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WEAK INTERACTION CORRECTIONS TO THE ANOMALOUS MAGNETIC MOMENTS
OF THE LEPTONS

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A B S T R A C T

The quadratic divergent contribution of the current current weak interactions to the anomaly of the leptons is calculated using the Pauli-Villars regularization prescription à la Gupta. This contribution is found not to be zero, contrary to what has been found in the literature.

As a by-product of the calculations, formulae for the weak contribution to the energy levels of the leptonic atoms are derived. It is noted incidentally that the neutrino loop exhibits an anomaly similar to that arising in the study of the divergence of the axial vector current.

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1. - INTRODUCTION

One of the most precise experiments in physics is the determination of the anomalous magnetic moments of the muon and the electron ¹⁾. A new CERN experiment ²⁾ plans to measure the anomaly of the muon to an accuracy of at least 15 times the present result. It is known that at such a level of accuracy the comparison of the experiment with the theory will require the knowledge of the sixth order quantum electrodynamical corrections and the strong interaction effects. The electrodynamical corrections have been calculated ³⁾ and the hadronic effects are measurable ⁴⁾ in the colliding beam experiments if certain assumptions are made. Moreover, the weak interaction corrections could become important. These corrections are interesting for two reasons. Firstly, if the intermediate vector bosons exist, they may be revealed in the "g-2 experiments". Secondly, if the current current interaction is the correct description, one could perhaps look for the second order weak interaction effects. The main part of this paper will be concerned with the latter possibility.

By power counting arguments, the leading (most divergent) contribution of the leptonic four-fermion interactions to the anomaly is expected to be of the order $G^2 \Lambda^2 m_\ell^2$, where G is the weak interaction constant and m_ℓ and Λ are the lepton mass and cut-off masses respectively. An earlier calculation of this quantity finds ⁵⁾ that the "coefficient" of this term mysteriously vanishes. However, we find that this mysterious cancellation does not occur if the divergent quantities are regularized. In Section 2 A, we calculate the quadratic divergent contribution of leptonic weak interactions to the anomaly. It is also interesting to note that the muon anomaly (because of the finite width of the muon) has a tiny imaginary part which we compute in Section 2 A. Next we estimate the quadratic divergent contribution to the anomaly due to semi-leptonic weak interactions under the assumption of constant form factors. We find that the inclusion of these processes does not affect the order of magnitude of the correction to the anomaly obtained from the leptonic interactions.

The most divergent contribution to the anomaly due to intermediate bosons has also been found to vanish. Since we have not found the reason for this phenomenon in the literature (and would like to know whether it depends on regularization) we discuss it in Section 2 B.

As a by-product of the calculations, we give in Section 3 the four-fermion corrections to the energy levels of leptonic hydrogen like atoms. These corrections are extremely small for "reasonable" values of the cut-off. What these reasonable values for the cut-off are is not clear to us. In the literature, one often takes the point of view that there exists some universal weak interaction cut-off and tries ⁶⁾ to put an upper bound on it from the experimental information. The most favourable processes are found to be the $K_L^0 - K_S^0$ mass difference and $K_L^0 \rightarrow \mu^+ \mu^-$. There are two objections to these calculations. Firstly, since the cut-off is not introduced in a dynamical manner (but only to cut-off different integrals) there is really no reason why a cut-off introduced for one process should be related to the cut-off for some other process. Secondly,

the way in which a cut-off is introduced could be very important. E.g., one could conceive of a cut-off prescription for which the leading divergent contribution for some process vanishes. This would drastically alter the bounds on the cut-off. We give an example of such a possibility in the trivial case of the electromagnetic form factor of the neutrino (Section 4). Conversely, a divergent quantity which without regularization seems to vanish could be non-vanishing if one regularizes. This happens in the case of the leptonic four-fermion corrections to the anomaly (Section 2 A). In our calculations we employ the regularization prescription of Pauli-Villars ⁷⁾ as given by Gupta ⁸⁾. This regularization procedure has three appealing features, viz., (i) it is in principle well defined because the cut-off masses are introduced in the Lagrangian from the very beginning ; (ii) it is simple in the sense that we need not bother to calculate the so-called "shift" of the integrals ⁹⁾ and (iii) in the presence of the electromagnetic field the gauge invariance is preserved. The third property is nice because, e.g., in the case of the electromagnetic form factor of the neutrino we need not "define" an infinite quantity to be zero (Section 4). Unfortunately, as soon as there is more than one mass in the theory, there are clearly several ways of regularizing and the answer is not unique. E.g., for the cases we have considered, in the calculation of the leptonic corrections to the $g-2$, the result can vary by about a factor of three and for more complicated diagrams larger variations could easily occur. Finally, in Section 4, we also discuss the anomalous behaviour of the neutrino loop which is similar to the anomaly of some commutators and axial vector current discussed in the literature.

2. - WEAK INTERACTION CORRECTIONS TO THE ANOMALY

The weak interaction corrections to the anomaly of the leptons, to the order G , due to the as yet hypothetical intermediate vector boson, have been calculated, recalculated and corrected by several authors ^{10), 11)}. The essential feature of these calculations is that the correction is proportional to Gm_ℓ^2 , m_ℓ being the lepton mass. Therefore, only the correction to the muon anomaly is considered.

The authors of Ref. 10) employ ordinary vector boson theory. Although, by power counting, one expects the correction to be quadratically divergent, they find that it is only logarithmically divergent. Since no explanation of this phenomenon is given, we discuss it in Section 2 B.

The authors of Ref. 11) use the so-called ξ limiting formalism ¹²⁾. One finds

$$\delta a_\mu = \frac{g_w^2}{8\pi^2} \left(\frac{\mu}{M_w} \right)^2 \left[2(1-x_w) \ln \xi + \frac{10}{3} \right] \quad (2.1)$$

where the notation is : $a_\mu = \frac{1}{2}(g_\mu - 2)$ = muon anomaly, μ = muon mass, M_W = boson mass, χ_W = boson anomalous magnetic moment, $(g_W/M_W)^2 = G/\sqrt{2}$ and ξ is a parameter.

Intuitively $1/\xi = \Lambda'^2/M_W^2$, where Λ' is a cut-off mass. Further in Eq. (2.1) one assumes that ξ and (μ/M_W) are small. Putting $\chi_W = 0$ and choosing ξ according to a certain prescription ¹³⁾ one finds

$$\delta a_\mu \cong -1.0 \times 10^{-8} \quad (2.2)$$

If instead one puts $\Lambda' \sim 300$ GeV and $M_W = 2$ GeV one finds ¹⁴⁾

$$\delta a_\mu \cong -2 \times 10^{-8} \quad (2.3)$$

The contribution to the anomaly due to the ordinary current-current interactions is generally considered to be negligible because it occurs first in order G^2 . This reasoning is not convincing because one expects the contribution to be proportional to $G^2 \Lambda'^2 m_\mu^2$ where Λ' is some cut-off [as compared to $G m_\mu^2 \ln(\Lambda'/M_W)$ in Eq. (2.1)]. Therefore, for the "unitarity value" of the cut-off $G \Lambda'^2 \sim 1$, the two contributions are of the same order of magnitude. One needs therefore to calculate the coefficient of $G^2 \Lambda'^2 m_\mu^2$ before deciding which one is the largest. A calculation of this quantity finds ⁵⁾ that the "coefficient" mysteriously vanishes. We find, however, that this is not the case if one regularizes [cf. Eq. (2.19)].

A. - Current Current Weak Interaction Corrections

We will assume that the weak interaction Lagrangian density has the form

$$\mathcal{L}_{int} = \frac{G}{\sqrt{2}} g_\lambda \cdot \bar{g}_\lambda \quad (2.4)$$

$$g_\lambda = l_\lambda + J_\lambda, \quad \bar{g}_\lambda = (-1)^{\delta_{\lambda 4}} g_\lambda^+, \quad (2.5)$$

$$l_\lambda = i \bar{\mu} O_\lambda \nu_\mu + i \bar{e} O_\lambda \nu_e, \quad (2.6)$$

$$O_\lambda = \gamma_\lambda (1 + \gamma_5), \quad (2.7)$$

where our notation and metric are the same as in Ref. 15). Further, J_λ is the hadronic current of the Cabibbo theory¹⁶⁾. As is often done in the recent literature, we distinguish between the so-called non-diagonal and diagonal leptonic interactions¹⁷⁾ (the terms $\bar{e} 0_\lambda \nu_e \bar{\nu}_e 0_\lambda e$ and $\bar{\mu} 0_\lambda \nu_\mu \bar{\nu}_\mu 0_\lambda \mu$ are diagonal and the muon decay terms are non-diagonal). First we consider the leptonic four-fermion corrections to the anomaly. The relevant diagram is shown in Fig. 1, where ℓ and ℓ' denote charged negative leptons and ν and ν' are neutrinos. By power counting, the diagram is quartically divergent, therefore we expect the correction to the anomaly to be at most quadratically divergent. To see this, consider the matrix element of the electromagnetic current between two one-lepton states

$$\langle \ell(\vec{p}') | J_\lambda(0) | \ell(\vec{p}) \rangle = \frac{i e}{V} \bar{u}(\vec{p}') \left[F_1 \gamma_\lambda - F_2 \frac{\sigma_{\lambda\nu} q_\nu}{2 m_\ell} \right] u(\vec{p}), \quad (2.8)$$

where $F_i = F_i(t)$, $t = -q^2$, $q_\nu = p'_\nu - p_\nu$ and V is the volume of quantization. Further, m_ℓ is the lepton mass; $u(\vec{p})$ and $\bar{u}(\vec{p})$ are the usual Dirac spinors. The anomaly is defined by

$$a_\ell = F_2(0). \quad (2.9)$$

Because the most divergent piece (quartic) of the diagram in Fig. 1 does not depend on external momenta it can only contribute to F_1 . The cubic term vanishes by symmetric integration and thus the most divergent contribution to the anomaly is quadratic. In order to calculate this contribution we employ the regularization prescription of Gupta⁸⁾. Let $I(m^2)$ be a divergent integral, $I(m^2) = \int d^4 k R(m^2, k)$ where m is some mass. Regularization is achieved by replacing $I(m^2)$ with $I^R(m^2)$,

$$I^R(m^2) = (-1)^n \int_0^{\Lambda^2} dz_1 \dots \int_0^{\Lambda^2} dz_n \int d^4 k R^{(n)}(m^2 + z_1 + \dots + z_n, k), \quad (2.10)$$

where $R^{(n)}$ is the n^{th} derivative of R with respect to m^2 and Λ is the cut-off mass which tends to infinity at the end of calculations. The power n must be chosen such that the $d^4 k$ integral becomes convergent.

The diagram shown in Fig. 1a is proportional to a quantity D_λ defined by

$$D_\lambda = \bar{u}(\vec{p}') \gamma_\rho \int d^4 Q d^4 \ell \frac{i \gamma(P+Q) - m}{(P+Q)^2 + m^2} \gamma_\lambda \frac{i \gamma(P+Q) - m}{(P+Q)^2 + m^2} \left[\ell_\rho (\ell - Q)_\sigma + \right. \\ \left. + \ell_\sigma (\ell - Q)_\rho - \delta_{\rho\sigma} \ell(\ell - Q) \right] [\ell^2 + m_\nu^2]^{-1} [(\ell - Q)^2 + m_\nu^2]^{-1} \gamma_\sigma u(\vec{p}). \quad (2.11)$$

The quantity $\ell_\rho (\ell - Q)_\sigma + \ell_\sigma (\ell - Q)_\rho - \delta_{\rho\sigma} \ell(\ell - Q)$ comes from the neutrino loop when one, by Fierz transformation, writes the interaction Lagrangian in the charge retention form. Because of the V-A form of the interaction, the neutrino masses in the numerator of the neutrino propagators have dropped out. Since we are interested in the quadratically divergent contribution to F_2 , we may put the lepton mass m ,

appearing in the numerator of Eq. (2.11) equal to zero. The reason being that because of the factors $1 + \gamma_5$ the terms linear in m vanish and the term quadratic in m gives at most logarithmically divergent contribution to F_2 .

In order to regularize the correction to F_2 we need to introduce at least two auxiliary masses z_1 and z_2 . We have at our disposal two masses (viz. m_ν and m) if we are considering the diagonal contribution and three masses (viz. m_ν , m_ν , and m) if we deal with the non-diagonal terms. There are clearly several ways of regularizing. Below, we list a few possibilities.

(a) We may put $m_\nu = m_\nu$ and regularize twice with respect to m_ν^2 and at the same time we may regularize as many times as we wish (say, n times) with respect to m^2 . After performing the $d^4\ell$ integration we can put $n = 0$ in the resulting formula without changing the leading behaviour in Λ .

(b) If the neutrinos are of different kind, we may choose to do as in (a) with the exception that instead of regularizing twice with respect to m_ν^2 we regularize once with respect to m_ν^2 and once with respect to $m_\nu'^2$.

(c) We could regularize once with respect to m^2 and once with respect to m_ν^2 and as many times as we wish (n' times) with respect to $m_\nu'^2$. If we now perform the d^4Q integration first we may put $n' = 0$.

We find that the result depends on the choice of the regulators. E.g., for the case (b) the regularized quantity D_λ^R is given by

$$D_\lambda^R = \bar{u}(\bar{p}) Q_p \int_0^{\Lambda^2} \prod_{i=1}^n dz_i \cdot (n+1)! \int_0^{\Lambda^2} dz' \int_0^{\Lambda^2} d\bar{z}' \int_0^1 dx \int_0^1 dy \gamma(1-y) dy \times \\ \times \int d^4Q d^4\ell \ i\delta(p'+Q) \gamma_\lambda \ i\delta(p+Q) [l_\rho(l-Q)_\rho + l_\sigma(l-Q)_\sigma - \delta_{\rho\sigma} l(l-Q)] \times \\ \times G_1^{(n)}(Q, x, m^2, z_i) G_2(l, Q, m_\nu^2, m_\nu'^2, z', z'', y) O_\sigma u(\bar{p}).$$

(2.12)

In Eq. (2.12)

$$G_1^{(n)}(Q, x, m^2, z_i) = [Q^2 + S + 2Q\xi]^{-n-2}, \quad (2.13)$$

$$\xi_\alpha = P_\alpha x + P'_\alpha (1-x), \quad (2.14)$$

$$S = m^2 + \sum_{i=1}^n z_i - \mu^2, \quad (2.15)$$

$$G_2(l, \dots, y) = [l^2 + (Q^2 - 2lQ)y + R]^{-4} \quad (2.16)$$

$$R = (m_\nu^2 + z'')y + (m_\nu^2 + z')(1-y). \quad (2.17)$$

Further $\mu(m)$ is the mass of the external (internal) charged lepton and we have taken all the cut-off masses equal. If we regularize according to (a) we must replace the factor $y(1-y)$ in Eq. (2.12) by unity and take R to be $m_\nu^2 + z' + z''$.

We perform the $d^4\ell$ integration in Eq. (2.12) and find

$$\begin{aligned} & \int d^4\ell [\ell_\mu (\ell - Q)_\mu + \ell_\mu (\ell - Q)_\mu - \delta_{\mu\mu} \ell (\ell - Q)] G_2(\ell, \dots) = \\ & = -\frac{i\pi^2}{6} [M(Q^2, y, z', z'') \delta_{\mu\mu} + N(Q^2, y, z', z'') Q_\mu Q_\mu], \end{aligned} \quad (2.18)$$

$$M = \frac{R}{y^2(1-y)^2(Q^2 + \tilde{R})^2}, \quad N = \frac{2}{y(1-y)(Q^2 + \tilde{R})^2},$$

where $\tilde{R} = R/y(1-y)$.

In order to see how the authors of Ref. 5) without regularizing found a vanishing "coefficient" and how regularization will affect their result, following them we expand $G_1^{(n)}$ in powers of $(2Q\xi/Q^2 + S)$ and isolate what they call the quadratically divergent terms. We find

$$\begin{aligned} D_\lambda^R = & -4\mu \cdot i\pi^2 \cdot i(P+P')_\lambda \bar{u}(\bar{P}') u(\bar{P}) \int_0^{\Lambda^2} \prod_{i=1}^n dz_i (n+1)! \int_0^{\Lambda^2} dz' \int_0^{\Lambda^2} dz'' \times \\ & \times \int_0^1 y(1-y) dy \int \frac{d^4Q}{(Q^2+S)^{n+2}} \left\{ M \left[2 - \frac{3}{2}(n+2) \frac{Q^2}{Q^2+S} + \right. \right. \\ & \left. \left. + \frac{(n+2)(n+3)}{6} \left(\frac{Q^2}{Q^2+S} \right)^2 \right] + Q^2 N \left[\frac{1}{2} - \frac{n+2}{4} \frac{Q^2}{Q^2+S} \right] \right\} + \dots, \end{aligned} \quad (2.19)$$

where the dots stand for terms which do not interest us (i.e., they are either less divergent or do not contribute to F_2).

Suppose that we had not regularized, i.e., in Eq. (2.19) we put $n = 0$, omit all the integrations over the regulators and replace S by $m^2 - \mu^2$. Moreover, in the unregularized version M and N are no longer given by the above expressions [after Eq. (2.18)], in fact they are very ill-defined but we know that they are functions of Q^2 and y . Further by dimensional arguments M has the dimension of Q^2 and N that of a constant. If we now write $(1/Q^2 + S) = (1/Q^2) + O((Q^2)^{-2})$, we see clearly that the quadratic divergent term (in Q^2) comes from the constant terms in the square brackets of Eq. (2.19). Since these constant terms vanish the authors of Ref. 5) conclude that the quadratic divergent anomaly has a vanishing "coefficient".

However, one must be very careful when dealing with two loops, i.e., the inner (neutrino) loop is by itself quadratically divergent. This quadratic divergence will multiply even the finite part of the outer loop to produce a quadratic divergent result. The fact that the d^4Q integrand in Eq. (2.19) is less divergent than expected is therefore irrelevant. Therefore, the above expansion does not make sense and before calculating, the functions M and N of the neutrino loop must be defined by some regularization procedure. Incidentally, it is also seen from Eq. (2.19) that for $n \neq 0$ (i.e., when one regularizes with respect to m^2) the constant coefficients of M and N are no longer zero, i.e., the high Q behaviour of the d^4Q integrand also depends on regularization.

Going back to Eq. (2.12) and putting $n = 0$, after some calculations, we find

$$D_{\lambda}^R = -\frac{4}{3} (i\pi^2)^2 \Lambda^2 \beta \, i\mu (P+P')_{\lambda} \bar{u}(\bar{p}') u(\bar{p}) + \dots \quad (2.20)$$

where the dots stand for the terms which do not concern us here. The constant β depends on the method of regularization

$$\beta \approx 4 \ln 2 \quad \text{case a)} \quad (2.21)$$

$$\beta \approx 1 \quad \text{case b)} \quad (2.22)$$

These equations clearly demonstrate the choice of regulators matters. If we regulate according to the prescription (c) above, after some calculations, we find

$$D_{\lambda}^R = \frac{-6 + \pi^2}{9} \Lambda^2 \cdot i\mu (P+P')_{\lambda} (i\pi^2)^2 \bar{u}(\bar{p}') u(\bar{p}) + \dots \quad (2.23)$$

where again the dots stand for the terms which do not concern us. Comparing our results we see that the answer varies by about a factor of three for these three cases.

The correction to the anomaly is obtained, from equations above, by multiplying the coefficients of $(p+p')_{\lambda} \bar{u}(\bar{p}') u(\bar{p})$ in D_{λ}^R by $(i\mu/2)(G/4\pi^4)^2$. Our result for the case a) is

$$\delta F_2(0) \stackrel{\text{non-diagonal}}{=} \frac{1}{2} \delta(g-2) \approx -\frac{\ln 2}{6\pi^4} G^2 \Lambda^2 \mu^2. \quad (2.24)$$

If also the diagonal interactions are considered the correction becomes twice as large

$$\delta F_2(0) \stackrel{\text{leptonic}}{=} -\frac{\ln 2}{3\pi^4} G^2 \Lambda^2 \mu^2. \quad (2.25)$$

Numerically, for the muon $\delta_{F_2(0)}^{\text{leptonic}} \approx -2.10^{-10}$ for $\Lambda \sim 300$ GeV as compared to $\delta_{F_2(0)} \approx -2.10^{-8}$ for $\Lambda' \sim 300$ GeV and $M_W \approx 2$ GeV of the vector boson theory ¹⁴⁾.

It is interesting to note that the F_2 form factor of the muon has a tiny imaginary part ^{*}) for space-like photon in diagram of Fig. 1. This "imaginary anomaly" which arises because of the finite width of the muon can be calculated from the unitarity equation for the current ¹⁸⁾ to the order eG^2 . After some very lengthy calculations we find

$$\lim_{t \rightarrow 0^-} \text{Im} F_2(t) = -\frac{\pi}{3} \left(\frac{G\mu^2}{8\pi^2} \right)^2 = -\frac{1}{\mu\tau} = -\frac{\Gamma}{\mu}, \quad (2.26)$$

where μ (τ, Γ) is the muon mass (lifetime, width).

In Eq. (2.26) we have dropped terms of the order $(m_e/\mu) \sim 5.10^{-3}$ and $(m_e/\mu) \ln(\mu/m_e) \sim 3.10^{-2}$ because none of these terms has an anomalously large coefficient. Since the magnitude of the imaginary anomaly is minute, $\text{Im} F_2(0) \sim -10^{-18}$, we do not discuss it any further.

We now turn to the question of semileptonic weak interaction corrections to the anomaly. The main difficulty in this connection is that we do not really know how to calculate these corrections. If we neglect the influence of the form factors and treat fermions as if they were leptons we can immediately generalize our results to the octet of baryons by replacing the virtual (l', ν') in Fig. 1 by $(\bar{p}, n)$, $(\Sigma^-, \bar{\Sigma}^0)$, $(\Xi^-, \bar{\Xi}^0)$ and $(\Xi^-, \bar{\Xi}^0)$. Regularizing twice with respect to m_ν^2 we find that each intermediate state contributes by about $\frac{1}{2} \delta_{F_2(0)}^{\text{non-diagonal}}$. Our rough estimate for the strangeness conserving "baryonic" weak interaction is therefore

$$\delta_{F_2(0)}^{\text{baryonic}} \approx \delta_{F_2(0)}^{\text{leptonic}}. \quad (2.27)$$

Next we consider the quadratic divergent contribution due to "mesonic" weak interactions, i.e., when (l', ν') is replaced by (π^-, π^0) or $(K^-, \bar{K}^0)$.

Again treating the pions as point particles and regularizing twice with respect to m_ν^2 after some calculations, we find

$$\delta_{F_2(0)}^{(\pi)} \sim -\frac{2}{3} \Lambda^2 \ln 2 \cdot \frac{G^2 \mu^2}{(4\pi)^4}, \quad (2.28)$$

^{*}) I wish to thank J.S. Bell for suggesting this calculation. The appearance and the physical meaning of the imaginary moment has been discussed by : J.S. Bell - Nuovo Cimento 24, 453 (1962).

i.e., the "pionic correction" is surprisingly small, $\delta F_2^{(\pi)}(0) \sim 1/64 \delta F_2^{\text{non-diagonal}}(0)$. In our approximation the $\delta F_2^{(K)}(0)$ is of the same order of magnitude as $\delta F_2^{(\pi)}(0)$ and is therefore negligible.

Another class of diagrams contributing to the anomaly is shown in Fig. 1b, where M denotes a meson. However, it is easily seen that contribution to the anomaly is only logarithmically divergent and therefore can be neglected.

B. - Vanishing of the Most Divergent Contribution to Anomaly in the Vector Boson Calculations

We have seen that the quadratic divergent contribution to the anomaly due to the four-Fermion interaction does not vanish if the divergent integrals are regularized. We will now ask why does the quadratic divergent contribution to the anomaly vanish in the calculations with vector bosons (Fig. 2).

The reason here is as follows : although the diagram in Fig. 2, by power counting, seems to be quartically divergent, it is in fact only quadratically divergent. Simplest way of seeing this is by considering the external field vertex shown again in Fig. 2a.

In this figure, Q and Q' stand for the momenta of the virtual bosons. The simplest vertex function (up to a constant) is given by ¹⁹⁾

$$\Gamma_{\lambda\eta\tau}(Q, Q') = \delta_{\eta\tau}(Q+Q')_\lambda - \delta_{\lambda\eta} Q_\tau - \delta_{\lambda\tau} Q'_\eta \quad (2.29)$$

Contracting with $Q_\eta Q'_\tau$ we find

$$\Gamma_{\lambda\eta\tau} Q_\eta Q'_\tau = 0 \quad (2.30)$$

The quadratically divergent term in the diagram of Fig. 2 is proportional to the left-hand side of Eq. (2.30) and thus vanishes. Equation (2.30) has its origin in the gauge invariant coupling of the vector boson to the external field, i.e., in the co-ordinate space we find expressions of the type

$$E = \partial_\lambda \Delta_{\eta\tau}^F(x-x') - \partial_\eta \Delta_{\lambda\tau}^F(x-x') \quad (2.31)$$

where

$$\Delta_{\eta\tau}^F(x) = \left[\delta_{\eta\tau} - \frac{1}{M_W^2} \partial_\eta \partial_\tau \right] \Delta_F(x),$$

$$\Delta_F(x) = \frac{-i}{(2\pi)^4} \int d^4q \frac{e^{iqx}}{q^2 + M_W^2 - i\epsilon}, \quad \partial_\eta = \frac{\partial}{\partial x_\eta}.$$

The vanishing of the quadratically divergent term can be traced to the fact that the most divergent term in E has the form

$$\partial_\lambda \partial_\eta \partial_\epsilon \Delta_F(x-x') - \partial_\eta \partial_\lambda \partial_\epsilon \Delta_F(x-x') = 0.$$

Any gauge invariant regularization, such as à la Gupta, will not affect Eq. (2.30) and the correction to the anomaly due to the vector boson calculations will be only logarithmically divergent.

3. - FOUR-FERMION CORRECTIONS TO THE ENERGY LEVELS OF THE LEPTONIC ATOMS

The loop shown in Fig. 3 ($l, l' = \text{neutrinos}$) is expected to shift the energy levels of the leptonic atoms, i.e., positronium and muonium. Here, we are only interested in the quadratic divergent contribution to the energy levels which we calculate using the non-relativistic first order perturbation theory. After some calculations we find the energy shift ΔE is given by

$$\Delta E = \frac{G^2 \Lambda^2}{\pi^2} \ln 2 |\varphi_n(0)|^2 \rho, \quad (3.1)$$

$$\rho = -1 \text{ for muonium, } \rho = 2(1 - \bar{S}^2) \text{ for positronium.}$$

Here, G = weak interaction constant, Λ = cut-off parameter, $\varphi_n(0)$ = wave function of the atom at the origin and $\bar{S}$ = total spin of the positronium. The factor $\ln 2$ in Eq. (3.1) comes from the Gupta formalism [cf. Eq. (2.10)]. From the muon we may compare ΔE with the finite width of the muon $\Gamma = 1/\tau_\mu = (G^2 m_\mu^5)/(192 \pi^3)$. For the ground state of the muonium $|\Delta E| \sim \Gamma$ if $\Lambda \gtrsim 4 \cdot 10^4$ GeV. The magnitude of ΔE for $n = 2$ state of the muonium is about $-3 \cdot 10^{-12} (\Lambda^2/M_p^2) (\text{Mc/sec})$ to be compared with the lowest order Lamb shift (which is about 10^3 Mc/sec). Thus, the current current weak interaction corrections to the energy levels of the leptonic atoms are negligible for "reasonable" values of the cut-off.

4. - ANOMALOUS BEHAVIOUR OF THE NEUTRINO LOOP AND A REMARK ON THE ELECTROMAGNETIC FORM FACTOR OF THE NEUTRINO

In this Section we discuss two interesting features of the lepton loops, Figs. 3 and 4, which to our knowledge have not been discussed in the literature.

A. - Anomalous Behaviour of the Neutrino Loop

The ν loop (Fig. 3 with l and l' neutrinos) has an anomalous behaviour similar to anomaly of some commutators ²⁰⁾ and of axial vector current ²¹⁾ discussed in the literature. In the co-ordinate space the lepton loop is proportional to a quantity $L_{\rho\sigma}(x-x')$ defined by

$$L_{\rho\sigma}(x-x'; m, m') = C \text{Tr} \left[S_F(x-x', m) \gamma_\rho (1 + \gamma_5) S_F(x'-x, m') \gamma_\sigma (1 + \gamma_5) \right] \quad (4.1)$$

$$C = \frac{(2\pi)^8}{32}, \quad S_F(x, m) = \frac{2}{i(2\pi)^4} \int d^4k e^{ikx} \frac{i\gamma k - m}{k^2 + m^2 - i\epsilon} \quad (4.2)$$

$$(\gamma \frac{\partial}{\partial x} + m) S_F(x) = -\frac{2}{i} \delta^4(x). \quad (4.3)$$

By naive use of Eq. (4.3), we find for the neutrino loop

$$\frac{\partial L_{\rho\sigma}(x-x', 0, 0)}{\partial x_\rho} = 0 \quad (4.4)$$

The Fourier transform of $L_{\rho\sigma}$, on general grounds, must be of the form

$$L_{\rho\sigma}(x-x', m, m') = A(Q^2, m, m') \delta_{\rho\sigma} + B(Q^2, m, m') Q_\rho Q_\sigma \quad (4.5)$$

where, by power counting, $A(Q^2, \dots)(B(Q^2, \dots))$ is quadratically (logarithmically) divergent. However, if Eq. (4.4) is valid we must have

$$L_{\rho\sigma}(Q, 0, 0) = B(Q^2, 0, 0) (Q_\rho Q_\sigma - Q^2 \delta_{\rho\sigma}), \quad (4.6)$$

i.e., the loop is only logarithmically divergent. A simple calculation gives

$$L_{\rho\sigma}(Q, m, m') = \int d^4l \frac{l_\rho(l-Q)_\sigma + l_\sigma(l-Q)_\rho - \delta_{\rho\sigma} l(l-Q)}{[l^2 + m^2][(l-Q)^2 + m'^2]} \quad (4.7)$$

The only essential difference between the expression (4.7) in the case $m = m'$ and the expression for the second order photon self energy diagram is that because of factors $1 + \gamma_5$ here the lepton mass in the numerator of (4.7) has dropped out. I.e., in quantum electrodynamics the term $\delta_{\mu\nu} \ell(\ell - Q)$ is replaced by $\delta_{\mu\nu} [\ell(\ell - Q) + m^2]$. Because of the presence of the m^2 , the photon self-energy diagram is only logarithmically divergent when regularized à la Gupta⁹⁾. However, the neutrino loop $L_{\mu\nu}(Q, 0, 0)$, which is quadratically divergent before regularization, remains quadratically divergent after regularization. In fact it is not possible to get rid of the quadratic divergent term in $L_{\mu\nu}(Q, 0, 0)$ by any Pauli-Villars regularization with two auxiliary masses. Our conclusion is that Eq. (4.6) is incorrect. For the sake of completeness we give the neutrino loop when regularized by Pauli-Villars method⁸⁾. We find

$$L_{\mu\nu}^{P.V.}(Q, 0, 0) = i\pi^2 [C(Q^2) \delta_{\mu\nu} + D(Q^2)(Q_\mu Q_\nu - Q^2 \delta_{\mu\nu})], \quad (4.8)$$

$$C(Q^2) = - \sum_i c_i M_i^2 [\ln M_i^2 + H(\frac{Q^2}{M_i^2})], \quad (4.9)$$

$$D(Q^2) = \frac{1}{3} \sum_i c_i [\ln M_i^2 + (1 - 2 \frac{M_i^2}{Q^2}) H(\frac{Q^2}{M_i^2})], \quad (4.10)$$

$$H(z) = R \ln \left(\frac{R+1}{R-1} \right), \quad R = \left[1 + \frac{4}{z} (1 - i\epsilon) \right]^{1/2}, \quad \epsilon \rightarrow 0^+, \quad (4.11)$$

where $M_0 = m_\nu = 0$, $C_0 = 1$, $\sum c_i = 0$ and $\sum c_i M_i^2 = 0$. Further M_i , $i \neq 0$ tend to infinity. Note that the Gupta regularization with $n = 2$ [cf. Eq. (2.10)] is a special case of the Pauli-Villars regularization with two masses $M_1 = \Lambda$ and $M_2 = \sqrt{2}\Lambda$.

B. - The Electromagnetic Form Factor of the Neutrino

A massless two-component neutrino has only one electromagnetic form factor²²⁾. The lowest order relevant diagram is shown in Fig. 4. It contributes²²⁾

$$\langle \nu(\vec{p}') | j_{\lambda}^{em}(0) | \nu(\vec{p}) \rangle = C e G J_{\lambda\sigma}(Q) \bar{u}(\vec{p}') \gamma_\sigma (1 + \gamma_5) u(\vec{p}), \quad (4.12)$$

where C is a constant, $Q = p' - p$ and $J_{\mu\nu}(Q)$ is obtained from $L_{\mu\nu}$, Eq. (4.7) by putting $m = m'$ and replacing $\delta_{\mu\nu} \ell(\ell - Q)$ with $\delta_{\mu\nu} [\ell(\ell - Q) + m^2]$. One writes²²⁾

$$J_{\lambda\sigma}(Q) = A(Q^2) \delta_{\lambda\sigma} + B(Q^2) Q_\lambda Q_\sigma. \quad (4.13)$$

The second term in Eq. (4.13) does not contribute to Eq. (4.12). Since $A(Q^2)$ (which is found to be quadratically divergent) gives non-vanishing electromagnetic form factor for the neutrino at $Q = 0$ one simply defines $A(0) = 0$. However, as mentioned before [cf. Eq. (4.7)] by Gupta regularization $J_{\lambda\sigma}^G(Q)$ has the form $B^G(Q^2)(Q_\lambda Q_\sigma - Q^2 \delta_{\lambda\sigma})$, the diagram is only logarithmically divergent and the neutrino form factor at $Q = 0$ vanishes automatically.

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FIGURE CAPTIONS

Figure_1 The lowest order leptonic diagram contributing to the anomaly.

Figure_2 The lowest order vector boson diagram contributing to the anomaly.

Figure_3 The lepton loop ; P_i , ℓ and ℓ' are leptons.

Figure_4 The diagram for the electromagnetic form factor of the neutrino.

FIG.1

FIG.1a

FIG.2

FIG.2a

FIG.3

FIG.4