{k}, \vec{n}) \) (however, expressed through the variable \( \vec{k}' \)) in the system where \( k'_1 + k'_2 = 0 \), but not in the system where \( k_1 + k_2 = 0 \). This fact is marked by a “prime”. The function \( \Psi^\lambda_{\sigma_1 \sigma_2} \) is obtained from \( \Phi^\lambda_{\sigma_1 \sigma_2}(k_1', k_2', p, \omega \tau') \) by simply expressing the arguments \( k_1', k_2', p, \omega \tau' \) through \( \vec{k}, \vec{k}', \vec{n} \). The relation between these variables has the form [18]:

\[
\vec{k}_1' = \vec{k}' + \frac{\varepsilon_{k}^2 - \varepsilon_{k'}^2}{2\varepsilon_k} \vec{n} + \frac{(\varepsilon_{k'} - \varepsilon_k)^2}{2\varepsilon_k \varepsilon_{k'}} (\vec{n} \cdot \vec{k}') \vec{n}, \quad k_{10}' = \frac{\varepsilon_{k'}^2 - \varepsilon_k^2}{2\varepsilon_k \varepsilon_{k'}} \vec{n} \cdot \vec{k}',
\]

(3.43)

\footnote{The coefficient in the relation (3.39) differs from previous references (see [1]) due to the different normalization coefficient in (3.1) and the different normalization of the amplitudes given by the rules of the graph technique.}
\[ k_{2}' = -k' + \frac{\varepsilon_{k}' - \varepsilon_{k}}{2\varepsilon_{k}} \vec{n}, \quad k_{20}' = \frac{\varepsilon_{k}' + \varepsilon_{k}}{2\varepsilon_{k}} - \frac{\varepsilon_{k}'}{2\varepsilon_{k}} \vec{n}, \quad k_{20}' = \frac{\varepsilon_{k}'}{2\varepsilon_{k}} - \frac{\varepsilon_{k}^2}{4\varepsilon_{k}^2} \vec{n}, \quad (3.44) \]

\[ \bar{p} = -\bar{\omega} = -\frac{4\varepsilon_{k}^2 - M^2}{4\varepsilon_{k}} \vec{n}, \quad p_0 = \frac{4\varepsilon_{k}^2 + M^2}{4\varepsilon_{k}}, \quad (3.45) \]

\[ \bar{\omega} = \frac{4\varepsilon_{k}^2 - M^2}{4\varepsilon_{k}} \vec{n}, \quad \omega_0 = \frac{4\varepsilon_{k}^2 + M^2}{4\varepsilon_{k}}, \quad (3.46) \]

Since the formulae (3.43-3.46) are the Lorentz transformations between the systems with \( \vec{k}_1 + \vec{k}_2 = 0 \) and \( \vec{k}_1' + \vec{k}_2' = 0 \), the function \( \psi_{\sigma_1\sigma_2}^{\lambda}(\vec{k}', \vec{n}) \) can also be obtained from \( \psi_{\sigma_1\sigma_2}^{\lambda}(\vec{k}, \vec{n}) \) by a rotation of spins by means of the \( D \)-functions:

\[ \psi_{\lambda}^{\lambda} = D^{(ij)}\{ R(L^{-1}(P'), k_2') \} \psi_{\lambda}^{\lambda} D^{(ij)}\{ R(L^{-1}(P'), k_1') \}. \]

(3.47)

For spin 1/2 particles, the matrix \( D^{(ij)}\{ R(L^{-1}(P), k) \} \) is given explicitly in eq.(3.37).

In the simple case of a scalar particle, the equation for the wave-function in terms of the variables \( \vec{k}, \vec{n} \) has the following form:

\[ (4(\vec{k}^2 + m^2) - M^2) \psi(\vec{k}, \vec{n}) = -\frac{m^2}{2\pi^2} \int \psi(\vec{k}', \vec{n}) V(\vec{k}', \vec{k}, \vec{n}, M^2) \frac{d^3k'}{\varepsilon_{k'}}. \]

An equation of such a type was also considered in refs. [32, 19, 51, 71, 52, 73].

In the non-relativistic limit, equation (3.48) turns into the Schrödinger equation in momentum space, the kernel \( V \) being the non-relativistic potential in momentum space, and the wave function no longer depends on \( \vec{n} \).

We emphasize that the wave function, which is an equal-time wave function on the light front, turns into the ordinary wave function in the non-relativistic limit where \( c \to \infty \). This reflects the fact that in the non-relativistic limit two simultaneous events in one frame are simultaneous in all other frames.

In the variables \( \vec{R}_\perp \) and \( x \), eq.(3.48) can be rewritten in the form:

\[ \left( \frac{\vec{R}_\perp^2 + m^2}{x(1 - x)} - M^2 \right) \psi(\vec{R}_\perp, x) = -\frac{m^2}{2\pi^3} \int \psi(\vec{R}_\perp', x') V(\vec{R}_\perp', \vec{R}_\perp, x, M^2) \frac{d^3R'_\perp dx'}{2x'(1 - x')} . \]

(3.49)

In this form, this equation is nothing else than the Weinberg equation [3].

The advantages of the equation for the wave function in the form (3.48) compared with (3.49) are its similarity to the non-relativistic Schrödinger equation in momentum space, and its simplicity in the case of particles with spin. These properties make eq.(3.48) very convenient for practical calculations.

The kernel of eq.(3.48) depends on the vector variable \( \vec{n} \). We shall see that this dependence, especially the part which depends on \( M^2 \), as given by eqs.(3.45), is associated with the retardation of the interaction. From this point of view, the dependence of the wave function \( \psi(\vec{k}, \vec{n}) \) on \( \vec{n} \) is a consequence of retardation.

In absence of explicit calculations, a very useful information on the global contribution of many-body Fock components can be obtained by considering the dependence of
the interaction on the total mass \( M \) of the system. The current normalization procedure is then to insert in the norm operator the derivative of the potential with respect to \( M^2 \). In the case of a \( J = 0 \) state made of two spinless particles, the normalization is defined by:

\[
\frac{1}{(2\pi)^3} \int \frac{d^3k}{\varepsilon_k} \frac{d^3k'}{\varepsilon_{k'}} \psi^*(\vec{k}', \vec{n}) \left[ \varepsilon_k \delta(\vec{k} - \vec{k}') - \frac{4m^2}{(2\pi)^3} \frac{\partial V(\vec{k}', \vec{k}, \vec{n}, M^2)}{\partial M^2} \right] \psi(\vec{k}, \vec{n}) = 1 , \tag{3.50}
\]

where the second term accounts for the many-body contribution to the norm, \( \sum_{n>2} N_n \). The extension to states with \( J \neq 0 \) is straightforward. There are many justifications for the introduction of the derivative of the potential \( V \) in eq.(3.50). A general derivation can be done similarly to the normalization condition of the BS function \[54\]. Apart from the fact that it appears in the description of a system of two coupled components in which one component is explicitly retained, it is suggested by the requirement that two-state vectors describing two infinitesimally close states should be orthogonal. Examination of eq.(3.48) then shows that such a property is fulfilled by introducing in the scalar product of the two state vectors the derivative of \( V \) with respect to \( M^2 \) as it is done in (3.50). The extension to the normalization immediately follows from the fact that the orthogonality and the renormalization are defined from the same operator, namely the time component of a conserved current.

The consistency of (3.50) with the general normalization condition (3.22) supposes that the potential \( V \) is determined to all orders in the coupling constant entering the interaction. In practice, this is limited to the exchange of one or two bosons, which may be already enough for the applications considered in the following. Let us just mention here that the examination of a few examples (Wick-Cutkosky model, model-perturbative calculation) shows that the \( \vec{n} \) dependence of the integrand in (3.50) due to the derivative of the potential tends to cancel that of the first term. This is in accordance with the general discussion given above in section 3.1.3.

When the state is described in relativistic quantum mechanics with a fixed number (two) of particles with \( \partial V/\partial M^2 = 0 \), the normalization for \( J = 0 \) is reduced to \( N_2 = 1 \).

### 3.3 Relation with the Bethe-Salpeter function

To find the relation between the light-front wave function and the Bethe-Salpeter function we should start from the integral that restricts the variation of the arguments of the Bethe-Salpeter function to the light-front plane:

\[
I = \int d^4x_1 \ d^4x_2 \ \delta(\omega \cdot x_1) \ \delta(\omega \cdot x_2) \ \Phi(x_1, x_2, p) \exp(ik_1 x_1 + ik_2 x_2) , \tag{3.51}
\]

where \( k_1, k_2 \) are the on-shell momenta: \( k_1^2 = k_2^2 = m^2 \), and \( \Phi(x_1, x_2, p) \) is the Bethe-Salpeter function \([\]\),

\[
\Phi(x_1, x_2, p) = \langle 0 | T(\phi(x_1)\phi(x_2)) | p \rangle . \tag{3.52}
\]
Representing the $\delta$-functions in (3.51) by the integral form
\[
\delta(\omega x_1) = \frac{1}{2\pi} \int \exp(-i\omega x_1 \alpha_1) d\alpha_1, \quad \delta(\omega x_2) = \frac{1}{2\pi} \int \exp(-i\omega x_2 \alpha_2) d\alpha_2,
\]
introducing the Fourier transform of the Bethe-Salpeter function $\Phi(k, p)$,
\[
\Phi(x_1, x_2, p) = (2\pi)^{-3/2} \exp\left[-ip(x_1 + x_2)/2\right] \tilde{\Phi}(x, p), \quad x = x_1 - x_2,
\]
where $l = (l_1 - l_2)/2$, $p = l_1 + l_2$, $l_1$ and $l_2$ are off-mass shell four-vectors, and making the change of variables $\alpha_1 + \alpha_2 = \tau$, $(\alpha_2 - \alpha_1)/2 = \beta$ we obtain:
\[
I = \sqrt{2\pi} \int_{-\infty}^{+\infty} \delta^{(4)}(k_1 + k_2 - p - \omega \tau) d\tau \int_{-\infty}^{+\infty} \Phi(l_1 = k_1 - \omega \tau/2 + \omega \beta, l_2 = k_2 - \omega \tau/2 - \omega \beta) d\beta.
\] (3.54)

On the other hand, the integral (3.51) can be expressed in terms of the two-body light-front wave function. We assume that the light-front plane is the limit of a space-like plane, therefore the operators $\varphi(x_1)$ and $\varphi(x_2)$ commute, and, hence, the symbol of the Heisenberg operators $\varphi(x)$ in (3.52) are identical on the light front $\omega x = 0$ to the Schrödinger operators (just as in the ordinary formulation of field theory the Heisenberg and Schrödinger operators are identical for $t = 0$). The Schrödinger operator $\varphi(x)$ (for the spinless case for simplicity), which for $\omega x = 0$ is the free field operator, is given by (2.51):
\[
\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \left[a(\vec{k}) \exp(-ik \cdot x) + a^\dagger(\vec{k}) \exp(ik \cdot x)\right] \frac{d^3k}{\sqrt{2\varepsilon_k}}.
\] (3.55)

We represent the state vector $|\phi\rangle$ in (3.52) in the form of the expansion (3.1). Since the vacuum state on the light front is always "bare", the creation operator, applied to the vacuum state $|0\rangle$ gives zero, and in the operators $\varphi(x)$ the part containing the annihilation operators only survives. This cuts out the two-body Fock component in the state vector. We thus obtain:
\[
I = \frac{(2\pi)^{3/2}(\omega p)}{2(\omega k_1)(\omega k_2)} \int_{-\infty}^{+\infty} \psi(k_1, k_2, p, \omega \tau) \delta^{(4)}(k_1 + k_2 - p - \omega \tau) d\tau.
\] (3.56)

Comparing (3.54) and (3.56), we find:
\[
\psi(k_1, k_2, p, \omega \tau) = \frac{(\omega k_1)(\omega k_2)}{\pi(\omega p)} \int_{-\infty}^{+\infty} \Phi(k + \beta \omega, p) d\beta,
\] (3.57)

where the argument $p$ in (3.57) is expressed through the on-shell momenta $k_1, k_2$ as $p = k_1 + k_2 - \omega \tau$, in contrast to off-mass shell relation $p = l_1 + l_2$. For the Bethe-Salpeter function $\Phi(l_1, l_2)$ parametrized in terms of the momenta $l_1, l_2$ the formula (3.57) reads:
\[
\psi(k_1, k_2, p, \omega \tau) = \frac{(\omega k_1)(\omega k_2)}{\pi(\omega p)} \int_{-\infty}^{+\infty} \Phi(l_1 = k_1 - \omega \tau/2 + \omega \beta, l_2 = k_2 - \omega \tau/2 - \omega \beta) d\beta.
\] (3.58)
In ordinary LFD, eqs. (3.57) and (3.58) correspond to the integration over $dk_-$. This equation makes the link between the Bethe-Salpeter function $\Phi$ and the wave function $\psi$ defined on the light front specified by $\omega$. It should be noticed however that eq. (3.58) is not necessarily an exact solution of eq. (3.48), since, as a rule, different approximations are made for the Bethe-Salpeter kernel and for the light-front one. In the ladder approximation, for example, the Bethe-Salpeter amplitude contains the box diagram, including the time-ordered diagram with two exchanged particles in the intermediate state, as indicated in fig. 41 in appendix B.3. This contribution is absent in the light-front ladder kernel.

The quasipotential type equations for the light-front wave function derived by restricting arguments of the Bethe-Salpeter amplitude to the light-front plane $z + t = 0$ and corresponding electromagnetic form factors were studied in refs. 55, 56.

### 3.4 Application to the Wick-Cutkosky model

#### 3.4.1 Solution in the covariant formulation of LFD

As a simple, but nevertheless instructive, example, we shall derive in this section the light-front wave function of a system consisting of two scalar particles with mass $m$ interacting through the exchange of a massless scalar particle calculated in the ladder approximation. This is the so-called Wick-Cutkosky model. The diagrams that determine the kernel are shown in fig. 3. The kernel is given by eq. (2.67) with $\mu = 0$. Going over from the kernel $K$ to the potential $V = -K/(4m^2)$, introducing the constant $\alpha = g^2/(16\pi m^2)$, and expressing (2.67) by means of the relations (3.43-3.46) in terms of $\vec{k}, \vec{k}', \vec{n}$, we obtain [57]:

$$V = -\frac{4\pi\alpha}{\bar{K}^2},$$  (3.59)

where

$$\bar{K}^2 = (\vec{k}' - \vec{k})^2 - (\vec{n} \cdot \vec{k}')^2 = (\vec{n} \cdot \vec{k}')^2 + (\vec{n} \cdot \vec{k})^2 - \frac{1}{2} M^2 \left| \frac{\vec{n} \cdot \vec{k}'}{\varepsilon_{k'}} - \frac{\vec{n} \cdot \vec{k}}{\varepsilon_k} \right|.$$  (3.60)

For $k, k' \ll m$, eq. (3.59) turns into the Coulomb potential in momentum space

$$V(\vec{k}', \vec{k}) \simeq -\frac{4\pi\alpha}{(k' - k)^2}. $$  (3.61)

For $\alpha \ll 1$, $|\epsilon_b| = |M - 2m| = ma^2/4 \ll m$, the wave function is concentrated in the non-relativistic region of momenta. The non-relativistic wave function of the ground state in the Coulomb potential has the form:

$$\psi(\vec{k}) = \frac{8\sqrt{\pi m} k^{5/2}}{(k^2 + \kappa^2)^2},$$  (3.62)
where \( \kappa = \sqrt{m|\varepsilon_0|} = m\alpha/2 \) (it is normalized, however, according to (3.25) with \( \varepsilon_k \approx m \)). The integral over \( d^3k' \) in (3.48) is concentrated in the region \( k' \approx \kappa \). Therefore, at \( k \gg \kappa \) the momentum \( \tilde{k}' \) in \( V(k',k,Q) \) can be ignored, and from (3.48) we find:

\[
\psi(k,\mathbf{n}) = -\frac{mV(0,\tilde{k},\mathbf{n},M^2)}{(2\pi)^3(\tilde{k}'^2 + \kappa^2)} \int \psi(\tilde{k}') d\tilde{k}'.
\] (3.63)

Note, that in (3.63) \( \int \psi(\tilde{k}') d^3k' = (2\pi)^3\psi(r = 0) \). The wave function (3.62) is normalized to 1. Substituting in the r.h.s. of eq. (3.63) the expressions (3.59,3.60) for \( V \) and (3.62) for \( \psi \), we obtain

\[
\psi(k,\mathbf{n}) = \frac{8\sqrt{\pi}m\kappa^{5/2}}{(\tilde{k}'^2 + \kappa^2)^2 \left(1 + |\mathbf{n} \cdot \mathbf{k}|/\varepsilon_k\right)}.
\] (3.64)

This relativistic wave function of the ground state with zero total angular momentum is a good approximation of a more exact one in the range \( k > \kappa \). Corrections of order \( \alpha \log(\alpha) \) should be considered in the range \( k < \kappa \) (see [58]). Although the kernel (3.59), (3.60) contains the modulus \( |\mathbf{n} \cdot \mathbf{k}'/\varepsilon_k - \mathbf{n} \cdot \tilde{k}/\varepsilon_k| \), one can show that the solution has no “cusp” at \( \mathbf{n} \cdot \tilde{k} = 0 \). This cusp in (3.64) appears due to our approximations.

One can check on this simple example that it is the retardation of the interaction that is the dynamical reason for the dependence of the wave function on the variable \( \mathbf{n} \). The non-relativistic Coulomb expression for the kernel (3.61) does not contain retardation and does not depend on \( \mathbf{n} \) while the relativistic kernel (3.59) contains retardation and depends on \( \mathbf{n} \). This leads to the dependence of the wave function on the argument \( \mathbf{n} \). It may seem at first sight that the dependence of the wave function on \( \mathbf{n} \) is not due to the contribution of the many-body sectors. Indeed, the coupling of the two-body sector with the other sectors must contain a coupling constant, whereas the parameter that determines the dependence of the wave function on \( \mathbf{n} \) is the nucleon mass and does not contain the coupling constant. This argument should be taken with some care however. It can be seen from eq. (3.48) that the operator \( k^2 + m^2 - M^2/4 \) acting on the wave function is equivalent to the potential acting on this same wave function. It is thus proportional to the coupling constant. Such a result underlies the disappearance of the \( \mathbf{n} \) dependence of the integrand of eq. (3.50) previously mentioned.

We emphasize that retardation leads to both the \( \mathbf{n} \)-dependence and the presence of the carriers of the interaction in the intermediate state, which contribute to the many body sectors. However, these two effects, being important in full measure in a truly relativistic system, can manifest themselves in a different way in weakly bound systems. Neglecting the many-body sectors does not necessarily entails to neglect the \( \mathbf{n} \)-dependence of the wave function at \( k \approx m \). The above mentioned \( \alpha \log(\alpha) \) correction to eq. (3.64) for instance, which originates from the last \( \mathbf{n} \)-dependent term in (3.60), is \( \mathbf{n} \) independent and has no counterpart from the many-body sector. As we shall see also in chapter [4], it is necessary to take into account the \( \mathbf{n} \)-dependence of the wave function even when one restricts to the two-nucleon sector.
The wave function of the 2p state can be found analogously. In the representation (3.33), it has the form [57]:

\[
\psi_\lambda(k, n) = \frac{\sqrt{6} \kappa^{7/2} m^{1/2}}{(k^2 + \frac{1}{4} \kappa^2)^3 \left(1 + \frac{|\vec{n} \cdot \vec{k}|}{\varepsilon_k}\right)^2} \times \left\{ k Y_1(\vec{k}/k) + Y_1(\vec{n}) \left[ \frac{(2 \varepsilon_k - M)^2}{4 \varepsilon_k M} (\vec{n} \cdot \vec{k}) - \frac{(k^2 + \frac{1}{4} \kappa^2)}{2m} \left( \theta(-\vec{n} \cdot \vec{k}) - \theta(\vec{n} \cdot \vec{k}) \right) \right] \right\}.
\]

The wave function corresponding to the angular momentum \( l = 1 \) contains the spherical function \( Y_1 \). This is an illustration of the fact that the vector \( \vec{n} \) participates in the construction of the total angular momentum on the same ground as the relative momentum \( \vec{k} \). The dynamical difference between the solution with \( \vec{k} \parallel \vec{n} \) and \( \vec{k} \perp \vec{n} \) is obviously related to the property that some of the components of the angular momentum \( \vec{J} \), before using the angular condition, depend on the interaction.

### 3.4.2 Solution in the Bethe-Salpeter approach

The exact expression for the Bethe-Salpeter function in the Wick-Cutkosky model is found in the form of the integral representation [59, 54] and, for zero angular momentum, reads:

\[
\Phi(l, p) = -\frac{i}{\sqrt{4\pi}} \int_{-1}^{+1} \frac{g(z, M) dz}{(m^2 - M^2/4 - l^2 - zp - i\epsilon)^3}.
\]

Substituting (3.66) in (3.57), and making the change of variable \( \beta \to \beta/\omega_p \), we find:

\[
\psi = -\frac{ix(1 - x)}{2\pi^{3/2}} \int_{-\infty}^{+\infty} d\beta \int_{-1}^{+1} \frac{g(z, M) dz}{[(k^2 + \kappa^2)(1 - z z_0) + \beta(z_0 - z) - i\epsilon]^3},
\]

where \( z_0 = 1 - 2x, \kappa^2 = m^2 - M^2/4 \).

For fixed \( z \neq z_0 \), the integrand, as a function of \( \beta \), has one pole of third order, and therefore the integral is zero. For \( z = z_0 \), the integrand does not depend on \( \beta \) and the integrand over \( \beta \) diverges. To find the value of the integrand (3.67), we first calculate the integral over \( dz \) from \( z_0 - \epsilon \) to \( z_0 + \epsilon \) at \( \epsilon \to 0 \), retaining only the term proportional to \( \epsilon \), and then over \( d\beta \). As a result, one obtains [30]:

\[
\psi = \frac{g(1 - 2x, M)}{2^5 \sqrt{\pi} x (1 - x)(k^2 + \kappa^2)^2}.
\]

The spectral function \( g(z, M) \) is determined by a differential equation [58, 54] and has no singularity at \( z = 0 \). The exact wave function (3.68) is therefore analytic at \( x = 1/2 \). The approximate explicit solution found in [58] for \( g(x, M) \) has the form:

\[
g(z, M) = 2^6 \pi \sqrt{m\kappa^{5/2}} (1 - |z|).
\]
Inserting \( (3.69) \) in \( (3.66) \) and integrating over \( z \), one can recover the approximate solution of the Bethe-Salpeter equation given by eq.\( (7.2) \). The discontinuity of the spectral function \( g(z, M) \) at \( z = 0 \) is again the result of our approximation, since the solution \( (3.69) \) corresponds to an asymptotically small binding energy. Substituting \( (3.69) \) in \( (3.68) \), we reproduce the expression \( (3.64) \) for the wave function. The discontinuity of \( (3.69) \) at \( z = 0 \) results in the “cusp” of \( (3.64) \) at \( \vec{n} \cdot \vec{k} = 0 \).

### 3.5 Angular momentum and angular condition

The wave function \( (3.65) \) is consistent with the general structure of the light-front wave function of a system with total angular momentum equal to 1 (for spinless constituents):

\[
\psi^\lambda(\vec{k}, \vec{n}) = f_1 Y_{1\lambda}(\vec{k}/k) + f_2 Y_{1\lambda}(\vec{n}) , \tag{3.70}
\]

with \( f_{1,2} = f_{1,2}(\vec{k}^2, \vec{n} \cdot \vec{k}) \). The states with higher angular momentum are constructed similarly, using Clebsch-Gordan coefficients. In this case, it is clear that the angular momentum operator in the representation \( (3.33) \) has the form:

\[
\vec{J} = -i[\vec{k} \times \partial/\partial \vec{k}] - i[\vec{n} \times \partial/\partial \vec{n}] . \tag{3.71}
\]

The same expression for \( \vec{J} \) was found in refs. [32]. It can be shown to be quite general for spinless particles. Acting on the wave function \( (3.64) \) for zero angular momentum and on any function depending on the scalars \( \vec{k}^2 \) and \( \vec{n} \cdot \vec{k} \), the operator \( (3.71) \) gives zero. The generalization for the case of constituents with non-zero spins is evident:

\[
\vec{J} = -i[\vec{k} \times \partial/\partial \vec{k}] - i[\vec{n} \times \partial/\partial \vec{n}] + \vec{s}_1 + \vec{s}_2 , \tag{3.72}
\]

where \( \vec{s}_1, \vec{s}_2 \) are the spin operators of the constituents.

As we explained in sect. 3.1.4, the relative momentum \( \vec{k} \) and the spin operators \( \vec{s}_1, \vec{s}_2 \) have identical transformation properties. In the standard approach, the angular momentum operator obtains the same form after performing the Melosh transformation \([46, 29]\), once the \( \vec{n} \)-dependent term is neglected in \( (3.71) \). In addition, the factor \( \vec{n} \cdot \vec{k} \) in \( f_{1,2}(\vec{k}^2, \vec{n} \cdot \vec{k}) \) turns into \( k_z \). At first glance, the dependence of the wave function on \( k_z \) can be interpreted as a violation of rotational invariance (see ref. [61]). Indeed, this wave function is not eigenfunction of the operator \( \vec{L}^2 \), where \( \vec{L} \) is the usual angular momentum operator:

\[
\vec{L} = -i[\vec{k} \times \partial/\partial \vec{k}] . \tag{3.73}
\]

The rotations changing the position of the light-front plane \( t + z = 0 \) are dynamical, the corresponding angular momentum operator contains the interaction, and does not therefore coincide with eq. \( (3.73) \).

In the covariant formulation, the action of the dynamical angular momentum operator has to coincide with the action of the operator \( (3.71) \). This is provided by the so called angular condition \( (2.17) \) derived in section 2.1.
As explained in section 2.1, the solutions of eq. (3.48) with non-zero angular momentum are always degenerate if one does not apply the angular condition. This condition eliminates this unphysical degeneracy. This can be seen easily in the example of the wave function (3.70).

For this aim, it is convenient to represent (3.70) in the form:

$$\vec{\psi}(\vec{k}, \vec{n}) = f_1 \vec{k} + f_2 \vec{n}.$$  (3.74)

By analogy with eq. (2.39), one can construct the operator:

$$A = (\vec{n} \cdot \vec{J})^2.$$  (3.75)

Due to eq. (3.71), this operator can be represented in the form: $A = (\vec{n} \cdot \vec{L})^2$. Any $\vec{n}$-dependence of the kernel does not violate conservation of the projection of the operator $\vec{L}$ on the direction $\vec{n}$. Hence, the operator $A$ is conserved. As a scalar, it commutes with the angular momentum operator $\vec{J}$: $[\vec{J}, A] = 0$. Therefore, any solution of eq. (3.48) with definite angular momentum is characterized also by the eigenvalues of the operator $A$:

$$A \psi_\alpha = \alpha \psi_\alpha.$$  (3.76)

For $J = 1$, there are two eigenstates with $\alpha = 0$ and $\alpha = 1$ respectively:

$$\vec{\psi}_0 = \vec{n}(\vec{n} \cdot \vec{k}) h_0(\vec{k}^2, \vec{n} \cdot \vec{k}) ,$$  (3.77)

$$\vec{\psi}_1 = [\vec{k} - \vec{n}(\vec{n} \cdot \vec{k})] h_1(\vec{k}^2, \vec{n} \cdot \vec{k}) ,$$  (3.78)

where $h_{0,1}$ are arbitrary scalar functions. In the case where $h_{0,1}$ do not depend on $\vec{n} \cdot \vec{k}$, the states $\vec{\psi}_{0,1}$ are eigenstates of the usual angular momentum operator squared: $\vec{L}^2 \vec{\psi}_{0,1} = 1(1 + 1) \vec{\psi}_{0,1}$. Rewritten in terms of the spherical functions, they still have no definite projection of the usual angular momentum on the $z$-axis. Thus, $\vec{\psi}_0$ is proportional to $\sum_m a_m Y_{1m}(\vec{k}/k)$ (with $a_m = Y_{1m}^*(\vec{n})$) and, hence, is a superposition of states with different projections $m$. In the non-relativistic limit $h_0 = h_1$, and the degeneracy of the states (3.77), (3.78) is nothing but the usual degeneracy relative to projections of the angular momentum.

In the practical case of the approximate two-body equation (3.48), the states (3.77) and (3.78) are not degenerate but correspond to two different energies. Both of them are nonphysical. It is difficult to project explicitly the general angular condition (2.18) on the two-body sector. However, the model solution (3.65), found from the known Bethe-Salpeter function, does not contain any ambiguity, and hence, satisfies the angular condition. It has the form of eq. (3.74), i.e., is a combination of two states with $\alpha = 0$ and $\alpha = 1$ (neglecting at this level of approximation the energy splitting). This example illustrates the result, proved in section 2.1, that the angular condition allows to construct the physical solution from unphysical degenerate solutions. Another example is given in [32]. The angular condition does not eliminate the $\vec{n}$-dependence of the wave function. The angular condition in the ladder approximation was found in [33].
Chapter 4
The nucleon-nucleon potential

The understanding of the $NN$ interaction should ultimately rely on QCD. However, the non-perturbative character of this theory in the low energy range makes this goal quite far away. For many purposes, using effective degrees of freedom such as mesons in the $t$-channel or nucleons and baryon resonances in the $s$-channel in order to describe this interaction will still be relevant, provided it is used in a given domain of energy and momentum. In the present context, this approach is natural because $i)$ the meson exchange picture provides a reasonably good description of the $NN$ interaction and $ii)$ it allows one to consider specific relativistic corrections. The effects due to a deeper understanding of hadronic physics will have to be considered, involving in the simplest case off-shell effects, the effective character of some meson exchanges or of hadronic form factors.

4.1 Mesonic degrees of freedom in nuclei

The relevance of mesonic degrees of freedom in describing the $NN$ interaction stems from the study of peripheral partial waves phase shifts at low energy. These ones are dominated by the exchange of the pion, due to both its low mass and the low energy domain. In related area, dealing for instance with electromagnetic properties of light nuclei, the relevance of the pion degree of freedom relies on meson exchange current contributions to the deuteron electrodisintegration near threshold \cite{62}, where it allows to achieve agreement with experiment at momentum transfers around $Q^2 \sim 0.5 \text{ (Gev/c)^2}$ \cite{63}. As this contribution is not required by a pure phenomenological description of the $NN$ interaction, it is a clear evidence for the role of the pion in the description of nuclear forces. In both cases ($NN$ interaction and deuteron electrodisintegration cross-section), the role of the pion is related to the peculiar nature of this particle within QCD. The pion is the Goldstone boson associated to the spontaneous breaking of chiral symmetry which provides its low mass. The $\pi$-exchange is now a common component of all the models of the $NN$ interaction.

Beyond single pion exchange, one expects contributions due to $2, 3, \ldots$ pion exchanges. They have shorter and shorter ranges, $1 \text{ fm}$ or less, and it becomes more and
more difficult to identify their contribution to the NN interaction as the number of exchanged pions increases. At this point, models differ from each other.

Some models \[64\], following the philosophy of the Reid Soft Core (RSC) potential \[65\], simply parametrize this part in terms of Yukawa potentials, but without precise relation to meson exchanges. As a support for such an approach, one may argue that nucleons have a radius of the order of 0.8 \(fm\) (as far as this radius can be identified to the proton charge radius) and consequently, below a distance of 1.5 \(fm\), nucleons began to overlap, making the description of \(NN\) interaction in this range in terms of exchange of mesons quite doubtful.

A second approach relies on the use of dispersion relations \[66\] to calculate the irreducible \(2\pi\)-exchange contribution to the \(NN\) scattering amplitudes. The interaction potential is obtained by removing the \(2\pi\)-exchange contribution resulting from the iteration of one \(\pi\)-exchange which is automatically generated by the dynamical equation used to calculate the two-body wave function. This allows one to extend the potential to distances as low as 0.8 \(fm\). Notice that the full \(2\pi\)-exchange scattering amplitude has covariance properties that are lost by the separate \(\pi\)-exchange and \(2\pi\)-exchange contribution to the \(NN\) potential. The short range part is largely phenomenological, with the consequence that the Lorentz structure of the \(NN\) amplitude may not be correctly determined, quite similarly to the previous approach. An important point to be noticed is that this approach relies on physical \(\pi N\) scattering amplitudes with the consequence that no form factor is introduced at the \(\pi NN\) \((\pi N\Delta, \ldots)\) vertices. Off-shell effects, which have a shorter range, are incorporated in the phenomenological part. On the other hand, the \(\pi N\) scattering amplitude satisfies properties expected from chiral symmetry, although a pseudoscalar \(\pi NN\) coupling is used. The large contribution from the excitations of \(NN\) pairs (Z-diagrams) to the \(\pi N\) scattering amplitude expected in this case is cancelled here by a contribution involving mainly the \(\Delta\) resonance intermediate state (calculated via dispersion relations). Finally, although it does not incorporate explicit \(\rho\)-exchange contribution, this approach can account for it, through the \(2\pi\) interaction in a P-state in the \(t\)-channel.

A third approach is based on a field-theoretical description of the meson-nucleon interaction, including vertex form factors \[7, 67, 68\]. In the simplest approximation, the \(NN\) interaction results from a sum of contributions involving single meson exchanges: \(\pi\) for the long range part, "\(\sigma\)" giving attraction at intermediate distances (1 \(fm\)) and accounting for the exchange of \(2\pi\) in a S-state, \(\rho\) accounting for the exchange of \(2\pi\) in a P-state, and \(\omega\) for the short-range repulsive part. Other well known mesons such as \(\eta, a_1, \delta\) may also be considered, but are quantitatively less important. While some phenomenology is involved in the vertex form factors or through adjustments of coupling constants, this approach has the great interest to provide a parametrization of the \(NN\) interaction in terms of explicit exchanges of mesons, which are expected to be important, if not dominant, in some channels. The fit to \(NN\) scattering is quite satisfactory and there is no real indication at present that such an approach breaks down, even at short distances where it could fail. Because it keeps the full Lorentz structure attached to the different exchanged mesons, this approach is quite appropriate to the present purpose.
The above approach can be improved by considering two-meson exchanges. This is especially important for a better account of the "$\sigma$" exchange contribution, whose effective character is well known. As in the dispersion relations approach, the iterated one meson (pion in practice) exchange has to be removed from the full $2\pi$-exchange contribution to the $NN$ amplitude. The above contribution also involves $\Delta$ resonances (and other higher resonances) in intermediate states in a quite similar way as Van der Waals forces in atomic physics are generated. The non-local character attached to such contributions is likely to be accounted for only in an approximate way by single meson exchanges (as "$\sigma$" and $\rho$ mesons).

4.2 The non-relativistic $NN$ potential

The non-relativistic $NN$ potential is to be used with the Schrödinger equation. In momentum space, which is more appropriate for an extension to relativistic calculations, it reads:

$$\frac{\vec{k}^2}{m}\psi(\vec{k}) + \int \frac{d^3k'}{(2\pi)^3}\langle \vec{k}|V|\vec{k}'\rangle\psi(\vec{k}') = E\psi(\vec{k}) ,$$

(4.1)

where $\vec{k}$ represents one half of the relative momentum of the two nucleons, $\vec{k} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$, and similarly for $\vec{k}'$. In $r$-space, where many models were developed, the corresponding equation would be:

$$-\frac{\vec{\nabla}^2}{m}\psi(\vec{r}) + \int d^3r'\langle \vec{r}|V|\vec{r}'\rangle\psi(\vec{r}') = E\psi(\vec{r}) .$$

(4.2)

Typically, in a meson exchange theory, $\langle \vec{k}|V|\vec{k}'\rangle$ takes the following form in the simplest case:

$$V(\vec{k},\vec{k}') \equiv \langle \vec{k}|V|\vec{k}'\rangle \propto \frac{g^2}{\mu^2 + (\vec{k} - \vec{k}')^2} ,$$

(4.3)

while in $r$-space, one would get

$$V(\vec{r},\vec{r}') \equiv \langle \vec{r}|V|\vec{r}'\rangle \propto g^2\delta^{(3)}(\vec{r} - \vec{r}')\exp\left(-\frac{\mu r}{4\pi}\right) .$$

(4.4)

where $\vec{r}$ represents the relative distance between the two nucleons, $\vec{r} = \vec{r}_1 - \vec{r}_2$. The same interaction that is non-local in momentum space, eq.(4.3), is local in $r$-space. This explains why the last one is often preferred in practical calculations.

4.2.1 Spin structure of the non-relativistic $NN$ potential

Quite generally, the $NN$ non-relativistic Galilean invariant potential contains 5 terms with different spin structure (assuming that strong interactions conserve isospin). It may
be written in momentum space as:

\[ V(\vec{k}, \vec{k}') = V_C(\vec{k}, \vec{k}') + V_{SS}(\vec{k}, \vec{k}')\vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_{SO}(\vec{k}, \vec{k}')\frac{i}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot [\vec{k} \times \vec{k}'] \]

\[ + V_T(\vec{k}, \vec{k}')\left\{ (\vec{k} - \vec{k}')^2\vec{\sigma}_1 \cdot \vec{\sigma}_2 - 3(\vec{k} - \vec{k}').\vec{\sigma}_1 (\vec{k} - \vec{k}').\vec{\sigma}_2 \right\} \]

\[ + V_{SO2}(\vec{k}, \vec{k}')\vec{\sigma}_1 [\vec{k} \times \vec{k}']\vec{\sigma}_2 [\vec{k} \times \vec{k}'] , \tag{4.5} \]

where \( V_C(\vec{k}, \vec{k}'), V_{SS}(\vec{k}, \vec{k}'), V_{SO}(\vec{k}, \vec{k}'), V_T(\vec{k}, \vec{k}') \) and \( V_{SO2}(\vec{k}, \vec{k}') \) stand respectively for central, spin-spin, spin-orbit, tensor and quadratic spin-orbit forces respectively. They are predominantly scalar functions of \( (\vec{k} - \vec{k}')^2 \) (with further slight dependence on \( \vec{k}^2 \) and \( \vec{k}'^2 \)) and contain isospin independent and dependent terms:

\[ V(\vec{k}, \vec{k}') = V^0(\vec{k}, \vec{k}') + V^1(\vec{k}, \vec{k}')\vec{\tau}_1 \cdot \vec{\tau}_2 . \tag{4.6} \]

Other structures, which have an off-energy shell character, are discarded on the basis that their effect can be re-absorbed in the shorter range part of the interaction model, mostly phenomenological. Such terms have been considered in ref. [69].

In \( r \)-space, and assuming a dependence of \( V(\vec{k}, \vec{k}') \) on \( (\vec{k} - \vec{k}')^2 \) only, the above interaction takes the following form:

\[ V(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}')\left\{ V_C(r) + V_{SS}(r)\vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_{SO}(r)\frac{\vec{\sigma}_1 + \vec{\sigma}_2}{2} \right\} \]

\[ + V_T(r)\left( 3\frac{3\vec{\sigma}_1 \vec{r} \cdot \vec{\sigma}_2 \vec{r} - \vec{\sigma}_1 \cdot \vec{\sigma}_2}{r^2} \right) + V_{SO2}(r)\left( \frac{\vec{\sigma}_1 \vec{l} \cdot \vec{\sigma}_2 \vec{l} + \vec{\sigma}_2 \vec{l} \cdot \vec{\sigma}_1 \vec{l}}{2} + \cdots \right) , \tag{4.7} \]

where the points indicate the existence of other terms involving \( V_{SO2} \), but usually neglected. The potentials \( V_C(r) \) and \( V_{SS}(r) \) can have a linear \( k^2 \) dependent term [68, 70], which, by using an appropriate transformation [70], can be dealt with in coordinate space without too much difficulty. In momentum space, such terms can be kept to any order without further difficulty. These terms have quite different origins. Some simply come from the mathematical structure of Dirac spinors. Other ones come from the dependence on the \( s \)-variable arising from two-meson exchange (\( \sqrt{s} \) is the energy of the system in its c.m. system). This \( s \)-dependence can be turned into a \( k^2 \) dependence if one assumes \( s \approx 4(k^2 + m^2) \). This procedure amounts to neglect off-shell effects that can be incorporated phenomenologically through a fit to experimental data of the short range part of the \( NN \) interaction.

Typically, the effects under consideration are of order \( \vec{k}^2//m^2 \) and look like kinematical relativistic corrections. Knowing their origin is however not sufficient to fix them in the \( NN \) potential. The equation, which the \( NN \) potential has to be associated with, has to be precise in order to have an unambiguous prescription. Thus, for the Schrödinger equation (or equivalent ones such as the Lippmann-Schwinger equation for the \( T \)-matrix for instance), the appropriate definition of \( V(\vec{k}, \vec{k}') \) is:

\[ V_{\sigma_2 \sigma_1}^\prime(\vec{k}, \vec{k}') = \sqrt{\frac{m}{\epsilon_k}} \frac{\bar{u}^{\sigma_1}(\vec{k})O_1u^{\sigma_2}(\vec{k}')}{4m^2(\mu^2 + (\vec{k} - \vec{k}')(2))} \sqrt{\frac{m}{\epsilon_k'}} , \tag{4.8} \]
where the spinors are normalized as $\bar{u}u = 2m$, hence the factor $4m^2$ in the denominator in (4.8), and $\hat{O}$ represents the vertex describing the interaction of nucleons with the mesons of interest. The factors $\sqrt{m/\varepsilon_k}$ and $\sqrt{m/\varepsilon_{k'}}$, that are neither 1, as in the non-relativistic limit nor $m/\varepsilon_k$ and $m/\varepsilon_{k'}$ as one may expect from the normalization factors relative to the two nucleons in the initial and final states, are required to satisfy a unitarity condition [66]. Furthermore, their omission in earlier calculations of the $NN$ potential was responsible for a large energy dependence of the $2\pi$-exchange contribution (obtained by removing the iterated one pion exchange from the full $2\pi$-exchange contribution to the $NN$ scattering amplitude). The presence of the factors $\sqrt{m/\varepsilon_k}$ and $\sqrt{m/\varepsilon_{k'}}$ in (4.8) is important in comparing relativistic calculations to non-relativistic ones. They already include some sizeable relativistic kinematical corrections.

Anticipating a comparison with the light-front equation for the $NN$ system, as given by eq.(3.48) for instance, we can rewrite the Schrödinger equation (4.1) with eq.(4.8) for the potential and $E \equiv k^2/m$, as follows:

$$
(k^2 - k_0^2) \left( \sqrt{\frac{\varepsilon_k}{m}} \bar{\psi}(\vec{k}) \right) = -m^2 \int \frac{d^3k'}{(2\pi)^3\varepsilon_{k'}} \frac{\bar{u}(\vec{k})O_1u(\vec{k}')}{4m^2(\mu^2 + (\vec{k} - \vec{k'})^2)} \left( \sqrt{\frac{\varepsilon_{k'}}{m}} \psi(\vec{k'}) \right).$$

(4.9)

Quite generally, the on-shell $NN$ amplitude, from which the $NN$ potential is derived, can be expressed as a sum of independent invariants. Their number, five, is in relation with the number of independent terms in the non-relativistic limit given by eq.(4.5). Their choice is not unique however. They may be built from the different tensors $(i) \gamma_5(PS)$, $(ii) \gamma_\mu(V)$, $(iii) \gamma_\mu\gamma_5(A)$ and $(iv) \sigma_{\mu\nu}(T)$. Other choices, in closer relation to the $2\pi$-exchange contribution can also be made [66]. They differ by off-shell effects, which may be accounted for in a phenomenological way by the fit of shorter range contributions to experimental data, as already mentioned.

### 4.2.2 Physical inputs

The $NN$ potential due to the exchange of single mesons which we consider here can be calculated from the following Lagrangian densities describing the meson-nucleon couplings:

(i) Pseudoscalar mesons ($\pi$, $\eta$) (PS coupling):

$$
\mathcal{L}^{\text{int}} = i \bar{\psi}\gamma_5\psi \phi^{(ps)};
$$

(4.10)

(ii) Scalar mesons ($\sigma$, $\delta$):

$$
\mathcal{L}^{\text{int}} = g \bar{\psi}\psi\phi^{(s)};
$$

(4.11)

(iii) Vector mesons ($\rho$, $\omega$):

$$
\mathcal{L}^{\text{int}} = \bar{\psi} \left[ g\gamma_\mu \phi^{(v)} + (f/4m)\sigma^{\mu\nu}(\partial_\nu \phi^{(v)} - \partial_\nu \phi^{(v)}) \right] \psi.
$$

(4.12)

\footnote{In the present paper we use the definition of coupling constants corresponding to $g_{\pi NN}^2/4\pi \approx 14$, in contrast to refs. [48, 71, 72], where the definition with $g_{\pi NN}^2 \approx 14$ has been used.}
For simplicity, the isospin degrees of freedom have been omitted.

Other types of coupling may be considered, such as the pseudo-vector one for the \( \pi \) (or \( \eta \)) meson for instance:

\( (iv) \) Pseudoscalar mesons (\( \pi, \eta \)) (PV coupling):

\[
L^{\text{int}} = -\frac{g}{2m} \bar{\psi} \gamma^\mu \gamma_5 \psi \partial_\mu \phi^{(\text{bos})}.
\]

For a transition involving on-mass-shell nucleons described by positive energy spinors, couplings \( (i) \) and \( (iv) \) are equivalent. They strongly differ off-mass shell as it is well known. The first one can give rise to larger effects due to the so-called \( Z \)-diagram it leads to, while the second one does not. The choice of the coupling is deeply connected with the way in which chiral symmetry is chosen to be realized.

As far as nucleons described by positive energy spinors are retained in the description of the NN system, the potential is unique. As will be seen later in section 4.3.1, different potentials can be obtained on the light front depending on the choice of the particular coupling used at the meson-nucleon vertex (like for instance the PS or PV \( \pi NN \) couplings). Using the standard approximation made in the meson propagator consisting in taking \( k_0 - k'_0 = 0 \), neglecting therefore retardation effects, one gets in the NN c.m.s.:

\[
V(k, k') = \sqrt{m \varepsilon_k} \left\{ g_{\pi NN}^2 \frac{[\bar{u}(k)\gamma_5 u(k')]_1 [\bar{u}(-k)\gamma_5 u(-k')]_2}{4m^2 (\mu_\pi^2 + (k - k')^2)} \tau_1 \cdot \tau_2 
- g_{\sigma NN}^2 \frac{[\bar{u}(k)u(k')]_1 [\bar{u}(-k)u(-k')]_2}{4m^2 (\mu_\sigma^2 + (k - k')^2)} 
+ \frac{g_{\omega NN}^2}{4m^2 (\mu_\omega^2 + (k - k')^2)} \bar{u}(k) \left[(g + f)\gamma_\mu - \frac{f}{2m} (k_1 + k'_1)\gamma_\mu\right]_1 u(k') 
\times \bar{u}(-k) \left[(g + f)\gamma_\mu - \frac{f}{2m} (k_2 + k'_2)\gamma_\mu\right]_2 u(-k') \right\} \sqrt{m \varepsilon_{k'}},
\]

where \( k_1, k_2, (k'_1, k'_2) \) are the initial (final) on-shell nucleon momenta. Only a few significant exchanges have been displayed in (4.14). Other ones, \( \eta, \delta \) or \( \rho \) can easily be obtained by removing or inserting the appropriate isospin dependence. Hadronic form factors may also be introduced by giving to the meson-nucleon coupling constants a dependence on the meson four-momentum.

### 4.2.3 Choice of the parametrization

There are various single meson exchange models in the literature. They are not all equivalent, with the consequence that they should be used within a defined scheme. One of the most ambitious model is the field theory motivated Bonn model (Bonn-E) \[7\]. Its energy dependence, which has a relationship with retardation effects included here in the kernel of the interaction, has prompted the authors to derive energy independent models (now denoted QA and RA) that can be used more easily. The emphasis on reproducing the
deuteron D state properties of the original model has biased this derivation. The model is unable to account for the observable mixing angle $\epsilon_1$ beyond 100 MeV. These drawbacks have been corrected in later versions (QB, QC, RB) [67] by relaxing the constraints on the deuteron D state properties that are known to be model dependent [63].

Other models differ by the values of the coupling constants, including the associated form factors. This is expected for short-range contributions which have an effective character anyway, but this also occurs for the $\pi NN$ coupling constant, which may be sensitive to isospin breaking effects [68]. The last one is important for a fine description of the $NN$ interaction but unimportant in the present review. The difference with Bonn QB and QC models [67], which motivation was shortly explained above, involves changes in both the $\pi NN$ coupling and those determining short range contributions.

The difference between momentum and coordinate space models has conceptually another origin. Working in one or the other should indeed be equivalent. In practice however, this is true as far as the linearly $k^2$-dependent terms, which are tractable in $r$-space models, represent an accurate description of the full momentum dependence of the $NN$ interaction model. Recent developments, dealing with the well known $\pi$-exchange, show that off-shell effects involving fourth order terms in the nucleon momentum are relevant for describing the $NN$ interaction in coordinate space [64]. Their neglect explains a large part of the differences in the bare predictions obtained from the Bonn QB [67] and Paris [70] models, which reproduce the same $NN$ scattering data otherwise. In particular, the deuteron D-state probability of coordinate space models is expected to be systematically between 0.5 % and 1 % higher than in momentum space models.

Other differences may be mentioned. The genuine term, $\vec{\sigma}_1 \cdot (\vec{k} \times \vec{k}^\prime) \cdot \vec{\sigma}_2 \cdot (\vec{k} \times \vec{k}^\prime)$, in eq.(4.5) is approximately replaced by the term $\vec{\sigma}_1 \cdot \vec{l} \cdot \vec{\sigma}_2 \cdot \vec{l} + \vec{\sigma}_2 \cdot \vec{l} \cdot \vec{\sigma}_1 \cdot \vec{l}$ in coordinate space models. This can introduce some bias in fitting the potential parameters to $NN$ scattering data. The full $2\pi$-exchange contribution considered in the Paris model does not seem to bring significant improvement upon a single meson exchange model, probably because these ones contain enough parameters (coupling constants together with vertex form factors) to accurately account for these $NN$ data. This is indirectly confirmed by the Argonne model [64] which also does well with these data although it does not incorporate $k^2$ dependent terms as in the Bonn R, Paris or Nijmegen models and, on the other hand, fit the deuteron quadrupole moment to the observed one without any regards to the contribution of meson exchange currents.

Altogether, it seems that $NN$ scattering data put strong constraints on the $NN$ interaction models and within a given scheme do not allow for much freedom. Differences are closely related to the scheme under consideration and, most often, involve terms of order $(\vec{k}/m)^2$, that are of relativistic origin. For some part, the present models may be unitary equivalent, implying the existence of corrections pertinent to the model under consideration for any calculation of the physical observables. The choice of some particular model depends on a prejudice [63], models that keep track of these theoretically motivated $(\vec{k}/m)^2$ terms should deserve a special attention however.
4.3 Meson exchange interaction on the light front

As already mentioned, the approach to the non-relativistic $NN$ interaction is semi-phe-
nomenological. This means that the meson-nucleon field theory is used to obtain a form
of the $NN$ interaction kernel (OBEP, as a rule), and the parameters of this kernel are
found from fitting the experimental data in the framework of the Schrödinger equation.

A similar approach can be based on the relativistic quantum mechanics in the front
form. Besides the Hamiltonian, one should introduce the interaction in all the generators
of the Poincaré group, which change the orientation of the light-front plane [2].

This approach has been developed in a number of studies (see, in particular, [26,
27, 75, 76, 77, 78, 79, 80]). We do not describe them here in details (see [28, 29] for a
review). One should mention that in this approach the full program of fitting the phase
shifts and the relativistic kernel has not yet been realized.

The kernel in these references is considered as purely phenomenological, i.e. it is
not derived from light-front field theory, even in the framework of OBEP. It is assumed
that it does not depend on the orientation of the light-front plane. However, if one
uses the field theory to construct the off-energy shell light-front kernel, this one always
gets a dependence on the light front through the four-vector $\omega$ or its spatial part $\vec{n}$ (see
examples in sect. 4.3.1). It seems reasonable therefore to use this kernel as an input in the
light-front version of relativistic quantum mechanics. This cannot be done by a simple
replacement of the $\vec{n}$-independent kernel in the above versions of the relativistic quantum
mechanics by the $\vec{n}$-dependent one. This would destroy the commutation relations, found
for the $\vec{n}$-independent mass operator. In order to restore the covariance, one should find
a solution of the Poincaré group algebra with this new $\vec{n}$-dependent mass operator. Due
to covariance, the exact on-shell amplitudes in this approach will not depend on $\omega$ (with
the accuracy of numerical calculations).

It is quite probable that in both phenomenological light-front approaches (with the
$\omega$ - independent and dependent kernels), in spite of the completely different behavior of
the off-shell amplitudes, the phase shifts and the binding energies can be equally fitted, at
least in the two-body problem. This implies that there exists an unitary transformation
between corresponding sets of the Poincaré generators. However, this does not imply
their practical equivalence, since this unitary operator has to be dynamical and highly
complicated, since it has to change the off-shell behaviour of all the partial amplitudes.
It hardly can be useful in practice.

It seems therefore reasonable to develop a phenomenological scheme inspired from
field theory and to start with the OBEP kernel in the light-front relativistic approach. This
way seems rather promising, since it reconciles the main features of the field-theoretical
kernel on the light front with relativistic covariance. It has not yet been developed in its
full form.
4.3.1 OBEP on the light front

The contribution due to single meson exchange, which is shown in fig. [5], can be obtained from the meson-nucleon interaction described by the Lagrangians given by eqs. (4.10)-(4.13). The analytical expression for the pion exchange in the pseudo-scalar coupling, for instance, is a straightforward generalization of eq. (2.67). It has the form:

\[ V_{\sigma_2 \sigma_1}^{\sigma_2} = \frac{g_{\pi NN}^2}{4m^2} \tau_1 \tau_2 \left\{ \frac{\theta(\omega(k'_1 - k_1))}{\mu^2 - (k'_1 - k_1)^2 + 2\tau\omega(k'_1 - k_1) - i\epsilon} F^2 \right. \]

\[ + \frac{\theta(\omega(k_1 - k'_1))}{\mu^2 - (k_1 - k'_1)^2 + 2\tau\omega(k_1 - k'_1) - i\epsilon} F^2 \]

\[ \times \left[ \bar{u}^\sigma(k_2)\gamma_5 u^\sigma(k'_2) \right] \left[ \bar{u}^\sigma(k_1)\gamma_5 u^\sigma(k'_1) \right], \quad (4.15) \]

where \( F \) is a phenomenological form factor, assumed to depend on the same argument as the denominator of the kernel. Here and below it is introduced "by hand".

The \( \eta \) exchange contribution can be obtained from (4.15) by removing the isospin factor and replacing \( g_{\pi NN}^2 \) by \( g_{\eta NN}^2 \). The contributions due to \( \sigma \) and \( \delta \)-exchanges are obtained from the previous ones by removing the \( \gamma_5 \) matrices and changing the overall sign. As for the vector meson exchange, it contains a peculiarity because of the contact terms due to the vector meson propagator and the coupling with derivative. We therefore explain this point in more details. The starting expression for the vector meson exchange contribution, obtained by applying the rules of the graph technique to the diagram of fig. [3], has the form:

\[ \mathcal{K} = \int \bar{u}(k_1) \left[ g_\gamma^\alpha + \frac{f}{2m} \sigma^{\alpha\alpha}(i)(k - \omega \tau_1)_\alpha' \right] u(k'_1) \]

\[ \times \left[ -g_{\alpha\beta} + \frac{(k - \omega \tau_1)_\alpha(k - \omega \tau_1)_\beta}{\mu^2} \right] \]

\[ \times \bar{u}(k_2) \left[ g_\gamma^\beta + \frac{f}{2m} \sigma^{\beta\beta}(-i)(k - \omega \tau_1)_\beta' \right] u(k'_2) \]

\[ \times \delta \left[ (k'_1 - k_1 + \omega \tau_1 - \omega \tau')^2 - \mu^2 \right] \theta(\omega(k'_1 - k_1)) \frac{d\tau_1}{\tau_1 - i\epsilon} \]

\[ + \frac{1}{(2\pi)^2} \int \bar{u}(k_1) \left[ g_\gamma^\alpha + \frac{f}{2m} \sigma^{\alpha\alpha}(-i)(k - \omega \tau_1)_\alpha' \right] u(k'_1) \]

\[ \times \left[ -g_{\alpha\beta} + \frac{(k - \omega \tau_1)_\alpha(k - \omega \tau_1)_\beta}{\mu^2} \right] \]

\[ \times \bar{u}(k_2) \left[ g_\gamma^\beta + \frac{f}{2m} \sigma^{\beta\beta}(i)(k - \omega \tau_1)_\beta' \right] u(k'_2) \]

\[ \times \delta \left[ (k_1 - k'_1 + \omega \tau_1 - \omega \tau)^2 - \mu^2 \right] \theta(\omega'(k_1 - k'_1)) \frac{d\tau_1}{\tau_1 - i\epsilon}, \quad (4.16) \]

where \( k \) is the meson momentum. The factors \((i)\) and \((-i)\) at \((k - \omega \tau_1)\) are associated with outgoing and incoming lines, according to sect. 2.2.4. The factor \( \omega \tau_1 \), everywhere in
the difference \((k - \omega \gamma)\) in the expression (1.19), incorporates the contact terms for both the vector meson exchange and for the coupling with derivative.

After integrating over \(\tau\) and introducing the form factor \(F\), we get for the potential \(V = -K/4m^2\):

\[
V_{\sigma_2 \sigma_1}^{\sigma_2' \sigma_1'} = \frac{F^2}{4m^2(K^2 + \mu^2)} \bar{u}^{\sigma_1}(k_1) \left\{ (g + f)\gamma_\alpha - \frac{f}{2m}(k_1 + k_1')_\alpha \right. \\
+ \frac{i f}{2m} \sigma_\alpha' \omega_\alpha [\tau \theta(\omega(k_1 - k_1')) - \tau' \theta(\omega(k_1' - k_1))]) \right\} u^{\sigma_1'}(k_1') \\
\times \bar{u}^{\sigma_2}(k_2) \left\{ (g + f)\gamma_\alpha - \frac{f}{2m}(k_2 + k_2')_\alpha \right. \\
+ \frac{i f}{2m} \sigma_\alpha' \omega_\alpha [\tau \theta(\omega(k_1' - k_1)) - \tau' \theta(\omega(k_1 - k_1'))]) \right\} u^{\sigma_2'}(k_2') \\
- \frac{g^2}{4m^2\mu^2(K^2 + \mu^2)} \left\{ \bar{u}^{\sigma_1}(k_1) \omega u^{\sigma_1'}(k_1') \right\} \left\{ \bar{u}^{\sigma_2}(k_2) \omega u^{\sigma_2'}(k_2') \right\} \tau \tau', \quad (4.17)
\]

where \(K^2\) is given by eq.(3.60).

The contribution of the isovector exchange (\(\rho\)-meson) is obtained by multiplying (4.17) by the factor \(\tilde{\tau}_1 \tilde{\tau}_2\). The terms in (4.17) proportional to \(\tau\), \(\tau'\) and \(\tau \tau'\) were omitted in ref. [48].

For reasons already explained, the pseudovector \(\pi NN\) coupling is sometimes preferred to the pseudoscalar one. To illustrate the treatment of terms corresponding to derivative couplings of the meson field, we just give the result here:

\[
V_{\sigma_2 \sigma_1}^{\sigma_2' \sigma_1'} = \frac{g^2_{\pi NN}}{4m^2} \left\{ \frac{\theta(\omega(k_1' - k_1)) F^2}{\mu^2 - (k_1' - k_1)^2 + 2\tau \omega(k_1' - k_1) - i\epsilon} \right\} \\
\times \left\{ \bar{u}^{\sigma_2}(k_2) \left( \gamma_5 - \frac{\omega_5 \gamma_5}{2m} \right) u^{\sigma_2'}(k_2') \right\} \left\{ \bar{u}^{\sigma_1}(k_1) \left( \gamma_5 + \frac{\omega_5 \gamma_5}{2m} \right) u^{\sigma_1'}(k_1') \right\} \\
+ \frac{\theta(\omega(k_1 - k_1')) F^2}{\mu^2 - (k_1 - k_1')^2 + 2\tau \omega(k_1 - k_1') - i\epsilon} \right\} \\
\times \left\{ \bar{u}^{\sigma_2}(k_2) \left( \gamma_5 + \frac{\omega_5 \gamma_5}{2m} \right) u^{\sigma_2'}(k_2') \right\} \left\{ \bar{u}^{\sigma_1}(k_1) \left( \gamma_5 - \frac{\omega_5 \gamma_5}{2m} \right) u^{\sigma_1'}(k_1') \right\} \tilde{\tau}_1 \tilde{\tau}_2, \quad (4.18)
\]

The comparison with the expression obtained with the pseudo-scalar \(\pi NN\) coupling evidences differences which depend on the four-vector \(\omega_\mu\) and vanish in the limit of on-energy shell nucleons \((\tau = \tau' = 0)\).

It is also interesting to compare the interaction on the light front, eq.(4.13), with the Bonn-Q model (4.14), which contains some relativity. Differences involve again terms depending on the four-vector \(\omega_\mu\), which vanish on-energy shell. Further differences, due to the time component of the four-momentum of the meson in its propagator (retardation
effects), vanish on energy shell. The difference in the normalization factors $\sqrt{m/\varepsilon_k}$ and $\sqrt{m/\varepsilon'_k}$, discussed in sect. 4.2.1, is related to the particular equation for the wave function.

4.3.2 Beyond OBEP on the light front

The interaction due to single meson exchanges can produce contributions depending on the arbitrary four vector $\omega$. While this dependence vanishes for the Born amplitude (which is calculated for on-shell particles), there is no guarantee that this property holds to all orders. This implies an asymptotic scattering behavior, and therefore a scattering amplitude which depends on $\omega$. A quite similar problem arises in other approaches based on a description referring to the plane determined by $\sigma = \lambda \varepsilon$ with the particular choice $\lambda = (1, 0, 0, 0)$, but here the difficulty is more striking as results may depend on the arbitrary orientation of the three-vector $\vec{\omega}(\vec{n})$ while physical ones should not. For example, the relativistic wave function originating from the $\pi$-exchange, calculated in a perturbative way in the first $1/m$ order, reads \[72\]:

\[
\delta \psi^{(1)}_{\pi}(\vec{k}, \vec{n}) = -\frac{g_{\pi NN}^2}{8m^2} \vec{n}_1 \cdot \vec{n}_2 \frac{m}{k^2 + \kappa^2} \int \frac{d^3 k'}{(2\pi)^3} \frac{m^2}{\mu^2 + (k - k')^2} \left[ (\vec{k} - \vec{k}') \times \vec{n} \right] \psi^{(0)}(k') .
\]

(4.19)

The quantity $\kappa^2$ is positive in the case of a bound state such as the deuteron, and it is negative for a scattering state.

The defect of the wave function (4.19) is that it has a $\vec{n}$-dependent residue at the pole at $k = i\kappa$, that can contribute to an observable amplitude. A similar defect in the scattering state wave function appears in the dependence of asymptotical states on $\vec{n}$, i.e., in the on-energy-shell amplitude, in contradiction with the results of sect. 2.2.

Knowing that the sum of the contributions of all irreducible diagrams to all orders should reproduce results obtained from Feynman diagrams, which are independent of $\omega$, one can guess that higher order terms in the interaction are essential for removing the above unpleasant feature. In this section, we explore their contributions and the role they may play. As will be shown below, the defect of the wave function (4.19) is corrected by incorporating the contact term in the $NN$ kernel [71].

Fig.12. Irreducible diagram with contact term.
To give some insight about the relevance of these extra terms in the interaction, we will concentrate on a selected one, where the contact term discussed for the spin 1/2 particle (eq. (2.60)) is operating (see fig. 12). This will be done by retaining the relevant lowest 1/m order terms, and for contributions in relation with the single π-exchange. At this order, only σ- and ω-exchanges contribute, so in fig. 12 one wavy line corresponds to π-exchange and the other wavy line corresponds to σ and ω. The corresponding contribution to the kernel has the form:

\[
\delta V^{\pi,\sigma+\omega}(\vec{k}, \vec{k}''; \vec{n}) \equiv \delta V_{1}^{\pi,\sigma+\omega} + \delta V_{2}^{\pi,\sigma+\omega} = -\frac{g_{\pi NN}^{2}}{8m^{2}} \vec{r}_{1} \cdot \vec{r}_{2} [\vec{\sigma}_{1} \times \vec{\sigma}_{2}] \tag{4.20}
\]

\[
\times \int \frac{d^{3}k'}{(2\pi)^{3}} \left[ \frac{[(\vec{k} - \vec{k}') \times \vec{n}]}{\mu^{2} + (\vec{k} - \vec{k}')^{2}} V^{\pi+\omega}(\vec{k}' - \vec{k}'') - V^{\sigma+\omega}(\vec{k} - \vec{k}'') \right] \frac{[(\vec{k}' - \vec{k}'') \times \vec{n}]}{\mu^{2} + (\vec{k}' - \vec{k}'')^{2}},
\]

where \( V^{\pi+\omega} \) represents the dominant contribution to the interaction at zeroth order in \( 1/m \).

The insertion of \( \delta V^{\pi,\sigma+\omega} \) in an equation such as (3.41) provides an extra, \( \vec{n} \)-dependent, contribution to the wave function. In a perturbative calculation it is given by:

\[
\delta \psi_{\pi,\sigma+\omega}(\vec{k}, \vec{n}) \equiv \delta \psi_{1,\pi,\sigma+\omega}(\vec{k}, \vec{n}) + \delta \psi_{2,\pi,\sigma+\omega}(\vec{k}, \vec{n})
\]

\[
= -\frac{m}{k^{2} + \kappa^{2}} \int \frac{d^{3}k''}{(2\pi)^{3}} \left[ \delta V_{1}^{\pi,\sigma+\omega}(\vec{k}, \vec{k}'', \vec{n}) + \delta V_{2}^{\pi,\sigma+\omega}(\vec{k}, \vec{k}'', \vec{n}) \right] \psi^{(0)}(\vec{k}'') \tag{4.21}
\]

where \( \psi^{(0)}(\vec{k}'') \) is a zeroth-order non-relativistic wave function.

One should also take into account all the iterations of the above contributions with the non-relativistic potential \( V^{\pi+\omega} \), since these iterations remain in the first 1/m order. They are represented as

\[
\delta \psi_{3,\pi,\sigma+\omega}(\vec{k}, \vec{n}) = -\frac{m}{k^{2} + \kappa^{2}} \int \frac{d^{3}k'}{(2\pi)^{3}} V^{\pi+\omega}(\vec{k} - \vec{k}') \left[ \delta \psi_{1}^{(1)}(\vec{k}', \vec{n}) + \delta \psi_{1,\pi,\sigma+\omega}(\vec{k}', \vec{n}) \right]
\]

\[
+ \delta \psi_{2,\pi,\sigma+\omega}(\vec{k}', \vec{n}) + \delta \psi_{3,\pi,\sigma+\omega}(\vec{k}', \vec{n}) \right], \tag{4.22}
\]

where the integrand incorporates the function \( \delta \psi_{3}^{\pi,\sigma+\omega} \) itself. Therefore eq.(4.22) is an inhomogeneous equation for \( \delta \psi_{3}^{\pi,\sigma+\omega} \).

It turns out (see below) that \( \delta \psi_{2}^{\pi,\sigma+\omega} + \delta \psi_{3}^{\pi,\sigma+\omega} = 0 \). Hence, the full correction to the wave function reads:

\[
\delta \psi^{\pi}(\vec{k}, \vec{n}) = \delta \psi_{1}^{(1)} + \delta \psi_{1,\pi,\sigma+\omega}. \tag{4.23}
\]

It can be written as:

\[
\delta \psi^{\pi}(\vec{k}, \vec{n}) = \frac{g_{\pi NN}^{2}}{8m^{2}} \vec{r}_{1} \cdot \vec{r}_{2} \left[ \frac{m}{k^{2} + \kappa^{2}} [\vec{\sigma}_{1} \times \vec{\sigma}_{2}] \int \frac{d^{3}k'}{(2\pi)^{3}} \frac{[(\vec{k} - \vec{k}') \times \vec{n}]}{\mu^{2} + (\vec{k} - \vec{k}')^{2}} \right.
\]

\[
\times \left[ \frac{\vec{k}^{2}}{m} \psi_{0}(\vec{k}') - \int d^{3}k'' \left( \frac{\vec{k}^{2}}{m} \delta^{(3)}(\vec{k}' - \vec{k}'') + \frac{1}{(2\pi)^{3}} V^{\sigma+\omega}(\vec{k}' - \vec{k}'') \right) \psi_{0}(\vec{k}'') \right]. \tag{4.24}
\]
Applying eq.(3.41) at the lowest order considered here allows to replace the integral in the last bracket by $-\kappa^2 m \psi_0(\vec{k}')$. It is then easily seen that this term adds to the first one in the bracket to cancel the front factor $\frac{m}{k^2 + \kappa^2}$ at the r.h.s. of (4.24). One thus gets:

$$\delta \psi^\pi(\vec{k}, \vec{n}) = -\frac{g_2^2 N N}{8m^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \int \frac{d^3 k'}{(2\pi)^3} \frac{[\vec{\sigma}_1 \times \vec{\sigma}_2] \cdot [\vec{m} \times \vec{n}]}{\mu^2 + (\vec{k} - \vec{k}')^2} \psi_0(\vec{k}') . \quad (4.25)$$

The pole in eq.(1.19) at $k = i\kappa$ ($k = k_0$ in continuous spectrum case) has now disappeared. This result is important since it guarantees that the observables, such as asymptotic normalization or phase shifts, do not depend on the arbitrary direction $\vec{n}$, though the wave function 4.25 depends on it.

In $r$-space eq.(4.25) takes a simpler form:

$$\delta \psi^\pi(\vec{r}, \vec{n}) = i \frac{g_2^2 N N}{8m^2} \vec{\tau}_1 \cdot \vec{\tau}_2 \left[ \vec{\nabla}_r \left( \frac{\exp(-\mu r)}{r} \right) \times \vec{n} \right] \psi_0(\vec{r}) , \quad (4.26)$$

which evidences a radial part in close correspondence with that appearing in the pair term contribution to meson exchange currents or also in relativistic components in other approaches.

To calculate the contribution $\delta \psi^\pi_{\sigma,\omega}$ one has to substitute for $\delta \psi^{(1)}_2(\vec{k}, \vec{n}) + \delta \psi^\pi_{\sigma,\omega}$ in the integrand in eq.(4.22) the wave function (4.25). The result of this substitution coincides with $-\delta \psi^\pi_{\sigma,\omega}$. Hence, the equation (4.22) turns into

$$\delta \psi^\pi_{\sigma,\omega}(\vec{k}, \vec{n}) + \delta \psi^\pi_{\sigma,\omega}(\vec{k}, \vec{n}) =$$

$$-\frac{m}{k^2 + \kappa^2} \int \frac{d^3 k'}{(2\pi)^3} V^\sigma + \omega(\vec{k} - \vec{k}') \left[ \delta \psi^\pi_{\sigma,\omega}(\vec{k}', \vec{n}) + \delta \psi^\pi_{\sigma,\omega}(\vec{k}', \vec{n}) \right] . \quad (4.27)$$

This is the homogeneous equation relative to $\delta \psi^\pi_{\sigma,\omega} + \delta \psi^\pi_{\sigma,\omega}$. However it does not determine any eigenvalue, since the energy $E$ in $\kappa^2 = -mE$ is already fixed by the equation for the complete wave function. Therefore its solution is zero:

$$\delta \psi^\pi_{\sigma,\omega}(\vec{k}, \vec{n}) + \delta \psi^\pi_{\sigma,\omega}(\vec{k}, \vec{n}) = 0 . \quad (4.28)$$

The significance of this result may be obtained from a comparison with other relativistic approaches [12]. In these approaches, a term like (4.20) is associated with a pair of nucleons with positive and negative energies. Their interaction due to $\sigma$ and $\omega$ exchanges produces a correction proportional to $V^\sigma - V^\omega$ (instead of $V^\sigma + V^\omega$ as in eq.(4.22)). The second term in eq.(1.20) has therefore the effect to remove the contribution that should be absent in any case.

In this section, we have considered some of the second order $\vec{n}$ dependent contributions to the irreducible interaction $V$. This has been done to lowest $1/m$ order. These contributions which are the first ones of an infinite series seem to play an important role to make the theory consistent, accounting in particular for the $\vec{n}$ independence of the asymptotic wave function, which are related to physical properties. They also show
that solving the equation of motion without using the appropriate interaction may lead to wrong results. This consideration makes the relationship with other relativistic approaches closer. The present study is, to a large extent, exploratory. It suggests that the interaction should fulfill some constraints whose determination may be helpful for further work.
Chapter 5

Applications to the nucleon-nucleon system

As a direct application of the formalism developed above, we consider in this chapter the two nucleon system. Since its non-relativistic phenomenology has been developed over more than twenty years, it allows a sensible discussion of the new features brought about by relativity.

5.1 The deuteron wave function

5.1.1 Spin structure

The construction of the deuteron wave function is a nice illustration of how to construct a state with non-zero angular momentum in the covariant formulation of LFD. For this nucleus, \( J^p = 1^+ \) (pseudovector particle), the decomposition of the wave function \( \Phi_{\sigma_2 \sigma_1}^{\lambda} \) in independent spin structures has the general form [48]:

\[
\Phi_{\sigma_2 \sigma_1}^{\lambda}(k_1, k_2, p, \omega \tau) = \sqrt{m} e_\mu^\lambda(p) \bar{u}^{\sigma_2}(k_2) \phi_\mu U_c \tilde{u}^{\sigma_1}(k_1),
\]

(5.1)

with

\[
\phi_\mu = \varphi_1 \frac{(k_1 - k_2)_\mu}{2m^2} + \varphi_2 \frac{1}{m} \gamma_\mu + \varphi_3 \frac{\omega_\mu}{\omega \cdot p} + \varphi_4 \frac{(k_1 - k_2)_\mu \omega}{2m \omega \cdot p} + \varphi_5 \frac{i}{2m^2 \omega \cdot p} \gamma_5 \epsilon_{\mu \nu \rho \gamma} (k_1 + k_2)_\nu (k_1 - k_2)_\rho \omega_\gamma + \varphi_6 \frac{m \omega_\mu \omega}{(\omega \cdot p)^2},
\]

(5.2)

and \( m \) is the nucleon mass. The light-front deuteron wave function is determined by six invariant functions \( \varphi_{1-6} \), depending on two scalar variables, e.g. \( s = (k_1 + k_2)^2 \) and \( t = (p - k_1)^2 \). The extra spin structures in front of \( \varphi_{3-6} \) are constructed by means of the four-vector \( \omega \). Other possible structures (e.g., \( e_\mu \pi_2 \omega \gamma_\mu U_c \pi_1 \), etc.) are expressed through the six structures given by (5.2). The wave function (5.1) is kinematically transformed under rotations and Lorentz transformations \( g \) of its arguments according to eq. (3.7).
The transformation to the new representation (5.3) has the form:

\[
\Psi^\lambda_{\sigma_2\sigma_1}(k_1, k_2, p, \omega \tau) = \sum_{\lambda'\sigma_1'\sigma_2'} \frac{1}{V} D_{\lambda\lambda'}^{(1)} \{ R(L^{-1}(P), p) \} D_{\sigma_1\sigma_1'}^{(2)} \{ R(L^{-1}(P), k_1) \} \times D_{\sigma_2\sigma_2'}^{(2)} \{ R(L^{-1}(P), k_2) \} \Phi_{\sigma_2\sigma_1'}(k_1, k_2, p, \omega \tau),
\]

(5.3)

where, e.g., \( R(L^{-1}(P), p) \) is given by (3.3), in which \( g = L^{-1}(P) \), and \( P \) is given by (3.13). The matrix \( D_{\sigma_2\sigma_1'}^{(2)} \) in (5.3) is defined by eq. (3.37).

The transformation law of the wave function (5.3) is given by eq. (3.36). In this new representation given by (5.3), the rotation operator is the same for all the spin indices and coincides with the operator rotating the variables \( \vec{k}, \vec{n} \) in eqs. (3.14).

In the variables \( \vec{k} \) and \( \vec{n} \), the problem of constructing the wave function is the same as in the non-relativistic case: we have to construct the general form of a pseudovector from the variables \( \vec{k}, \vec{n} \), the Pauli matrices and two-component spinors. Since \( \sigma_1, \sigma_2 = \pm 1/2 \), the wave function is a \( 2 \times 2 \)-matrix. In this respect, the only difference from the non-relativistic case is the presence of an additional vector \( \vec{n} \) that increases the number of independent structures. The decomposition of the wave function (5.3) takes therefore the form [17]:

\[
\Psi^\lambda_{\sigma_2\sigma_1}(\vec{k}, \vec{n}) = \sqrt{m w_{\sigma_2}^\dagger} \psi^\lambda(\vec{k}, \vec{n}) \sigma_2 w_{\sigma_1}^\dagger,
\]

(5.4)

with

\[
\vec{\psi}(\vec{k}, \vec{n}) = f_1 \frac{1}{\sqrt{2}} \vec{\sigma} + f_2 \frac{1}{2} \left( \frac{3\vec{k}(\vec{k} \cdot \vec{\sigma})}{k^2} - \vec{\sigma} \right) + f_3 \frac{1}{2} \left( 3\vec{n}(\vec{n} \cdot \vec{\sigma}) - \vec{\sigma} \right) + f_4 \frac{1}{2} \left( 3\vec{\sigma}(\vec{k} \cdot \vec{\sigma}) + 3\vec{n}(\vec{n} \cdot \vec{\sigma}) - 2(\vec{k} \cdot \vec{n}) \vec{\sigma} \right) + f_5 \sqrt{\frac{3}{2}} \frac{1}{k} \left( \vec{k} \times \vec{n} \right) + f_6 \sqrt{\frac{3}{2}} \frac{1}{2k} \left( [\vec{k} \times \vec{n}] \times \vec{\sigma} \right),
\]

(5.5)

where \( w \) is the two-component nucleon spinor normalized to \( w^\dagger w = 1 \). The relation between \( \psi^\lambda \) and \( \vec{\psi} \) is the same as the relation between the spherical function \( Y_1^\lambda(\vec{n}) \) and \( \vec{n} \). The scalar functions \( f_{1-6} \) depend on the scalars \( \vec{k}^2 \) and \( \vec{n} \cdot \vec{k} \) or \( k \equiv |\vec{k}| \) and \( z = \cos(\vec{n} \cdot \vec{k}) \).

For an isospin zero state, the Pauli principle results into:

\[
f_{1,2,3,5}(k, z) = f_{1,2,3,5}(k, -z), \quad f_{4,6}(k, z) = -f_{4,6}(k, -z).
\]

(5.6)

The reduction of the number of variables can be summarized as follows. The initial wave function (3.4) depends on 4 four-vectors, i.e. on 16 variables. Four mass-shell constraints reduce this number to 12. The four-dimensional conservation law (3.8) reduces it to 8. Lorentz-invariance eliminates the dependence on the velocity \( \vec{v} \) of the system of reference, i.e., reduces the number of variables to 5. The wave function (5.3) just depends on the 5 independent components of two vectors \( \vec{k}, \vec{n} \), \( (\vec{n}^2 = 1) \). Because the rotational invariance eliminates their dependence on the three Euler angles, the scalar functions \( f_i \) depend only on the two variables \( k \) and \( z \) or \( \hat{R}^2 \) and \( x \).
In the region \( k \ll m \), the functions \( f_{3-6} \) which are of relativistic origin become negligible, \( f_{1,2} \) do not depend anymore on \( z \) and turn into the S- and D-waves: \( f_1 \approx u_S, f_2 \approx -u_D \). From the general decomposition (5.3), one recovers the usual non-relativistic wave function:

\[
\tilde{\psi}_{NR}(\vec{K}) = u_S(k) \frac{1}{\sqrt{2}} \vec{\sigma} - u_D(k) \frac{1}{2} \left[ 3\vec{\sigma}(\vec{k} \cdot \vec{\sigma}) \vec{k}^2 - \vec{\sigma} \right].
\] (5.7)

The representation (5.2) is however convenient in the calculation of physical observables, like electromagnetic form factor for instance. The decomposition (5.3) is more adapted for comparison with the non-relativistic limit and will be chosen in this chapter to present numerical results. The relations between the functions \( \varphi_{1-6} \) and \( f_{1-6} \) can be found by comparing eqs.(5.1),(5.2) with eqs.(5.4),(5.5) in the reference system where \( \vec{k}_1 + \vec{k}_2 = 0 \). Indeed, both representations coincide in this system, since all the D-matrices in (5.3) turn into unit matrices. These relations are given in appendix C.

The number of components of the wave function is a priori not the same in different relativistic approaches. For example, the Bethe-Salpeter amplitude of the deuteron is determined by eight components [21] (see sect. 5.2), whereas the Gross wave function [13] contains four components. It does not follow, however, that these components do not lead to physical observable consequences. In a given framework, they are unambiguously related to measured cross sections. The convenience of one approach or another depends on the problem under consideration. The relation between the Bethe-Salpeter and light-front components of the deuteron functions is given in section 5.2. We emphasize that within the LFD the number of components is the same both in the non-covariant approach and in the covariant formulation. In the non-covariant approach, these components manifest themselves as six independent matrix elements of the wave function \( \psi_{m}^{\sigma_{1},\sigma_{2}} \) with \( m = 0, \pm 1 \), \( \sigma_{1,2} = \pm 1/2 \) instead of two matrix elements in the non-relativistic case. The representation of the wave function (5.3) in terms of the well defined spin structures and corresponding components \( f_{1-6} \) is however much more transparent than in terms of those matrix elements.

Note that when a deuteron state at rest, and described from two positive energy spinors for its constituents, is seen from a different inertial frame, it automatically acquires four extra components depending on the velocity of this frame. This also holds for the \(^1 S_0\) scattering state (one extra component in this case). The equality of the total number of six components thus obtained with the number of components in the light-front approach is not a coincidence, since it is related in both cases to the appearance of an extra vector. The extra components in the light-front approach just represent the dynamical counterpart arising from the boost of the system at rest to some velocity \( \vec{v} \), and the vector \( \vec{n} \) is the counterpart of the direction \( \vec{v} \) when \( v \to c \).


5.1.2 Two-body contribution to the deuteron normalization

For the contribution of the two-body Fock component, the normalization integral (3.23) is rewritten as:

$$N_{2}^{\lambda'\lambda} = \frac{1}{(2\pi)^3} \int \sum_{\sigma_1 \sigma_2} \Phi_{j_2 \sigma_2 j_1 \sigma_1}^{* \lambda'} \Phi_{j_2 \sigma_2 j_1 \sigma_1}^{\lambda} \delta^{(4)}(k_1 + k_2 - p - \omega \tau) \frac{d^3k_1}{2\varepsilon_{k_1}} \frac{d^3k_2}{2\varepsilon_{k_2}} 2(\omega \cdot p) d\tau. \quad (5.8)$$

Substituting the wave function (3.1) into the integral (5.8), we get:

$$N_{2}^{\lambda'\lambda} = e_{\mu}^{* \lambda'}(p) I^{\mu\nu} e_{\nu}(p)$$

where

$$I^{\mu\nu} = \frac{m}{(2\pi)^3} \int Tr\{\phi^{\mu}(k_2 + m) \phi^{\nu}(k_1 - m)\} \delta^{(4)}(k_1 + k_2 - p - \omega \tau) \frac{d^3k_1}{2\varepsilon_{k_1}} \frac{d^3k_2}{2\varepsilon_{k_2}} 2(\omega \cdot p) d\tau, \quad (5.9)$$

and $\phi$ is given by eq.(5.2).

Substituting in (3.31) the integral given by (5.9) with the wave function (5.2), one can obtain the normalization condition for the deuteron wave function in terms of the components $\varphi_{1-6}$. Expressing the functions $\varphi_{1-6}$ through $f_{1-6}$ by the formulae (C.3) - (C.8) given in appendix C we can represent it in terms of the $f_{1-6}$ functions. It is however simpler to do this directly in terms of $f_{1-6}$. For this goal, consider the integral (5.8) in the reference system where $\vec{k}_1 + \vec{k}_2 = 0$. The deuteron wave function is given by (5.5). We then get for $N_{2}^{\lambda'\lambda}$, in the basis where $\lambda', \lambda = x, y, z$:

$$N_{2}^{ij} = \frac{m}{(2\pi)^3} \int \left( \frac{\psi_i^\dagger(k, \vec{n}) \psi_j(k, \vec{n})}{\varepsilon_k} \right) \frac{d^3k}{\varepsilon_k} . \quad (5.10)$$

The general structure of the integral (5.10) is given by (3.30) and $A_2$ is found from (3.31):

$$A_2 = \frac{1}{3} \delta_{ij} N_{2}^{ij} . \quad (5.11)$$

Substituting in (5.10) the wave function (5.5), we get:

$$A_2 = \frac{m}{(2\pi)^2} \int_0^{\infty} \frac{k^2 dk}{\varepsilon_k} \int_{-1}^{1} I dz , \quad (5.12)$$

where

$$I = f_1^2 + f_2^2 + f_3^2 + f_4^2 + f_5^2 + f_6^2(3 + z^2) + f_7^2(1 - z^2) + f_8^2(1 - z^2) - f_2 f_3 (1 - 3z^2) + 4 f_2 f_4 z + 4 f_3 f_4 z \quad (5.13)$$

and $z = \cos(\vec{q} \vec{n})$. In the case where the kernel of the interaction, $V$, is independent of $M$, one has $A_2 = 1$. Keeping the dominating contribution in the region $k \ll m$, where the functions $f_{3-6}$ are negligible and $f_1 = u_S$; $f_2 = -u_D$, we recover the non-relativistic normalization condition for the deuteron wave function:

$$\frac{1}{2\pi^2} \int (u_S^2 + u_D^2) k^2 dk = 1 .$$
5.1.3 Equation for the wave function

Keeping all the spin indices, the equation for the wave function $\Phi_{\sigma_1\sigma_2}$ has the form already indicated in eq. (3.40). In terms of the variables $\vec{k}$ and $\vec{n}$, this equation can be rewritten as:

$$\left[4(\vec{k}^2 + m^2) - M^2 \right] \tilde{\psi}_{\sigma_1\sigma_2} (\vec{k}, \vec{n}) = -\frac{m^2}{2\pi^3} \sum \int \left[ \tilde{\psi}' (\vec{k}', \vec{n}) \sigma_y \right]_{\alpha_1\alpha_2} V^{\sigma_1\sigma_2}_{\alpha_1\alpha_2} (\vec{k}'', \vec{k}, \vec{n}, M) (\sigma_y)_{\sigma_1\sigma_2} \frac{d^3k'}{\varepsilon_{k'}}$$

(5.14)

According to our discussion of section 3.2, the function $\tilde{\psi}'_{\sigma_1\sigma_2}$ is obtained from $\Phi^\lambda_{\sigma_1\sigma_2} (k_1', k_2', p, \omega \tau')$ simply by expressing the arguments $k_1', k_2', p, \omega \tau'$ through $\vec{k}, \vec{k}', \vec{n}$, as given by eqs. (3.43-3.46). Expressing the function $\tilde{\psi}$ through $\Phi^\lambda_{\sigma_1\sigma_2}$ with the help of the $D$-matrices, in eq. (5.3), the wave function $\tilde{\psi}'(\vec{k}', \vec{n})$ can then be obtained from $\tilde{\psi}(\vec{k}', \vec{n})$ by a rotation of spins according to:

$$\tilde{\psi} = D^{(\frac{1}{2})^\dagger} \{ R(L^{-1}(\mathcal{P}'), k'_1) \} \tilde{\psi} D^{(\frac{1}{2})} \{ R(L^{-1}(\mathcal{P}'), k'_1) \},$$

(5.15)

where the matrix $D^{(\frac{1}{2})} \{ R(L^{-1}(\mathcal{P}), k) \}$ is defined by (3.37).

5.2 Connection with the Bethe-Salpeter amplitude

The Bethe-Salpeter function of the deuteron $\Phi(p, k_1, k_2)$ is a $4 \times 4$ matrix depending on the two off-shell nucleon momenta $k_1$ and $k_2$ and the on-shell deuteron momentum $p = k_1 + k_2$ with $p^2 = M^2$. Both nucleons in the vertex $d \rightarrow np$ are final, whereas, as a convention, the index $\alpha$ in the matrix $\Phi_{\alpha\beta}$ corresponds to the final nucleon and the index $\beta$ corresponds to the initial nucleon. It is therefore convenient to separate from $\Phi^{BS}$ at the right the charge conjugation matrix $U_c$:

$$\Phi = \psi^{BS} U_c$$

(5.16)

In order to construct the Bethe-Salpeter function $\psi^{BS}$, we will start with the “initial” expression $a\hat{\xi} + b(k \cdot \xi)/m$ and then multiply it by $\hat{k}_1$ and $\hat{k}_2$ from right and left respectively and from both sides. Here $\xi \equiv e^{\lambda}(p)$ is the deuteron polarization vector with $p\xi = 0$, and $k = (k_1 - k_2)/2$. We thus obtain:

$$\psi^{BS} = \left[ a_1\hat{\xi} + b_1 \left( \frac{k \cdot \xi}{m} \right) \right] + \left[ a_2\hat{\xi} + b_2 \left( \frac{k \cdot \xi}{m} \right) \right] \frac{\hat{k}_1}{m} + \left[ a_3\hat{\xi} + b_3 \left( \frac{k \cdot \xi}{m} \right) \right] \frac{\hat{k}_2}{m} + \left[ a_4\hat{\xi} + b_4 \left( \frac{k \cdot \xi}{m} \right) \right] \frac{\hat{k}_1}{m}$$

(5.17)

The functions $a_i, b_i$ depend on the scalars $k \cdot p$ and $k^2$. One can see from this equation that the deuteron Bethe-Salpeter function, $\psi^{BS}$, contains eight spin structures. This fact agrees with another classification [17, 82, 83] using the partial waves $2S+1L^\rho_j$, where the index $\rho$ corresponds to the so called $\rho$-spin, which operates on the positive- and negative-energy states exactly in the same manner as usual spin operates on the usual spin states.
In these terms, the Bethe-Salpeter function of the deuteron is determined by the following eight coupled waves [17]:

\[ \begin{align*}
3S^+_1, & \quad 3D^+_1, \quad 3S^-_1, \quad 3D^-_1, \quad 1P^+_1, \quad 3P^0_1, \quad 1P^-_1, \quad 3P^e_1.
\end{align*} \] (5.18)

The indices “+” and “-” mean “up” (+) and “down” (−) \( \rho \)-states of both nucleons, the indices \( e \) (even) and \( o \) (odd) mean the triplet and singlet \( \rho \)-spin states. From the Pauli principle it follows that the function \( \Phi \) satisfies the symmetry relation: \( \Phi^\dagger (\mathbf{k}, \mathbf{p}) = \Phi_{\alpha \beta} (k_1, k_2, p) \equiv \Phi_{\beta \alpha} (k_2, k_1, p) \). Hence, \( \psi^{BS} (1, 2) = - U_c [\psi^{BS} (2, 1)]^T U_c \). To provide the symmetry relation in a simplest way, we transcribe (5.17) in the form:

\[
\psi^{BS} (k_1, k_2, p) = g_1 \left( \frac{k \cdot \xi}{m} \right) + g_2 \xi + g_3 \left( \frac{\hat{k} \cdot \xi - 2m \xi}{m} \right) + g_4 \left( \frac{k \cdot \xi}{m^2} \right) (\hat{k}_1 - \hat{k}_2 + 2m)
+ g_5 \left( \frac{\hat{k}_1 + \hat{k}_2 \xi}{m} \right) + (\hat{k}_2 - m) \left( \frac{g_6 \xi + g_7 (k \cdot \xi)}{m} \right) \frac{(\hat{k}_1 + m)}{m} + g_8 \left( \frac{k \cdot \xi}{m^2} \right) (\hat{k}_1 + \hat{k}_2). \] (5.19)

The symmetry implies: \( g_i (k \cdot p, p^2) = g_i (-k \cdot p, p^2) \) for \( i = 1, 4, 6, 7 \) and \( g_{5,8} (k \cdot p, p^2) = - g_{5,8} (-k \cdot p, p^2) \).

The relation between the Bethe-Salpeter and the light-front wave function is given by eq.(3.58). We show below that (3.58) together with the deuteron Bethe-Salpeter function (5.19), coincides with the deuteron light-front wave function (1.2).

We substitute (5.19) in (3.58). The momenta \( k_1, k_2 \) in eq.(5.19) are off the mass shell. However, after the following substitution in (3.58):

\[
k_1 \rightarrow l_1 = k_1 - \omega \tau/2 + \omega \beta, \quad k_2 \rightarrow l_2 = k_2 - \omega \tau/2 - \omega \beta
\]

we have to put \( k_1, k_2 \) on mass shell. Due to that, we may use the Dirac equation \( \bar{u} (k_2) (\hat{k}_2 - m) = (k_1 + m) U_c \bar{u} (k_1) = 0 \). After applying the Dirac equation, the function \( \psi^{BS} \) in the integrand of (3.58) writes:

\[
\psi^{BS} (k_1 - \omega \tau/2 + \beta \omega, k_2 - \omega \tau/2 - \beta \omega, p) \rightarrow g_1 \frac{1}{m} [(k \cdot \xi) + (\omega \xi) \beta] + g_2 \xi
+ g_3 \frac{1}{m} [2 (\omega \xi) \beta - \frac{1}{2} (\hat{\xi} \omega - \hat{\omega} \xi) \tau] + g_4 \frac{1}{m^2} [(k \cdot \xi) + (\omega \xi) \beta] 2 \beta \omega
+ g_5 \frac{1}{m} [(\hat{\xi} \omega - \hat{\omega} \xi) \beta - (\omega \xi) \tau] - g_6 \frac{2 (\omega \xi) \omega (\beta^2 - \tau^2/4)}{m^2} - g_8 \frac{1}{m} [(k \cdot \xi) + (\omega \xi) \beta] \omega \tau. \] (5.20)

The function \( g_7 \) does not contribute. One can see that the structures at \( g_{1,2,4,6,8} \) in (5.20) coincide with the corresponding structures at \( \varphi_{1-4,6} \) in (5.2). The expression \( (\hat{\xi} \omega - \hat{\omega} \xi) \beta \) at \( g_3 \) and \( g_5 \) in (5.20) can be transformed to the form \( \gamma_5 \epsilon_{\mu \nu \rho \tau} \xi_\mu p_\nu k_\rho \omega_\tau \) at \( \varphi_5 \) in (5.2). For this aim we decompose \( \hat{k}_2 (\hat{\xi} \omega - \hat{\omega} \xi) \hat{k}_1 \) in the complete set of 4 \( \times \) 4 matrices:

\[
M \equiv \hat{k}_2 (\hat{\xi} \omega - \hat{\omega} \xi) \hat{k}_1 = A + B \gamma_5 + C \mu \gamma_\mu + D \mu i \gamma_\mu \gamma_5 + E^{\mu \nu} i \sigma_{\mu \nu}. \] (5.21)

The coefficients are given by: \( A = \frac{1}{4} Tr (M), \quad B = \frac{1}{4} Tr (\gamma_5 M), \quad 2 E_{\mu \nu} = \frac{1}{4} Tr (i \sigma_{\mu \nu} M) \) and \( C_\mu = D_\mu = 0 \). Sandwiching eq.(5.21) between the spinors \( \bar{u} (k_2) \) and \( U_c \bar{u} (k_1) \), and using
again the Dirac equation, we find:

\[
\bar{u}_2(\hat{\xi}\hat{\omega} - \hat{\omega}\hat{\xi})U_c\bar{u}_1 = \frac{4}{s}\bar{u}_2[i\gamma_5\epsilon_{\mu\nu\rho\gamma}\xi_\mu k_{1_\rho} k_{2_\omega} - (k\cdot\xi)(\omega\cdot p) - \hat{\xi}m(\omega\cdot p)]
+ \frac{1}{2}(s - M^2)(x - \frac{1}{2})(\omega\cdot \xi) + \frac{1}{2}(s - M^2)m\frac{\hat{\omega}(\omega\cdot \xi)}{\omega\cdot p}]U_c\bar{u}_1 ,
\]

(5.22)

where, as usual, \(s = (k_1 + k_2)^2 = 4(\vec{k}^2 + m^2)\). This structure contributes to \(\varphi_5\) and to other structures. By this way, we reproduce exactly the six structures in the light-front deuteron wave function (5.20). Comparing their coefficients, one can easily find the expressions for \(\varphi_i\) in terms of the integral of \(g_i\) over \(\beta\). The arguments of the functions \(g_i = g_i(k^2, p\cdot k)\) are replaced in the integrand by:

\[
\begin{align*}
k^2 &\rightarrow -\vec{k}^2 + 2m^2(x - \frac{1}{2})\beta', \\
p\cdot k &\rightarrow -\frac{1}{2}(x - \frac{1}{2})(s - M^2) + m^2\beta',
\end{align*}
\]

where we introduce the dimensionless variable \(\beta' = \beta(\omega\cdot p)/m^2\), and, hence, depend on \(\vec{k}^2\), \(\vec{n}\cdot \vec{k} = -2\varepsilon_k(x - 1/2)\). This provides the explicit link between Bethe-Salpeter and light-front wave functions.

### 5.3 Two nucleon wave function in the \(J^\pi = 0^+\) scattering state

This relativistic scattering state wave function has been calculated in ref. [71]. Before coming to the relativistic case we remind some definitions for the non-relativistic continuous spectrum wave function in coordinate and momentum spaces.

#### 5.3.1 Non-relativistic wave function

We start with the known S-wave continuous spectrum wave function in coordinate space, \(\psi_p(r)\), having the usual asymptotical behavior:

\[
\psi_p(r)|_{r\rightarrow \infty} \approx \psi_p^A(r) = \frac{\sin(pr)}{pr} + f e^{ipr} = e^{i\delta}\frac{\sin(pr + \delta)}{pr} ,
\]

(5.23)

where \(f = e^{i\delta} \sin \delta/p\) is the scattering amplitude. The corresponding energy is \(E_p = p^2/m\).

The Fourier transform \(\hat{\psi}_p(k)\) of \(\psi_p(r)\) is expressed through the off-shell scattering amplitude \(f(k, p)\):

\[
\psi_p(k) = \int \psi_p(r) \exp(-i\vec{k}\cdot \vec{r})d^3r = \frac{2\pi^2}{p^2}\delta(k - p) + \frac{4\pi f(k, p)}{k^2 - p^2 - i\epsilon} ,
\]

(5.24)

\(^1\text{We use the same notation both for the wave function }\psi_p(r)\text{ in coordinate space and for its Fourier transform }\psi_p(k).\text{ The difference is indicated by the arguments.}\)
where the S-wave off-shell amplitude \( f(k, p) \) satisfies the Lippmann-Schwinger equation. The variable \( k \) is the argument of the Fourier transform, whereas \( p \) is related to the eigenvalue \( E_p \).

Because of the asymptotical behavior (5.23), valid for any short-range interaction, the momentum space wave function (5.24) has a singular term at \( k = p \) and a quickly decreasing off-energy shell dynamical part \( f(k, p) \).

It is convenient to extract the global phase factor from the wave function:

\[
\psi_p(r) = e^{i\delta(p)} \varphi_p(r) \quad (5.25)
\]

and from the amplitude:

\[
f(k, p) = e^{i\delta(p)} g(k, p) \quad (5.26)
\]

The function \( \varphi_p(r) \) is then real and has the asymptotical form:

\[
\varphi_p(r)|_{r \to \infty} \approx \varphi^A_p(r) = \frac{\sin(pr + \delta)}{pr} \quad (5.27)
\]

The function \( g(k, p) \) is real with \( g(p, p) = \sin \delta/p \).

The Fourier transform of \( \varphi^A_p \) thus gives:

\[
\varphi^A_p(k) = \int \varphi^A_p(r) \exp(-ik \cdot r) d^3r = \frac{2\pi^2}{k^2 - p^2 - i\epsilon} \delta(k - p) + \frac{4\pi f}{k^2 - p^2 - i\epsilon} \quad (5.28)
\]

where \( f = f(p, p) \) is the on-shell scattering amplitude given above. Subtracting \(5.28\) from \(5.24\), one finds:

\[
g(k, p) = \frac{\sin \delta}{p} + (k^2 - p^2) \int_0^\infty (\varphi_p(r) - \varphi^A_p(r)) \frac{\sin(kr)}{k} r dr \quad (5.29)
\]

By the relations \(5.24\) and \(5.26\), \( g(k, p) \) determines the continuous spectrum wave function in momentum space. The difference, \( \varphi_p(r) - \varphi^A_p(r) \), decreases faster for \( r \) greater than the interaction range (exponentially for the Yukawa-type potentials) and the integral \(5.29\) can be easily computed.

### 5.3.2 Spin structure of the relativistic wave function in LFD

This wave function is the relativistic light-front generalization of the non-relativistic \( np \) wave function in the \( ^1S_0 \) state. The partial scattering amplitude (scattering through \( J^\pi = 0^+ \) state) can be factorized as follows:

\[
\mathcal{F}^{\alpha_1 \alpha_2}_{\sigma_1 \sigma_2} = e^{i\delta} A_{\sigma_1 \sigma_2} \chi_{\alpha_1 \alpha_2}^\dagger (1^S_0) \quad (5.30)
\]

where \( \chi(1^S_0) \) is the final \( ^1S_0 \)-state spin function on energy shell normalized to 1:

\[
\chi_{\alpha_1 \alpha_2} (1^S_0) = \frac{1}{2\sqrt{2m}} \pi^1(p_1) \gamma_5 U \pi^2(p_2) \quad (5.31)
\]
In the non-relativistic limit in the center-of-mass system of the final nucleons, \(\chi(1S_0)\) turns into its non-relativistic counterpart:

\[
\chi(1S_0) \approx \frac{i}{\sqrt{2}} w_1^\dagger \sigma_y w_2^\dagger = C^{00}_{\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2} w_1^\dagger (1) w_2^\dagger (2), \tag{5.32}
\]

where \(w_\sigma\) represents the two-component nucleon spinor (see appendix A), and \(C^{00}_{\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2}\) is the Clebsh-Gordan coefficient.

The amplitude \(A\) does not depend on the spin projections of the final nucleons \(\alpha_1\) and \(\alpha_2\). We have separated the phase factor \(e^{i\delta}\), where \(\delta\) is the \(1S_0\) phase shift. We expand the off-energy shell amplitude \(A\) for forming the \(0^+\)-state in the most general invariant amplitudes [71]:

\[
A = \frac{1}{\sqrt{2}} w_\sigma^\dagger \left(g_1 + \frac{i\tilde{\sigma} \cdot \hat{k} \times \hat{n}}{k} g_2\right) \sigma_- w_\sigma^\dagger, \tag{5.34}
\]

where the scalar functions \(g_{1,2}\) depend on \(k\) and \(z = \cos \hat{n}\hat{k}\). This procedure is completely analogous to the one used above for the deuteron wave function. The relativistic amplitude (5.34) contains not only the \(1S_0\) state with spin zero (the first item), but also a state with spin 1 (the second item). Therefore, it can be misleading to use the notation \(1S_0\). We refer here to the \(J^\pi = 0^+\) state. From the Pauli principle it follows:

\[
g_{1,2}(k, z) = g_{1,2}(k, -z). \tag{5.35}
\]

The relations between the invariant amplitudes in the representations (5.33) and (5.34) are the following:

\[
a_1 = \frac{\sqrt{2}}{\pi} g_1, \quad a_2 = \frac{\sqrt{2} \varepsilon_2}{\pi m k} g_2. \tag{5.36}
\]

### 5.3.3 Equation for the wave function

The inhomogeneous equation for the relativistic light-front amplitude has a form similar to that determining the bound state wave function (3.40):

\[
\mathcal{F}(p_1, p_2; k_1, k_2, \omega \tau) = \mathcal{K}(p_1, p_2; k_1, k_2, \omega \tau) + \frac{1}{(2\pi)^3} \int \mathcal{F}(p_1, p_2; k_1', k_2', \omega \tau') \times \mathcal{K}(k_1', k_2', \omega \tau'; k_1, k_2, \omega \tau) \delta^4(k_1' + k_2' - \omega \tau' - k_1 - k_2 + \omega \tau) \frac{d\tau'}{(\tau' - i\epsilon)} \frac{d^3k_1'}{2\varepsilon_{k_1'}} \frac{d^3k_2'}{2\varepsilon_{k_2'}}. \tag{5.37}
\]
The delta-function, together with the integration on one of the momenta, is here kept for symmetry. Substituting here the amplitude (5.30) and using the variables \(\vec{k}, \vec{n}\) in the system of reference where \(\vec{P} = 0\) we find:

\[
\frac{1}{\sqrt{2}}(g_1 + i\vec{\sigma} \cdot [\vec{k} \times \vec{n}])g_2 = -\frac{m^2}{4\pi\varepsilon_k}e^{-i\phi}V - \frac{m^2}{32\pi^2\varepsilon_k} \int \frac{AV}{(k'^2 - p^2 - i\epsilon)\varepsilon_{k'}} \, d^3k'.
\]

(5.38)

with \(V = -\mathcal{K}/4m^2\). The products \(\chi V\) and \(\mathcal{A} V\) contain summation over spin indices.

### 5.4 Numerical results

In order to have a first estimate of the relativistic corrections to the deuteron wave function, one can calculate the components \(f_1 - f_6\) perturbatively, starting from the non-relativistic solution \(u_S\) and \(u_D\):

\[
\tilde{\psi}_{\sigma_1\sigma_2}(\vec{k}, \vec{n}) = \frac{-m^2}{2\pi^3(4(\vec{k}^2 + m^2) - M^2)} \sum_{\sigma_1'\sigma_2'} \int (\tilde{\psi}_{NR}^\sigma_{y} \sigma_1'\sigma_2') V_{\sigma_1'\sigma_2'}(\vec{k}', \vec{k}, \vec{n}, M)(\sigma_y)_{\sigma_2'\sigma_2} \frac{d^3k'}{\varepsilon_{k'}}.
\]

(5.39)

where \(\tilde{\psi}_{NR}^\sigma_{y}\) is given by eq.(5.15) with the replacement of \(\tilde{\psi}^\sigma_{y}\) by \(\tilde{\psi}_{NR}^\sigma_{y}\) defined by (5.7). The approximation (5.39) is based on the fact that due to the small deuteron binding energy, the deuteron wave function is concentrated at non-relativistic values of \(k'\) and the integral in the r.h.s. of (5.14) is dominated by the same domain. From eq.(5.39) one can find \(f_1 - f_6\) (see ref. [48] for more details).

The comparison of eq.(5.39) with the equation to be used with the Bonn-Q models (as written in eq.(4.9) for instance) indicates that they are identical in the limit where \(\vec{n}\) dependent terms and boost effects under the integral are omitted. In this limit, the light-front wave function should be equal to \(\sqrt{\varepsilon_{k}/m}\psi_{Bonn-Q}(k)\). This one could therefore provide an approximate zeroth order in solving eq.(5.39).

The same approximate solution can be found for the \(1S_0\) scattering state [71]. In this case, one replaces \(A\) in the integrand of eq.(5.38) by:

\[
\mathcal{A} = \frac{\sqrt{2}}{\pi} \bar{u}_{1'} \gamma_5 U_c \bar{u}_{2'} g_1(k', p)
\]

(5.40)

with \(g_1(k', p) = (\varepsilon_{k'}/m)g(k', p)\) and \(g(k', p)\) given in (5.29).

We use the one-boson-exchange interaction with the parameters of the Bonn-QA model [7]. This one was used in earlier calculations [48, 71] and in view of the exploratory character of the present work, we have not used a more realistic model (section 4.3.2). The non-relativistic deuteron wave function was also taken from ref. [7]. For the \(1S_0\) scattering state, we use the Paris potential [70] and the wave function was taken from [64].

Results for the deuteron components [48] are displayed in figs.13-15. All the wave functions are normalized by \(\sqrt{A_2}\), as given by (5.12). In figs.13 and 14 the functions...
Fig. 13. The function $f_1$, in $(\text{MeV/c})^{-3/2}$, in absolute value, as a function of $k$ at a fixed value of $z = \cos \vec{n} \vec{k} = 0.6$. $f_1$ is positive at the origin and changes sign twice. The dashed line is the non-relativistic S wave function calculated with the Bonn potential, and the solid line is the relativistic result calculated in perturbation theory (see text).
$f_1(k, z)$ and $f_2(k, z)$ in (MeV/c)$^{-3/2}$ are shown for a fixed value of $z = 0.6$, with $z = \cos(\vec{k} \cdot \vec{n})$, in the region $0 \leq k \leq 1.5$ GeV/c. They are compared to their non-relativistic counterparts (dashed lines). All the functions are shown together in fig. 13 as a function of $k$.

The wave functions $g_{1,2}$ [71] for the $^1S_0$ scattering state, for $E_p = 1.5$ MeV ($p = 37.52$ MeV/c), in (MeV/c)$^{-1}$, are shown in fig. 16.

In view of the results given in refs. [48, 71], one can make the following remarks.

1. In the region $k \geq 0.4-0.5$ GeV/c, the components $f_5$ of the deuteron wave function, and $g_2$ of the $^1S_0$ scattering state are comparable to the non-relativistic components. They are dominant above $k = 0.5$ GeV/c. These components are calculated here.

Fig.14. The same as in fig. 13, but for the function $f_2$. 

- The wave functions $g_{1,2}$ [71] for the $^1S_0$ scattering state, for $E_p = 1.5$ MeV ($p = 37.52$ MeV/c), in (MeV/c)$^{-1}$, are shown in fig. 16.

- In view of the results given in refs. [48, 71], one can make the following remarks.

1. In the region $k \geq 0.4-0.5$ GeV/c, the components $f_5$ of the deuteron wave function, and $g_2$ of the $^1S_0$ scattering state are comparable to the non-relativistic components. They are dominant above $k = 0.5$ GeV/c. These components are calculated here.
Fig. 15. The relativistic components $f_1 - f_6$, in $(\text{MeV/c})^{-3/2}$, as a function of $k$, at $z = 0.6$. 
Fig. 16. The relativistic wave functions $g_{1,2}$ of the $^1S_0$ scattering state, in MeV/c$^{-1}$, for $E_p = 1.5$ MeV, at $z = 0.6$ (solid line). The dashed line is the non-relativistic $^1S_0$ scattering state wave function.
with the pseudo-scalar representation at the $\pi NN$ vertex. We shall come back to this point in the next chapter.

2. The functions $f_1, -f_2$ and $g_1$ are close to the input non-relativistic wave functions in the region $0 < k < 0.5$ GeV/c. They are however strongly modified for larger momenta. This indicates that relativistic corrections are important in this region. Some of the differences at low $k$, for $f_2$ for instance, may disappear once the low-energy $NN$ scattering data or the asymptotic normalization, $A_D$ and $A_S$, are required to be produced.

3. Calculating the phase shift by means of the relativistic function, $g_1(p, p) = \sin \delta/p$, we get $\delta = 48^\circ$ in comparison with the value $61^\circ$ found from the non-relativistic input Paris wave function. The deviation of $13^\circ$ indicates that the relativistic effects on the low-energy phase shift are sizeable and advocate for a new fit of the parameters in the relativistic kernel. This deviation shows the actual uncertainties of the relativistic kernels.

4. The relative contribution of the relativistic components $f_3-6$ to the normalization integral is of the order of 1%.

5. At $k$ of the order of $1$ GeV/c all the components of the wave function strongly depend on $z$. This is in contrast with the components $f_1, f_2$ and $g_1$ in the non-relativistic, small $k$, region.

6. The $\vec{n}$ dependence of the wave function is a property of the wave function related to the off-energy shell amplitude. At $k = p$ the scattering amplitude should not depend on $\vec{n}$, otherwise it would violate rotational invariance. In practice this dependence could appear due to approximations. It was found in [71] that, at $k = p \approx 37.5$ MeV/c, $g_2$ is negligible compared to $g_1$ and the $z$-dependence of $g_1$ is of the order of $10^{-3}$. These facts indicate that in this respect these calculations are self-consistent.

The existence of the relativistic extra components $f_5$ and $g_2$ which dominate already at moderate momenta may seem surprising given the success of the non-relativistic phenomenology in the two nucleon systems. This fact is indeed related to the dominance of meson exchange currents in the deuteron electrodisintegration amplitude. It shows that the calculation of the light-front wave function omitting its $n$-dependence cannot be considered as realistic. We shall come back to this point in section 7.3.1, where the physical significance of these components will be investigated in leading order in a $1/m$ expansion.
Chapter 6

The electromagnetic amplitude

The physical amplitude and electromagnetic form factors should not depend on a particular choice of orientation of the light-front plane. This is however only the case in a given order of perturbation theory or in an exact calculation. In any approximate calculation, the light-front electromagnetic amplitude depends on the orientation of the light front, i.e., on the four-vector $\omega$. This dependence is non-physical for any on-shell amplitude. We will show in this chapter, for systems with spins 0, 1/2, 1 and transitions between them, how to extract the physical form factors from the $\omega$-dependent electromagnetic vertex. The covariant formulation of LFD will prove to be very powerful in resolving the ambiguities arising from the light-front orientation.

6.1 Factorization of the electromagnetic amplitude

The form factors are experimentally extracted from the scattering amplitude of a charged particle (usually the electron) from a bound system. One of the contribution to the amplitude is shown graphically in fig. 17.

For simplicity, we consider a two-body system to start with, but all our results are obviously applicable to many-body systems. In the Feynman approach, the amplitude of this process is factorized in the product of the electromagnetic vertex of the projectile, the propagator $1/q^2$ of the virtual photon and the electromagnetic vertex of the bound system. This latter is then expressed through the physical form factors. The complete set of light-front diagrams should, of course, be factorized and reproduces this structure. However, the separate amplitude represented in fig. 17 does not in general factorize in this formalism. Indeed, the expression for this amplitude, corresponding to the photon propagator and the adjoining spurion lines has the form:

$$M = \int (\cdots) \theta (\omega' (p' - p)) \delta \left( (p' - p + \omega \tau_3 - \omega \tau_2)^2 \right) \frac{d\tau_2}{\tau_2 - i\epsilon} \frac{d\tau_3}{\tau_3 - i\epsilon}. \quad (6.1)$$

The dots represent the parts of the amplitude which contains the matrix element of the current $J_\rho$. They will be discussed in the next section.
For definiteness, we suppose that $\omega \cdot (p' - p) > 0$. Integrating eq. (6.1) over $d\tau_3$ by means of the $\delta$-function, we get, with $q \equiv p' - p$:

$$M = \int \cdots \frac{d\tau_2}{[2(\omega q)\tau_2 - q^2 - i\epsilon] (\tau_2 - i\epsilon)}.$$  \hfill (6.2)

This amplitude is not a product of the $1/q^2$ term with the form factors.

However, as seen from (6.2), the factorization can be obtained under the special condition

$$\omega \cdot (p' - p) = \omega \cdot q = 0,$$  \hfill (6.3)

implied as the limit $\omega q \to +0$. Under this condition, the amplitude (6.2) obtains the form $M \sim f(q)/q^2$. The integration over $d\tau_2$ in (6.2) is absorbed by the form factors. The corresponding electromagnetic vertex can then be represented by fig. 18 with the
off-mass-shell photon momentum $q^2 \neq 0$. In the standard formulation of LFD with
$\omega = (1, 0, 0, -1)$, the condition (6.3) turns into the usual condition $q_+ = 0$. This condition
can only be achieved for $q^2 \leq 0$.

### 6.2 Extracting the physical form factors

The problem of extracting the physical form factors from the elementary electromagnetic
amplitude in the LFD is rather general and appears for a system with any spin. As
it was discussed in chapter 2, the sum of all the on-energy-shell amplitudes in a given
order of perturbation theory does not depend on the four-vector $\omega$, whereas any given
amplitude can be $\omega$-dependent. This is so also for the on-energy shell electromagnetic
amplitudes. The amplitude corresponding to a separate diagram can indeed depend on
$\omega$. The diagram of fig. [17] for instance is not the only one contributing to the form factor.
Other contributions are shown in fig. [18].

![Diagram](image)

Fig.19. Amplitude which cannot be expressed through the bound state wave function.

The matrix element of the exact current, $J_\rho = \langle p' | J_\rho(0) | p \rangle$, can be represented as
the sum of all possible contributions:

$$J_\rho = J_\rho^{(1)} + J_\rho^{(2)} + \cdots + J_\rho^{(n)} + \cdots,$$

where the superscripts is a short notation for indicating the various levels of approxima-
tion. In the case of the deuteron form factors for instance, the terms in (6.4) correspond to
the impulse approximation, meson exchange current contribution, etc. Any contribution
in (6.4) consists of the sum of two parts: a $\omega$ independent one denoted by $F_\rho^{(n)}$, and a $\omega$
dependent one denoted by $B_\rho^{(n)}(\omega)$:

$$J_\rho^{(n)} = F_\rho^{(n)} + B_\rho^{(n)}(\omega).$$

(6.5)

The explicit form of eq.(6.4) for systems with spin 0, 1/2 and 1 is given below in sect. 6.3.
Since the matrix element of the exact current does not depend on \( \omega \), the \( \omega \)-dependent parts cancel each other:

\[
B^{(1)}_\rho(\omega) + B^{(2)}_\rho(\omega) + \cdots + B^{(n)}_\rho(\omega) + \cdots = 0 ,
\]

and, hence, the matrix element is determined by the sum of the \( \omega \)-independent parts only:

\[
J_\rho = F^{(1)}_\rho + F^{(2)}_\rho + \cdots + F^{(n)}_\rho + \cdots.
\]

(6.6)

(6.7)

It follows that the problem of restoring the independence of any physical amplitude on \( \omega \) is reduced to the separation, and omission, of the “pure” \( \omega \)-dependent terms.

We emphasize that this definition of the form factors, after separating out \( \omega \)-dependent contributions, still leads to approximate form factors. However, our procedure guarantees that the form factors do not receive any spurious, non-physical contributions. Since these spurious, \( \omega \)-dependent, contributions can be as large as the physical ones, in particular at high momentum transfer, our procedure is the most adequate when one wants to draw any conclusion on a particular model, or to compare various relativistic approaches. Moreover, it leads to an unambiguous determination of the form factors, independently of the orientation of the light front or of the choice of the component of the current which is used to calculate the amplitude. This is not the case in the usual formulation on the light-front plane \( t + z = 0 \), as it is already known for the deuteron form factors [85, 86].

In the case of spin 0 and 1 systems, the criterion of irreducibility for the \( \omega \)-dependent contribution is rather evident (though with a subtle point discussed in sect. 6.3.3). It can be read directly from the decomposition of the current: the terms proportional to \( \omega \) do not contribute to the physical form factors. In the case of spin 1/2 particle, the unambiguous separation of the purely \( \omega \)-dependent part requires some care and is discussed in section 6.3.2.

In a more general language, the extraction of the physical form factors can be explained as follows. The physical electromagnetic amplitude \( J_\rho \equiv \langle p'|J_\rho(0)|p \rangle \) does not depend on the hypersurface where the state vector \( |p \rangle \) is defined and, hence, does not contain any term proportional to \( \omega_\rho \), provided the current operator \( J_\rho(0) \) is the full operator. This current operator has to contain the interaction. Indeed, the commutation relation between \( J_\rho(0) \) and the four-dimensional angular momentum \( \hat{J}_{\alpha\beta} \) is similar to (2.33), since it is universal for any four-vector:

\[
\frac{1}{i}[J_\rho(0), \hat{J}_{\alpha\beta}] = g_{\rho\beta}J_\alpha(0) - g_{\rho\alpha}J_\beta(0) .
\]

(6.8)

Since \( \hat{J}_{\alpha\beta} \) in (6.8) contains the interaction, the current \( J_\rho(0) \) has to contain the interaction too. The consistency of the transformation properties of the current and of the state vector ensures the independence of the matrix element of the current \( J_\rho(0) \) on the quantization plane orientation.

In practical calculations however, this consistency is violated twice: i) the diagram of fig. 15 corresponds to the free current and ii) the state vector is approximated by its
two-body component. Due to these reasons, the vertex $J_\rho$ obtains spurious contributions proportional to $\omega_\rho$. They are however completely explicit in our covariant formulation, while they are hidden in the usual formulation of the LFD with $t + z = 0$.

6.3 Light-front electromagnetic vertex

The calculation of any amplitude with composite systems in the initial and final states is done according to the rules of the graph technique discussed in section 2.2, with the coupling constants at the vertices replaced by the vertex functions $\Gamma$ and $\Gamma^\dagger$ introduced in section 3.2 for the final and initial systems respectively. We detail in this section how to perform the calculations for spin 0, 1/2 and 1 bound states.

6.3.1 Spin 0 system

Let us consider the structure of the vertex of fig. 18 for a system of spin 0. It has the general form:

$$\tilde{J}_\rho = (p + p')_\rho F(Q^2) + \omega_\rho \frac{(p + p')^2}{2\omega \cdot p} B_1(Q^2) , \quad (6.9)$$

in which the factor $(p + p')^2/(2\omega \cdot p)$ is separated for convenience. The tilde on top of the electromagnetic amplitude $\tilde{J}_\rho$ here and below indicates that this amplitude is an approximate one and contains $\omega$-dependent non-physical contributions, as compared to the exact physical amplitude with no tilde. As usual, we have $Q^2 = -q^2 \equiv -(p' - p)^2$. The invariant functions $F$ and $B_1$ could depend in principle on $\omega \cdot p$ and $\omega \cdot p'$. However, the four-vector $\omega$ is defined up to an arbitrary number, and, hence, the theory is invariant relatively to the replacement $\omega \to \alpha \omega$, where $\alpha$ is a number. The form factors $F$ and $B_1$ can therefore depend only on the ratio $\omega \cdot p'/\omega \cdot p$. But since $\omega q = 0$, we have $\omega p'/\omega p = 1$, and the functions $F$ and $B_1$ (and similar functions for the case of spin $1/2$ and $1$) depend only on $Q^2$ only.

The general physical electromagnetic amplitude of a spinless system is given by:

$$J_\rho \equiv \langle p'|J_\rho(0)|p \rangle = (p + p')_\rho F(Q^2) . \quad (6.10)$$

The main difference of the amplitude $(6.9)$ with respect to $(6.10)$ is the presence of an additional contribution, proportional to $\omega_\rho$. Note that the expression $(6.9)$ satisfies the conservation of the current $\tilde{J}_\rho \omega^\rho = 0$ since $\omega \cdot q = 0$. This conservation is automatically fulfilled in this particular case due to the simplicity of the structure $(6.9)$, but it cannot be imposed a priori. We will see below that for a system with spin $1$, some $\omega$-dependent terms violate indeed current conservation.

In the simple case where the constituents are also spinless (Wick-Cutkosky model), the amplitude indicated in fig. 18 reads:

$$\tilde{J}_\rho = \int \frac{d^4k}{(2\pi)^4} \theta(\omega \cdot (p - k_1)) \delta \left((p + \omega \tau_2 - k_1)^2 - m^2\right)$$
\[
\times \theta(\omega(p' - k_1)) \delta \left( (p + \omega \tau_1 - k_1)^2 - m^2 \right) \theta(\omega k_1) \delta(k_1^2 - m^2)
\]
\[
\times \frac{\Gamma_1 \Gamma_2 \, d\tau_1 \, d\tau_2 \, d^4k_1}{(\tau_1 - i\epsilon)(\tau_2 - i\epsilon) \, (2\pi)^3}, \tag{6.11}
\]
where the vertex functions \( \Gamma_1, \Gamma_2 \) depend on the momenta in the vertices 1 and 2. Integrating by means of the \( \delta \)-functions over \( d\tau_1 \, d\tau_2 \, dk_0 \), and expressing \( \Gamma \) through \( \psi \) by means of eq.(3.39), we get:
\[
\tilde{J}_\rho = \frac{1}{(2\pi)^3} \int \frac{(p + p' + \omega \tau_1 + \omega \tau_2 - 2k_1)\rho}{(1 - \omega \cdot k_1/\omega \cdot p)^2} \psi^2 \psi \theta(\omega(p - k_1)) \frac{d^3k_1}{2\epsilon k_1}, \tag{6.12}
\]
where \( \tau_1 = [m^2 - (p - k_1)^2] / (2\omega(p - k_1)) \) and \( \tau_2 = [m^2 - (p' - k_1)^2] / (2\omega(p' - k_1)) \).

From (6.12) one clearly sees that \( \tilde{J}_\rho \) contains terms proportional to \( \omega \rho \) and, hence, has the structure of eq.(6.9). However, to avoid any misunderstanding, we emphasize that the contribution \( \omega \tau_1 + \omega \tau_2 \) is not the only source of \( \omega \)-dependence. Even in the absence of this term and in the case where the wave function \( \psi \) does not depend on \( \vec{n} \), the term proportional to \( \omega \rho \) survives because of momentum conservation at the electromagnetic vertex.

In the spinless case, the prescription to extract the physical contribution to the form factor, \( F(Q^2) \), from the \( \omega \)-dependent amplitude \( \tilde{J}_\rho \) can be obtained immediately by multiplying both sides of eq.(6.9) by \( \omega \rho \). We thus get:
\[
F(Q^2) = \frac{\tilde{J}_\rho}{2\omega p}. \tag{6.13}
\]
With (6.12), (6.13) we obtain:
\[
F(Q^2) = \frac{1}{(2\pi)^3} \int \psi(\vec{R}_\perp, x) \psi((\vec{R}_\perp - x\vec{\Delta})^2, x) \frac{d^2R_\perp \, dx}{2x(1 - x)}. \tag{6.14}
\]
We have represented here, and in the following, the four-momentum transfer \( q \) by \( q = (q_0, \vec{\Delta}, \vec{q}_\parallel) \) with \( \vec{\Delta} \cdot \vec{\omega} = 0 \) and \( \vec{q}_\parallel \) is parallel to \( \vec{\omega} \). Since \( \omega \cdot q = 0 \), we have \( Q^2 = -q^2 = \vec{\Delta}^2 \).

In the usual light-front formulation, with \( \omega = (1, 0, 0, -1) \), eq.(6.13) corresponds to expressing the form factor through the \( \tilde{J}_+ \) component. This is well known, and eq.(6.14) has been found in ref. [87]. However, this procedure cannot be extended to the calculation of physical form factors of systems with total spin 1/2 and 1, as we shall see in the next sections.

One can similarly calculate for comparison the non-physical form factor \( B_1 \):
\[
B_1(Q^2) = \frac{2}{(2\pi)^3} \int \frac{(\vec{R}_\perp^2 + m^2 - M^2(1 - x)^2 - x\vec{R}_\perp \cdot \vec{\Delta} + Q^2/2)}{(4M^2 + Q^2)} \times \psi(\vec{R}_\perp, x) \psi((\vec{R}_\perp - x\vec{\Delta})^2, x) \frac{d^2R_\perp \, dx}{2x(1 - x)^3}. \tag{6.15}
\]
Here and below we use \( M \) for the total mass of the system, \( m \) being used for the mass of its constituents.

87
For a system with small binding energy $|\epsilon_b| \ll m$, the wave function is mainly concentrated at $x \sim 1/2, \vec{R}_2^2 \sim m|\epsilon_b|$. For such a system, the function $B_1$ at $|q^2| \ll m^2$ has smallness $\sim |\epsilon_b|/m$ in comparison with $F$ or $\sim Q^2/M^2$, if $Q^2/M^2 > |\epsilon_b|/m$. However, at $Q^2 \geq m^2$ the function $B_1$ becomes comparable with $F$. We will illustrate this in section 7.1 in the Wick-Cutkosky model.

6.3.2 Spin 1/2 system

The electromagnetic physical amplitude for a spin 1/2 fermion is determined by the two standard form factors:

$$J_\rho = F_1(Q^2)\bar{u}'\gamma_\rho u + \frac{iF_2(Q^2)}{2M}\bar{u}'\sigma_{\rho\nu}q^\nu u \equiv \bar{u}'\Gamma_\rho u .$$

(6.16)

Here $\bar{u}' = \bar{u}\sigma'(p')$, $u = u'(p)$ are the spin 1/2 fermion bispinors and $M$ is the fermion mass. The charge ($G_E$) and magnetic ($G_M$) form factors are expressed through $F_1$ and $F_2$ as follows:

$$G_E = F_1 - \frac{Q^2}{4M^2}F_2 , \quad G_M = F_1 + F_2 .$$

(6.17)

The covariant light-front spin 1/2 electromagnetic vertex was firstly considered in ref. [85] and investigated in detail in ref. [88] (see also ref. [89]).

Like in the case of spin 0, the dependence of the corresponding amplitude on the extra four-vector $\omega$ increases the number of independent terms in the decomposition of the amplitude. Besides the two structures entering in eq. (6.16), one can construct the following three structures:

$$\frac{1}{\omega \cdot p}\bar{u}'\hat{\omega}u P_\rho , \quad \frac{1}{\omega \cdot p}\bar{u}'u \omega_\rho , \quad \frac{1}{(\omega \cdot p)^2}\bar{u}'\hat{\omega}u \omega_\rho ,$$

(6.18)

where $P = p + p'$, $\hat{\omega} = \omega_{\mu}\gamma^\mu$.

Other structures, like non-gauge-invariant ones: $\bar{u}u q_\rho$, $\bar{u}\hat{\omega}u q_\rho$, $\bar{u}'\gamma_\rho u \omega_\nu$ and also $i\bar{u}'\gamma_5 u \epsilon_{\rho\nu\omega}\gamma P^\mu q^\nu\omega^\gamma$ (not independent from the previous three), contradict $T$-invariance.

At first glance, it seems that all the structures in (6.18) exhibit no $\omega$-independent parts, since, like in the previous section, they are proportional to $\omega/\omega \cdot p$ in the first and second degree. This is indeed true for the last two structures which are explicitly proportional to $\omega_\rho/\omega \cdot p$.

This is however not the case for the first structure $\bar{u}'\hat{\omega}u P_\rho/\omega \cdot p$ which is directly proportional to $P_\rho$, as one of the physical amplitudes (after Gordon decomposition). It could in principle contain a part which has to contribute to the physical form factors and therefore cannot be simply omitted together with the full structure. This becomes evident in the particular situation where $p' = p$. In this case, due to the identity $\bar{u}(p')\gamma_\mu u(p) = \bar{u}u P_\rho/M$, we get $\bar{u}\hat{\omega}u P_\rho/\omega \cdot p = \bar{u}u P_\rho/M$, i.e., a non-zero $\omega$-independent contribution only. It indicates that this part exists also when $p' \neq p$.

In order to construct the appropriate structure, we demand that it is orthogonal to the physical contribution given in (6.16). Since there are only two physical contributions,
any other contribution **proportional to** $P_\rho$ and **orthogonal to them** is completely independent and therefore non-physical. Thus, eliminating this structure from the vertex, we shall not loose any physical contribution to the form factors.

For the two structures $\bar{u}'U^\mu u$ and $\bar{u}'V^\nu u$, we define here the orthogonality by the following condition:

$$Tr[\hat{(p' + M)}U^\rho(\hat{p} + M)\hat{V}_\rho] = 0$$

with $\hat{V}_\rho = \gamma_0 V_\rho^\dagger \gamma_0$. This definition is motivated by the procedure, detailed in the next subsection, to calculate the form factors.

The only alternative to the structure $\bar{u}'\hat{\omega}u/\omega \cdot p$ is the term $\bar{u}'(\hat{\omega}/\omega \cdot p - a)u$ with arbitrary $a$, since all the possible scalar structures sandwiched with the spinors are proportional either to $\bar{u}'\hat{\omega}u$ or to $\bar{u}'u$. The value of $a$ is then determined by the orthogonality conditions. This gives:

$$a = \frac{4M}{P^2} = \frac{1}{(1 + \eta)M}, \quad \text{where} \quad \eta = \frac{Q^2}{4M^2}.$$ (6.20)

Indeed, at this value of $a$, the following two orthogonality conditions are simultaneously satisfied:

$$Tr\left[(\hat{p' + M})\gamma^\rho(\hat{p} + M)\left(\hat{\omega}/\omega \cdot p - \frac{4M}{P^2}\right)\right] P_\rho = 0 ,$$

$$Tr\left[(\hat{p' + M})i\sigma^{\rho\nu}q_\nu(\hat{p} + M)\left(\hat{\omega}/\omega \cdot p - \frac{4M}{P^2}\right)\right] P_\rho = 0 .$$ (6.21)

The reason why a unique value of $a$ satisfies two conditions is a consequence of the fact that the expression $(\hat{\omega}/\omega \cdot p - 1/((1 + \eta)M))$, being substituted in (6.21), is proportional to $\hat{e}$, where $e$ is the following four-vector:

$$e = \frac{\sqrt{P^2}}{2\omega \cdot p} \left(\omega - 2P \omega \cdot p/P^2\right) ,$$ (6.22)

orthogonal to all available four momenta:

$$e \cdot p = e \cdot p' = e \cdot q = 0 .$$ (6.23)

For convenience, we shall normalize $e$ by the condition $e^2 = -1$. Note also that the term $\bar{u}'(\hat{\omega}/\omega \cdot p - 1/((1 + \eta)M))u$ is proportional to $\bar{u}'\hat{e}u$.

The two last structures in (6.18) are not orthogonal to $\gamma_\rho$ and $i\sigma_\rho q^\nu$. However, they are proportional to $\omega_\rho$, and, hence, do not contribute to the $\omega$-independent structures associated with $F_1$, $F_2$ and vice versa. In this respect, the situation does not differ from the spin 0 and 1 cases. We emphasize that in this way the structures proportional to $\omega_\rho/\omega \cdot p$ and $P_\rho$ are separated by different formal criterions.

Another way to separate the physical and non-physical structures is the following. One can simply subtract the two physical structures with indefinite coefficients from
\( \bar{u}' \hat{\omega} u \ P_\rho/(\omega \cdot \rho) \) and then multiplying at the left by \( u' \) and at the right by \( \bar{u} \) and taking sum over polarizations get a \( 4 \times 4 \) - matrix \( M_\rho = (\hat{p}' + M) U_\rho (\hat{p} + M) \). This matrix can be decomposed in the full set of \( 4 \times 4 \) -matrices, similarly to (5.21):

\[
M_\rho = A_\rho + B_\rho \gamma_5 + C_\rho \gamma^\mu + D_\rho i \gamma_5 \gamma^\mu + E_\rho \omega_i \gamma^\mu.
\]

Now the matrix \( M_\rho \) can be analyzed in terms of the tensors \( A_\rho, B_\rho, \) etc. We require them to be irreducible or proportional to \( \omega \). The tensor \( A_\rho \) turns out to be proportional to \( \bar{P}_\rho \), and therefore is set to zero. The tensor \( B_\rho \) equals to zero. The trace of \( C_\rho \) is set to zero in order to construct an irreducible tensor. The antisymmetric part of it is proportional to \( \omega \). These two conditions fix the coefficients, and we obtain the final structure \( U_\rho \) which exactly coincides with \( \hat{\omega}/(\omega \cdot \rho - 1/(1 + \eta)M) \). With these coefficients, the tensor \( D_\rho \) turned out to be proportional to \( \omega \rho \), and the tensor \( E_\rho \) is irreducible and proportional to \( \omega \) for the antisymmetric part. In this analysis, all the three nonphysical structures in (6.18) are treated on an equal footing, since the tensors obtained after decomposition of the last two are proportional to \( \omega_\rho \) and therefore are non-physical.

It should also be mentioned that in the Breit system, where \( \vec{P} = 0 \), the term \( \bar{u}' \hat{e} u \) obtains a particularly transparent form. In this system, the spatial part of \( e \) is the unit vector directed along \( \vec{\omega} / \omega_0 = \vec{n} \), whereas \( e_0 = 0 \). Therefore, the term \( \bar{u}' \hat{e} u \) becomes proportional to \( \vec{n} \):

\[
\bar{u}' \hat{e} u = -\bar{u}' (p') \vec{n} \cdot \vec{\gamma}_u (p),
\]

with \( \vec{n} \) orthogonal to both \( \vec{p} \) and \( \vec{p}' = -\vec{p} \).

In an arbitrary system of reference one can find in \( \bar{u}' \hat{e} u \) an \( \omega \)-independent part. Omitting this full structure, we inevitably omit also this \( \omega \)-independent part. Its form depends on the criterions we used above (orthogonality and irreducibility). These criterions, strictly speaking, are however a matter of convention. They are imposed as natural and reasonable, but are not mathematically derived from some more general principles.

The above consideration shows that we achieve (with the structure \( \bar{u}' \hat{e} u P_\rho \)) optimal separation of nonphysical and physical contributions.

Collecting the various possible contributions, we obtain the following expression for the spin 1/2 light-front electromagnetic amplitude\(^1\) [88]:

\[
\bar{J}_\rho = \bar{u}' \tilde{\Gamma}_\rho u, \\
\tilde{\Gamma}_\rho = F_1 \gamma_\rho + \frac{iF_2}{2M} \sigma_{\rho \mu} q^\mu + B_1 \left( \frac{\hat{\omega}}{\omega \cdot p} - \frac{1}{(1 + \eta)M} \right) P_\rho + B_2 \frac{M}{\omega \cdot p} \omega_\rho + B_3 \frac{M^2}{(\omega \cdot p)^2} \hat{\omega} \omega_\rho .
\]

The electromagnetic vertex (6.25) is gauge invariant since \( \bar{J}_\rho q^\rho = 0 \) (with \( \omega \cdot q = 0 \)). As mentioned above, the possible non-gauge-invariant terms are forbidden by T-invariance.

\(^1\)A similar expression for \( \tilde{\Gamma}_\rho \) was given in [88]. In this reference however, as well as in [89], the structure \( \hat{\omega} P_\rho / \omega \cdot p \) was used instead of the structure proportional to \( B_1 \) in (6.25). In this case, the form factor \( G_E(0) \) does not coincide with the normalization integral.
The physical form factors

We can now express the physical form factors $F_1$ and $F_2$ through the full vertex function $\bar{\Gamma}_\rho$. We multiply $\bar{J}_\rho$ by $[\bar{u}^{\sigma'}(p')\gamma^\rho u^\sigma(p)]^*$, $[\bar{u}^{\sigma'}(p')i\sigma_{\rho\nu}q_\nu/(2M)u^\sigma(p)]^*$, etc. and sum over polarizations. We thus obtain the following quantities:

\[
c_1 = Tr[O_\rho \gamma^\rho], \quad c_2 = Tr[O_\rho i\sigma_{\rho\nu}q_\nu]/(2M), \quad c_3 = Tr[O_\rho(\omega/\omega \cdot p - 1/(1 + \eta)M)]P^\rho, \quad c_4 = Tr[O_\rho \omega^\rho M/\omega \cdot p], \quad c_5 = Tr[O_\rho \omega^\rho M^2/(\omega \cdot p)^2], \tag{6.26}
\]

where

\[
O_\rho = (\hat{p}^\prime + M)\bar{\Gamma}_\rho(\hat{p} + M)/(4M^2). \tag{6.27}
\]

By this way, we get a linear system of five equations for $F_1, F_2, B_{1-3}$ with the inhomogeneous part determined by $c_{1-5}$. The way of calculating $c_{1-5}$ through traces with the matrix $O_\rho$ just corresponds to the operation used in the definition of the orthogonality condition (6.19). The values of $c_{1-5}$ can be calculated in specific models for the structure of the composite system, as shown in the next chapter.

Solving the above mentioned system of equations relative to $F_1$ and $F_2$, we find:

\[
F_1 = \frac{1}{4(1 + \eta)^2} \left[(c_3 + 4c_4 - 2c_1)(1 + \eta) + 2(c_1 + c_2) - 2c_5(1 + \eta)^2\right], \quad \tag{6.28}
\]

\[
F_2 = \frac{1}{4\eta(1 + \eta)^3} \left[(c_3 + 4c_4 - 2c_1)(1 + \eta) + 2(c_1 + c_2) - 2(c_3 + c_4)(1 + \eta)^2\right]. \tag{6.29}
\]

In spite of $\eta$ in the denominator in eq.(6.29) and below, there is no singularity at $Q^2 = 0$. From eqs.(6.17), we can now easily obtain $G_E$ and $G_M$:

\[
G_E = c_4/2, \quad \tag{6.30} \\
G_M = \frac{1}{4\eta(1 + \eta)^3}[(c_3 + 4c_4 - 2c_1)(1 + \eta) + 2(c_1 + c_2) - 2c_5(1 + \eta)^2], \quad \tag{6.31}
\]

with $c_{1-5}$ given in (6.26). These formulae are quite general in the light-front approach. They are applicable to any model of the spin 1/2 fermion structure.

Comparison with other approaches

In the usual formulation of LFD on the plane $t + z = 0$, the form factors of spin 1/2 systems are found from the plus-component of the current (see, e.g., [89, 90, 91, 92]), i.e., with our notation, from the contraction of $\bar{J}_\rho$ in eq.(6.25), with $\omega_\rho$. This contraction eliminates the contributions of $B_{2,3}$, but does not eliminate the term with $B_1$. The form factors $F_1'$ and $F_2'$ deduced in this way are thus given by:

\[
\bar{J}_\omega = \bar{u}'[F_1' \gamma_\rho + \frac{iF_2'}{2M} \sigma_{\rho\nu}q_\nu + 2B_1(\hat{\omega} - \frac{\omega \cdot p}{(1 + \eta)M})]u \\
\equiv \bar{u}'[F_1' \gamma_\rho + \frac{iF_2'}{2M} \sigma_{\rho\nu}q_\nu]u \omega^\rho. \tag{6.32}
\]
where
\[ F'_1 = F_1 + \frac{2\eta B_1}{1 + \eta}, \quad F'_2 = F_2 + \frac{2}{1 + \eta} B_1. \]  

Equation (6.32) shows that the structure of \( \tilde{J}_\omega \) (or \( \tilde{J}_\omega \)) indeed coincides with the standard representation of the nucleon electromagnetic vertex, eq. (6.16). However, \( F'_1 \) and \( F'_2 \) in (6.32) are not the physical form factors, but their superposition (6.33) with the non-physical contribution \( B_1 \).

The expression for \( B_1 \) can be found from the above mentioned system of equations, leading to:
\[ B_1 = -\frac{1}{8\eta(1 + \eta)}[(c_3 + 4c_4 - 2c_1)(1 + \eta) + 2(c_1 + c_2) - 4c_5(1 + \eta)^2]. \]  

Together with eqs. (6.33), one has for \( F'_1 \) and \( F'_2 \):
\[ F'_1 = c_5/2, \quad F'_2 = (c_5 - c_4)/(2\eta). \]  

From eqs. (6.33) and (6.17) one can find the corresponding \( G'_E \) and \( G'_M \):
\[ G'_E = G_E, \quad G'_M = G_M + 2B_1 = [c_5(1 + \eta) - c_4]/(2\eta). \]  

The difference between the two approaches – the first one separating out all the \( \omega \)-dependent terms to define the physical form factors, and the second based on the component \( \tilde{J}_\omega \) of the electromagnetic current and incorporating non-physical item with \( B_1 \) – is in the magnetic form factor. This difference is of relativistic origin and has to disappear for form factors of extremely non-relativistic systems (i.e. in the limit where the average internal momenta go to zero) at finite \( Q^2 \). It also disappears in an exact calculation leading to all \( B_i = 0 \). We shall estimate this difference in a simple quark model for the nucleon structure in the next chapter.

**Off-energy shell effects in electromagnetic form factors**

Though all the four-momenta are on the mass shell, this does not mean that there is no off-shell effects in the form factors at all. If, for example, the nucleon form factors are used in the calculation of the deuteron form factors, they are part of a larger diagram and contain external spurion lines. They are off-energy shell. Such a form factor is shown in fig. 20.

In the standard formulation of LFD this corresponds to the fact that the minus-components of the nucleon momenta and the photon momentum are not related by the conservation law. Hence, the off-energy shell form factor can depend on the minus-components of the momenta, in addition to its \( Q^2 \)-dependence. In the covariant formulation of LFD the graph for the off-energy shell form factor contains external spurion lines.
The off-shell effects are parametrized in terms of the corresponding spurion four-momenta. Besides $Q^2$, the off-shell form factor $F$ depends therefore on two variables [43]:

$$F = F(Q^2, s, s'),$$

where $s = (p - \omega \tau)^2$, $s' = (p' - \omega \tau')^2$. (6.38)

Since $\omega \cdot q = 0$, no other independent scalar products can be constructed.

The spin structure of the off-energy shell electromagnetic vertex differs from (6.25) by three extra items:

$$\Gamma_{\rho}^{off} = F_1\gamma_\rho + \frac{iF_2}{2M}\sigma_{\rho\nu}q^\nu + B_1\left(\frac{\hat{\omega}}{\omega \cdot p} - \frac{1}{(1 + \eta)M}\right)P_\rho + B_2\frac{M}{\omega \cdot p}\omega_\rho + B_3\frac{M^2}{(\omega \cdot p)^2}\hat{\omega}\omega_\rho$$

$$+ B_4\left(\frac{s' - s}{M^3}\right)q_\rho + B_5\left(\frac{s' - s}{M^2(\omega \cdot p)}\right)\hat{\omega}q_\rho + B_6\frac{i(s' - s)}{2M(\omega \cdot p)}\sigma_{\rho \nu}\omega^\nu, \quad (6.39)$$

where all the form factors depend on $Q^2, s, s'$ and are symmetric relative to the permutation $s \leftrightarrow s'$. The last three spin structures have been already mentioned after eq. (6.18). Their opposite $T$-parity is corrected in (6.39) by the factor $(s' - s)$, changing the sign due to permutation of initial and final momenta. We emphasize that, in contrast to (6.25), the $\omega$-dependent items in (6.39) appear not because of approximation, but due to the off-energy shell continuation. Of course, in an approximate calculation of the off-energy shell form factors these item appear due to both reasons. The form factors $B_{1-3}$ in an exact electromagnetic amplitude turn into zero on the energy shell $\tau = 0 \rightarrow s = s' = M^2$.

The off-shell electromagnetic vertex of a spin-zero particle differs from eq. (6.9) by the extra term proportional to $(s' - s)q_\rho/M^2$.

The extra form factors and the dynamical dependence of all the form factors on the off-shell variables $s, s'$ are usually unknown and are neglected in almost all calculations.

### 6.3.3 Spin 1 system

The electromagnetic form factors of a system with spin 1 (the deuteron, for instance) are defined by the following decomposition [93] of the electromagnetic vertex:

$$\langle \lambda'|J_\rho|\lambda \rangle = e_{\mu}^*\lambda'(p') \left\{ P_\rho \left[ F_1(q^2)g^{\mu\nu} + F_2(q^2)\frac{q^\mu q^\nu}{2M^2} \right] + G_1(q^2)(g_\mu^\rho q_\nu - g_\nu^\rho q_\mu) \right\} e_\nu^\lambda(p)$$

$$\equiv e_{\mu}^*\lambda'(p') T_{\rho\nu}^\mu e_\nu^\lambda(p). \quad (6.40)$$
Here $e^\lambda_\mu(p)$ is the deuteron polarization vector, $p$ and $p'$ are the initial and final deuteron momenta, $\lambda$ and $\lambda'$ are the corresponding helicities, $P = p + p'$ and $q = p' - p$. Charge ($F_C$), magnetic ($F_M$) and quadrupole ($F_Q$) form factors are expressed through $F_1$, $F_2$ and $G_1$ as follows:

\[
F_C = -F_1 - \frac{2\eta}{3}[F_1 + G_1 - F_2(1 + \eta)], \tag{6.41}
\]

\[
F_M = G_1, \tag{6.42}
\]

\[
F_Q = -F_1 - G_1 + F_2(1 + \eta), \tag{6.43}
\]

where $\eta = Q^2/4M^2$.

In the absence of polarization, the cross section of electron-deuteron scattering has the form:

\[
\left( \frac{d\sigma}{d\Omega} \right)_0 = \frac{\alpha^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)}. \tag{6.45}
\]

The functions $A(q^2)$, $B(q^2)$ are expressed through the form factors as follows:

\[
A(q^2) = F_C^2(q^2) + \frac{8}{9} \eta^2 F_Q^2(q^2) + \frac{2}{3} \eta F_M^2(q^2), \tag{6.46}
\]

\[
B(q^2) = \frac{4}{3} \eta(1 + \eta) F_M^2(q^2). \tag{6.47}
\]

Unpolarized experiments are able to separate $A$ and $B$, but do not allow one to extract all three form factors. To do this, one should measure a polarization observable. Due to its spin one, the deuteron can have, besides vector polarization, quadrupole polarization. The corresponding observable is $T_{20}$ given by:

\[
T_{20} \left( A(q^2) + \tan^2\frac{1}{2} \theta B(q^2) \right) = - \frac{1}{\sqrt{2}} \left[ \frac{8}{3} \eta F_C F_Q + \frac{8}{9} \eta^2 F_Q^2 + \frac{1}{3} \eta \left( 1 + 2(1 + \eta) \tan^2\frac{1}{2} \theta \right) F_M^2 \right]. \tag{6.48}
\]

The deuteron electromagnetic properties are reviewed in ref. [94].

The decomposition (6.40) is valid for the full electromagnetic current operator. Considering the one-body current $J^0_\rho$, as indicated in fig. [18], we get the following analytical expression:

\[
\langle \lambda' | J^0_\rho | \lambda \rangle = e^\ast_\mu(p') \tilde{J}^{\mu\nu}_\rho e_\nu(p),
\]

\[
\tilde{J}^{\mu\nu}_\rho = \frac{m}{(2\pi)^3} \int Tr[\phi''(\hat{k}_2 + m)\Gamma_\rho(\hat{k}_2 + m)\phi'(\hat{k}_1 - m)] \frac{\theta(\omega \cdot k_1)\theta(\omega \cdot k_2)\theta(\omega \cdot k'_2) d^3k_1}{(1 - \omega \cdot k_1 / \omega \cdot p)^2} 2\varepsilon_{k_1}, \tag{6.49}
\]

94
where $\Gamma_\rho$ is the electromagnetic vertex of the nucleon given by eq. (6.16) (with $M$ replaced by $m$). The matrix $\phi^\mu$ is given by eq. (5.2), while the matrix $\phi^{\nu'}$ is obtained from $\phi_\nu$ by the replacement $k_2 \rightarrow k'_2$. This expression is derived using the graph technique presented in chapter 4. It is similar to the calculation indicated in sect. 6.3.1 for spinless particles. The difference with (6.11), (6.12) is due to the non-zero spins. We explain below, how eq. (6.49) is derived.

According to the remark in the end of sect. 2.2.3, we start in the graph 18 from the outgoing deuteron line (i.e., from the vertex $NN \rightarrow d'$), follow the line in the direction opposite to the orientation of any nucleon line and reach its end, i.e., the vertex $d \rightarrow NN$. Then we start again from the vertex $NN \rightarrow d$, and follow another nucleon line, also in the direction opposite to its orientation. We thus obtain:

$$
e^\lambda(p') \left\{ [\gamma_0 \phi^\mu U_c \gamma_0]^\dagger (\hat{k}_2 + m) \Gamma_\rho (\hat{k}_2 + m) \phi^{\nu'} U_c \right\}_{\beta \alpha} e^\lambda(p) \times (\hat{k}_1 + m)_{\beta \alpha}. \tag{6.50}
$$

The first row in (6.50) corresponds to the upper nucleon line in the diagram, the second one to the lower line. We attach the deuteron wave function to the upper nucleon line. The first row contains the factor $[\gamma_0 \phi^\mu U_c \gamma_0]^\dagger$ which originates from the complex conjugated final wave function of the deuteron:

$$[e^\lambda(p') \bar{u}(k'_2) \phi^\mu U_c \bar{u}(k_1)]^* = e^\lambda(p') \bar{u}(k_1) [\gamma_0 \phi^\mu U_c \gamma_0]^\dagger \bar{u}(k_2).$$

We keep in (6.50) the matrix indices $\alpha$ and $\beta$ explicitly. Since both nucleon lines are equivalent, the order of indices $\beta, \alpha$ is the same. This means that one of the factors in (6.50) (we take the second one) is the transposed matrix. With the wave function (5.2) we get:

$$[\gamma_0 \phi^\mu U_c \gamma_0]^\dagger = -U_c \phi^\mu$$

and, hence, obtain the factor:

$$-U_c (\hat{k}_1 + m)^\dagger U_c = (\hat{k}_1 - m),$$

that gives the trace in eq. (6.49).

Like in the case of a spinless system, the light-front vertex (6.49) depends on the four-vector $\omega$. Its expansion in terms of form factors contains therefore extra terms. This expansion has the general form [53]:

$$\langle \lambda | \tilde{J}_\rho | \lambda \rangle = \frac{1}{2 \omega \cdot p} e^\lambda(p') \tilde{J}_\rho^{\mu \nu} e^\lambda(p), \quad \text{where} \quad \tilde{J}_\rho^{\mu \nu} = T_\rho^{\mu \nu} + B_\rho^{\mu \nu}(\omega), \tag{6.51}
$$

where $T_\rho^{\mu \nu}$ has the structure given by eq. (6.41) and is determined by the physical form factors, $F_1$, $F_2$ and $G_1$. We recall that tilde is used to distinguish approximate amplitudes and matrix elements, containing $\omega$-dependent non-physical contributions, from the $\omega$-independent ones. The tensor $B_\rho^{\mu \nu}$ contains the $\omega$ dependent terms:

$$B_\rho^{\mu \nu} = \frac{M^2}{2(\omega \cdot p)} \omega_\rho \left[ B_1 g^{\mu \nu} + B_2 q^\mu q^\nu M^2 + B_3 M^2 \frac{\omega^\mu \omega^\nu}{(\omega \cdot p)^2} + B_4 q^\mu \omega^\nu - q^\nu \omega^\mu \right]$$

95
Here $P = p + p'$ and $B_1, ..., B_8$ are invariant functions. The tensor $B_{\rho \mu \nu}$ as well as $T_{\rho \mu \nu}$ is symmetrical with respect to the simultaneous permutations $p \leftrightarrow p' \ (q \rightarrow -q)$ and $\mu \leftrightarrow \nu$. The structures at $B_7$ and $B_8$ in eq. (6.52) do not satisfy the conservation of current, that is $q^\rho J_\rho^{\lambda \nu} \neq 0$.

The total number of independent spin structures can be calculated without their explicit construction. Owing to the $\omega$-dependence, the helicity indices in the vertex are not constrained by the equality $\lambda = \lambda_1 + \lambda_2$ ($\lambda$ is the photon helicity, $\lambda_1, \lambda_2$ are the hadron helicities). Therefore we have $3 \times 3 \times 3 = 27$ initial matrix elements. $P$-invariance reduces this number down to 14. $T$-invariance (or $C$-invariance in annihilation channel) eliminates in addition 4 structures and finally 10 independent structures remain. This calculation automatically takes into account the conservation of the electromagnetic current which would lead to a relation between $B_7$ and $B_8$. However, $\omega$-dependent terms in eq. (6.52) do not obligatory satisfy this condition. The functions $B_7$ and $B_8$ are therefore independent and we obtain 11 structures (or invariant form factors): three of them are in $T_{\rho \mu \nu}$ and the remaining eight ones are in $B_{\rho \mu \nu}$.

From the point of view of the usual light-front approach with $t + z = 0$, the dependence of the vertex (6.51) on the orientation of the light front means lack of relativistic covariance. Hence, the formula (6.40), based on this covariance, is not any longer valid. Therefore, the number of independent matrix elements in eq. (6.51) is not reduced to 3, but is larger. The increase of the number of matrix elements just corresponds to the increase of the number of independent spin structures in eq. (6.52). In the formulation of the light-front with $t + z = 0$, the contribution of these structures is not separated out from the physical form factors. In the covariant formulation, the dependence on the light front is parametrized explicitly by eqs. (6.51), (6.52).

However, like in the spin 1/2 case, and in the absence of explicit evaluation of the $\omega$-dependent terms, one can see some ambiguity in the procedure: what has to be considered as a non-physical contribution? Consider for example the term proportional to $B_5$: $P_{\rho} e_{\mu}^{\lambda}(p') \omega_{\mu} \omega_{\nu} e_{\nu}^{\lambda}(p)/(\omega p)^2$. Multiplying it by $e'_{\rho}$ and summarizing over $\lambda', \lambda$, we get:

$$P_{\rho} \sum_{\lambda', \lambda} e_{\mu}^{\lambda}(p') e_{\mu}^{\lambda'}(p') \frac{\omega_{\mu} \omega_{\nu}}{(\omega p)^2} e_{\nu}^{\lambda}(p) = P_{\rho} \left( \frac{\omega_{\mu} \omega_{\nu}}{(\omega p)^2} - \frac{p_{\mu} \omega_{\nu}}{(\omega p) M^2} - \frac{\omega_{\mu} p_{\nu}}{(\omega p) M^2} + \frac{p_{\mu} p_{\nu}}{M^4} \right).$$

(6.53)

It contains, apart from $\omega$-dependent contributions, the term $p_{\mu} p_{\nu}/M^4$. This one cannot be removed from (6.53) and be included in the physical part, since it is required by the transversality of the tensor (6.53). The orthogonality condition, similar to the spin-1/2 case (obtained by the generalization of eq. (6.19)), results in the charge form factor $F_C$, which does not coincide at $q = 0$ with the normalization integral, in contrast to the spin-1/2 case. We therefore do not impose this condition in the spin-1 case, and define the
non-physical structures as proportional to the first and higher degrees of $\omega$, including the terms where $\omega$ is contracted with the polarization vector. After that, the physical form factors can be unambiguously separated from the spurious $\omega$-dependent contributions. This allows one to obtain the contribution to the physical form factors in terms of the electromagnetic amplitude in an explicit form by a generalization of the formula (6.13). This approach is confirmed by comparison with the Bethe-Salpeter solution in the Wick-Cutkosky model (see sect. 7.1.2).

Contributions to physical form factors

The explicit expressions for the form factors are obtained by solving eq. (6.54) relative to $F_1$, $F_2$ and $G_1$. The procedure is described in ref. [85]. The result is given below:

$$ F_1 = \frac{j_{\mu\nu}}{2} \frac{\omega}{2\omega p} \left[ g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{P_\mu \omega_\nu + P_\nu \omega_\mu}{2\omega p} + \frac{p^2 (\omega_\mu \omega_\nu)}{4(\omega p)^2} \right], $$ (6.54)

$$ \frac{F_2}{2M^2} = -\frac{j_{\mu\nu}}{2(\omega p)q^2} \left[ g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{P_\mu \omega_\nu + P_\nu \omega_\mu}{2\omega p} + \frac{M^2 \omega_\mu \omega_\nu}{(\omega p)^2} - \frac{q_\mu q_\nu - q_\nu q_\mu}{2\omega p} \right], $$ (6.55)

$$ G_1 = \frac{1}{4} j_{\rho} \left\{ \frac{2}{q^2} \left[ g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - \frac{P_\mu \omega_\nu + P_\nu \omega_\mu}{2\omega p} + \frac{M^2 \omega_\mu \omega_\nu}{(\omega p)^2} - \frac{q_\mu q_\nu - q_\nu q_\mu}{2\omega p} \right] \right\}, $$ (6.56)

These expressions determine the electromagnetic form factors. In spite of the fact that $\omega$ enters the r.h.s. of eqs. (6.54-6.56), these expressions do not depend on $\omega$.

The form factor $F_1$, eq. (6.54) which coincides at $Q^2 = 0$ with $F_C(0)$, does not coincide with the normalization integral (3.31). This is due to the fact that the normalization is obtained on the basis of the decomposition (3.29), where the non-physical structure proportional to $E$, contains $g^{\mu\nu}$ besides $\omega^{\mu}\omega^{\nu}$. This is required by the irreducibility of this structure and is compatible with the angular condition. In the decomposition of the electromagnetic vertex, we have not separated out this term from the terms proportional to $\omega^{\mu}\omega^{\nu}$, as we explained above. We believe that the concordance of $F_C(0)$ with the normalization integral is beyond the accuracy of the current calculations which omits all the Fock components except for the two-body one. In practical calculations of the deuteron elastic form factors, the difference is very small.

Comparison with other approaches

The deuteron electromagnetic form factors were already calculated in LFD in refs. [95, 96, 97, 98, 99, 100, 101]. More generally, spin-1 form factors were considered in refs.
Using the exact decomposition (6.40), one can express the matrix elements \( <\lambda|J_+|\lambda> \equiv J_{\lambda\lambda} \) in terms of the form factors:

\[
J_{11} = -\mathcal{F}_1 + \eta\mathcal{F}_2 \\
J_{1\bar{1}} = -\eta\mathcal{F}_2 \\
J_{00} = -(1 - 2\eta)\mathcal{F}_1 - 2\eta^2\mathcal{F}_2 + 2\eta\mathcal{G}_1 .
\]

(6.57) (6.58) (6.60)

The four matrix elements in (6.57-6.60) can be expressed through three form factors and, hence, are not independent from each other. They satisfy the relation [98]:

\[
(1 + 2\eta)J_{11} + J_{1\bar{1}} - 2\sqrt{2\eta}J_{10} - J_{00} = 0 ,
\]

(6.61)

which can be checked by direct substitution. This condition is called “angular condition” (do not confuse with the angular condition for the state vector discussed in sect. 21.3).

In principle, with the exact matrix elements \( J_{\lambda\lambda} \) one can calculate any triplet of matrix elements and find the form factors. It is not so however for the approximate ones \( \tilde{J}_{\lambda\lambda} \). The following triplets of the matrix elements were used in the literature: \( \tilde{J}_{11}, \tilde{J}_{1\bar{1}}, \tilde{J}_{00} \) and \( \tilde{J}_{11}, \tilde{J}_{1\bar{1}}, \tilde{J}_{10} \) in ref. [102] (the so called solutions A and B respectively); \( \tilde{J}_{11}, \tilde{J}_{10}, \tilde{J}_{00} \) in ref. [102]. For example, taking the triplet \( \tilde{J}_{11}, \tilde{J}_{1\bar{1}}, \tilde{J}_{00} \) and solving the system of equations (6.57), (6.58) and (6.60) (with \( J_{\lambda\lambda} \rightarrow \tilde{J}_{\lambda\lambda} \)) relative to \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1 \), one finds:

\[
\mathcal{F}_1^{\text{GK-A}} = -\tilde{J}_{11} - \tilde{J}_{1\bar{1}}, \\
\mathcal{F}_2^{\text{GK-A}} = -\tilde{J}_{1\bar{1}}/\eta, \\
\mathcal{G}_1^{\text{GK-A}} = -\frac{1}{2\eta} [(1 - 2\eta)\tilde{J}_{11} + \tilde{J}_{1\bar{1}} - \tilde{J}_{00}] .
\]

(6.62)

In refs. [103, 104] the form factors were expressed through different combinations of four matrix elements.

The matrix elements in the r.h.s. of eqs. (6.62) calculated in LFD are however not given by the decomposition (6.40), but should be calculated according to eqs. (6.51), (6.52). The matrix elements of (6.51) have the form:

\[
\tilde{J}_{1,1} = -\mathcal{F}_1 + \eta\mathcal{F}_2 , \\
\tilde{J}_{1\bar{1}} = -\eta\mathcal{F}_2 , \\
\tilde{J}_{1,0} = -\sqrt{2\eta}(\mathcal{F}_1 - \eta\mathcal{F}_2 + \mathcal{G}_1/2) + \sqrt{\eta/2}\mathcal{B}_6 , \\
\tilde{J}_{0,0} = -(1 - 2\eta)\mathcal{F}_1 - 2\eta^2\mathcal{F}_2 + 2\eta\mathcal{G}_1 - 2\eta\mathcal{B}_6 + \mathcal{B}_5 + \mathcal{B}_7 .
\]

(6.63)

The matrix elements \( \tilde{J}_{11}, \tilde{J}_{1\bar{1}} \) have the same form as \( J_{11}, J_{1\bar{1}} \), whereas \( \tilde{J}_{10}, \tilde{J}_{00} \) differ from \( J_{10}, J_{00} \) by the items containing the nonphysical form factors \( \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_7 \). Other nonphysical form factors \( B_{1-4} \) and \( B_8 \) do not contribute to these matrix elements.

The matrix elements \( \tilde{J}_{\lambda\lambda} \) do not satisfy the condition (6.61). Substituting \( \tilde{J}_{\lambda\lambda} \) in eq.(6.61) instead of \( J_{\lambda\lambda} \), we get:

\[
\Delta \equiv (1 + 2\eta)\tilde{J}_{11} + \tilde{J}_{1\bar{1}} - 2\sqrt{2\eta}\tilde{J}_{10} - \tilde{J}_{00} = -(\mathcal{B}_5 + \mathcal{B}_7) .
\]

(6.64)
Table 6.1: The set of form factors $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1$ calculated in six versions discussed in the section 6.3.3.

This function was calculated for the deuteron in ref. [98] and for $\rho$-meson in refs. [90, 103]. Note that $\Delta$ in (6.64) should not be confused with the perpendicular component of the momentum transfer $q$.

The substitution in (6.62) of the matrix elements $\tilde{J}_{\lambda',\lambda}$ from (6.63) gives:

$$
\mathcal{F}_{1,2}^{GK-A} = \mathcal{F}_1, \quad \mathcal{F}_{2}^{GK-A} = \mathcal{F}_2, \\
\mathcal{G}_{1}^{GK-A} = \mathcal{G}_1 - B_6 - \frac{1}{2\eta}\Delta,
$$

(6.65)

This solution gives $\mathcal{F}_1, \mathcal{F}_2$ without any nonphysical contribution (though still approximate), but $\mathcal{G}_{1}^{GK-A}$ contains the nonphysical form factor $B_6$ and, in addition, the function $\Delta = -(B_5 + B_7)$, eq. (6.64), responsible for the violation of the condition (6.61).

The different solutions were compared numerically in ref. [103] (for the case of $\rho$-meson form factor) and analytically in ref. [104]. The results are summarized in table 6.1. Calculations by analytical formulae from this table exactly coincide with the numerical calculations [103].

From the table 6.1, one can see that all the methods give superpositions of the physical form factors with the nonphysical ones. Most of them contain the nonphysical contribution $\Delta$ and differ from each other by its coefficient. This is so not only for the methods considered above, but also for any other method based on the $\tilde{J}_{\lambda}$ component. If $\Delta = 0$, the form factors do not depend on the prescription, as it was pointed out in [90, 103]. But $\mathcal{G}_1$ still contains $B_6$.

Note that the terms $B_{\mu}^{\mu}$ contribute to the matrix elements of the $J_{+}^{(0)}$ current component if at least one of the projections $\lambda, \lambda'$ equals to zero. In ref. [97] it was proposed to avoid using the zero projections of $\lambda, \lambda'$, but consider the matrix elements with $\lambda, \lambda' = \pm 1$ of the current components $J_{+}^{(0)}$ and $\tilde{J}_{\pm}^{(0)} = \{J_{x}^{(0)}, J_{y}^{(0)}\}$. Taking into account that $\omega_+ = 0$ for $\omega = (1, 0, 0, -1)$, one gets:

$$
e^{\lambda'}(p')B_{+}^{\mu}\epsilon_{\mu}(p) = 0, \quad e^{\lambda'}(p')B_{-}^{\mu}\epsilon_{\mu}(p) = 0 \quad \text{for } \lambda, \lambda' = \pm 1.
$$
We see that this method indeed gives the solution for the form factors, in which the nonphysical contributions vanish entirely.

In both methods the complete separation of the physical form factors from nonphysical ones is realized at the expense of incorporating, besides $\tilde{J}_+$, other components of the current. A similar situation takes place with the form factors of a spin 1/2 system \[88\]. These other components can receive contributions not contained in $\tilde{J}_+$ (for example from the instantaneous part of the fermion propagators, which do not contribute to $\tilde{J}_+$ due to $\gamma^2 = 0$). The instantaneous contributions (contact terms) have been already taken into account in the deuteron electrodisintegration \[72\]. As we shall see in the following section, these contributions have a very important physical interpretation in terms of meson exchange currents in the non-relativistic approach. They can therefore be interpreted also as a two-body current. Considering the $\tilde{J}_+$-components only, these contributions would be lost. This also shows that the consideration of other components of the current, beside $\tilde{J}_+$, is inevitable if one wants to keep all physical contributions, and only these, to the form factors.

### 6.3.4 The transitions $1^+ - 0^+$ and $1^- - 0^-$

The pseudo-vector – scalar transition $1^+ - 0^+$ corresponds to the deuteron electrodisintegration amplitude near threshold (a few MeV in the center-of-mass system of the np pair). In this case, the final state is dominated by the singlet S-wave and the amplitude does not depend on the direction of the relative np-momentum. The amplitude of the vector – pseudoscalar transition $1^- - 0^-$, corresponds, for example, to the $\rho - \pi$ transition, and has the same structure. For definiteness we will discuss here the deuteron electrodisintegration.

In the impulse approximation, LFD was applied to the deuteron electrodisintegration amplitude in ref. \[105\] and, in its explicitly covariant form, in ref. \[106\]. The corresponding amplitude can be written as:

$$M_{2\to 3} = \frac{4\pi\alpha}{q^2} \bar{u}(k_e')\gamma^\mu u(k_e)e_\mu^\lambda(p)F_\mu^\rho\chi(1S_0), \quad (6.66)$$

where $F_{\mu\rho}$ describes the vertex $d + \gamma^* \to 1S_0$. The function $\chi(1S_0)$ is the spin function of the final nucleons in the $1S_0$-state we discussed in section 5.3. The vertex $F_{\mu\rho}$ can be represented as follows:

$$F_{\mu\rho} = \frac{1}{4m^2}e_{\rho\mu}\gamma^\nu P\gamma A, \quad (6.67)$$

where $P = p + p'$, $p' = p_1 + p_2$ and $A$ is a dimensionless scalar function: $A = A(Q^2, \nu)$. The tensor (6.67) is the only one which has the appropriate properties expected from covariance, transversality and parity conservation to describe the vertex $d\gamma^* \to np(1S_0)$. Let us emphasize that the vertex (6.67) automatically satisfies the transversality condition $F_{\mu\rho}q^\rho = 0$. In this particular case, the Lorentz invariance leads to gauge invariance for the amplitude. This is a consequence of the particular $1S_0$ final state. The cross-section
of inclusive electron-nucleus scattering is usually represented in terms of the structure functions $W_1$ and $W_2$ (see, e.g., [107]):

$$\frac{d\sigma}{d\Omega_e dE'} = \left(\frac{d\sigma}{d\Omega_0}\right)_0 (W_2 + 2 \tan^2(\frac{\theta}{2}) W_1) .$$  \hspace{1cm} (6.68)

The structure functions $W_2$ and $W_1$ can be expressed through the unique scalar amplitude $A$ [106]:

$$W_2 = \frac{p^* Q^2}{3 \cdot 2^6 \pi^2 m^4} |A|^2 ,$$  \hspace{1cm} (6.69)

$$W_1 = (1 + \nu^2/Q^2) W_2 .$$  \hspace{1cm} (6.70)

Here $p^*$ is the c.m.-momentum of the final nucleons. All the dynamical information about the process is therefore contained in the transition form factor $A$. We stress that this result follows only from the assumption that the final $^1S_0$ state dominates and is valid independently of the presence or absence of final state interaction, meson exchange currents and isobar configurations. Their contribution only influences the amplitude $A$.

The particular amplitude for the vertex $d\gamma^* \rightarrow np(^1S_0)$ in any approximate calculation, denoted by $\tilde{F}_{\mu\rho}$, does not coincide, however, with $F_{\mu\rho}$ in (6.67). The tensor $\tilde{F}_{\mu\rho}$ depends in that case on $\omega$. Following the arguments presented in section 6.2, we can decompose $\tilde{F}_{\mu\rho}$ on the general invariant amplitudes. It has the form:

$$\tilde{F}_{\mu\rho} = \frac{1}{2m^2} e_{\rho\mu\gamma\rho'} q^{\rho'} p^\gamma A + e_{\rho\mu\gamma\rho'} q^{\rho'} \omega^\gamma B_1 + e_{\rho\mu\gamma\rho'} p^{\rho'} \omega^\gamma B_2$$  \hspace{1cm} (6.71)

$$+ (V_\mu q_\rho + V_\rho q_\mu) B_3 + (V_\mu \omega_\rho + V_\rho \omega_\mu) B_4 + \frac{1}{2m^2 \omega_\rho} (V_\mu p_\rho + V_\rho p_\mu) B_5 ,$$

where $V_\mu = e_{\mu\alpha\beta\gamma} \omega^\alpha q^\beta p^\gamma$. The decomposition (6.71) contains the symmetrical structures in front of the functions $B_{3,4,5}$. Corresponding antisymmetric terms (like $V_\mu q_\rho - V_\rho q_\mu$) are not independent and can be expressed through the first three contributions. We emphasize that $\tilde{F}_{\mu\rho}$ depends on $\omega$ even for the components of the wave function which do not depend on $\omega$. In the latter case, $\omega$ enters through the rules of the graph technique. Similarly to the case of the deuteron form factors, the decomposition (6.71) enables us to separate the $\omega$-independent parts from nonphysical $\omega$-dependent ones so that one can unambiguously extract the physical form factors from the initial tensor $\tilde{F}_{\mu\rho}$. From eq.(6.71), we immediately find:

$$A = -\frac{m^2}{Q^2 (\omega p)} e^{\mu\rho\nu\gamma} q_\nu \omega_\gamma \tilde{F}_{\mu\rho} .$$  \hspace{1cm} (6.72)

This simple formula is the analogue of formulae (6.54) - (6.56) for the deuteron form factors.

Comparison with other approaches

The transition form factor $A$ given by (6.72) cannot be reduced to the $\rho = +$ component of the amplitude $\tilde{F}_{\mu\rho}$. For example, in the reference system where $q_0 = 0$, and $\vec{q}$ is parallel
to the x-axis, eq. (6.72) is reduced to:

\[ A = -\frac{m^2 q_x}{Q^2 p_+} (\tilde{F}_{y+} - \tilde{F}_{+y}) . \]  

This formula includes \( \tilde{F}_{y+y} \) with \( \rho = y \).

The contribution of the plus-component is given, as usual, by the contraction \( \tilde{F}_{\mu\rho} \omega^\rho \).

If \( \tilde{F}_{\mu\rho} \) would coincide with \( F_{\mu\rho} \) in eq. (6.67), i.e., would not contain any non-physical contributions, we indeed could find the form factor from the plus-component by the equation:

\[ A = -\frac{2m^2}{Q^2 (\omega \cdot p)^2} e^{\alpha\mu\nu\gamma} \omega_\alpha q_\nu p_\gamma (F_{\mu\rho} \omega^\rho) . \]  

(6.74)

However, applying this equation to \( \tilde{F}_{\mu\rho} \), we get:

\[ A' = -\frac{2m^2}{Q^2 (\omega \cdot p)^2} e^{\alpha\mu\nu\gamma} \omega_\alpha q_\nu p_\gamma (\tilde{F}_{\mu\rho} \omega^\rho) = A - B_5 . \]  

(6.75)

Like in the cases considered above, we get a superposition of the physical form factor \( A \) and nonphysical one \( B_5 \). Rewritten in terms of the components, the contraction (6.75) obtains the form:

\[ A' = -\frac{2m^2 q_x}{Q^2 p_+} \tilde{F}_{y+} . \]  

(6.76)

The difference with eq. (6.72) is reduced to the replacement of \( \tilde{F}_{y+y} \) in (6.76) by the antisymmetric combination \( (\tilde{F}_{y+} - \tilde{F}_{+y})/2 \) in (6.73). This seems very natural since the physical amplitude (6.67) is antisymmetric relative to permutation of the indices \( \rho, \mu, \) and, hence, any symmetrical contribution is a spurious one and has to be excluded (independently of the fact that it depends on \( \omega \) or not). This is another reason to separate it. We emphasize that the difference between the amplitudes \( A \) and \( A' \) is of relativistic origin and disappears in the non-relativistic limit.

6.3.5 The transitions \( 0^- - 1^+ \) and \( 0^+ - 1^- \)

The pseudo-scalar – pseudo-vector transition \( 0^- - 1^+ \) corresponds, for example, to \( \pi - A_1 \) and to \( K - K_1 \) transitions. The scalar – vector transition amplitude \( 0^+ - 1^- \) has the same form.

The general transition amplitude reads:

\[ \langle PS(p') | J_\rho | PV(p, \lambda) \rangle = [F_1((pq)g_\rho^\mu - p_\rho q^\mu) + F_2(q_\rho q^\mu - q^2 g_\rho^\mu)] e^\lambda_\mu(p) . \]  

(6.77)

It is determined by two form factors \( F_1 \) and \( F_2 \).

In an approximate calculation, this amplitude should incorporate the \( \omega \)-dependent contributions. These are given by:

\[ \langle PS(p') | \tilde{J}_\rho | PV(p, \lambda) \rangle = \tilde{G}_\rho^\mu e^\lambda_\mu(p) , \]  

(6.78)
where

\[
\tilde{G}_{\rho\mu} = F_1((p\cdot q)g_{\rho\mu} - p_\rho q_\mu) + F_2(q_\rho q_\mu - q^2 g_{\rho\mu}) \\
+ \ A'p_\rho p'_\rho - q_\rho (q^2 p')/q^2 + C p'_\rho q_\rho + C' p_\rho q_\rho \\
+ \ B_1 q^2 \omega_\mu p'_{\rho}/\omega p + B_2 p'_\mu \omega_\rho + B'_2 p_\mu \omega_\rho + B_3 \omega_\mu q_\rho + B_4 \omega_\mu \omega_\rho .
\] (6.79)

The terms \(A', C', B_2\) do not contribute to eq. (6.4), since \(p^\mu e^\lambda_\mu(p) = 0\). They appear however before contraction with \(e^\lambda_\mu(p)\). The term \(C p'_\mu q_\rho\) appears, since an approximate amplitude is not gauge invariant. It should be also separated. Its appearance has nothing to do with the \(\omega\)-dependence or LFD. In the Feynman approach its separation is equivalent to the construction, by hand, of a gauge invariant amplitude by the following subtraction:

\[ G_{\rho\mu} \rightarrow G_{\rho\mu} - (G_{\rho'\mu} q^\rho')q_\rho/q^2. \]

One can construct other gauge non-invariant terms, proportional to \(p'_\mu p_\rho\), \(q_\mu p_\rho\) and \(q'_\mu p_\rho\), but they are expressed through the structures included in (6.79). The term \(V_\mu V_\rho\) with \(V_\mu = e_\mu_{\alpha\beta\gamma}\omega^\alpha q^\beta q^\gamma\) is also expressed through them.

Solving eq. (6.79) relative to \(F_1, F_2\) one gets:

\[
F_1 = \frac{\tilde{G}_{\rho\mu}}{q^2} \left[ - q^\mu \omega^\rho \right. + (m_{ps}^2 - m_{pv}^2 - q^2) \frac{\omega^\mu \omega^\rho}{2(\omega p)^2} \left. \right], \] (6.80)

\[
F_2 = \frac{\tilde{G}_{\rho\mu}}{(q^2)^2} \left\{ \left[ (m_{ps}^2 - m_{pv}^2)^2 - q^2 (2m_{ps}^2 - q^2) \right] \frac{\omega^\mu \omega^\rho}{2(\omega p)^2} + (m_{pv}^2 - m_{ps}^2 + q^2) \frac{q^\mu \omega^\rho}{2(\omega p)} \right. \\
+ \left. (m_{pv}^2 - m_{ps}^2) \frac{\omega^\mu q^\rho}{(\omega p)} + q^2 \frac{\omega^\mu p^\rho + \omega^\rho p^\mu}{(\omega p)} + q^\mu q^\rho - q^2 g^{\mu\rho} \right\}. \] (6.81)

Here \(m_{ps}\) and \(m_{pv}\) are the masses of pseudo-scalar and pseudo-vector mesons.

**Comparison with other approaches**

In ref. [108] the form factors are found from the component \(\rho = +\). Contracting (6.77) with \(\omega_\rho\), we get:

\[
\langle PS(p')|J_\rho|PV(p, \lambda)\rangle \omega^\rho = [F_1((p\cdot q)\omega^\mu - (\omega \cdot p)q^\mu) - F_2 q^2 \omega^\mu] e^\lambda_\mu(p). \] (6.82)

Contracting (6.78) with \(\omega_\rho\), we reproduce the equation (6.82) with new form factors \(F'_1, F'_2\) given by:

\[
F'_1 = F_1, \quad F'_2 = F_2 - B_1 .
\]

Hence, the \(J^+\) component provides \(F_1\) without any spurious contributions, while the form factor \(F'_2\) is determined by:

\[
F'_2 = F_2 - B_1 = \frac{\tilde{G}_{\rho\mu}\omega^\rho}{q^2(\omega p)} \left[ m_{pv}^2 \omega^\mu - p^\mu \right]. \] (6.83)

It includes the spurious contribution \(B_1\).
Chapter 7

Electromagnetic observables

The formalism and methods presented above are applied to different physical systems and processes: pion, nucleon and deuteron electromagnetic form factors, $\rho - \pi$ and $ed - enp$ transitions. With standard assumptions and approximations, we easily reproduce results given in the literature. Some of these results, however, like the asymptotic behavior of the pion form factor and meson-exchange current contributions in $ed - enp$ amplitude, obtain a new and clear physical interpretation in our approach, since they directly reflect the presence of extra relativistic components in the corresponding wave functions.

7.1 Electromagnetic form factors in the Wick-Cutkosky model

The Wick-Cutkosky model\(^{[59]}\) allows to get the explicit form of both the Bethe-Salpeter functions and the light-front wave functions for states with angular momentum $J = 0$ and $J = 1$. This gives us the opportunity to compare the electromagnetic form factors calculated by two methods: by the standard one, with the use of the Feynman rules and through the Bethe-Salpeter amplitude, and by means of the light-front graph technique through the light-front wave function. This comparison is especially non-trivial in the $J = 1$ case, where the physical form factors should be calculated after separating out the non-physical ($\omega$-dependent) amplitude from the physical one.

7.1.1 Spin 0

For a spin-0 system consisting of spinless constituents, the form factor is expressed through the light-front wave function by eq. (6.14). Its expression through the Bethe-Salpeter function $\Phi(k, p)$ corresponding to the Feynman graph of fig. 21 reads:

$$
(p + p')_{\rho} F(t) = i \int (p + p' - 2k)_{\rho} \Phi(\frac{1}{2}p - k, p) \Phi(\frac{1}{2}p' - k, p')(m^2 - k^2) \frac{d^4k}{(2\pi)^4},
$$

(7.1)
Fig. 21. Feynman electromagnetic vertex of a two-body system.

| $|t|/m^2$ | $F(t)$  | BS    | LF    | NR    |
|---------|---------|-------|-------|-------|
| 0       | $1.00E+00$ | $9.67E-01$ | $1.00E+00$ |
| 0.01    | $5.14E-01$ | $4.98E-01$ | $5.17E-01$ |
| 1       | $6.02E-04$ | $5.75E-04$ | $6.23E-05$ |
| 25      | $1.01E-06$ | $9.76E-07$ | $1.05E-06$ |
| 100     | $6.40E-08$ | $6.20E-08$ | $6.55E-08$ |
| 225     | $1.28E-08$ | $1.24E-08$ | $1.29E-08$ |

Table 7.1: Electromagnetic form factor of a system with spin $J = 0$ in the Wick-Cutkosky model for $\alpha = 0.08$. Columns BS, LF and NR represent the results of the calculation using the Bethe-Salpeter amplitude (7.2), the light-front wave function (3.64) and non-relativistic wave function (3.62) respectively.

The Bethe-Salpeter function is given by eq. (3.60). Substituting here the spectral function (3.69), one can find the explicit form of the Bethe-Salpeter amplitude:

$$
\Phi(k,p) = -ic \left[ \left( m^2 - \frac{1}{2}M^2 - k^2 \right) \left( m^2 - \frac{1}{2}(p+k)^2 - i0 \right) \left( m^2 - \frac{1}{2}(p-k)^2 - i0 \right) \right]^{-1},
$$

(7.2)

where $c = 2^5 \sqrt{\pi}m\kappa^5$ with $\kappa = \sqrt{m|\epsilon_b|} = ma/2$.

The analytical (in the asymptotical region) and numerical comparison of the form factors calculated through the light-front wave function, eq. (6.14), and through the Bethe-Salpeter amplitude, eq. (7.1), was carried out in ref. [85]. The form factors calculated by these two approaches are compared in table 7.1.

Both approaches give the same asymptotical behavior of the form factors at $|t| \gg m^2$:

$$
F(t) \approx \frac{16\alpha^4m^4}{t^2} \left[ 1 + \frac{\alpha}{2\pi} \log \left( \frac{|t|}{m^2} \right) \right],
$$

(7.3)
where $\alpha = g^2/(16\pi m^2)$, $g$ is the coupling constant in the Wick-Cutkosky model.

In contrast to ref. [85] the LF form factor in table 7.1 has not been renormalized at $t = 0$. In the light-front approach, the form factor at low $t$, as well as the normalization, is smaller than the Bethe-Salpeter one by $4\alpha/3\pi$. For the normalization, the discrepancy has its origin in the contribution of the derivative of potential to the norm operator, as indicated in eq. (3.50). For the form factor, this corresponds to omitted contributions indicated in fig. 22 (the so-called recoil contributions). If corrections to the form factor and to the norm (which coincide with each other at $Q^2 = 0$) are included, then the renormalization of the light-front result, as done in ref. [85], is not required.

At high $t$, the results obtained in LFD and in the Bethe-Salpeter approach are very similar. Non-relativistic calculation is also close to them, when $(\alpha/2\pi) \log(|t|/m^2) \ll 1$. This is a peculiarity of the Wick-Cutkosky model. In the asymptotical region where $(\alpha/2\pi) \log(|t|/m^2) > 1$ LFD and Bethe-Salpeter form factors are still close to each other, while they differ from non-relativistic one.

The asymptotics of the non-physical form factor $B_1$, which appears in the decomposition (6.3) and is given by eq.(6.13), has the form:

$$B_1(t) \approx \frac{32\alpha^4 m^4}{t^2}. \quad (7.4)$$

In the intermediate region, $B_1$ exceeds $F : B_1(t)/F(t) \approx 2$ while it is negligible in the non-relativistic region. According to [85], $B_1$ becomes comparable with the physical form factor $F(t)$ at $|t| \sim m^2$. Therefore its separation is important. In the spin-0 case it is separated automatically, if one proceeds from the $\rho = +$ component of the current. However, this is not so for the spin-1 case.

### 7.1.2 Spin 1

For a system with angular momentum $J = 1$ made out from two spinless constituents, the light-front matrix elements $\tilde{J}_\mu^{\nu\rho}$ of the current are obtained from the spin-0 case equation (6.12) with the replacement of the spin-0 wave functions $\psi', \psi$ by the spin-1 wave functions $\psi_{1\mu}, \psi_{2\nu}$. In four-dimensional notations, $\psi_{1\mu}$ has the form:

$$\Phi^\lambda(k_1, k_2, p, \omega, \tau) = e^{(\lambda)}(p) \psi_{1\mu}, \quad \psi_{1\mu} = \varphi_1 \frac{(k_1 - k_2)^\mu}{2} + \varphi_2 \omega^\mu \cdot p. \quad (7.5)$$

The explicit form of the components $\varphi_{1,2}$ in (7.3) can be found from the comparison of (7.3) with (3.65) in the reference frame where $\vec{k_1} + \vec{k_2} = 0$. Substituting $\tilde{J}_\mu^{\nu\rho}$ in eqs.(6.54)-(6.56), in order to separate the nonphysical contributions, we find three form factors $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1$.

On the other hand, in the Bethe-Salpeter approach, the electromagnetic vertex is directly related to the physical form factors by the decomposition (5.40) and can be obtained from (7.1) by the replacement of the spin-0 Bethe-Salpeter amplitude by the
Fig. 22. Amplitudes corresponding to admixtures of the three-body sector in the state vector. These amplitudes correspond to retardation effects.
spin-1 function. The latter has the following form:

\[
\Phi^\lambda(k, p) = \epsilon_\mu^\lambda(p) k^\mu \Phi_1(k, p),
\]

\[
\Phi_1(k, p) = -ic_1 \left[ \left( m^2 - \frac{1}{2} M^2 - k^2 \right)^2 \left( m^2 - \left( \frac{1}{2} p + k \right)^2 - i0 \right) \right]^{-1},
\]

where \( c_1 = 2^{7/2} \sqrt{\pi} \kappa_1^{7/2} \), \( \kappa_1 = \sqrt{m|\epsilon_b|} = m\alpha/4 \).

Fig. 23. The form factor \( G_1 \) in the Wick-Cutkosky model. The solid line represents the calculation with the Bethe-Salpeter function (7.6) and by eq. (6.5) with the light-front wave function (3.6). Dashed line illustrates the form factor incorporating the non-physical contribution by eq. (6.6) (solution A from ref. [98]).

The form factor \( G_1 \) calculated in the two approaches in ref. [56] is shown by the solid line in fig. 23. The two calculations give the curves indistinguishable from each other. In order to show the importance of a proper separation of the non-physical contribution we show also in fig. 23 the form factor corresponding to the solution A from ref. [98] (dashed line). It is given by eq. (6.6) and contains the non-physical contributions \( \Delta \) and \( B_6 \).
considerably differs from the correct form factor. According to ref. [86], this difference in
the model is mainly due to $\Delta$, whereas the $B_6$ contribution is negligible.

In the asymptotical region, these Bethe-Salpeter and light-front form factors are
given by the following analytical expressions [85]:

$$
\mathcal{F}_1 \approx -\frac{\alpha^6 m^6}{|t|^3} \left[ 1 + \frac{\alpha}{4\pi} \log \left( \frac{|t|}{m^2} \right) \right],
$$

$$
\mathcal{F}_2 \approx -\frac{48\alpha^6 m^8}{t^4} \left[ 1 + \frac{\alpha}{4\pi} \log \left( \frac{|t|}{m^2} \right) \right],
$$

$$
\mathcal{G}_1 \approx \frac{\alpha^6 m^6}{|t|^3}.
$$

We emphasize that, generally speaking, using two-body Bethe-Salpeter function
does not correspond to two-body Fock component, but incorporates implicitly many-body
sectors of the state vector.

### 7.2 Applications to the quark model

We shall investigate in this section a few simple examples on the application to the quark
model of the formalism developed in this review. We shall discuss in some details the
pion wave function and form factors, and give some results for other observables.

#### 7.2.1 The pion form factor

**Pion wave function**

The spin structure of the pion wave function as a system consisting of quark and antiquark
in the $J^\pi = 0^-$ state is identical to the $np$ wave function in $J^\pi = 0^+$ state, eq.(5.33). The
negative parity is generated automatically by opposite internal parities of the quark and
antiquark and therefore does not change the spin structure. The pion wave function has
thus the form:

$$
\psi = \frac{1}{\sqrt{2}} \bar{u}(k_2) \left[ A_1 \frac{1}{m} + A_2 \frac{\hat{\omega}}{\omega \cdot p} \right] \gamma_5 v(k_1) = \bar{u}(k_2)Ov(k_1),
$$

(7.10)

where $O = (A_1/m + A_2\hat{\omega}/\omega \cdot p)\gamma_5$, and $m$ is the quark mass. For simplicity, we do not take
into account isospin.

The representation of this wave function in terms of the variables $\vec{k}$ and $\vec{n}$ is almost
identical to (5.34):

$$
\psi = \frac{1}{\sqrt{2}} w'_2 \left( g_1 + \frac{i\hat{\sigma} \cdot [\vec{n} \times \vec{k}]}{k} g_2 \right) w_1,
$$

(7.11)

with the following relations between the invariant functions:

$$
A_1 = -\frac{m}{2\varepsilon_k} (g_1 + \frac{m}{k} g_2), \quad A_2 = \frac{\varepsilon_k}{k} g_2.
$$

(7.12)
The normalization condition is a particular case of (3.23):

$$N_2 = \int \sum_{\lambda_1,\lambda_2} \psi_{\lambda_1,\lambda_2}^* \psi_{\lambda_1,\lambda_2} D,$$

(7.13)

where

$$D = \frac{1}{(2\pi)^3} \frac{d^3 k_1}{(1-x)2\varepsilon_{k_1}} = \frac{1}{(2\pi)^3} \frac{d^3 k}{\varepsilon_k} = \frac{1}{(2\pi)^3} \frac{d^2 R_\perp dx}{2x(1-x)}.$$

(7.14)

Substituting in eq. (7.13) the wave function represented by (7.10) and (7.11), we get:

$$N_2 = \int \left[ \frac{(m^2 + \vec{R}_2^2)}{m^2x(1-x)} A_1^2 + 4A_1A_2 + 4x(1-x)A_2^2 \right] D = \int [g_1^2 + (1-z^2)g_2^2] D,$$

(7.15)

where $z = \cos(\vec{k} \vec{n})$. Obviously, in the case where the interaction potential between quarks is independent of $M$, as often assumed in phenomenological approaches, $N_2$ should be equal to one.

**Pion form factor**

The diagrammatical representation of the pion electromagnetic current in the impulse approximation almost coincides with fig. 18. For the interaction of a virtual photon with a quark, the horizontal line has to be replaced by the double line of antiquark. This current has the form:

$$\tilde{J}_{\gamma q}^\rho = \int Tr[-\vec{\sigma}(\vec{k}_2' + m)j_\rho(\vec{k}_2 + m)O(m - \hat{k}_1)] \frac{1}{(1-x)} D,$$

(7.16)

where $O$ depends on the initial momenta, and $O'$ depends on the final momenta. The quark current $j_\rho$ is taken as:

$$j_\rho = f_1 \gamma_\rho + i f_2 \sigma_{\rho\nu} q^{\nu}.$$

(7.17)

In this equation $f_1$ and $f_2$ are the quark form factors. The minus sign in $Tr[-\vec{\sigma} \cdots]$ is from the fermion loop. We remind that both lines in the loop, single and double, corresponding to the quark and antiquark, are in the same direction. But since, according to the rules of the graph technique, they are followed in the same and opposite orientations respectively, one makes a loop when going through them.

The pion form factor for the $\gamma q$ interaction is given by:

$$F_{\gamma q}(Q^2) = \int Tr[-\vec{\sigma}(\vec{k}_2' + m)(\omega \cdot j)(\vec{k}_2 + m)O(m - \hat{k}_1)] \frac{1}{2\omega p(1-x)} D.$$

(7.18)

The pion form factor for the $\gamma \bar{q}$ interaction is found similarly:

$$F_{\gamma \bar{q}}(Q^2) = \int Tr[-O(m - \hat{k}_2)(\omega \cdot j)(m - \hat{k}_2')\vec{\sigma}(m + \hat{k}_1)] \frac{1}{2\omega p(1-x)} D.$$

(7.19)
The momenta $k_2, k'_2$ in (7.16) are the quark momenta, whereas they are the antiquark momenta in (7.19).

After a trivial generalization for the case of different masses of the quark and antiquark, these formulas can be applied to the $K$-meson form factors. The final form factor is expressed through the momentum transfer $\vec{\Delta}$ (with $Q^2 = \vec{\Delta}^2$) and the integration variables $\vec{R}_\perp, x$ with the use of the invariants indicated in appendix D.

Let us for a moment put $A_2 = 0$. We thus have:

$$F^{\gamma q}(Q^2) = 2 \int \left[ (m^2 + \vec{R}_\perp \vec{R}'_\perp) f_1 - \frac{\vec{\Delta}^2}{2} x f_2 \right] \frac{A_1 A'_1}{x(1-x)m^2} D. \tag{7.20}$$

With $A_1$ expressed through $g_1$ by (7.12) the form factor (7.20) exactly coincides with the expressions given in ref. [109]. We emphasize that the calculation of (7.20), and more generally of any form factor or transition amplitude is, due to the covariance of our approach, a simple routine for analytical computer calculations.

When $A_2$ is different from zero, but for pointlike quarks, i.e. with $f_1 = 1, f_2 = 0$ in the current (7.17), the calculation of (7.18) gives:

$$F^{\gamma q}(Q^2) = \int \left[ \frac{m^2 + \vec{R}_\perp^2 - x \vec{R}_\perp \cdot \vec{\Delta}}{x(1-x)m^2} A_1 A'_1 + 2(A_1 A'_2 + A'_1 A_2) + 4x(1-x)A_2 A'_2 \right] D, \tag{7.21}$$

where $A'_{1,2} = A_{1,2}(\vec{R}_\perp - x \vec{\Delta}, x)$. We shall use this expression in the following to evaluate the asymptotical behaviour of the pion form factor.

Note that for the interaction of the virtual photon with the antiquark, in eq.(7.19), we get: $F^{\gamma \bar{q}} = -F^{\gamma q}$. Hence, the full form factor of a system consisting of identical $q\bar{q}$ is zero: $F^{\gamma q}(t) + F^{\gamma \bar{q}} = 0$. We get opposite sign of $F^{\gamma \bar{q}}$ automatically, without introducing negative charge of antiquark. This negative charge is automatically taken into account in this formalism. It can be obtained from the antiquark contribution, multiplying it by a sign factor depending on the process under consideration.

For the $\rho - \pi$ transition, the quark and antiquark contributions have the same sign. The transition amplitude $\tilde{F}_{\mu \rho}$ can be obtained in this case from the $\pi$ elastic amplitude $J_\rho$ by replacing in eqs.(7.10) the initial pion wave function $O$ by the wave function of the vector meson $\phi_\mu$. The form of this wave function coincides with the deuteron wave function (5.1).

### Asymptotical behaviour of the pion form factor

It is well known that QCD provides a $1/Q^2$ asymptotical behaviour for the pion form factor [40]. As it was shown above, the pion wave function contains in general two components. The first one leads to the usual non-relativistic wave function, while the second one is of purely relativistic origin and $\omega$-dependent. We shall show in this section that the asymptotical $1/Q^2$ behaviour of the pion form factor is entirely determined by this last
\(\omega\)-dependent component of the pion wave function [110]. This piece is essential in order to explain the difference with the form factor of the \(J = 0\) state in the Wick-Cutkosky model, which asymptotical behaviour is \(1/Q^4\).

The asymptotical behaviour of the form factor can be related to the asymptotical behaviour of the wave function. We shall calculate the latter in a \(qq\) model with a one-gluon exchange kernel. The equation for the wave function corresponding to the diagram indicated in fig. [11] has the form:

\[
\psi(k_1, k_2, p, \omega \tau) = -\frac{1}{s - M^2} \int \frac{1}{(2\pi)^3} \delta^{(4)}(p + \omega \tau' - k'_1 - k'_2)2(\omega p) d\tau' \\
\times \bar{u}(k_2) \gamma_{\mu}(\hat{k}'_2 + m)\theta(\omega \cdot k'_2)\delta(k'_2^2 - m^2)d^4k'_2 O'(m - \hat{k}'_1)\theta(\omega \cdot k'_1)\delta(k'_1^2 - m^2)d^4k'_1 \gamma^{\mu}v(k_1) \\
\times K(k'_1, k'_2, \omega \tau'; k_1, k_2, \omega \tau).
\] (7.22)

Here \(M\) is the pion mass. We use the gluon propagator in the Feynman gauge and \(K\) is the scalar part of the gluon propagator (after separation of \(-g_{\mu\nu}\)). It coincides with the scalar \(t\)-channel exchange amplitude, given by eq. (2.67) (where one should put \(\mu = 0\)). The matrix \(O'\) is the “inner” part of the pion wave function as defined in eq. (7.10). Prime means that it depends on the momenta in the loop. For simplicity, we omit here any color degrees of freedom which are irrelevant for our consideration.

To find the asymptotical behaviour of the wave function, we shall use the iterative procedure already explained in chapter 3 to find the deuteron wave function. We suppose that \(O'\) is concentrated in a finite region of the quark momenta and the integral (7.22) is dominated by this domain. These momenta are negligible relative to the asymptotical ones we are interested in. This means that we can evaluate all the factors except for \(O'\) and the delta-functions at zero relative quark momentum. This corresponds to \(\bar{R}'_\perp = 0\) and \(x'_1 = x'_2 = 1/2\). In that case, the quark four-momenta are equal to each other:

\[k'_1 = k'_2 = (p + \omega \tau_0)/2,\]

where \(\tau_0 = (4m^2 - M^2)/(2\omega p)\). We thus get the following replacement in (7.22):

\[
\int \frac{1}{(2\pi)^3} \delta^{(4)}(p + \omega \tau' - k'_1 - k'_2)2(\omega p) d\tau' \\
\times \theta(\omega \cdot k'_2)\delta(k'_2^2 - m^2)d^4k'_2 O' \theta(\omega \cdot k'_1)\delta(k'_1^2 - m^2)d^4k'_1 \Rightarrow O_0,
\] (7.23)

where \(O_0\) is the integrated value of the wave function.

We substitute in (7.22) the one-component wave function obtained from (7.10) with \(A_2 = 0\). Together with the replacement (7.23), this gives in (7.22) \(O' = A_0\gamma_5/m\), where \(A_0\) is a constant. The second component \(A_2\) of the wave function is then automatically generated by eq. (7.22).

We thus find for the pion wave function:

\[
\psi = -\frac{2A_0}{s - M^2} \bar{u}(k_2)(\hat{p} + \omega \tau_0 - 4m)\gamma_5 v(k_1)K.
\] (7.24)
Substituting here $p$ from the relation $p + \omega \tau = k_1 + k_2$ and using the Dirac equation, we finally obtain:

$$
\psi = \frac{2A_0}{s - M^2} \bar{u}(k_2)(\hat{\omega}(\tau - \tau_0) + 2m)\gamma_5 v(k_1)K. \tag{7.25}
$$

The pion wave function (7.25) has the form of eq. (7.10) with the following two components:

$$
A_1 = \frac{4m^2}{s - M^2}A_0K, \quad A_2 \approx A_0K \tag{7.26}
$$

with $s = \frac{\vec{R}_1^2 + m^2}{x(1 - x)}$. The kernel $K$ given in (2.67) can be rewritten in terms of the variables $\vec{R}_\perp, x$. For $x \leq x'$, it reads:

$$
K(\vec{R}'_\perp, x'; \vec{R}_\perp, x, M^2) = g^2 \left[ \mu^2 + \frac{x'}{x} \left( 1 - \frac{x}{x'} \right)^2 m^2 + \frac{x'}{x} \left( \frac{\vec{R}'_\perp - x'}{x'} \frac{\vec{R}_\perp}{x' - M^2} \right)^2 \right]^{-1}, \tag{7.27}
$$

with $\mu = 0$. For $x \geq x'$, $K(\vec{R}'_\perp, x'; \vec{R}_\perp, x, M^2)$ is obtained from (7.27) by the replacement: $x \leftrightarrow x'$, $\vec{R}_\perp \leftrightarrow \vec{R}'_\perp$. As indicated above, one should then put $\vec{R}'_\perp = 0, x' = 1/2$.

We will see below that this wave function enters in the form factor at $x = 1/2$. Then $s \approx 4\vec{R}_1^2$. As follows from (7.27), the kernel $K$ at $\vec{R}'_\perp = 0, x' = 1/2, x = 1/2$ obtains the simple form: $K = g^2/\vec{R}_1^2$. The components of the wave function which dominate in the asymptotical region are then given by:

$$
A_1 = \frac{g^2m^2A_0}{\vec{R}_1^4}, \quad A_2 = \frac{g^2A_0}{\vec{R}_1^2}. \tag{7.28}
$$

In this region, the component $A_2$ decreases more slowly than $A_1$.

The form factor is now easily calculated through the components $A_1, A_2$ according to eq. (7.21). The integral for the form factor is dominated by the region where one of the two wave functions (initial or final ones) takes its non-relativistic value, whereas the other wave function corresponds to the asymptotical one. This is equivalent to the standard calculation discussed in the literature in the usual formulation of LFD [11]. Hence, similarly to eq. (7.22), we can replace $A'_1, A'_2$ in (7.21) by

$$
A'_1 = A_0\gamma_5\delta(2)(\vec{R}_\perp)\delta(x - 1/2), \quad A'_2 = 0.
$$

and take the asymptotical values (7.28) for $A_1$ and $A_2$. We thus find:

$$
F(Q^2) = \frac{A_0}{\pi^3}(2A_1 + A_2) \approx \frac{A_0}{\pi^3}A_2. \tag{7.29}
$$
The substitution of $A_2$ from (7.28) in (7.29) with $\vec{R}_\perp = \vec{\Delta}/2$ and $\vec{\Delta}^2 = Q^2$ gives:

$$F(Q^2) \approx \frac{4Q^2A_0^2}{\pi^3} \frac{1}{Q^2}.$$  

(7.30)

From (7.29) and (7.30) we see that the asymptotical behaviour $\propto 1/Q^2$ of the pion form factor is indeed determined by the extra component $A_2$ of the pion wave function, related to the $\omega$-dependent spin structure. This originates from the slower decrease of $A_2 \propto 1/R^2_\perp$ in comparison to $A_1 \propto 1/R^4_\perp$ (see eq.(7.28)). The enhancement of $A_2$ is determined by the factor $\tau(\omega \cdot p) = (s - M^2)/2 \approx 2R^2_\perp$ in (7.25). This extra factor originates from the part $\omega \tau$ in the conservation law $p + \omega \tau = k_1 + k_2$ used to derive eq.(7.23) and therefore accompanies namely the $\omega$-dependent spin structure.

### 7.2.2 The nucleon form factors

As we have already seen in the case of the two-body systems, it is convenient to parametrize the covariant light-front wave function in terms of the following three sets of variables:

(i) The four-momenta $k_i$ of each constituent. In these variables, the wave function of any three-quark system has the following form:

$$\psi = \psi(k_1, k_2, k_3, p, \omega \tau; \sigma_1, \sigma_2, \sigma_3, \sigma),$$  

(7.31)

where $\sigma$ is the nucleon spin projection and $\sigma_{1,2,3}$ are the quark spin indices, with the following conservation law [3]:

$$k_1 + k_2 + k_3 = p + \omega \tau.$$  

(7.32)

(ii) The three-vector variables $\vec{q}_i$ which are identical, in the c.m.s., to the constituent momenta. They are constructed similarly to the two-body case, (3.11):

$$\vec{q}_i = L^{-1}(P)\vec{k}_i,$$  

(7.33)

where $P = p + \omega \tau = k_1 + k_2 + k_3$ and $L^{-1}(P)$ is the Lorentz boost operator defined in (3.11). The sum of $\vec{q}_i$ gives zero, as it should be for the relative momenta: $\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0$.

(iii) The four-vectors: $R_i = k_i - x_i p$, where $x_i = \omega \cdot k_i/\omega \cdot p$. They can be represented as $R_i = (R_0, \vec{R}_{i\perp}, \vec{R}_{i\parallel})$, where $\vec{R}_{i\perp} \cdot \vec{\omega} = 0$, $\vec{R}_{i\parallel}$ is parallel to $\vec{\omega}$. We thus have the following relations:

$$\vec{R}_{i\perp}^2 = -R_i^2,$$

$$\vec{R}_{1\perp} + \vec{R}_{2\perp} + \vec{R}_{3\perp} = 0,$$

$$x_1 + x_2 + x_3 = 1.$$  

We do not pretend here to a calculation of the form factors in a realistic model for the nucleon. In order to illustrate our approach and estimate the effect of eliminating the unphysical $B_1$ contribution from the physical form factors, we consider a simple model in
which two neutral “quarks” labeled 2 and 3 are coupled to zero total angular momentum, so that the spin of the nucleon is determined by the charged “quark” labeled 1. We put the masses of all the quarks equal to each other. The corresponding wave function reads:

$$\psi(k_1, k_2, k_3, p, \omega \tau; \sigma, \sigma_1, \sigma_2, \sigma_3) = [\bar{u}^{\sigma_2}(k_2)\gamma_5 U_c \bar{u}^{\sigma_3}(k_3)] [\bar{u}^{\sigma_1}(k_1)u^{\sigma}(p)] \psi_0(k_1, k_2, k_3, p, \omega \tau), \quad (7.34)$$

where $U_c = \gamma_2 \gamma_0$ is the charge conjugation matrix. The spinor structure of (7.34) coincides with one of the invariants given in [111]. In order to show the method more clearly, we do not symmetrize the wave function (7.34) and do not introduce isospin. So, we have only one (charged) nucleon – “proton”. In eq. (7.34) $\psi_0$ is a scalar function.

![Fig.24. Nucleon electromagnetic amplitude in the three-quark model.](image)

We consider the charged quark 1 as a structureless particle coupled to the electromagnetic current: $j_\rho = \bar{u}(k'_1)\gamma_\rho u(k_1)$. The electromagnetic nucleon vertex (1.24) is shown in fig. 24 and has the form:

$$\tilde{J}_\rho = \bar{u}' \Gamma_\rho u = \int [\bar{u}(p')(k'_1 + m)\gamma_\rho (k_1 + m)u(p)] Tr[(\hat{k}_2 + m)(\hat{k}_3 + m)] \psi_0 \psi'_0 D(R_i, x_i). \quad (7.35)$$

Here $\psi_0$ and $\psi'_0$ depend on the initial and final momenta respectively. Details of the calculation can be found in ref. [88].

At zero momentum transfer, the difference $G'_M(0) - G_M(0)$ obtains the simple form:

$$G'_M(0) - G_M(0) = 2B_1(0) = -\frac{\langle \vec{k}^2 \rangle}{3m^2}. $$

Here $\langle \vec{k}^2 \rangle$ is the average quark momentum. The difference between $G'_M$ and $G_M$ is of course of relativistic origin.

We indicate in fig. 24 the ratio $R = G'_M/G_M$ of the physical form factors $G_M$ obtained with our formalism and the form factor $G'_M$ which would be obtained with the usual procedure (using the $J_+$ component of the current), as discussed in section 3.3.
Fig. 25. The ratio $G_M/G_M'$ as a function of $Q^2$ in $(GeV/c)^2$ [88].
Both form factors are calculated with a simple harmonic oscillator radial wave function \cite{8}. We recall that the electric form factor $\tilde{G}_E$ is the same in both approaches.

The difference between $G_M$ and $G'_M$ is of relativistic origin. It is already sizable at $Q^2 = 0$ for the magnetic moment (enhancement of about 15\%). The two form factors largely differ at high momentum transfer, where corrections of the order of 50\% are to be expected. The physical form factor $G_M$, i.e. the form factor free from any ambiguity from the position of the light front, is always larger than $G'_M$ in this model. According to fig. 25, the magnetic radius is larger for $G_M$ than for $G'_M$.

### 7.3 Application to the nucleon-nucleon systems

#### 7.3.1 Light-front dynamics and meson-exchange currents

**Non-relativistic phenomenology**

The qualitative, and to a large extent also quantitative, understanding of electromagnetic observables in the few-body systems in terms of meson-exchange currents is now well established \cite{6}. This is particularly transparent for isovector magnetic transitions like the deuteron electrodisintegration cross-section near threshold. These two-body currents appear in that case already at order $1/m$, while they are of higher order for charge and isoscalar magnetic transitions. Last but not least, these currents are also required by the low-energy soft-pion theorems.

In order to extend these non-relativistic analyses to a wide range of momenta, let say for $Q^2 > M^2$, a relativistic framework is necessary. In order to be fully consistent, the relativistic analyses have to give, as a limiting case, the well-known results in terms of meson-exchange currents. While genuine two-body currents like the mesonic contribution indicated in fig. 26 has to be considered explicitly in both LFD and non-relativistic formalisms, some of them may appear differently depending on the underlying framework. Physical amplitudes should however be the same.

This is the case for the so-called pair term which, in the non-relativistic approach, is indicated on fig. 24 (we assume here that the $\pi NN$ vertex is of the pseudo-scalar type, we shall come back to this point at the end of this section). This contribution cannot survive on the light front because of the absence of vacuum fluctuation diagrams, together with the particular constraint $\omega q = 0$. A careful analysis of the deuteron electrodisintegration amplitude near threshold in LFD is therefore of particular interest.

**Relativistic impulse approximation**

The vertex $d + \gamma^* \rightarrow np(1S_0)$ is represented graphically in fig. 28. It corresponds to the following amplitude:

$$\tilde{F}_{\mu\rho} = c \int \frac{Tr[\gamma_5(\not{k}_2 + m)\Gamma_\rho(\not{k}_2 + m)\phi_\mu(\not{k}_1 - m)]}{(1 - \omega k_1/\omega p)^2} \varphi f \frac{d^3k_1}{\varepsilon_{k_1}}, \quad (7.36)$$
Fig. 26. Two-body meson exchange current: the mesonic current.

Fig. 27. Two-body meson exchange current: the pair term.
where \( c = i2^{-11/2}\pi^{-3}\sqrt{m} \), \( \phi_\mu \) is the deuteron wave function (5.2), \( \varphi_f \) is the final state wave function and \( \Gamma_\rho \) is the nucleon electromagnetic vertex (6.16). The superscript \( V \) refers to the isovector part of the form factors which is the only one which contributes here. In the plane wave approximation, we simply have:

\[
\bar{F}_{\mu\rho} = i\sqrt{2}m Tr[\gamma_5(\hat{k}_f + m)\Gamma^V_\rho(\hat{k}_2 + m)\phi_\mu(\hat{k}_1 - m)].
\] (7.37)

The expression of the electromagnetic current used here involves the Pauli and Dirac form factors \( F_1 \) and \( F_2 \). They differ off-energy-shell from the Sachs form factors used in ref. [106]. The present one for its \( F_1 \) part is consistent with the minimal substitution in the Dirac equation. Sachs form factors may be used, but extra terms have then to be introduced in the current to get the exact result.

The amplitude \( A \), determining the cross section (6.68) is found by the formula (6.72). In the plane wave approximation and for zero c.m.-energy in the final state, the amplitude \( A \) calculated with the deuteron wave function (5.2) has the form:

\[
A = \frac{m^{5/2}}{\bar{\varepsilon}_k}(2G_M^V\varphi_2 - G_E^V\varphi_5)
\] (7.38)

with \( k = \Delta/2 \). We introduce in (7.38) the charge and magnetic nucleon form factors according to (6.17). The deuteron components \( \varphi_{1,3,4,6} \) do not contribute to (7.38). Expressing \( \varphi_2, \varphi_5 \) through \( f_1, f_2, f_5 \) by eqs. (C.4), (C.7) (and neglecting the small contributions from \( f_3 \) and \( f_4 \)) we get:

\[
A = 2m^{3/2}\left\{ G_M^V[u_S(\Delta/2) + \frac{1}{\sqrt{2}}u_D(\Delta/2)] - G_E^V\frac{\sqrt{3}m}{\Delta}f_5(\Delta/2) \right\},
\] (7.39)
where $\Delta = \sqrt{Q^2}$.

Neglecting $f_5$ in eq. (7.39), we recover the usual expression in the plane wave approximation.

**Meson exchange currents and $1/m$ expansion**

In order to make the connection with the non-relativistic phenomenology, it is instructive to calculate the $f_5$ component in leading order in a $1/m$ expansion. Following the iterative procedure used in refs. 48, 71, and recalled in section 5.4, the $f_5$ component has the following form in momentum space and in $1/m$ order [72]:

$$f_5[k \times \vec{n}]_i = \frac{-3g_{\pi NN}^2\sqrt{2}}{8\sqrt{3}\pi m^2} \int \frac{d^3k'}{(k^2 + \kappa^2) \mu^2 + (\kbar - \kbar')^2} \left[ \frac{1}{\sqrt{2}} u_S(k')\delta_{ij} - \frac{1}{2} \frac{3k'k_j}{k^2} - \delta_{ij} u_D(k') \right] [(\kbar - \kbar') \times \vec{n}]_j . \quad (7.40)$$

In the limit where the momentum $k'$ is restricted by the integration domain to low momenta, one can take $(k^2 - \kbar'^2)/(k^2 - \kappa^2) \approx 1$. We shall come back to this approximation later on.

The coordinate space integral for the $f_5$ component has the form:

$$f_5(k) = \frac{-3g_{\pi NN}^2}{8\sqrt{3}\pi m^2} \int_0^\infty \frac{\exp(-\mu r)}{r} (\mu r + 1) j_0(\mu r) \left[ u(r) + \frac{1}{\sqrt{2}} w(r) \right] dr , \quad (7.41)$$

where $u$ and $w$ are the S and D state deuteron wave functions in coordinate space. The extra component $g_2$ of the final state wave function can be calculated simply from isospin arguments, leading to the replacement $-3g_{\pi NN}^2 \rightarrow -(3 + 1)g_{\pi NN}^2 = -4g_{\pi NN}^2$ in the amplitude.

The amplitude $A$ has thus the following form, in coordinate space:

$$A = 4\pi^{1/2}m^{3/2} \left\{ G_M \int_0^\infty \left[ u(r)j_0\left(\frac{r\kappa}{2}\right) - \frac{1}{\sqrt{2}} w(r)j_2\left(\frac{r\kappa}{2}\right) \right] r dr \right\} + \frac{G_V^L g_{\pi NN}^2}{m\Delta} \int_0^\infty \frac{\exp(-\mu r)}{r} (\mu r + 1) \left[ u(r) + \frac{1}{\sqrt{2}} w(r) \right] j_1\left(\frac{r\Delta}{2}\right) dr . \quad (7.42)$$

The structure of the second term is that of a two-body contribution of order $g_{\pi N N}^2/4\pi$. It is similar to the one given by the pair term in the non-relativistic MEC analyses 6, 62, 112, 113. However, its strength is smaller by a factor of 2.

**Contribution from the contact term**

The one to one correspondence with the non-relativistic phenomenology in terms of MEC is indeed recovered if one collects all the contributions of order $g_{\pi NN}^2/4\pi$ to the light-front amplitude. The missing contribution originates in this particular example from the
contact term which appears once we have singled out the one-pion exchange mechanism in generating the $f_5(g_2)$ component of the deuteron ($^1S_0$) wave function. It is indicated in fig. 29. There are four diagrams corresponding to the contact terms. Two of them, $29(a)$ and $(b)$, correspond to the contact interaction at the left of the photon line. Two other diagrams, $29(c)$ and $(d)$, correspond to the right contact term with the two possible time orderings for the lower vertex. Each diagram can be calculated in terms of the kinematical invariants given in section D.2 of appendix D. In leading order in $1/m$, the contact term contribution is exactly equal to the contribution from the $f_5(g_2)$ components, and therefore the total amplitude of the deuteron electrodisintegration near threshold is strictly equivalent in LFD and in the non-relativistic phenomenology including meson-exchange currents (more precisely the pair term ) [72].

This analysis can even be extended to the whole domain of momenta in eq.(7.40), i.e. without neglecting the factor $(\vec{k}^2 - \vec{k}'^2)/(\vec{k}^2 + \kappa^2)$ in the integrand. This term gives in fact a wrong asymptotical behaviour of the wave function. As shown above in sect. 4.3.2 and in ref. [81], incorporation of the complete contact interaction which arises with the other components of the kernel (NN potential) like $\sigma$ and $\omega$ exchange and of a higher order iterations, gives a correction which turns the factor $(\vec{k}^2 - \vec{k}'^2)/(\vec{k}^2 + \kappa^2)$ into 1.

For the deuteron electrodisintegration, the dominant component $\varphi_5$ of the deuteron wave function does not contribute to the amplitude corresponding to the contact term. For the elastic $ed$ scattering with the representation (6.16) for the nucleon electromagnetic vertex $\Gamma_\rho$, the contact terms contribute to the deuteron electromagnetic form factor $G_1$ only and do not contribute to $F_1$ and $F_2$.

The relevance of $\pi$ exchange. Numerical results

The $1/m$ analysis of the relativistic contributions to the deuteron electrodisintegration amplitude is based on the $\pi$-exchange interaction with the pseudo-scalar $\pi NN$ vertex. It is also well known that, in leading $1/m$ order, the pair contribution associated with the pseudo-vector $\pi NN$ coupling is zero. After minimal substitution in the $\pi NN$ vertex, generating a genuine $NN\pi\gamma$ current, the deuteron electrodisintegration amplitude is however equivalent to the one given in the pseudo-scalar representation. In LFD, the equivalence between the two representations is realized in the following way. In the pseudo-vector representation with $\pi$-exchange only, the $f_5$ component in the deuteron wave function is strictly zero in leading order. This is due to the off-shell condition at the $\pi NN$ vertex. The contribution from the light-front contact term is also of higher $1/m$ order for this representation. On the other hand, the current originating from the direct coupling of the photon to the pseudo-vector $\pi NN$ vertex has its analogue in LFD, providing the equivalence between relativistic and non-relativistic formulations in leading $1/m$ order.

The accuracy of the $1/m$ approximation and the contribution of the pion exchange relative to all mesons are shown in fig. 30. The solid curve in fig. 30 is the ratio of $f_5$ for the pion exchange only, calculated in the $1/m$ approximation by eq. (7.40), to the $f_5$, also for the pion exchange only, but calculated beyond the $1/m$ approximation. It shows the accuracy of the $1/m$ approximation, which is very good in the non-relativistic region.
Fig. 29. Contact terms contribution to the electromagnetic interactions with the deuteron.
Fig. 30. The solid curve is the ratio of $f_5$’s in $1/m$ approximation and beyond, both for the pion exchange only. The dashed curve is the ratio of $f_5$ for pion only to $f_5$ incorporating all the meson exchanges, both beyond $1/m$ approximation.
of \( k \), but changes the result by a large factor for higher \( k \). The dashed curve is the ratio of \( f_5 \) for pion only to \( f_5 \) incorporating all the meson exchanges, both beyond the \( 1/m \) approximation. The pion exchange contributes about 60 per cent in the region \( k \approx 0.4 \) GeV/c, where \( f_5 \) dominates.

The cross section of deuteron electrodisintegration is shown in fig.31. As it was explained above, there is equivalence, in \( 1/m \) order, between the relativistic impulse approximation (including all components of the wave functions, together with the contact interaction), and non-relativistic calculations including the dominant contribution to meson-exchange current (the so-called pair term). The comparison between meson exchange contributions (pair term) and the light-front approach is shown in fig. 31. The
dotted line is the non-relativistic impulse approximation without meson exchange currents. The dashed line is the sum of the non-relativistic impulse approximation and of the contribution from the pair term for \( \pi \)-exchange only. The dot-dashed line is the relativistic impulse approximation (with \( f_5 \) and \( g_2 \)) together with the contribution from the contact term. The relativistic components \( f_5, g_2 \) and the contact term are calculated with \( \pi \)-exchange only and with the pseudo-scalar \( \pi NN \) vertex, according to the iterative procedure indicated in sect.5.4. To be consistent with this iterative procedure, we omit in the amplitude terms of higher order than \( g_{\pi NN}^2 \). The solid curve differs from the dot-dashed one by the contribution of all mesons both in \( f_5, g_2 \) and in the contact term. For the nucleon electromagnetic form factors the dipole fit [114] was used.

The deviation between the dotted and dashed lines shows the well known influence of meson-exchange currents due to the pion pair term. The deviation between the dashed and dot-dashed lines indicates that the influence of relativistic effects to both the deuteron and the scattering state wave function, and to the electromagnetic operator (contact term) are important. The comparison between dash-dotted and solid curves shows that the contribution of other mesons, in addition to the pion, considerably influence the cross section at large \( Q^2 \).

The relativistic calculations in fig. 31 are higher than non-relativistics ones. This difference is related to a deviation of relativistic scattering state wave function \( g_1 \) from non-relativistic one (see fig. 16). As seen in fig. 16, the non-relativistic scattering state wave function in momentum space, in contrast to the non-relativistic one, changes sign. On the other hand, one can expect that the recoil-type diagram, indicated on fig. 22, partially cancels the difference between both calculations. These calculations should therefore be considered as an estimate of relativistic effects and are not compared with experimental data.

The difference between the non-relativistic impulse approximation (dotted curve) and the relativistic impulse approximation, calculated with the same wave functions (neglecting \( f_5, g_2 \)), is negligible. This was already noticed in ref. [105, 106].

7.3.2 The elastic deuteron form factors

The deuteron form factors can be calculated according to eqs. (6.54)-(6.56) which express them through the tensor \( \tilde{J}^{\mu\nu} \). This tensor is expressed through the deuteron wave function by means of the rules of the graph technique given in chapter 2. In the impulse approximation this expression is given by eq. (6.49). It includes the full deuteron wave function \( \phi^\mu \), with its six components. We shall take into account below the dominant ones, \( f_1, f_2 \) and \( f_5 \). The contribution of the contact term to the tensor \( \tilde{J}^{\mu\nu} \) is obtained by calculating the amplitude corresponding to the graph of fig. 29. All relevant kinematical invariants are given in appendix D.

Like in the case of the electrodisintegration cross-section, the component \( f_5 \) of the deuteron wave function was calculated by an iterative procedure, keeping the first degree of the relativistic kernel (of the order of \( g_2^2 \)). Relativistic contributions of order \( g_4 \), like terms in \( f_5^2 \), have been removed from the calculation of the form factor.
Fig. 32. The structure function $A(Q^2)$ for the deuteron. The dashed line is non-relativistic impulse approximation calculated with the Bonn wave function. The dot-dashed curve is the relativistic impulse approximation with relativistic deuteron components $f_1$ and $f_2$. The solid curve is the same as the dot-dashed one, but including the $f_5$ component (in first degree only). The dotted curve incorporates, in addition, the contact term. The nucleon form factor were taken in the dipole form \[114\].
The influence of relativistic effects on the structure function $A(Q^2)$ is shown in fig. 32. The dashed curve corresponds to the non-relativistic impulse approximation \cite{115, 116} with the S- and D-waves of the Bonn-QA wave function \cite{7}. The dot-dashed line is calculated in the light-front formalism with relativistic deuteron components $f_1$ and $f_2$ (solid lines in figs. 13 and 14), but without $f_3$. The solid curve is similar to the dot-dashed one, but includes the $f_5$ component (in first degree only). The dotted curve incorporates, in addition, the contact term (where the contributions of order $g^2$ are taken into account only).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig33}
\caption{Fig. 33. Same as fig. 32 but for the $B(Q^2)$ form factor of the deuteron.}
\end{figure}

Figure 33 shows the influence of the different approximations to $T_{20}$. The effect of incorporating the component $f_5$ is qualitatively similar to that obtained when adding the contribution of the pair current in non-relativistic calculations \cite{117}.

Figure 34 illustrates the influence of $B(Q^2)$ of the higher order terms than $g^2$. Their importance indicates the need to carry out the calculations beyond the iterative procedure.
Fig. 34. The deuteron tensor polarization $T_{20}$ at $\theta = 70^\circ$ calculated by relativistic formulas for form factors. The designations of the curves are the same as in figs. 32 and 33.
Fig. 35. The structure function $B(Q^2)$. The solid curve is the same as in fig. 33. The dashed curve incorporates the quadratic terms in $f_5$. 
The function $A(Q^2)$ is less sensitive to this contribution.

The results presented in this section should be considered only as indicative. They can be improved in many respects. An exact solution of the equation for the deuteron and scattering state wave functions is indeed feasible. It implies the redefinition of the parameters of the NN kernel. However, we believe a realistic estimate of relativistic corrections to the deuteron can be performed along the lines presented in this section. It necessitates a careful choice of the NN potential to generate the kernel (energy-dependent OBE potential for instance), and a good starting point for the non-relativistic wave functions (with kinematical relativistic corrections indicated in sect. 4.2). This may imply also the use of a PV $\pi NN$ coupling to be consistent with chiral symmetry constraints in leading order. This necessitates also a careful treatment of second order contributions, as discussed in sect.4.3.2. Finally, the calculation of electromagnetic observables should include retardation effects as indicated in fig.22.
Chapter 8

Concluding remarks

We have reviewed the general properties of the covariant formulation of LFD, and its application to relativistic few-body systems. We have first detailed the general transformation properties of the state vector, both kinematical and dynamical, and emphasized the most important features brought about by the covariant formulation of the state vector on the plane \( \omega \cdot x = 0 \). The explicit covariance provides similar advantages over the standard formulation, as the Feynman graph technique in comparison to old fashioned perturbation theory. We have presented a covariant light-front graph technique to calculate physical amplitudes.

The covariance enables us also to construct bound and scattering states of definite angular momentum, separating kinematical and dynamical aspects of this problem. The angular momentum of a system is a property of the wave function relative to transformation of the coordinate system and therefore has in our formalism a purely kinematical nature. Dynamics is involved in the composition of this angular momentum from the spins of its constituents.

We illustrate our formalism with the most simple and adequate system, the two nucleon system, and construct explicitly the deuteron wave function and the \( ^1S_0 \) scattering state. In both cases, the dominant relativistic components of the wave function originating from its dependence on the light front position (the variable \( \vec{n} \)) has been found. We also extend our formalism to the \( \pi \) and nucleon wave function in terms of valence quarks and antiquarks.

This formulation is essential in order to extract the physical amplitude for all electromagnetic transitions. This is mandatory in all practical calculations in order to avoid non-physical contributions coming from the definite choice of \( \omega \). We show how this applies to the electromagnetic form factors of spin 0, 1/2 and 1 systems, as well as for the deuteron electrodisintegration. We show also in this last example how one can recover easily, in a \( 1/m \) expansion, the well known results obtained in the non-relativistic phenomenology in terms of meson-exchange currents (the so-called pair term). The dominant \( \vec{n} \)-dependent components of the wave functions are essential in order to reproduce these results. This shows that the calculations which omit this dependence cannot be considered as realistic.

These developments have been made particularly simple, and intuitive, by the three-
dimensional nature of the formalism, and the absence of vacuum fluctuations. This enables a transparent link with the non-relativistic framework developed over the last twenty years in the microscopic structure of few-nucleon systems.

We expect many new developments in the near future along the lines presented in this review, both in nuclear and particle physics. One of them concerns the particular way to implement the independence of various results on the light-front orientation. Some restricted examples (determination of the NN-interaction, normalization of the wave function, ...) have been considered in this review to show how this can be done, but more extensive work is required. Another related aspect, which is related, concerns the construction of the generators satisfying, in the truncated Fock space, the commutation relation of the Poincaré group algebra, that also provides independence of observables on the light-front orientation.

This feature of LFD – the dependence of approximate on-shell amplitudes on $\omega$, which has to be excluded from observables, – has, however, a positive aspect. It gives a quantitative measure of the incompleteness of a given approximation. The model calculations in other approaches where this feature is absent are often still incomplete, but this does not manifest itself explicitly.

In nuclear physics, the exact calculation of the two-body wave function is worth to be done. This however necessitates to reparametrize the NN potential starting from the same relativistic formalism. This is essential in order to have quantitative predictions for electromagnetic observables in the few GeV range. In particle physics, the structure of the pion and nucleon can be investigated easily in this framework, and electromagnetic form factors calculated in the high momentum region. Application to virtual compton scattering can also be considered, as well as the extension to axial transitions. One other particularly interesting application is the structure of heavy quarkonium, like the $J/\psi$, in order to investigate the importance of relativistic dynamical corrections, i.e. the importance of higher Fock states in the $J/\psi$ wave function.

Last but not least, the application of the covariant formulation of LFD to field theory should be investigated.

Acknowledgement

One of the authors (V.A.K.) is sincerely grateful for the warm hospitality of the theory group at the Institut des Sciences Nucléaires, Université Joseph Fourier, in Grenoble and of Laboratoire de Physique Corpusculaire, Université Blaise Pascal, in Clermont-Ferrand, where part of this work was performed.
Appendix A

Notations

We use the standard covariant/contravariant notation, with implicit summation over repeated indices, according to:

\[ x^\mu = (t, \vec{x}), \quad x_\mu = g_{\mu\nu} x^\nu, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \] (A.1)

The Dirac matrices satisfy the anticommutation relation:

\[ \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}. \] (A.2)

The standard representation for the Dirac matrices and spinors is used:

\[ \gamma_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 \\ -\vec{\sigma} \\ 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \] (A.3)

where \( \vec{\sigma} \) are the usual Pauli matrices. Note the minus sign for the definition of \( \gamma_5 \).

The tensor \( \sigma_{\mu\nu} \) is:

\[ \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \] (A.4)

Throughout the review the following convention is taken:

\[ \hat{k} = k_\mu \gamma^\mu. \] (A.5)

The Dirac spinors \( u^\sigma(k) \) and \( v^\sigma(k) \) for spin 1/2 fermion and antifermion respectively satisfy the following Dirac equations:

\[ (\hat{k} - m)u^\sigma(k) = \bar{u}^\sigma(k)(\hat{k} - m) = 0, \quad (\hat{k} + m)v^\sigma(k) = \bar{v}^\sigma(k)(\hat{k} + m) = 0, \] (A.6)

with the normalization:

\[ \bar{u}^\sigma(k)u^\sigma'(k) = 2m\delta_{\sigma\sigma'} = -\bar{v}^\sigma(k)v^{\sigma'}(k). \] (A.7)
They are given by:

\[ u^\sigma(k) = \sqrt{\varepsilon_k + m} \left( \frac{1}{\bar{\sigma} \cdot \vec{k}} \right) w^\sigma, \quad v^\sigma(k) = \sqrt{\varepsilon_k + m} \left( \frac{\bar{\sigma} \cdot \vec{k}}{(\varepsilon_k + m) \sigma \cdot \vec{k}} \right) w^\sigma. \]  

(A.8)

where \( w^\sigma \) are the two-component spinors and \( \varepsilon_k = \sqrt{\vec{k}^2 + m^2} \). The summation over spin indices writes in that case:

\[ \sum_{\sigma} u^\sigma_{\alpha}(k) \bar{u}^\sigma_{\beta}(k) = (\hat{k} + m)_{\alpha\beta}, \quad \sum_{\sigma} v^\sigma_{\alpha}(k) \bar{v}^\sigma_{\beta}(k) = (\hat{k} - m)_{\alpha\beta}. \]  

(A.9)

The charge conjugation matrix is defined as: \( U_c = \gamma_2 \gamma_0 \).

We use the following properties: \( \sigma_y \sigma_y = -\sigma_y \sigma_y, \quad U_c \gamma_\mu = -\gamma_\mu U_c \), where the superscript \( t \) means transposition.

The polarization vector \( e_{\mu}^{(\lambda)}(p) \) of a spin 1 particle of mass \( m \) satisfies the following general relations:

\[ p^\mu e_{\mu}^{(\lambda)}(p) = 0, \quad e_{\mu}^{*\lambda}(p)e^\lambda_{\mu}(p) = -\delta_{\lambda\mu}, \quad \sum_{\lambda} e_{\mu}^{\lambda}(p)e^{*\lambda}_{\mu}(p) = -g_{\nu\mu} + \frac{p_\nu p_\mu}{m^2}. \]  

(A.10)
Appendix B

Relation to other techniques

B.1 Relation to the Weinberg rules

As we already mentioned, the approach considered in this report is a generalization of the usual LFD obtained by the replacement of the particular value of the light-front vector \( \omega = (1, 0, 0, -1) \) by the most general one \( \omega = (\omega_0, \vec{\omega}) \) with the only constraint \( \omega^2 = 0 \). Hence, the graph technique developed in section 2.2 is the direct generalization of the Weinberg rules [3]. These rules have been obtained from old fashioned perturbation theory in the infinite momentum frame. We shall show below for the case of a spinless particle how to transform the amplitudes calculated in the covariant light-front graph technique to the Weinberg amplitudes. This explicit transformation was found in [13].

Any diagram of the covariant light-front graph technique can be represented as a sequence of intermediate states, each of them containing, besides the particles, one spurion line and only one. An example of such contribution is indicated in fig. 36.

![Diagram](image)

Fig. 36. Amplitude with one intermediate state shown explicitly. Other intermediate states, if any, are absorbed by the amplitudes \( M_1 \) and \( M_2 \).

The expression for the corresponding amplitude has the form:

\[
M = \int M_1 M_2 \delta^{(4)}(\sum p_i - \sum k_i + \omega \tau) \delta^{(4)}(\sum p'_i - \sum k_i + \omega \tau)
\]
If a particle passes through a vertex without interaction, (like in fig. 18 the particle in the lower part of the diagram propagates without interaction in the upper vertex), the intermediate states can still be factorized. This is ensured by the following representation of the propagator:

\[
\int (\cdots) \theta(\omega\cdot k) \, \delta(k^2 - m^2) \, d^4k = \int (\cdots) (2\omega\cdot k_1) \, \theta(\omega\cdot k_1) \, \delta(k_1^2 - m^2) \, d^4k_1 \\
\times \delta^{(4)}(k_1 + \omega\tau - k_2) \, d\tau \, \theta(\omega\cdot k_2) \, \delta(k_2^2 - m^2) \, d^4k_2 ,
\]

which reduces the amplitude with the non-interacting particle to the form of eq. (B.1).

From the conservation equation \( \sum p_i + \omega\tau = \sum k_i \) in (3), it follows that:
\[
\tau = \left[ \left( \sum k_i \right)^2 - \left( \sum p_i \right)^2 \right] / \left( 2\omega \sum p_j \right).
\]

Integrating over \( dk_{0j} \) we get:
\[
M = \delta^{(4)}(\sum p_i - \sum p'_i) \int \frac{M_1 M_2 (-2\omega \cdot \sum p_i) \delta^{(4)}(\sum p_i - \sum k_i + \omega\tau) d\tau}{2\pi(\sum p_i)^2 - (\sum k_i)^2 + i\varepsilon} \prod_j \frac{d^3k_j}{2\varepsilon} .
\]

In order to transform the expression (B.3) to a form analogous to old fashioned perturbation theory in the infinite momentum frame, we introduce the variables:

\[
R_i^{\text{ext}} = p_i - y_i \sum p_j, \quad y_i = \omega \cdot p_i / (\omega \cdot \sum p_j) ,
\]
\[
R_i = k_i - x_i \sum p_j, \quad x_i = \omega \cdot k_i / (\omega \cdot \sum p_j) ,
\]

satisfying \( \sum y_i = \sum x_i = 1 \) and \( R_i \cdot \omega = R_i^{\text{ext}} \cdot \omega = 0 \) with \( R = (R_0, \vec{R}_\perp, \vec{R}_\parallel) \), where \( \vec{R}_\perp \cdot \vec{R}_\parallel = 0 \) and \( \vec{R}_\parallel \) is parallel to \( \vec{\omega} \). Taking into account that \( R^2 = -\vec{R}_\perp^2 \), we can write the denominator of eq. (B.3) into the form:
\[
\left[ \sum \frac{(R_i^{\text{ext}})^2 + m_i^2}{y_i} - \sum \frac{\vec{R}_\perp^2 + m_i^2}{x_i} + i\varepsilon \right] .
\]

This form coincides with the one given by the Weinberg rules [3]. The factor \( d^3k_j/2\varepsilon_j \) in these variables turns into \( d^2R_j dx_j/2x_j \), the domain of integration over \( dx_j \) being from 0 to 1. The expression
\[
(2\omega \cdot \sum p_i) \delta^{(4)}(\sum p_i - \sum k_i + \omega\tau) d\tau
\]
turns into
\[
2\delta^{(2)}(\sum \vec{R}_\perp) \, \delta(\sum x_i - 1) .
\]

By this way, one can transform the expression of any intermediate state. The equivalence between the expression for any amplitude in the covariant formulation of light front dynamics and the one given by the Weinberg rules is thus realized. The vectors \( \vec{R}_\perp \) play here
the role of the momenta transverse to the infinite momentum, and the variables \(x_i, y_i\) are analogous to the fractions of the particle momenta with respect to the infinite momentum.

Note that the factorization formula (B.2) is generalized for the propagator of a spin 1/2 particle as follows:

\[
\int (\ldots)(\hat{k} + m)\delta(k^2 - m^2)d^4k = \int (\ldots)\frac{\omega^k}{m}\theta(\omega \cdot k_1)\delta(k_1^2 - m^2)(\hat{k}_1 + m)d^4k_1 \\
\times \delta(\omega + \omega_2 - k_2)d\tau_2\theta(\omega \cdot k_2)\delta(k_2^2 - m^2)(\hat{k}_2 - \hat{\omega}_2 + m)d^4k_2 .
\] (B.6)

The part \(-\hat{\omega}_2\) in (B.6) corresponds to the instantaneous interaction which appears in light front dynamics, since the fermion line with the momentum \(k_2\) now “extends over a single time interval” (see section 2.2).

### B.2 Relation between the Feynman amplitudes and the Weinberg rules

It is also instructive to derive the Weinberg rules directly from the Feynman diagrams. This allows one to avoid some subtle problems of light-front quantization. The direct relation between Feynman and the light-front amplitudes was shown for a few cases in ref. [118] and investigated in detail in ref. [119]. The problem of renormalization was investigated in ref. [120]. We illustrate this relation by the simplest case of the triangle Feynman graph shown in fig. 21. This example has been considered also in ref. [121]. Up to a multiplicative factor, the corresponding Feynman amplitude for scalar particles of mass \(m\) has the form:

\[
F_\Delta = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon)((k - p)^2 - m^2 + i\epsilon)((k - p')^2 - m^2 + i\epsilon)} .
\] (B.7)

Introducing the new light front variables:

\[
k_+ = \frac{k_0 + k_3}{\sqrt{2}}, \quad k_- = \frac{k_0 - k_3}{\sqrt{2}}, \quad \vec{k}_\perp = (k_1, k_2),
\] (B.8)

we rewrite eq. (B.7) in the form:

\[
F_\Delta = \int \frac{dk_+d^2k_\perp}{(2\pi)^3} M_\Delta, \quad M_\Delta = \int \frac{1}{2\pi\phi_3(k_- - H_1)(k_- - H_2)(k_- - H_3)},
\] (B.9)

where \(\phi_3 = 8k_+(k_+ - p_+)(k_+ - p'_+)\) and

\[
H_1 = \frac{\vec{k}_\perp^2 + m^2 - i\epsilon}{2k_+} \\
H_2 = p_- + \frac{((\vec{k}_\perp - \vec{p}_\perp)^2 + m^2 - i\epsilon}{2(k_+ - p_+)} \\
H_3 = p'_- + \frac{((\vec{k}_\perp - \vec{p}'_\perp)^2 + m^2 - i\epsilon}{2(k_+ - p'_+)}. \] (B.10)
Let \( p^2 = p^2 = m^2 \) and \( q^2 = (p - p')^2 < 0 \). For simplicity we put \( q_+ = 0 \), i.e., \( p_+ = p'_+ \). At \( k_+ < 0 \) and \( k_+ > p_+ \) all the poles in \( k_- \) of the integrand in eq. (B.9) are above or below the real axis. Hence, the integral differs from zero only. Application of the residue theorem gives:

\[
M_\triangle = -\frac{i}{\phi_3 (H_1 - H_2)(H_1 - H_3)}.
\] (B.11)

Introducing \( x = k_+ / p_+ \), we get:

\[
F_\triangle(t) = -\frac{i}{(2\pi)^3} \int \frac{d^2k_\perp dx}{2x(1-x)^2},
\] (B.12)

where \( t = q^2 = -\Delta^2 \). The form (B.12) reproduces the amplitude given by the Weinberg rules.

![Box Feynman graph.](image)

The next step is the box diagram of fig. 37. This example is considered in ref. [119].

The amplitude has the form:

\[
F_\Box = \int D(k_1, m)D(k_2, \mu)D(k_3, m)D(k_4, \mu) \frac{d^4k}{(2\pi)^4},
\] (B.13)

with \( D(k, m) = i/(k^2 - m^2 + i\epsilon) \) and \( k_1 = k, k_2 = k - l, k_3 = p + q - k, k_4 = k - p \). Particles are supposed to have mass \( m \) or \( \mu \) depending on their appearance in the \( s \) or \( t \) channel. This amplitude can be represented analogously to (B.9):

\[
F_\Box = \int \frac{dk_+ k^2}{(2\pi)^3} M_\Box, \quad M_\Box = \int \frac{dk_-}{2\pi \phi_4} \prod_{i=1}^4 \frac{1}{(k_- - H_1)},
\] (B.14)

where \( \phi_4 = 16k_+ (k_+ - l_+) (k_+ - p_+ - q_+) (k_+ - p_+) \) and \( H_{1-4} \) are analogous to (B.10):

\[
H_1 = \frac{\vec{k}_\perp^2 + m^2 - i\epsilon}{2k_+}.
\]
\( H_2 = l_+ + \frac{(\vec{k}_\perp - \vec{l}_\perp)^2 + \mu^2 - i\epsilon}{2(k_+ - l_+)} \),
\( H_3 = p_- + q_- + \frac{(\vec{k}_\perp - \vec{p}_\perp - \vec{q}_\perp)^2 + m^2 - i\epsilon}{2(k_+ - p_+ - q_+)} \),
\( H_4 = p_- + \frac{(\vec{k}_\perp - \vec{p}_\perp)^2 + \mu^2 - i\epsilon}{2(k_+ - p_+)} \). (B.15)

Let \( p_+ > l_+ \). Then at \( k_+ < 0 \) and at \( k_+ > p_+ + q_+ \) the imaginary parts \( \text{Im}H_{1-4} \) have the same signs and, hence, the integral for \( M_\square \) is zero. At \( 0 < k_+ < l_+ < p_+ < p_+ + q_+ \) there is one pole in the variable \( k_- \) at the down half plane relative to the real axis and three poles are at upper half plane. The residue reproduces the expression given by the Weinberg rules for the diagram analogous to fig. 38, as in the case of the triangle graph indicated in fig. [21].

Fig.38. Time-ordered box graph.

At \( k_+ > p_+ > l_+ \), but \( k_+ < p_+ + q_+ \), the amplitude corresponds to a diagram similar to fig. 38 but up side down. Note that if the poles have imaginary parts of the same signs, the corresponding lines of the time-ordered graph have the same (clockwise or counter clockwise) directions. For the poles with opposite imaginary parts, the directions of lines are opposite.

New feature appears at \( l_+ < k_+ < p_+ \), when two pairs of poles are at the opposite sides of the real axis. In this case, any given residue does not reproduce any time-ordered diagram. However, the sum of them can be transformed [119] to a form consisting of two other parts, each of them coinciding with the amplitudes given by the Weinberg rules for the diagrams analogous to those shown below in figs. 40, 41. We explain this below in more detail in the case of the covariant formulation of LFD. The general algorithm is developed in ref. [119].
B.3 Relation between the Feynman amplitudes and the covariant light-front graph technique

B.3.1 Spin 0

Like in the case of the Weinberg rules, the Feynman approach and the covariant light-front graph technique give different ways of calculating one and the same $S$-matrix and, hence, give one and the same on-shell amplitude. However, these amplitudes are represented in different forms. It is therefore instructive to transform explicitly the Feynman amplitudes to the form of covariant light-front amplitudes.

We will explain this transformation in the simplest case of spinless particles for the box diagram indicated in fig. [37]. The analytical expression for this diagram is given by (B.13). For this transformation, we first represent the Feynman propagator in the following form [122]:

$$
iD(k,m) = -\frac{1}{k^2 - m^2 + i\epsilon} = \int \frac{\delta^4(\omega \tau + k - p)}{\tau - i\epsilon} \theta(\omega p) \left| \delta(p^2 - m^2) \right| \frac{d^4p}{d\tau}$$

Substituting (B.16) in (B.13), we get:

$$F = \int \frac{d^4k}{(2\pi)^4} \left\{ \begin{array}{c}
\frac{\delta^4(\omega \tau_1 + k - p_1)}{\tau_1 - i\epsilon} \theta(\omega p_1) \left| \delta(p_1^2 - m^2) \right| (1a) \\
+ \frac{\delta^4(\omega \tau_1 - k - p_1)}{\tau_1 - i\epsilon} \theta(\omega p_1) \left| \delta(p_1^2 - m^2) \right| (1b) \\
\times \left\{ \begin{array}{c}
\frac{\delta^4(\omega \tau_2 + k - l - p_2)}{\tau_2 - i\epsilon} \theta(\omega p_2) \left| \delta(p_2^2 - \mu^2) \right| (2a) \\
+ \frac{\delta^4(\omega \tau_2 - k + l - p_2)}{\tau_2 - i\epsilon} \theta(\omega p_2) \left| \delta(p_2^2 - \mu^2) \right| (2b) \\
\times \left\{ \begin{array}{c}
\frac{\delta^4(\omega \tau_3 + k - p - q - p_3)}{\tau_3 - i\epsilon} \theta(\omega p_3) \left| \delta(p_3^2 - m^2) \right| (3a) \\
+ \frac{\delta^4(\omega \tau_3 - k + p + q - p_3)}{\tau_3 - i\epsilon} \theta(\omega p_3) \left| \delta(p_3^2 - m^2) \right| (3b) \\
\times \left\{ \begin{array}{c}
\frac{\delta^4(\omega \tau_4 + k - p - p_4)}{\tau_4 - i\epsilon} \theta(\omega p_4) \left| \delta(p_4^2 - \mu^2) \right| (4a) \\
+ \frac{\delta^4(\omega \tau_4 - k + p + p_4)}{\tau_4 - i\epsilon} \theta(\omega p_4) \left| \delta(p_4^2 - \mu^2) \right| (4b) \\
\right) \right) \right) \right) \right) \right) \right) \right) \right)
$$

Let us first consider the product of the first items in the four braces in eq. (B.17). Using schematic notations, we have:

$$\int (1a)(2a)(3a)(4a) \ldots = \int \frac{d^4k}{(2\pi)^4} \frac{\delta^4(\omega \tau_1 + k - p_1)}{\tau_1 - i\epsilon} \frac{\delta^4(\omega \tau_2 + k - l - p_2)}{\tau_2 - i\epsilon}$$
\[
\begin{align*}
\times \frac{\delta^{(4)}(\omega \tau_3 + k - p - q - p_3)}{\tau_3 - i\epsilon} \frac{\delta^{(4)}(\omega \tau_4 + k - p - p_4)}{\tau_4 - i\epsilon} (\cdots) d\tau_1 d\tau_2 d\tau_3 d\tau_4.
\end{align*}
\] (B.18)

The functions \(\theta(\omega \cdot p_i), \delta(p_i^2 - \mu^2)\) and all other factors in (B.18) are absorbed in \((\cdots)\). To exclude \(\tau_1\) from the numerator, we make the following change of variables

\[
k + \omega \tau_1 = k', \quad \tau_2 - \tau_1 = \tau'_2, \quad \tau_3 - \tau_1 = \tau'_3, \quad \tau_4 - \tau_1 = \tau'_4,
\]

after which we find:

\[
\int (1a)(2a)(3a)(4a) \cdots = \int \frac{(\cdots) d\tau_1}{(\tau_1 - i\epsilon)(\tau'_2 + \tau_1 - i\epsilon)(\tau'_3 + \tau_1 - i\epsilon)(\tau'_4 + \tau_1 - i\epsilon)} = 0.
\] (B.19)

The numerator \((\cdots)\) in (B.19) does not depend on \(\tau_1\). All the poles of the integrand in (B.19) relative to \(\tau_1\) are above the real axis. Therefore the integral (B.19) equals to zero. Similarly we get \(\int (1b)(2b)(3b)(4b) = 0\).

We suppose that all the external momenta are on the mass shells and, for convenience, \(\omega \cdot p > \omega \cdot l\). The expression (B.17) contains 16 items. However, only three of them differ from zero, namely, \(\int (1a)(2b)(3b)(4b), \int (1a)(2a)(3b)(4a)\) and \(\int (1a)(2a)(3b)(4b)\). In all other items, except for the ones considered above \(\int (1a)(2a)(3a)(4a)\) and \(\int (1b)(2b)(3b)(4b)\), we get the products of the \(\theta\)-functions with mutually incompatible restrictions on the scalar products of \(\omega\) with four-momenta. Let assign to the items \((a)\) the arrow counter clockwise in the loop and to the items \((b)\) – the clockwise arrow. Put on the external lines with the momenta \(p\) and \(q\) the arrows incoming to the box and to the lines with momenta \(l\) and \(p + q - l\) – the outcoming arrows. Then the items which disappears due to the \(\theta\)-functions correspond to the diagrams containing at least one vertex with all incoming or outcoming lines, which are interpreted as vacuum fluctuations. One of these graphs, corresponding to the product \(\int (1b)(2a)(3b)(4b)\) is shown in fig. 39.

\[
\begin{align*}
\text{Fig.39. Time-ordered box graph with the vacuum vertex No. 2.}
\end{align*}
\]

The vertex 2 in fig. 39 corresponds to the creation of particles from the vacuum. The items \(\int (1a)(2a)(3a)(4a)\) and \(\int (1b)(2b)(3b)(4b)\), which are zero, correspond to the loops with all the lines directed clockwise or counter clockwise. Now consider the item

\[
\int (1a)(2b)(3b)(4b) \cdots = \int \frac{\delta^{(4)}(\omega \tau_1 + k - p_1)}{\tau_1 - i\epsilon} \frac{\delta^{(4)}(\omega \tau_2 + l - k - p_2)}{\tau_2 - i\epsilon}
\]
\[ \times \frac{\delta^{(4)}(\omega \tau_3 + p + q - k - p_3)}{\tau_3 - i\epsilon} \frac{\delta^{(4)}(\omega \tau_4 + p - k - p_4)}{\tau_4 - i\epsilon} (\cdots) d\tau_1. \]  

(B.20)

Again after replacing \( k + \omega \tau_1 = k', \tau_{2,3,4} + \tau_1 = \tau'_{2,3,4} \) we get:

\[ \int(1a)(2b)(3b)(4b) \ldots = \int \frac{(\cdots) d\tau_1}{(\tau_1 - i\epsilon)(\tau_2 - \tau_1 - i\epsilon)(\tau_3 - \tau_1 - i\epsilon)(\tau_4 - \tau_1 - i\epsilon)}, \]  

(B.21)

where (\( \cdots \)) does not depend on \( \tau_1 \). Calculating the residue at \( \tau_1 = i\epsilon \) and deleting the prime over \( \tau \)'s we find:

\[ \int(1a)(2b)(3b)(4b) \ldots = i \int \theta(\omega \cdot k) \delta(k^2 - m^2) \times \theta(\omega \cdot (l - k)) \delta((l - k + \omega \tau_2)^2 - \mu^2) \times \theta(\omega \cdot (p + q - k)) \delta((p + q - k + \omega \tau_3)^2 - m^2) \times \theta(\omega \cdot (p - k)) \delta((p - k + \omega \tau_4)^2 - \mu^2) \times \frac{d\tau_2}{(\tau_2 - i\epsilon)} \frac{d\tau_3}{(\tau_3 - i\epsilon)} \frac{d\tau_4}{(\tau_4 - i\epsilon)} \frac{d^4 k}{(2\pi)^3}. \]  

(B.22)

The expression (B.22), after separating the factor \( i \) (due to its presence in the definition (2.54) of the amplitude), exactly corresponds to the light-front amplitude indicated by the diagram in fig. 40. The integral \( \int(1a)(2a)(3b)(4b) \) is calculated analogously (by integrating over the variable \( \tau_3 \) instead of \( \tau_1 \)) and corresponds to the diagram having the form of the overturned trapezium (relative to fig. 38).

Fig. 40. Time-ordered box graph.

There are two other diagrams in the light-front dynamics shown in figs. 40 and 41. We can find both of them in the item \( \int(1a)(2a)(3b)(4b) \). After similar transformations we get:

\[ \int(1a)(2a)(3b)(4b) \ldots = \int \frac{(\cdots) d\tau_1}{(\tau_1 - i\epsilon)(\tau_2 + \tau_1 - i\epsilon)(\tau_3 - \tau_1 - i\epsilon)(\tau_4 - \tau_1 - i\epsilon)}, \]  

(B.23)
Fig. 41. Time-ordered box graph.

Taking the sum of the two residues at $\tau_1 = +i\epsilon$ and $\tau_1 = -\tau_2 + i\epsilon$, we obtain two items:

$$\int (1a)(2a)(3b)(4b)\ldots = \int \left[ \frac{1}{\tau_2 \tau_3 \tau_4} - \frac{1}{\tau_2 (\tau_2 + \tau_3) (\tau_2 + \tau_4)} \right] (\ldots).$$

(B.24)

Each of them does not correspond to any diagram. However, (B.23) indeed gives the sum of the diagrams in figs. 40 and 41.

To transform it to the appropriate form, we shall use the simple formula:

$$\frac{1}{\alpha \beta} = \frac{1}{\alpha} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right).$$

(B.25)

This formula should be applied recursively to pairs of factors in the integrand having the poles in the integration variables with the imaginary parts of opposite signs. For example, applying (B.25) to $1/\beta_2 \beta_3 \beta_4$, first to $1/\beta_2 \beta_3$ and then again to $1/\beta_2 \beta_4$, we get:

$$\frac{1}{\beta_2 \beta_3 \beta_4} = \frac{1}{(\beta_2 + \beta_3) \beta_3 \beta_4} + \frac{1}{(\beta_2 + \beta_3)(\beta_3 + \beta_4) \beta_4} + \frac{1}{(\beta_2 + \beta_3)(\beta_2 + \beta_4) \beta_2}. $$

(B.26)

We apply (B.26) to the integrand of (B.23), i.e., substitute $\beta_2 = \tau_2 + \tau_1 - i\epsilon$, $\beta_3 = \tau_3 - \tau_1 - i\epsilon$ and $\beta_4 = \tau_4 - \tau_1 - i\epsilon$. The factors $1/\beta_2$ and $\beta_3$ in $1/\beta_2 \beta_3$ have the poles relative to the integration variable $\tau_1$ with the imaginary parts of opposite signs. The same is true for the factors $\beta_2$ and $\beta_4$ in $1/\beta_2 \beta_4$. By this way, we transform the integrand of (B.23) to the form which after integration over $d\tau_1$ and standard replacement of variables leads to the expression:

$$\int (1a)(2a)(3b)(4b)\ldots = i \int \theta(\omega \cdot k) \delta(k^2 - m^2)$$

$$\times \theta(\omega \cdot (k - l)) \delta((k - l + \omega \tau_2 - \omega \tau_3)^2 - \mu^2)$$

$$\times \theta(\omega \cdot (p + q - k)) \delta((p + q - k + \omega \tau_3)^2 - m^2)$$

$$\times \theta(\omega \cdot (p - k)) \delta((p - k + \omega \tau_4)^2 - \mu^2)$$

143
\[ \times \frac{d\tau_2 d\tau_3 d\tau_4}{(\tau_2 - i\epsilon)(\tau_3 - i\epsilon)(\tau_4 - i\epsilon)} \frac{d^4 k}{(2\pi)^3} \]

+ \[ i \int \theta(\omega k) \delta(k^2 - m^2) \]

\[ \times \theta(\omega(k - l)) \delta((l - k + \omega \tau_3 - \omega \tau_4)^2 - \mu^2) \]

\[ \times \theta(\omega(p + q - k)) \delta((p + q - k + \omega \tau_2 + \omega \tau_4 - \omega \tau_3)^2 - m^2) \]

\[ \times \theta(\omega(p - k)) \delta((p - k + \omega \tau_4)^2 - \mu^2) \]

\[ \times \frac{d\tau_2 d\tau_3 d\tau_4}{(\tau_2 - i\epsilon)(\tau_3 - i\epsilon)(\tau_4 - i\epsilon)} \frac{d^4 k}{(2\pi)^3} . \] (B.27)

Two items in eq. (B.27) exactly correspond to the amplitudes for the diagrams of figs. 40 and 41. The method can be generalized to more complicated diagrams. The general case is considered in ref.[119] and the renormalization in ref. [120].

**B.3.2 Spins 1/2 and 1**

For spin 1/2, one should use the following representation of the propagator [38]:

\[ iS(k) = -\frac{\hat{k} + m}{k^2 - m^2 + i\epsilon} \]

\[ = \int \frac{\delta^{(4)}(\omega \tau + k - p)}{\tau - i\epsilon} (\hat{p} - \hat{\omega} \tau + m) \theta(\omega \cdot p) \delta(p^2 - m^2) \ d^4 p \ d\tau \]

\[ - \int \frac{\delta^{(4)}(\omega \tau - k - p)}{\tau - i\epsilon} (\hat{p} - \hat{\omega} \tau - m) \theta(\omega \cdot p) \delta(p^2 - m^2) \ d^4 p \ d\tau \]

\[ = \int \frac{\delta^{(4)}(\omega \tau + k - p)}{\tau - i\epsilon} (\hat{p} + m) \theta(\omega \cdot p) \delta(p^2 - m^2) \ d^4 p \ d\tau \]

\[ + \int \frac{\delta^{(4)}(\omega \tau - k - p)}{\tau - i\epsilon} (m - \hat{p}) \theta(\omega \cdot p) \delta(p^2 - m^2) \ d^4 p \ d\tau \]

\[ = \int \frac{\delta^{(4)}(\omega \tau + k - p)}{\tau - i\epsilon} (\hat{p} + m) \theta(\omega \cdot p) \delta(p^2 - m^2) \ d^4 p \ d\tau \]

\[ - \frac{\hat{\omega}}{2(\omega \cdot k)} . \] (B.28)

In the representation given by eq. (B.28), the propagator contains the spin parts (\( \hat{p} + m \)) and (\( m - \hat{p} \)) with the on-shell four-momentum \( p^2 = m^2 \) and the extra contact term \(-\hat{\omega}/2(\omega \cdot k)\), like in the rules of the graph technique, sect. 2.2.3. At \( \omega = (1, 0, 0, -1) \) this contact term coincides with the one given in refs. [39, 123, 124].

For the propagator with spin 1, we have the representation:

\[ iD_{\mu\nu}(k) = -\frac{g_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{m^2}}{k^2 - m^2 + i\epsilon} \]

\[ = \int \frac{\delta^{(4)}(\omega \tau + k - p)}{\tau - i\epsilon} \left( -g_{\mu\nu} + \frac{(p - \omega \tau)_\mu(p - \omega \tau)_\nu}{m^2} \right) \theta(\omega \cdot p) \delta(p^2 - m^2) \ d^4 p \ d\tau \]
\[ + \int \frac{\delta^{(4)}(\omega \tau - k - p)}{\tau - i\epsilon} \left( -g_{\mu\nu} + \frac{(p - \omega \tau)_{\mu}(p - \omega \tau)_{\nu}}{m^2} \right) \theta(\omega \cdot p) \delta(p^2 - m^2) \, d^4p \, d\tau \]

\[ = \int \frac{\delta^{(4)}(\omega \tau + k - p)}{\tau - i\epsilon} \left( -g_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{m^2} \right) \theta(\omega \cdot p) \delta(p^2 - m^2) \, d^4p \, d\tau \]

\[ + \int \frac{\delta^{(4)}(\omega \tau - k - p)}{\tau - i\epsilon} \left( -g_{\mu\nu} + \frac{p_{\mu}p_{\nu}}{m^2} \right) \theta(\omega \cdot p) \delta(p^2 - m^2) \, d^4p \, d\tau \]

\[ - \frac{k_\mu \omega_\nu + k_\nu \omega_\mu}{2(\omega \cdot k)m^2} - \frac{\omega_\mu \omega_\nu}{4(\omega \cdot k)^2m^2} (k^2 - m^2) \].

(B.30)

The item in the last line of (B.30) is the vector contact term.

Let now the particle 1 in fig. 37 has spin 1/2. Then the contact term for the particle 1 does not contribute to the amplitude indicated in fig. 38. Indeed, this amplitude was calculated in eq. (B.21), as the residue at \(\tau_1 = i\epsilon\) in the pole of the propagator of particle 1. Replacing this propagator by the contact term, we see that the only pole above the real axis disappears and the amplitude turns to zero. However, the contact terms for the particles 2,3 and 4 contribute to the diagram of fig. 38. According to (B.28) they can be incorporated by replacing the spin part of the propagators (\(\hat{p} \pm m\)) by (\(\hat{p} - \omega \tau \pm m\)).

These facts agree with the general rule given in section 2.2: the contact terms have to be associated with these particles which lines are directly connected by a spurion line. This also coincides with the rule given in ref. [33]: contact terms modify only the propagators corresponding to the “lines extending over a single time interval”. The lines of the particles 2,3,4 in fig. 38 extend over single time intervals and are associated with contact terms, while the line for the particle 1 extends over three time intervals and is not associated with any contact term. Another example is given by the diagram of fig. 40. In this case, the contact term contributes for the exchanged particles (if they have spin) and does not contribute for the particles shown by the horizontal lines. On the contrary, in fig. 41 the contact term contributes for the horizontal internal lines and does not contribute for the exchanged particles.
Appendix C

Relation between the deuteron components

We give here for completeness the relations between the six invariant functions coming in the decomposition of the wave function (5.1), (5.2) and (5.6). These two representations are connected by eq.(5.3) and coincide with each other in the system of reference where \( \vec{P} = 0 \). In this system, we have \( \vec{k}_1 = -\vec{k}_2 = \vec{k} \).

The deuteron polarization vector \( e_{\mu}^{(j)}(p) \) in the representation with \( \lambda = j = x, y, z \) in the rest system has the form: \( e_{i}^{(j)}(0) = \delta_{i}^{j}, \ e_{0}^{(j)}(0) = 0 \). In an arbitrary system where \( \vec{p} \neq 0 \), it can be obtained by the Lorentz transformation (cf.(3.11)):

\[
\vec{e}^{(j)}(p) = \vec{e}^{(j)}(0) + \left( \begin{array}{c} \vec{e}^{(j)}(0) \cdot \vec{p} \\ M(M + p_0) \end{array} \right), \quad e_{0}^{(j)}(p) = \frac{\vec{e}^{(j)}(0) \cdot \vec{p}}{M}. \tag{C.1}
\]

In the system of reference where \( \vec{P} = 0 \), we have:

\[
p_0 = \frac{4\varepsilon_k^2 + M^2}{4\varepsilon_k}, \quad \vec{p} = -\vec{n}\frac{\varepsilon_k^2 - m^2}{\varepsilon_k}.
\]

With these expressions for \( p \) we get from eq.(C.1):

\[
e_{i}^{(j)}(p) = \delta_{i}^{j} + \frac{(2\varepsilon_k - M)^2}{4M\varepsilon_k}n_in_j, \quad e_{0}^{(j)}(p) = -\frac{4\varepsilon_k^2 - M^2}{4M\varepsilon_k}n_j . \tag{C.2}
\]

The polarization vector (C.2) satisfies the general relations (A.10).

Substituting in eqs.(5.1,5.2) the expressions for the nucleon spinors and Dirac matrices given in appendix A and the deuteron polarization vector given above, we obtain the structure of the wave function given in eq.(5.5). Comparing the coefficients of identical spin structures, we can find the expressions of \( \varphi_i \) in terms of \( f_i \). In the approximation where \( M \approx 2m \), they have the form:

\[
\varphi_1 = \frac{m^2(2\varepsilon_k + m)}{4\varepsilon_k k^2}f_2 + \frac{m^2}{4\varepsilon_k(\varepsilon_k + m)}(\sqrt{2}f_1 - f_3 + zf_4 - \sqrt{3}zf_6) . \tag{C.3}
\]
\[ \varphi_2 = \frac{m}{4\varepsilon_k} (\sqrt{2}f_1 - f_2 - f_3 - 2zf_4), \quad (C.4) \]

\[ \varphi_3 = \frac{\sqrt{2}k^5}{4\varepsilon_k^2(\varepsilon_k + m)^3}zf_1 - \frac{(2\varepsilon_k + m)k^3}{4\varepsilon_k^2(\varepsilon_k + m)^2}zf_2 + \frac{(\varepsilon_k^2 + 4m\varepsilon_k + m^2)k}{4\varepsilon_k(\varepsilon_k + m)}zf_3 \\
+ \frac{3m}{2k} \left( 1 - \frac{z^2k^6}{6m\varepsilon_k^2(\varepsilon_k + m)^3} \right) f_4 + \frac{\sqrt{3}m}{2k} \left( 1 + \frac{z^2k^6}{2m\varepsilon_k^2(\varepsilon_k + m)^3} \right) f_6, \quad (C.5) \]

\[ \varphi_4 = -\frac{3m}{2k} f_4 + \frac{\sqrt{3}m}{2k} f_6, \quad (C.6) \]

\[ \varphi_5 = \frac{1}{2\sqrt{2}} \sqrt{\frac{3m^2}{k\varepsilon_k}} f_5, \quad (C.7) \]

\[ \varphi_6 = \frac{k^4}{2m\varepsilon_k(\varepsilon_k + m)^2} (\sqrt{2}f_1 - f_2 + zf_4 - \sqrt{3}zf_6) - \frac{(\varepsilon_k^2 + 4m\varepsilon_k + m^2)}{2m\varepsilon_k} f_3. \quad (C.8) \]
Appendix D

Two-body kinematical relations

In the calculation of the electromagnetic amplitude we have to express the scalar products of the four vectors $\omega, k_1, k_2, k'_1, k'_2, p, p', q$ with each other in terms of the integration variables and the four-momentum transfer squared. The variables and, hence, the kinematical relations for the one loop diagram of fig. 18 (impulse approximation) and for the two loop diagram of fig. 42 are different.

D.1 One loop diagram

The one loop diagram is shown in fig. 18. All four-momenta are on the corresponding mass shells:

$$k_1^2 = m_1^2, \quad k_2^2 = k'_2^2 = m_2^2, \quad p^2 = M^2, \quad p'^2 = M'^2, \quad q^2 = -\Delta^2 = -Q^2 = t, \quad \omega^2 = 0.$$  

We consider the general case of different masses, $M$ being the mass of the initial state, while $M'$ is the mass of the final state.

We start from the four-vector $R_1$ already defined in eq.(3.16):

$$R_1 = k_1 - xp, \quad x = \omega \cdot k_1 / \omega \cdot p.$$  

Since $R_1 \cdot \omega = 0$, it can be represented as $R_1 = (R_0, \vec{R}_\perp, \vec{R}_\parallel)$ with $\vec{R}_\perp \cdot \vec{\omega} = 0$ and $\vec{R}_\parallel$ is parallel to $\vec{\omega}$. Similarly we represent $q = (q_0, \vec{\Delta}, \vec{q}_\parallel)$ with $\vec{\Delta} \cdot \vec{\omega} = 0$. With these definitions, we immediately get (with $\omega \cdot q = 0$):

$$\omega \cdot k_1 = x \omega \cdot p, \quad \omega \cdot k_2 = (1 - x) \omega \cdot p, \quad \omega \cdot k'_2 = (1 - x) \omega \cdot p, \quad \omega \cdot p = \omega \cdot p'.$$  \quad (D.1)

We get for the various scalar products the following relations.

Scalar products with $k_1$:

$$k_1 \cdot k_2 = \frac{(s - m_1^2 - m_2^2)}{2}, \quad k_1 \cdot k'_2 = \frac{(s' - m_1^2 - m_2^2)}{2},$$  

$$k_1 \cdot p = xM^2/2 + (1 - x)s/2 + (m_1^2 - m_2^2)/2,$$

$$k_1 \cdot p' = k_1 \cdot p + k_1 \cdot q,$$

$$k_1 \cdot q = -\vec{R}_\perp \cdot \vec{\Delta} + xpq,$$  \quad (D.2)
with $s$ and $s'$ given by:

$$
s \equiv (k_1 + k_2)^2 = \frac{\vec{R}_1^2 + m_1^2}{x} + \frac{\vec{R}_1^2 + m_2^2}{1-x},
$$

$$
s' \equiv (k_1' + k_2')^2 = \frac{\vec{R}_1'^2 + m_1^2}{x} + \frac{\vec{R}_1'^2 + m_2^2}{1-x}.
$$

(D.3)

where $\vec{R}_\perp = \vec{R}_\perp - x\vec{\Delta}$.

Scalar products with $k_2$:

$$
k_2 \cdot k_2' = m_2^2 + k_2 q + (1-x)(s'-M^2-s+M^2)/2,
$$

$$
k_2 \cdot p = (1-x)M^2/2 + xs/2 + (m_2^2 - m_1^2)/2,
$$

$$
k_2 \cdot p' = k_2 p + k_2 q,
$$

$$
k_2 \cdot q = \vec{R}_\perp \cdot \vec{\Delta} + (1-x)p \cdot q.
$$

(D.4)

Scalar products with $k_2'$:

$$
k_2' \cdot p = k_2' p' - k_2' q,
$$

$$
k_2' \cdot p' = (1-x)M^2/2 + xs'/2 + (m_2^2 - m_1^2)/2,
$$

$$
k_2' \cdot q = k_2 q - \vec{\Delta}^2.
$$

(D.5)

Scalar products with $p$:

$$
p \cdot p' = (\vec{\Delta}^2 + M^2 + M^2)/2,
$$

$$
p \cdot q = (\vec{\Delta}^2 - M^2 + M^2)/2.
$$

(D.6)

Scalar product with $p'$:

$$
p' \cdot q = (M^2 - M^2 - \vec{\Delta}^2)/2.
$$

(D.7)

**D.2 Two-loop diagram**

The two-loop diagrams indicated on fig. [23] corresponds to the contribution of the contact term. One of them is reproduced in fig. [12], where all the momenta are explicitly indicated.

Some of the relations coincide with the ones given above. We repeat them here for completeness. For the scalar products of $\omega$ we immediately get:

$$
\omega \cdot k_1 = x \omega \cdot p, \ \omega \cdot k_2 = (1-x) \omega \cdot p, \ \omega \cdot k_1' = x' \omega \cdot p, \ \omega \cdot k_2' = (1-x') \omega \cdot p, \ \omega \cdot p = \omega \cdot p', \ \omega \cdot q = 0.
$$

(D.8)

All the following invariants can be transformed to the variables $\vec{k}, \vec{k}'$ and $\vec{\varepsilon}$ by the substitution:

$$
\vec{R}_\perp = \vec{k} - (\vec{n} \cdot \vec{k}) \vec{n}, \ x = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} (1 - \frac{\vec{n} \cdot \vec{k}}{\varepsilon_1}),
$$

(D.9)

$$
\vec{R}_\perp' = \vec{k} - (\vec{n} \cdot \vec{k}') \vec{n}, \ x' = \frac{\varepsilon_1'}{\varepsilon_1' + \varepsilon_2} (1 - \frac{\vec{n} \cdot \vec{k}'}{\varepsilon_1'}),
$$

(D.10)
taking into account that $\vec{\Delta} \cdot \vec{n} = 0$. We denote in this equation $\varepsilon_i = \sqrt{k^2 + m_i^2}$, and similarly for quantities with prime. We thus get the following scalar products.

Scalar products with $k_1$:

\[
\begin{align*}
  k_1 \cdot k_2 &= (s - m_1^2 - m_2^2)/2, \\
  k_1 \cdot k_1' &= -\vec{R}_1 \cdot \vec{R}_1' + x \vec{\Delta} \cdot \vec{R}_1' + x'(1 - x) s'/2 + x'(1 - x) s/2 + x x' \vec{\Delta}^2/2, \\
  k_1 \cdot k_2' &= k_1 \cdot p' + x(s' - \mathcal{M}^2)/2 - k_1 \cdot k_1', \\
  k_1 \cdot p &= (1 - x) s/2 + x \mathcal{M}^2/2, \\
  k_1 \cdot p' &= k_1 \cdot q + k_1 \cdot p, \\
  k_1 \cdot q &= -\vec{R}_1 \cdot \vec{\Delta} + x p \cdot q. 
\end{align*}
\]

(D.11)

Scalar products with $k_2$:

\[
\begin{align*}
  k_2 \cdot k_1' &= k_1' \cdot p + x'(s - \mathcal{M}^2)/2 - k_1 \cdot k_1', \\
  k_2 \cdot k_2' &= k_2 \cdot p' - k_2 \cdot k_1' + (1 - x)(s' - \mathcal{M}^2)/2, \\
  k_2 \cdot p &= x s/2 + (1 - x) \mathcal{M}^2/2, \\
  k_2 \cdot p' &= k_2 \cdot q + k_2 \cdot p, \\
  k_2 \cdot q &= \vec{R}_1 \cdot \vec{\Delta} + (1 - x) p \cdot q. 
\end{align*}
\]

(D.12)

Scalar products with $k_1'$:

\[
\begin{align*}
  k_1' \cdot k_2' &= (s' - m_1^2 - m_2^2)/2, \\
  k_1' \cdot p &= k_1' \cdot p' - k_1' \cdot q, \\
  k_1' \cdot p' &= (1 - x') s'/2 + x' \mathcal{M}^2/2, \\
  k_1' \cdot q &= -\vec{R}_1' \cdot \vec{\Delta} + x p' \cdot q. 
\end{align*}
\]

(D.13)

Scalar products with $k_2'$:

\[
\begin{align*}
  k_2' \cdot p &= k_2' \cdot p' - k_2' \cdot q. 
\end{align*}
\]
\[ k'_2 p' = x' s' / 2 + (1 - x') \mathcal{M}^2 / 2, \]
\[ k'_2 q = \vec{R}'_\perp \cdot \vec{\Delta} + (1 - x') p' \cdot q. \]

(D.14)

The scalar products with \( p \) and \( p' \) are the same as (D.6), (D.7).

The kinematics for the final plane wave contribution (neglecting the final state energy) is obtained from the above formulas by the substitution \( \mathcal{M} = 2m, \vec{R}'_\perp = 0, x' = 1/2 \) (or \( \vec{k}' = 0 \)). The kinematics for the elastic deuteron form factors is obtained by the replacement \( \mathcal{M} \to M \).
Appendix E

Three-body kinematical relations

The scalar products appearing in the calculation of the amplitude indicated in fig. 24 are expressed through the integration variables $\vec{R}_{i\perp}, x_i$ and $\vec{\Delta}$. We give them below in the general case of different quark masses. For $p \cdot p'$ we have:

$$p \cdot p' = M^2 + \vec{\Delta}^2 / 2.$$  \hfill (E.1)

In order to find $p \cdot k_i$ ($i = 1, 2, 3$), we calculate $(k_i - x_ip)^2 = m_i^2 - 2x_ipk_i + x_i^2M^2 = R_i^2 = -\vec{R}_{i\perp}^2$. From here we find:

$$p \cdot k_i = \frac{\vec{R}_{i\perp}^2 + m_i^2}{2x_i} + \frac{1}{2}M^2x_i.$$  \hfill (E.2)

Other scalar products are found analogously. We have:

$$p' \cdot k_i = p \cdot k_i - \vec{R}_{i\perp} \cdot \vec{\Delta} + x_i\vec{\Delta}^2 / 2,$$  \hfill (E.3)

$$p' \cdot k_i' = \frac{\vec{R}_{i\perp}^2 + m_i^2}{2x_i} + \frac{1}{2}M^2x_i,$$  \hfill (E.4)

$$p \cdot k_i' = p' \cdot k_i + \vec{R}_{i\perp}' \cdot \vec{\Delta} + x_i\vec{\Delta}^2 / 2,$$  \hfill (E.5)

$$k_i \cdot k_j' = -\vec{R}_{i\perp} \cdot \vec{R}_{j\perp} + x_ipk_j' + x_jp'k_i - x_ix_jpp' \quad \text{for } i, j = 1, 2, 3.$$  \hfill (E.6)

The scalar product $k_i \cdot k_j$ is obtained from eq.(E.6) by deleting primes at all variables, the scalar product $k_i' \cdot k_j'$ is obtained by the following replacement in eq.(E.6): $\vec{R}_{i\perp} \rightarrow \vec{R}_{i\perp}'$, $k_i \rightarrow k_i'$, $p \rightarrow p'$. The variables $x_i$ are the same for initial and final states.

Let us now express $\vec{R}_{i\perp}'$ through $\vec{\Delta}$ and the integration variables $\vec{R}_{i\perp}$ and $x_i$. The expressions are not symmetrical relative to the quark 1, which interacts with the photon, and to the spectator quarks 2 and 3. We have $R_{i\perp}' = k_i' - x_2p'$, $R_{i\perp}' = k_i' - x_3p'$. With $k_2' = k_2$, $k_3' = k_3$ we find $R_{i\perp}' = R_{i\perp} - x_2q$, $R_{i\perp}' = R_{i\perp} - x_3q$. With $R_{i\perp}' = k_i' - x_1p' = R_{i\perp} + (1-x_1)q + \omega(\tau' - \tau)$, we get:

$$\vec{R}_{1\perp}' = \vec{R}_{1\perp} + (1-x_1)\vec{\Delta}, \quad \vec{R}_{2,3\perp}' = \vec{R}_{2,3\perp} - x_{2,3}\vec{\Delta}.$$  \hfill (E.7)
Note that $\vec{R}'_{1\perp}$ can be also found from the relation $\vec{R}'_{1\perp} = -\vec{R}'_{2\perp} - \vec{R}'_{3\perp}$. The arguments $\vec{q}'_{i}^{2} (i = 1, 2, 3)$ of the initial wave function are given by

$$\vec{q}'_{i}^{2} = \varepsilon'_{q_{i}}^{2} - m_{i}^{2},$$

where $\varepsilon'_{q_{i}}$ is the energy of the particle in the reference system where $\vec{P} = \vec{k}_{1} + \vec{k}_{2} + \vec{k}_{3} = 0$. It can be represented in the invariant form as $\varepsilon'_{q_{i}} = P \cdot k_{i} / M$ with $P = p + \omega \tau$, $\tau = (M^{2} - M'^{2}) / (2\omega p)$, that gives:

$$\varepsilon'_{q_{i}} = [p \cdot k_{i} + \frac{1}{2} x_{i} (M^{2} - M^{2})] / M,$$  \hspace{1cm} (E.8)

where $M^{2}$ is the invariant mass squared:

$$M^{2} = (k_{1} + k_{2} + k_{3})^{2} = \sum_{i=1}^{3} \frac{\vec{R}_{i}^{2} + m_{i}^{2}}{x_{i}}.$$  \hspace{1cm} (E.9)

The arguments $\vec{q}_{i}^{2}$ of the final wave function are constructed similarly from the final momenta.
Bibliography


   24 (1977) 547].
   25 (1977) 347].


[84] B. Loiseau, private communication.


