Quantum mutual entropy for Jaynes-Cummins model

\[ S^Q = -\sum_i p_i \log p_i \]

Where \( S^Q \) is the quantum mutual entropy and \( p_i \) are the probabilities of the quantum states.

The Jaynes-Cummins model is a quantum extension of the classical mutual information and is given by

\[ I^Q = S^Q - S^{Q_1} - S^{Q_2} \]

Where \( I^Q \) is the quantum mutual information, and \( S^{Q_1} \) and \( S^{Q_2} \) are the quantum entropies of the subsystems.

The quantum mutual entropy is defined as the von Neumann entropy of the reduced density matrix of one of the subsystems.
mutual entropy is not suited to rigorously analyze the fully quantum system. In order to formulate the quantum mutual entropy, a compound state [4] was introduced in stead of the joint probability.

The quantum mutual entropy was defined in the following manner [4]. A certain initial state $\rho$ is decomposed as

$$\rho = \sum_k \lambda_k E_k$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots$$

$$E_i \perp E_j \quad (i \neq j)$$

where $\lambda_k$ is an eigenvalue of $\rho$ and $E_k$ is the associated one dimensional projection. Each $E_k$ can be considered as an elementary event composed of the initial state $\rho$. The above decomposition is called (von Neumann) Schatten decomposition [9]. The Schatten decomposition is unique if every $\lambda_k$ is nondegenerated. For a Schatten decomposition, the compound state $\sigma_E$ describing the correlation between an initial state $\rho$ and its final state $\Lambda^* \rho$ is defined by

$$\sigma_E \equiv \sum_k \lambda_k E_k \otimes \Lambda^* E_k.$$ 

This compound state $\sigma_E$ contains a connection among initial constituents $E_k$ and final ones $\Lambda^* E_k$ expressing a correlation between $\rho$ and $\Lambda^* \rho$. Then the quantum mutual entropy, $I(\rho; \Lambda^*)$ is defined by

$$I(\rho; \Lambda^*) \equiv \sup_{E} \{ S(\sigma_E, \sigma_0) \; ; \; E = \{E_k\} \}$$

where $\sigma_0 \equiv \rho \otimes \Lambda^* \rho$ is a trivial compound state and $S(\sigma_E, \sigma_0)$ is the quantum relative entropy [10] defined by

$$S(\sigma_E, \sigma_0) \equiv \text{tr} \sigma_E (\log \sigma_E - \log \sigma_0).$$

In the above definition of the quantum mutual entropy, we have to take “sup” over all Schatten decompositions when some eigenvalues are degenerated.

The quantum mutual entropy mathematically includes the mutual entropy in classical system. Moreover, this quantum mutual entropy with the lifting and the quantum mechanical channel mentioned above has been applied to quantum communication theory [7,11], quantum Markov chains [12], quantum teleportation processes [13] and so on. Especially, the irreversibility of a certain system was studied by the quantum mutual entropy in [12,14].

The quantum mutual entropy like the classical mutual entropy expresses how much information carried by an initial state $\rho$ is correctly transmitted to a final state $\Lambda^* \rho$ through a quantum mechanical channel. Therefore, since the decrease of the quantum mutual entropy means a loss of initial information, it can be considered a dissipative change of a state of a physical system [14]. It is a reason why we apply this quantum mutual entropy to investigate the irreversible behavior of the JCM.

The resonant JCM Hamiltonian can be expressed by using rotating-wave approximation in the following form:

$$H = H_0 + H_1 + H_{01}$$

$$H_0 = \hbar \omega \sigma_0^+ \sigma_0^-, \quad H_1 = \frac{1}{2} \hbar \omega_0 \sigma_z$$

$$H_{01} = \hbar g (a \otimes \sigma^+ + a^* \otimes \sigma^-)$$

where $g$ is a coupling constant, $\sigma^\pm$ are the pseudo-spin operators of two-level atom, $\sigma_z$ is the Pauli spin operator, $a$ is the annihilation operator of a photon and $a^*$ is the creation operator of a photon.

In order to derive the quantum mutual entropy, we have to give the quantum mechanical channel for the JCM. Now, we suppose that the initial state of the atom is a superposition of the upper level and the lower level;

$$\rho = \lambda_0 E_0 + \lambda_1 E_1 \in \mathcal{S}_A$$

where $E_0 = |1\rangle \langle 1|, \; E_1 = |2\rangle \langle 2|, \; \lambda_0 + \lambda_1 = 1$. We also suppose that a field state is a coherent state defined as:

$$\omega = |\theta\rangle \langle \theta| \in \mathcal{S}_F$$

$$|\theta\rangle = \exp \left[ -\frac{1}{2} |\theta|^2 \right] \sum \frac{\theta^n}{\sqrt{n!}} |n\rangle$$

Then we set the lifting $\mathcal{E}^*_f$ and the time-dependent quantum mechanical channel $\Lambda^*_t$ which describes the time evolution of the atom for the JCM as follows:

$$\Lambda^*_t : \mathcal{S}_A \rightarrow \mathcal{S}_A$$

$$\mathcal{E}^*_f : \mathcal{S}_A \rightarrow \mathcal{S}_A \otimes \mathcal{S}_F$$

Then, the final state $\mathcal{E}^*_f \rho \in \mathcal{S}_A \otimes \mathcal{S}_F$ after the interaction between the atom and the photons at the time $t$ is given as follows:

$$\mathcal{E}^*_f \rho = U_t (\rho \otimes \omega) U_t^*.$$ 

Moreover the quantum mechanical channel $\Lambda^*_t$ is written by this lifting $\mathcal{E}^*_f$ as follows:

$$\Lambda^*_t \rho = tr_\mathcal{H} \mathcal{E}^*_f \rho = tr_\mathcal{H} U_t (\rho \otimes \omega) U_t^*.$$ 

This quantum mechanical channel represents the final state of the atom at the time $t$. From the Hamiltonian form in (3),(4),(5), the following communication relation holds [15]:

$$[H_0 + H_1, H_{01}] = 0.$$ 

Therefore the time evolution of the system is determined by the interaction Hamiltonian $H_{01}$;

$$U_t = \exp (-itH_{01}/\hbar).$$
Take a dressed state:
\[
|\Phi_j^{(n)}\rangle = \frac{1}{\sqrt{2}} \left( |2 \otimes n\rangle + (-1)^j |1 \otimes n + 1\rangle \right), \quad (j = 0, 1)
\]
then the following eigen-equation holds:
\[
H_{01} \begin{pmatrix} |\Phi_j^{(n)}\rangle \\ |\Phi_{j'}^{(n')}\rangle \end{pmatrix} = \mathbb{H} \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix} \begin{pmatrix} |\Phi_j^{(n)}\rangle \\ |\Phi_{j'}^{(n')}\rangle \end{pmatrix}
\]
where \( \Omega = g \sqrt{n+1} \) is called the Rabi frequency. From (9), for \( j = 0, 1 \)
\[
H_{01} |\Phi_j^{(n)}\rangle = (-1)^j \mathbb{H} \Omega |\Phi_j^{(n)}\rangle
\]
Moreover, since for \( j = 0, 1 \) we have
\[
\langle \Phi_0^{(n)} | \Phi_1^{(n')} \rangle = \langle \Phi_1^{(n')} | \Phi_0^{(n)} \rangle = 0, \quad \| \Phi_j^{(n)} \| = 1.
\]
(10) can be written as;
\[
H_{01} = \sum_{n=0}^{\infty} H_{01}^{(n)} = \sum_{n=0}^{\infty} \sum_{j=0}^{1} (-1)^j \mathbb{H} \Omega \langle \Phi_j^{(n)} | \Phi_j^{(n)} \rangle \| \Phi_j^{(n)} \|,
\]
where \( E_{n,j} = \exp \left(-it(-1)^j \mathbb{H} \Omega \right) \).

We can compute the transition probability by using this unitary operator. For example, we now suppose that the initial state of the atom is the upper level. Then the probability \( c_n(t) \) of the atom being in the upper level at the time \( t \) is computed as follows;
\[
c_n(t) = \langle 2 \otimes n | U_t | 2 \otimes n \rangle^2 = \exp \left[-\frac{1}{2} \mathbb{H} \Omega t \right] \sum_{n=0}^{\infty} \frac{[\mathbb{H}^2]^{2n}}{n!} \cos^2 \mathbb{H} \Omega t.
\]
Contrary to this, the probability \( s_n(t) \) of the atom being in the lower level at the time \( t \) is also computed as follows;
\[
s_n(t) = \langle 1 \otimes n + 1 | U_t | 2 \otimes n \rangle^2 = \exp \left[-\frac{1}{2} \mathbb{H} \Omega t \right] \sum_{n=0}^{\infty} \frac{[\mathbb{H}^2]^{2n}}{n!} \sin^2 \mathbb{H} \Omega t.
\]
From the unitary operator given in (11), the final state of the atom of the JC model is given by:
\[
\Lambda \rho = tr_{H_R} E^* \rho = tr_{H_R} U_t (\rho \otimes \omega) U_t^*
\]
\[
= \sum_{m,n=0}^{\infty} \sum_{i,j=0}^{1} E_{n,j} E_{m,i} \langle \Phi_j^{(n)} | \rho \otimes \omega | \Phi_i^{(m)} \rangle \times tr_{H_R} \langle \Phi_i^{(m)} | \Phi_j^{(n)} \rangle
\]
Moreover, from some simple computations, we obtain the quantum mechanical channel such that:
\[
\Lambda \rho = \left( \lambda_0 \tilde{c}_1 \frac{t}{\lambda_0 \tilde{c}_1 (t) + \lambda_1 \tilde{s}_0 (t)} \right) \rho + \left( \lambda_1 \tilde{s}_1 \frac{t}{\lambda_0 \tilde{c}_1 (t) + \lambda_1 \tilde{s}_0 (t)} \right) \rho
\]
\[
\quad + \left( \lambda_0 \tilde{s}_0 \frac{t}{\lambda_0 \tilde{c}_1 (t) + \lambda_1 \tilde{s}_0 (t)} \right) \rho + \left( \lambda_1 \tilde{c}_0 \frac{t}{\lambda_0 \tilde{c}_1 (t) + \lambda_1 \tilde{c}_0 (t)} \right) \rho
\]
where, \( \tilde{c}_i \) and \( \tilde{s}_i \) are the reduced state for the atom after the interaction with the field. This expression can be seen in the result of another approach by Gea-Banacloche [16].

For the computation of the quantum mutual entropy, it is useful to apply the following identity [4]:
\[
S(\sigma_E, \sigma_0) = \sum_k \lambda_k S(\Lambda_k^E \rho, \Lambda_k^\rho)
\]
Since the initial state of the atom given by (6) is the non-degenerated Schatten decomposition, the quantum mutual entropy is uniquely given by:
\[
I(\rho; \Lambda_k^\rho) = S(\sigma_E, \sigma_0) = \sum_k \lambda_k S(\Lambda_k^E \rho, \Lambda_k^\rho)
\]
From the quantum mechanical channel given in (12), the following identities hold:
\[
\langle 1 | \Lambda_k^E | 1 \rangle = \langle 2 | \Lambda_k^E | 2 \rangle = 0 \quad (k = 0, 1)
\]
\[
\langle 1 | \Lambda_k^E | 2 \rangle = \langle 2 | \Lambda_k^E | 1 \rangle = 1
\]
\[
\langle 1 | \Lambda_k^E | 1 \rangle = \tilde{c}_1(t) = 0
\]
\[
\langle 2 | \Lambda_k^E | 2 \rangle = \lambda_0 \tilde{c}_1(t) + \lambda_1 \tilde{s}_0(t)
\]
\[
\langle 1 | \Lambda_k^E | 2 \rangle = \lambda_1 \tilde{s}_1(t)
\]
\[
\langle 2 | \Lambda_k^E | 1 \rangle = \lambda_0 \tilde{s}_1(t) + \lambda_1 \tilde{c}_0(t)
\]
Therefore, the quantum mechanical entropy can be computed as follows:
\[
I(\rho; \Lambda_k^\rho) = \lambda_0 \tilde{c}_1(t) \log \frac{\tilde{c}_1(t)}{\lambda_0 \tilde{c}_1(t) + \lambda_1 \tilde{s}_0(t)} + \lambda_1 \tilde{s}_1(t) \log \frac{\tilde{s}_1(t)}{\lambda_0 \tilde{c}_1(t) + \lambda_1 \tilde{s}_0(t)}
\]
\[
+ \lambda_0 \tilde{s}_0(t) \log \frac{\tilde{s}_0(t)}{\lambda_0 \tilde{c}_1(t) + \lambda_1 \tilde{s}_0(t)} + \lambda_1 \tilde{c}_0(t) \log \frac{\tilde{c}_0(t)}{\lambda_0 \tilde{c}_1(t) + \lambda_1 \tilde{c}_0(t)}
\]
In FIG.1, the quantum mutual entropy is plotted as a function of time \( t \). What is evident from Fig.1 is that the quantum mutual entropy decreases with time. Especially, it is obvious that the local maximum point arising in each “revival time” [17] (i.e. \( T_k = k T_e, T_e \equiv 2 \pi |\langle 2 | / | 1 \rangle | (k = 0, 1, 2, \cdots) \) becomes lower as time goes by. Therefore, we conclude that the time development of the quantum mutual entropy on the JC provides a dissipative change of the state of the atom.
As we have seen, we concretely gave the quantum mechanical channel representing the state change of the atom on the JCM and rigorously derived the quantum mutual entropy. Then, it is shown that the quantum mutual entropy explains the irreversible behavior of the JCM. The relation between the decrease of the quantum mutual entropy and the degree of entanglement can be also studied in the forthcoming paper.