FREE ENERGY OF GRAVITATING FERMIONS

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ABSTRACT

We calculate rigorously, in a suitable thermodynamic limit, the free energy of a system of nonrelativistic fermions which interact with attractive \( r^{-1} \) potentials. It is shown that the effective field approximation becomes exact in this limit and results in the temperature-dependent Thomas-Fermi equations.

*) On leave of absence from the University of Heidelberg, Germany.

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Ref.TH.1330-CERN
26 May, 1971
1. **INTRODUCTION**

The quantum mechanical Hamiltonian

\[ H = \frac{1}{\Sigma} \sum_{i=1}^{N} \frac{p_i^2}{2M_i} + \sum_{1 \leq i, j \leq N} \frac{e_i e_j - \kappa M_i M_j}{|x_i - x_j|} \]  

(1.1)

describing \( N \) particles interacting with \( r^{-1} \) potentials is the relevant quantity if weak and nuclear interactions as well as relativistic effects can be neglected. In spite of the vast domain of applicability only few results have been rigorously derived from it, if \( N > 2 \). Dyson and Lenard \(^1\) have shown that, for \( \kappa = 0 \), \( \Sigma e_i = 0 \) and certain combinations of statistics, the ground state energy of (1.1) for large \( N \) is proportional to \( N \). Lebowitz and Lieb \(^2\) announced a proof that the free energy \( F_N \) then is well-behaved.

Lévy-Leblond \(^3\) proved that, for \( \kappa > 0 \) and \( \Sigma e_i = 0 \), the ground state energy for identical fermions is proportional to \( \frac{1}{N} N^{7/3} \) for large \( N \).

We propose to calculate exactly the limit \( N \to \infty \) of \( N^{-7/3} F_N \) for nonrelativistic identical fermions interacting with their gravitational forces. The reason why this can be done is that, owing to the long range of the force, the temperature-dependent Thomas-Fermi equations become exact.

The system exhibits an interesting thermic behaviour which resembles certain features of stars and which has been discussed previously for simplified models \(^4\).

There is a region where the microcanonical heat capacity is negative. In the canonical ensemble that region is bridged by a phase transition.

In this paper we shall concentrate on the mathematical problem of the asymptotic equality of the exact and the Thomas-Fermi free energy.
We denote by $F(N, \beta, R)$ the free energy of a system of $N$ identical fermions enclosed within a spherical volume $\frac{4\pi}{3} R^3$ at temperature $T = 1/k_\beta$ (k = Boltzmann's constant). The fermions interact with their gravitational forces only. We will choose units $\hbar = 1$, Fermion mass = 1, and $\kappa = \text{gravitation constant} = 1$.

The free energy is defined by

$$e^{-\beta F(N, \beta, R)} = \text{Tr}_{\mathcal{H}(N, R)} e^{-\beta \left( \frac{1}{2} \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{4} \sum_{i+j=1}^{N} |x_i - x_j|^2 \right)}$$  \hspace{1cm} (1.2)$$

where $\mathcal{H}(N, R)$ is the Hilbert space of square integrable, complex valued, totally antisymmetric wave functions of $N$ arguments $x_1, x_2, \ldots, x_N$ which vanish if at least one $|x_i| > R$. By the unitary transformation $x \to R^{-1}x$, $p \to Rp$ expression (1.2) can be rewritten as

$$e^{-\beta F(N, \beta, R)} = \text{Tr}_{\mathcal{H}(N, 1)} e^{-\beta \left( \frac{1}{2} R^{-2} \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{4} R^{-1} \sum_{i+j=1}^{N} |x_i - x_j|^2 \right)}$$  \hspace{1cm} (1.3)$$

We will investigate the limit $\lambda \to \infty$ of

$$\lambda^{-7/3} F(\lambda N, \lambda^{-4/3} \beta, \lambda^{-1/3} R)$$  \hspace{1cm} (1.4)$$

for fixed $N, \beta, R$ and for $\lambda N \in \mathbb{N}$.

The limit along the particular "ray" (1.4) is dictated by the Thomas-Fermi equations and their law of corresponding states. It means that the system becomes hotter and contracts if $N$ is increased. The usual limit ($\beta$ constant, $R \sim N^{1/3}$) could be taken if we would choose $\kappa \sim N^{-2/3}$.

It should also be noted that for non-interacting particles the two limits coincide since

$$\lambda^{-1} F_{\kappa=0} (\lambda N, \beta, \lambda^{1/3} R) = \lambda^{-7/3} F_{\kappa=0} (\lambda N, \lambda^{-4/3} \beta, \lambda^{-1/3} R)$$

holds.
We define

\[ f(\lambda, V) = -\frac{1}{\beta \lambda} \log \text{Tr} \mathcal{P}(\lambda N, 1) e^{-\beta \lambda (K + V)} \]  

(1.5)

with

\[ K = \frac{1}{2} \lambda^{-5/3} R^{-2} \sum_{i=1}^{\lambda N} \frac{\bar{p}_i^2}{i} \]  

(1.6)

and for various interactions \( V \).

For

\[ V = -\frac{1}{2} \lambda^{-2} R^{-1} \sum_{i,j=1}^{\lambda N} |\vec{x}_i - \vec{x}_j|^{-1} \]  

(1.7)

we have

\[ f(\lambda, V) = \lambda^{-7/3} F(\lambda N, \lambda^{-4/3}, \beta, \lambda^{-1/3} R) \]  

(1.8)

i.e. the function of which we want to study the limit \( \lambda \to \infty \). If \( N \) has been chosen sufficiently large, \( \beta \) and \( R \) small \( \xi \) the limit is the desired free energy since then

\[ \lim_{\lambda \to \infty} f(\lambda, V) = f(1, V) = F(N, \beta, R) \]  

(1.9)

For technical reasons we cannot directly prove our assertion for the singular Newton potential

\[ v(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|^{-1} \]  

(1.10)
We shall have to replace it by

\[ \nu_{\mu s}(\vec{x}, \vec{y}) = \sum_{a=1}^{s} \nu_a \varphi_a(\vec{x}) \varphi_a(\vec{y}) \]  

(1.11)

where \( \varphi_a \) are the normalized and real eigenfunctions appearing in the expansion of the continuous potential \( (\mu > 0) \)

\[ \frac{1 - e^{-\mu |\vec{x} - \vec{y}|}}{|\vec{x} - \vec{y}|} = \sum_{a=1}^{\infty} \nu_a \varphi_a(\vec{x}) \varphi_a(\vec{y}) \]  

(1.12)

considered as an integral kernel operator.

The \( \varphi_a \) satisfy the equation

\[ \int_{|y| \leq 1} d^3 y \frac{1 - e^{-\mu |\vec{x} - \vec{y}|}}{|\vec{x} - \vec{y}|} \varphi_a(\vec{y}) = \nu_a \varphi_a(\vec{x}) \]  

(1.13)

with positive eigenvalues \( \nu_a \).

Again for technical reasons we include the self-interaction and define

\[ V_{\mu s} = -\frac{1}{2} \lambda^{-2 R - 1} \sum_{i,j=1}^{\lambda^N} \nu_{\mu s}(\vec{x}_i, \vec{x}_j) = -\lambda^{-1} \sum_{a=1}^{s} J_a \]  

(1.14)

where

\[ J_a = \left( \frac{\nu_a}{2 R \lambda} \right)^{\lambda N} \sum_{i=1}^{\lambda^N} \varphi_a(\vec{x}_i). \]  

(1.15)

In chapter 2 we shall prove that these approximations are arbitrarily good in the sense that

\[ \lim_{\mu \to \infty} \lim_{S \to \infty} \lim_{\lambda \to \infty} (f(\lambda, V_{\mu s}) - f(\lambda, V_{\mu})) = 0 \]  

(1.16)

holds. This also shows that our result does not depend on the singularity but on the long range of the Newton potential. In particular, the addi-
tion of sufficiently short range forces will not affect it.

Next we add to \( V_{\mu s} \) a term

\[
W_{\mu s} [\sigma] = \lambda^{-1} \sum_{a=1}^{s} (J_a - \sigma_a)^2
\]

\[(1.17)\]

where \((\sigma_1, \sigma_2, \ldots, \sigma_s) \in \mathbb{R}^s\).

It will turn out that for suitable \(\sigma\)'s the effect of this term is negligible: we prove in chapter 3 that

\[
\lim_{\lambda \to \infty} \left\{ \inf_{\sigma \in \mathbb{R}^s} f(\lambda, V_{\mu s} + W_{\mu s} [\sigma]) - f(\lambda, V_{\mu s}) \right\} = 0
\]

\[(1.18)\]

is true.

The interaction \( V_{\mu s} + W_{\mu s} [\sigma] \) is linear in the operators \( J_a \), it describes a system of non-interacting particles in the external field generated by \( \sigma \).

We will demonstrate in chapter 4 that the barometric formula results in the limit \( \lambda \to \omega \): If the external field \( U : [0,1] \to \mathbb{R} \) is a regulated function (i.e. the uniform limit of step functions, see ref. 9) and

\[
V = \lambda^{-1} \sum_{i=1}^{\lambda N} U(|x_i|)
\]

\[(1.19)\]

the corresponding interaction then

\[
\lim_{\lambda \to \infty} f(\lambda, V) = -\frac{N\alpha}{\beta} - \frac{1}{\beta} \int_{0}^{1} \! dr \! 4\pi r^2 g_{\beta}(-\alpha U(r))
\]

\[(1.20)\]

where

\[
g_{\beta}(z) = \int \frac{dP}{(2\pi)^3} \ln(1 + e^{-\frac{1}{2}P^2 + z})
\]

\[(1.21)\]
and $\alpha$ is the solution of

$$\int_{0}^{R^3} 4\pi r^2 \ g'_\beta(-\alpha - \beta U(r)) = N. \quad (1.22)$$

$\alpha$ is unique since $g'(z)$ is strictly monotonic.

In chapter 5 it will be shown that $\lim$ and $\inf$ in (1.18) can be interchanged, that the infimum is actually attained for a $\sigma^\epsilon_{\leq s}$, and that this $\sigma^{\mu s}$ is a solution of the self-consistency equation, i.e.

$$U^{\mu s}(x) = \sum_{a=1}^{s} \nu^{\mu s}_{a} \ U^{\mu s}(x) \quad \text{satisfies}$$

$$U^{\mu s}(x) = -\int d^3 y \nu^{\mu s}_{a}(x,y) \ g'_\beta(-\beta R^2 U^{\mu s}(y) - \mu^{\mu s}) \quad (1.23)$$

with

$$\int_{0}^{R^3} d^3 x \ g'_\beta(-\beta R^2 U^{\mu s}(x) - \mu^{\mu s}) = N.$$

These are the well known temperature dependent Thomas-Fermi equations for particles interacting with the potential $\nu^{\mu s}$. In ref. 5 we discuss uniqueness and properties of its solutions. In particular, we demonstrate that the solution is insensitive to small changes of $\nu$ (and can therefore be calculated on a computer): we shown that $U^{\mu s}$ tends with $\mu, s \to \infty$ to a solution $U$ of the Thomas-Fermi equation with the Newton potential. $\alpha$ and $F$ converge to the corresponding values.

Putting (1.16), (1.18), (1.20), and the results of chapter 5 and ref. 5 together, we arrive at the final result:

For all $N \in \mathbb{N}$, $\beta > 0$ and $R > 0$ we have
\[
\lim_{\lambda \to \infty} \lambda^{-7/3} F(\lambda N, \lambda^{-4/3} \beta, \lambda^{-1/3} R) = \\
- \frac{R^5}{2} \int_{|x| \leq 1} d^3x \frac{d^3p}{(2\pi)^3} \frac{1}{\beta \left( \frac{p^2}{2} + R^2 U(x) \right) + \alpha} - \frac{Na}{\beta} \\
- \frac{R^3}{\beta} \int_{|x| \leq 1} d^3x \int \frac{d^3p}{(2\pi)^3} \ln \left( 1 + e^{-\beta \left( \frac{p^2}{2} + R^2 U(x) \right) - \alpha} \right) \tag{1.24}
\]

where \(U(x)\) and \(\alpha\) are determined by

\[
U(x) = -\int_{|x| \leq 1} \frac{d^3x}{(x-x')^3} \frac{d^3p}{(2\pi)^3} \frac{1}{\beta \left( \frac{p^2}{2} + R^2 U(x) \right) + \alpha} \tag{1.25}
\]

and

\[
\int_{|x| \leq 1} d^3x \frac{d^3p}{(2\pi)^3} \frac{1}{\beta \left( \frac{p^2}{2} + R^2 U(x) \right) + \alpha} = N. \tag{1.26}
\]

If (1.25) and (1.26) admit, as is actually the case, for some values \(N, \beta, R\) several solutions, that one for which the right-hand side of (1.24) is smallest is to be chosen.
2. REPLACING $V_{\mathcal{W}}$ BY $V_{\mu s}$

The operator $V_{\mathcal{W}}$ of (1.7) is bounded with respect to $K$ of (1.6): for all $\Psi \in \mathcal{D}_K$ is the standard domain of $K$ such that $K$ is self-adjoint 7) - there exist positive numbers $a$ and $b$ such that

$$||V_{\mathcal{W}}\Psi|| \leq a ||\Psi|| + b ||K\Psi||$$

holds. The infimum of all such $b$, the $K$-bound of $V_{\mathcal{W}}$, is zero 7). Therefore, according to an investigation by Maison 8), $f(\lambda, \kappa V_{\mathcal{W}})$ of (1.5) exists, is entire in $\kappa$, and holomorphic in $\beta$ in the half-plane $\text{Re} \beta > 0$. The derivative with respect to $\kappa$ can be expressed as an expectation value of the interaction:

$$\frac{d}{d\kappa} f(\lambda, \kappa V_{\mathcal{W}}) = \langle V_{\mathcal{W}} \rangle_{\kappa} V_{\mathcal{W}}$$

(2.2)

where

$$< A >_{\mathcal{W}} = \frac{\text{Tr} \mathcal{F}(\lambda N, 1) A e^{-\lambda \beta (K+V)}}{\text{Tr} \mathcal{F}(\lambda N, 1) e^{-\lambda \beta (K+V)}}$$

(2.3)

The domain of the self-adjoint operator $K + V_{\mathcal{W}}$ is also $\mathcal{D}_K$. The eigenfunctions $\Psi_a$ of (1.13) are continuous, hence the operators $J_a$ are bounded, and so is $V_{\mu s}$. Thus, $f(\lambda, V_{\mu s})$ exists as well.

The difference between $V_{\mu s}$ and $V_{\mathcal{W}}$ is

$$V_{\mu s} - V_{\mathcal{W}} = V - \frac{1}{2} \lambda^{-1} R^{-1} N_{\mu} + \frac{1}{2} \lambda^{-2} R^{-1} \sum_{i,j=1}^{\lambda N} \gamma_i a_{\sigma_i} (x_i) a_{\sigma_j} (x_j)$$

(2.4)

with
\[ V_Y = \frac{1}{2} \lambda^{-2} R^{-1} \sum_{i \neq j=1}^{N} \frac{-\mu^{|x_i - x_j|}}{|x_i - x_j|} \]  \hspace{1cm} (2.5)

By Mercer's theorem \(^9\) the sum (1.12) converges uniformly in \( S_1 \times S_1 \), consequently the norm of the last term in (2.4) converges to zero uniformly with respect to \( \lambda \) if \( s \to \infty \). Therefore,

\[ \lim_{s \to \infty} \lim_{\lambda \to \infty} \left[ f(\lambda, V_{\mu s}) - f(\lambda, V_{\mu} + V_Y) \right] = 0 \]  \hspace{1cm} (2.6)

holds for all \( \mu > 0 \) in virtue of the general property

\[ |f(\lambda, A + B) - f(\lambda, A)| \leq ||B||. \]  \hspace{1cm} (2.7)

Now, \( V_Y \) is smaller than \( -V_{\mu} \) thus the \( (K + V_{\mu}) \) bound of \( V_Y \) is also zero. Consequently, the mapping \( t \to f(\lambda, V_{\mu} + t V_Y) \) is entire. For real \( t \) it increases with \( t \) since \( V_Y > 0 \) and it is concave (this is a general property of \( f(\lambda, A + tB) \)). We then find

\[ 0 \leq f(\lambda, V_{\mu} + V_Y) - f(\lambda, V_{\mu}) \]

\[ \leq \frac{d}{dt} f(\lambda, V_{\mu} + t V_Y) \big|_{t=0} = \langle V_Y \rangle \] \hspace{1cm} (2.8)

It is now our task to show that the expectation value of the Yukawa-interaction \( V_Y \) vanishes uniformly in \( \lambda \) if \( \mu \to \infty \). For this we will calculate a lower bound of \( \mu^{-1/5} K - V_Y \).

Following Dyson \(^1\) we decompose this Hamiltonian as follows:

\[ \mu^{-1/5} K - V_Y = \sum_{i=1}^{n} h_i; \quad h_i = \sum_{j=1}^{n} \frac{-\mu^{|x_i - x_j|}}{|x_i - x_j|} \] \hspace{1cm} (2.9)
where

\[ n = \lambda N, \quad M = \mu^{1/5} \lambda^{5/3} R^2 (\lambda N - 1) \quad \text{and} \]

\[ \alpha = \frac{1}{2} \lambda^{-2} R^{-1}. \]

Each of the \( h_i \) is an Hamiltonian describing the motion of \( n-1 \) particles of mass \( M \) in the attractive Yukawa-potential of the \( n' \)th. The particles do not interact. The ground state of \( h_i \) is obtained if the \( n \) particles are filled into the \( n \) lowest states (recall that we deal with fermions).

The single particle bound states lie certainly higher than those of the hydrogen atom, namely \( \epsilon_\nu = -M \alpha^2 / 2 \nu^2 \) with multiplicity \( \nu^2 (\nu = 1, 2, \ldots) \). However, the Yukawa-potential can bind at most \( n_c \) states, for which number the upper bound

\[
n_0 \leq 2 \left\{ \int_0^\infty \frac{dr}{r} \frac{2M_\alpha}{r} \frac{e^{-r}}{r} \right\} \left\{ \sup_{r \neq 0} r^2 \frac{2M_\alpha}{r} \frac{e^{-\frac{r}{r}}}{r} \right\}^{\frac{1}{2}}
\]

\[
\leq 2 \left( \frac{2M_\alpha}{\mu} \right)^{3/2}
\]

(2.10)

is known \(^{10}\). \( n_0 \) corresponds to a hydrogen atom principal quantum number \( \nu_0 \) with \( \nu_0^2 = n_0 \), therefore the ground state of \( h_i \) is higher than

\[-\frac{1}{5} M \alpha^2 \sqrt[3]{n_0} \geq -2\alpha^2 M(\lambda M/\mu)^{\frac{1}{2}}.\]

We conclude that \( \mu^{-1/5} K - V_Y \) is bounded from below by \(-2\alpha^2 M(\lambda M/\mu)^{\frac{1}{2}}\) so that we obtain

\[ 0 \leq V_Y \leq \mu^{-1/5} (K + R^5 N^{5/2}). \]  

(2.11)
It remains to show that \( \langle K \rangle \) is bounded independently of \( \lambda \).

The mapping

\[
t \rightarrow - \frac{2}{\beta \lambda} \log \text{Tr} \frac{\mathcal{F}(\lambda N, 1)}{\mathcal{F}(\lambda N, 1)} e^{-\frac{\beta \lambda}{2} (tK + 2V)}
\]

(2.12)

is analytic in the half-plane \( \text{Re } t > 0 \), increases with real \( t \) and is concave. Hence the derivative with respect to \( t \) for \( t = 2 \), which is just \( \langle K \rangle \), is smaller than

\[
2f(\lambda, V) - 2f(\lambda, 2V) = \beta/2
\]

where the suffix \( \beta/2 \) means that definition (1.5) applies with \( \beta \) being replaced by \( \beta/2 \).

The first term can be bounded above by

\[
- \frac{1}{\beta \lambda} \log \text{Tr} \mathcal{F}(\lambda N, 1) e^{-\beta \lambda K} = - \frac{1}{\beta \lambda} \log \text{Tr} \mathcal{F}(\lambda N, 1/3) e^{-\beta \lambda K}
\]

which is nothing else but the usual free energy for non-interacting fermions. It is known \(^{11}\) that the free energy of a system of \( \lambda N \) non-interacting particles within a spherical volume \( \lambda \frac{4\pi}{3} R^3 \), if divided by \( \lambda \), converges with \( \lambda \rightarrow \infty \) towards the well-known limit. Since \( \lambda N \in \mathbb{N} \) there is an upper and a lower bound, \( f_L(N, \beta, R) < f(\lambda, 0) < f_U(N, \beta, R) \), so that

\[
f(\lambda, V) < f(\lambda, 0) < f_U(N, \beta, R)
\]

holds for all \( \lambda \).

The second term can be bounded from below since with Lévy-Leblond's estimate \(^{3}\) for the ground state of identical fermions with gravitational interaction, we find \( K + 2\kappa V \geq -\kappa^2 N^{7/3} \) for all \( R \) and \( \lambda \) so that
K + 2V \geq \frac{1}{2} K - 2N^{7/3} \text{ and}

f(\lambda, 2V_{\beta/2}) \geq \frac{1}{2} f(\lambda, 0)_{\beta/4} - 2N^{7/3} \geq \frac{1}{2} f_{L}(N, \frac{\beta}{4}, R) - 2N^{7/3} \quad (2.14)

holds for all \lambda.

With (2.13) and (2.14) we have established an \lambda-independent bound for \langle K \rangle_{V_{\beta}}. This result together with (2.11), (2.8) and (2.6) complete the proof of equation (1.16):

$$\lim_{\mu \to \infty} \lim_{S \to \infty} \lim_{\lambda \to \infty} \{f(\lambda, V_{\mu S}) - f(\lambda, V_{\beta})\} = 0.$$
3. THE EFFECTIVE FIELD APPROXIMATION

It has been demonstrated in the preceding section that, for the purpose of calculating the free energy, the original interaction $V$ can be approximated by a finite sum of squares of bounded hermitian operators $J_a$. In this chapter we shall show how a further simplification is achieved if a product of operators is replaced by the product of the operator and its expectation value. The justification of replacing a field by what is usually called the effective field was originally demonstrated by Bogoliubov jr.\textsuperscript{12}) in connection with the BCS theory of superconductivity.

Let us define

$$
\phi(t,j) = -\frac{1}{\beta} \ln \text{Tr} \ e^{-\beta(\lambda K + \sum_{a=1}^{s} [-J_a^2 + t(J_a - (J_a))^2 + J_a J_a^*]})
$$

(3.1)

where

$$
\langle A \rangle_j = \frac{\text{Tr} \ A e^{-\beta(\lambda K + \sum_{a=1}^{s} [-J_a^2 + J_a J_a^*]})}}{\text{Tr} \ e^{-\beta(\sum_{a=1}^{s} [-J_a^2 + J_a J_a^*])}} \quad (3.2)
$$

for $j \in \mathbb{R}^s$.

$\phi(t,j)$ is increasing and concave in $t$, hence

$$
0 \leq \phi(1,j) - \phi(0,j) = \delta(j) \leq \sum_{a=1}^{s} \langle (J_a - (J_a))^2 \rangle_j
$$

(3.3)

holds.

The fluctuation on the right-hand side can be further estimated\textsuperscript{13}) by
\[ \delta(j) \leq \sum_{a=1}^{s} \left\{ \frac{1}{\beta} \left( -\frac{2}{3} \phi(O,j) \right) + \sqrt{\frac{2}{3} \phi(O,j)} \right\} < \left[ J_a, \left[ \lambda_k, J_a \right] \right] > \right\}. \quad (3.4) \]

Now, the integral

\[ \delta(\xi) = \int_0^{1/s} \ldots \int_0^{1/s} \delta(j) \]

defines a \( \xi \in \mathbb{R}^s \) with \( 0 \leq \xi_a \leq 1 \), and since \( |\delta(j)| \leq 2 ||J_a|| \) holds - recall that \( \delta(j) \) is the difference between two \( \phi_a \)'s the derivative of which with respect to \( j_a \) is an expectation value of \( J_a \) - we arrive at

\[ \delta(0) \leq \delta(\xi) + \sum_{a=1}^{s} 2 ||J_a||. \quad (3.6) \]

\( \delta(\xi) \) can now easily be estimated by

\[ \delta(\xi) \leq \sum_{a=1}^{s} \left\{ \frac{1}{\beta} 2 ||J_a|| + \sqrt{2 ||J_a|| \cdot ||\left[ J_a, \left[ \lambda_k, J_a \right] \right]||} \right\} \]

so that

\[ 0 \leq \phi(1,0) - \phi(0,0) \quad (3.8) \]

\[ \leq \sum_{a=1}^{s} \left\{ 2^{1/2} \frac{1+3}{\beta} v_a^{1/2} R^{-\frac{3}{2}} N \lambda_a^{1/2} \lambda_k^{3/4} \lambda_a \right\} \]

\[ + 2^{-k} v_a^{3/4} R^{-7/4} N \lambda_a^{1/2} \lambda_a^{-1/12} \]

has been established. We have used that the double commutator is equal to

\[ \frac{1}{2} \lambda^{-5/3} R^{-3} v_a^N \sum_{i=1}^{\lambda_a} \left( \nabla_a^{(x_i)} \right)^2 \]

and that

\[ \nabla_a = \sup_{x \in S_1} |\nabla_a(x)| \quad (3.9) \]

\[ \nabla_a' = \sup_{x \in S_1} |\nabla_a(x)| \quad (3.10) \]
are both finite since the eigenfunctions $\psi_a$ are continuous and continuously differentiable in $S_1$ (cf. equation (1.13)).

With the positive definite operator $W_{\mu S}[\sigma]$ as defined in (1.17) one finds by comparing (1.5) with (3.1)

$$\lambda^{-1} \phi(0,0) = f(\lambda, V_{\mu S})$$

$$\leq \inf_{\sigma \in \mathbb{R}^S} f(\lambda, V_{\mu S} + W_{\mu S}[\sigma])$$

$$\leq f(\lambda, V_{\mu S} + W_{\mu S}[<J>]_{\sigma}) = \lambda^{-1} \phi(1,0)$$

which, together with (3.8) proves equation (1.18), namely

$$\lim_{\lambda \to \infty} \inf_{\sigma \in \mathbb{R}^S} f(\lambda, V_{\mu S} + W_{\mu S}[\sigma]) - f(\lambda, V_{\mu S}) = 0.$$
4. THE BAROMETRIC FORMULA

With

\[ \sigma(x) = \sum_{a=1}^{\infty} \sqrt{\frac{2}{s \cdot R^5 \cdot \nu_a}} \sigma_a \varphi_a(x) \] (4.1)

and

\[ U(x) = -\sum_{a=1}^{\infty} \frac{2}{s \cdot R^5 \cdot \nu_a} \sqrt{\frac{2}{s \cdot R^5 \cdot \nu_a}} \sigma_a \varphi_a(x) \] (4.2)

we may write for the interaction appearing in equation (1.18):

\[ V_{\mu \nu} + W_{\mu \nu} = \int_{S_1} d^3x \sigma(x) U(x) + \lambda \sum_{i=1}^{\infty} U(x_i) \] (4.3)

The first term on the r.h.s. of (4.3) is a n-number and will appear as an additive contribution to the free energy. It presents no problem for the limit \( \lambda \to \infty \). In this chapter we concentrate on the second term, in particular, we want to study the limit \( \lambda \to \infty \) of

\[ \psi(\lambda, U, N) = f(\lambda, \lambda^{-1} \sum_{i=1}^{\infty} U(x_i)) \] (4.4)

\[ = -\frac{1}{\beta \lambda} \log \text{Tr} e^{\lambda K + \sum_{i=1}^{\infty} U(x_i)} - \beta \sum_{i=1}^{\infty} \frac{\langle p_i^2 \rangle}{2} + U \left( \frac{x_i}{\lambda^{1/3} R} \right) \]

for fixed \( \beta, R \) and various \( \lambda \)-independent external potentials \( U(x) \).

Note that fixed \( U \) corresponds to \( \sigma_a - \frac{1}{\lambda} \), but this is no difficulty for equation (1.18) since the infimum there extends over all of \( \mathbb{R}^s \).
Another remark concerns rotational symmetry: the truncation of the eigenfunction expansion (1.12) to (1.11) can always be done in such a way that the rotational symmetry of $v_{\mu s}(\vec{x},\vec{y})$ is preserved. Since then $V_{\mu s}$ of (1.14) is also invariant under rotations the expectation value appearing in equation (3.11) of the J's will define a spherically symmetric $\sigma(x)$. Therefore, the infimum in (3.11) needs to be with respect to spherically symmetric $\sigma(x)$ only. Since $U(x)$ of equation (4.2) equals $-\int R^2 f d^3 y v_{\mu s}(\vec{x},\vec{y}) \sigma(y)$ we have to consider spherically symmetric external potentials only if $s$ is chosen such that $v_{\mu s}$ is spherically symmetric.

The problem thus separates into a radial and an angular part. Correspondingly, the eigenvalues $\epsilon$ of

$$H = \beta \frac{P^2}{2} + U \left( -\frac{x}{1/3 R} \right)$$

(4.5)

can be labelled by a radial quantum number $n$ and the angular quantum number $\ell$. A lower bound for the $\epsilon$'s is readily available

$$\epsilon_{n, \ell} \geq \beta (R^{-2} - 2^{2/3} \frac{1}{2}(\ell+1) + v)$$

(4.6)

where

$$v = \inf_{0 \leq r \leq 1} U(r)$$

(4.7)

is finite since $U$ is a finite sum of functions which are continuous in the unit ball $S_1$.

For this reason $U$ can be approximated by a piecewise constant potential $U_T$ with a finite number $g$ of steps: for all $\eta > 0$ there is an integer $g$ such that for
$$U_T(r) = U_i = U\left(\frac{i-1/2}{g}\right) \quad \text{if} \quad \frac{i-1}{g} \leq r < \frac{i}{g}, \quad i=1,...,g$$ (4.8)

we have

$$\sup_{0 \leq r \leq 1} |U(r) - U_T(r)| \leq \gamma.$$  

We furthermore consider a potential $U_W$, which is $U_T +$ infinite walls at $r=i/g$. This means that we impose in addition the restriction that the wave functions have to vanish at $r=i/g$. Both $U_T$ and $U_W$ are extensions of the same potential $U_D$ defined on the dense set of wave functions vanishing at $r=i/g$. The intersection of this domain with the domain of $K$ gives $\mathcal{D}_D$, the domain of $H_D = \beta\left(p^2/2 + U_D(r/\lambda^{1/3} R)\right)$. $H_D$ is not self-adjoint but has defect indices $(g,g)$. Its self-adjoint extension $H_W$ and $H_T$ have domains $\mathcal{D}_D + \mathcal{D}_W$ and $\mathcal{D}_D + \mathcal{D}_T$, respectively where $\mathcal{D}_W$ and $\mathcal{D}_T$ are $g$-dimensional subspaces. Clearly $\varepsilon_{n,\lambda} < \varepsilon_{n,\lambda}$ and from the minimax-principle ($E_n$ is an $n$-dimensional subspace)

$$\varepsilon_{n,\lambda} = \inf_{E_n \subset \mathcal{D}_D + \mathcal{D}_W, T} \sup_{X \in E_n} (\chi(r) \frac{u^m}{\mathcal{D}_W T X(r) u^m})$$ (4.9)

we learn

$$\varepsilon_{n-g,\lambda} - \eta < \varepsilon_{n,\lambda} < \varepsilon_{n,\lambda} + \eta < \varepsilon_{n,\lambda} + \eta$$ (4.10)

($\forall n \geq g, \forall \lambda$).

This implies for the partition functions the following inequalities

$$\varepsilon_{n,\lambda} - N\lambda \eta - \varepsilon_{n,\lambda} - \sum_{n=1}^{g-1} \varepsilon_{n,\lambda} + N\lambda \eta$$ (4.11)

$$\leq \varepsilon_{n,\lambda} - N\lambda \eta - \sum_{n=g+1}^{\infty} \varepsilon_{n,\lambda}$$
\[ \Sigma' \] indicates the sum over all occupation numbers compatible with

\[ \Sigma \nu_{n,\xi} = \lambda N. \]

In terms of the corresponding free energies (4.4) we deduce from (4.6) and (4.11)

\[ -\phi(\lambda, U, N) - N\eta \leq -\phi(\lambda, U, N) \]

\[ \leq \lambda^{-1} (\beta^{-1} \log g - g\nu) + N\eta - \inf_{\lambda N - g\xi N' \leq \lambda N} \phi(\lambda, U, N'). \]  

Now, the eigenfunctions of \( H_W \) have their support in one of the shells \( i-1 \leq rg \leq i \), we can therefore replace the radial quantum number \( n \) by the pair \( (i, m) \) where \( i \) labels the shell and \( m \) the radial excitation in this shell. We shall require the following estimate later on:

\[ \frac{1}{2} \beta R^{-2} \lambda^{-2/3} \left[ (\pi mg)^2 + \ell(\ell+1) \left( \frac{\ell}{i-1} \right)^2 + 2U_i \right] \]

\[ \leq e_i, m, \lambda \leq \]

\[ \frac{1}{2} \beta R^{-2} \lambda^{-2/3} \left\{ \begin{array}{ll}
(\pi mg)^2 + \ell(\ell+1) \left( \frac{\ell}{i-1} \right)^2 + 2U_i & \text{for } i > 1 \\
(\pi mg)^2 + 4\ell(\ell+1)g^2 + 2U_i & \text{for } i = 1
\end{array} \right\}. \]

Introducing the partition function of the \( i \)'th shell

\[ e^{-\lambda \beta \phi(i)(\lambda, N)} = \sum_{\nu} e^{-\Sigma \nu_{n,\xi} \nu_{i, m, \xi} e^W_{i, m, \xi}} \]

we obtain for the partition function with walls

\[ e^{-\lambda \beta \phi(\lambda, U, N)} = \sum_{(N_1, \ldots, N_g)} e^{-\lambda \beta \frac{\xi}{i=1} \phi(i)(\lambda, N_i)} \cdot \sum_{N_i = N} \]

\[ e^{-\sum_{\nu} \nu_{n,\xi}}. \]
Since $U_N$ is constant inside a shell the free energy $\phi_{(i)}(\lambda, N)$ is the free energy of non-interacting particles plus $NU_1$. One knows (11) that $\phi_{(i)}(\lambda, N)$ decreases with increasing $\lambda$. By standard arguments (11) one can demonstrate that this property also holds for $\phi(\lambda, U_N, N)$. Since the latter is bounded below by $\phi(\lambda, 0, N) = vN$ which is known to converge for $\lambda \to \infty$ we conclude that $\phi(\infty, U_N, N) = \lim_{\lambda \to \infty} \phi(\lambda, U_N, N)$ exists.

From (4.12) we deduce that $\phi(\infty, U_N, N)$ also exists and is arbitrarily close to $\phi(\infty, U_N, N)$ for $\eta$ sufficiently small.

The explicit form of $\phi(\infty, U_N, N)$ can be calculated by studying the grand canonical ensemble.

The standard proof of the equivalence of the canonical and the grand canonical ensemble can easily be formulated to apply to the case at hand. The grand canonical partition function is the sum of those for the individual shells which are the usual expressions for non-interacting particles (this can be seen by inspecting the limits (4.13) of the eigenvalues). In the limit $\lambda \to \infty$ the sums over eigenvalues approach integrals in momentum space, and with $\eta \to 0$ ($g \to \infty$) the sum over shells becomes a space integral. We will not write down all the necessary epsilonics since this is an exercise in elementary analysis.

The result is

$$\lim_{\lambda \to \infty} \phi(\lambda, U, N) = -\frac{N \alpha}{\beta} - R^3 \left[ \int_0^1 \int_0^{1 \sqrt{2} \pi} \frac{d\epsilon}{\sqrt{2} \pi} \right]_0^1 \times$$

$$\ln (1 + e^{-\beta(\epsilon + U(r))}) - \alpha$$

with $\alpha$ being the unique solution of

$$N = R^3 \left[ \int_0^1 \int_0^{1 \sqrt{2} \pi} \frac{d\epsilon}{\sqrt{2} \pi} \right]_0^1 \left(1 + e^{\beta(\epsilon + U(r))} + a^{-1}\right)$$

(4.17)
This completes the proof of the barometric formula (1.20) to (1.22).
5. THE FINAL RESULT

It is a by-product of the investigations in the preceding section that \( \psi(\lambda, U, N) \) converges from above towards the limit. Therefore, the limit and the infimum operation in equation (1.18) can be interchanged.

We have thus to investigate the infimum of

\[
\frac{R^5}{2} \sum_{a=1}^{s} \nu_a \sigma_a^2 - \frac{Na_0}{\beta} - \frac{1}{\beta} R^3 \int_{|x| < 1} d^3 x \, g^\beta_{\beta}(\beta R^2 \sum_{a=1}^{s} \nu_a \sigma_a \psi_a(x) - \sigma_0) \quad (5.1)
\]

where \( \sigma_0 \) is a solution of

\[
R^3 \int_{|x| < 1} d^3 x \, g^\beta_{\beta}(\beta R^2 \sum_{a=1}^{s} \nu_a \sigma_a \psi_a(x) - \sigma_0) = N \quad (5.2)
\]

Since \( \sigma \) depends on \( \nu \), this solution need no longer be unique.

The derivative of (5.1) with respect to \( \sigma_b \) is

\[
\frac{R^5}{2} \nu_b \sigma_b - \int_{|y| < 1} d^3 y \, g^\beta_{\beta}(\beta R^2 \sum_{a=1}^{s} \nu_a \sigma_a \psi_a(y) - \sigma_0) \psi_b(y) \quad (5.3)
\]

Note that the derivative of \( \sigma_0 \) does not appear because of the subsidiary condition (5.2).

We see that the free energy (5.1) increases with \( |\sigma_b| \) if \( |\sigma_b| > R^{-N} \psi_b \).

Hence the infimum in equation (1.18) can be restricted to an infimum over the compact cube \( |\sigma| \leq 2R^{-N} \max_{1 \leq b \leq s} \psi_b \).

Since (5.1) is continuous and continuously differentiable with respect to any \( \sigma_b \), the infimum is attained at a point \( \sigma^{\mu^s}(x) = \sum_{a=1}^{s} \sigma^{\mu^s}_a \psi_a(x) \) where all partial derivatives (5.3) vanish.
(5.2) and (5.3) are exactly the self-consistency equations (1.23) we referred to in chapter 1.

Since the limit \( s \to \infty, \mu \to \infty \) of (5.1) is identical with the right-hand side of (1.24) we have completed our proof.

ACKNOWLEDGEMENT

We would like to thank Dr. D. Maison and Dr. A. Wehrl for helpful discussions.
REFERENCES

1) F.J. Dyson in Statistical Physics, Phase transitions and Superfluidity, 1966 Brandeis University Summer School in Theoretical Physics, lecture notes;


6) A typical "neutron star" of $10^{57}$ particles at a temperature of 5 MeV and enclosed into a sphere of 100 km radius corresponds to $(\lambda N, \lambda^{-4/3}_R, \lambda^{-1/3}_N)$ with $N = 1$, $\beta = 60 \hbar^2 \chi^2 \approx 2 m^2_N$, $R = 2.9 \times 10^{-1} m_N^1$ and $\lambda = 10^{57}$. Since $N$, $\beta$ and $R$ are of order unity (if measured in their natural units) and since $\lambda = 10^{57}$ is sufficiently large, we will describe the above "neutron star" by the limit $\lambda \to \infty$. For $N = 10^{57}$, $\beta = (5 \text{ MeV})^{-1}$ and $R = 100 \text{ km}$ we would have reached the same accuracy for $\lambda = 1$.

   There the infinite volume case is studied, however, the result also holds for finite volume.


    Again this estimate for infinite volume is a fortiori also valid for finite volume.

