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ON THE DIMENSION OF CHIRAL AND CONFORMAL SYMMETRY BREAKING

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ABSTRACT

We investigate the general consequences of Gell-Mann's decomposition of the energy density $\theta_{00}(x) = \theta_{00}^*(x) + \sum_n \delta_n(x) + u(x)$, where u is the $SU(3) \times SU(3)$ breaking term transforming according to a $(3\bar{3}) + (\bar{3}3)$ representation and is a Lorentz scalar of dimension d . We show that there must be at least two terms among θ_{00}^* , δ_n with non-vanishing vacuum expectation values. If we further assume that there are only two such terms with dimensions d' and d'' we can obtain a new sum rule involving the spectral functions of the propagators of θ_μ^μ , $\partial^\mu A_\mu^\pi$ and $\partial^\mu A_\mu^K$, which reads

$$\int \rho_0(m^2) \frac{d^4 m^2}{m^2} = X \left[\frac{r-2}{2r} \int \rho_\pi(m^2) \frac{d^4 m^2}{m^2} + \frac{r+2}{r+1} \int \rho_K(m^2) \frac{d^4 m^2}{m^2} \right]$$

where $r = -\frac{2}{3}(c+\sqrt{2})/c$ and $X = (d-d')(d''-d)$. Saturating with σ , π , K one finds, using an earlier result on

$$\frac{m_\sigma}{f_{\sigma\pi\pi}} = \frac{32\pi}{3m_\sigma^2} X \left[\frac{r-2}{2r} f_\pi^2 m_\pi^2 + \frac{r+2}{r+1} f_K^2 m_K^2 \right] (1 + [d-2] \frac{m_\pi^2}{m_\sigma^2})^{-2}$$

We use the experimental values $m_\sigma \approx 700$ and $f_{\sigma\pi\pi} \approx 400$ to estimate X in both the Gell-Mann, Oakes, Renner and the Brandt, Preparata $SU(3) \times SU(3)$ symmetry breaking schemes. Unfortunately for reasonable values of f_K/f_π our sum rule does not distinguish between these two schemes, e.g., for $f_K/f_\pi = 1.25$, both schemes give $X \approx 3$. Nevertheless, this typical value allows us to conclude that σ dominance is consistent with many models not involving operators of anomalously high dimensions and at the same time it allows us to exclude many other models.

1. INTRODUCTION

Recently, a great deal of work has been invested in linking up chiral and dimensional information and extracting its physical consequences ^{1),2)}. Chiral Lagrangians provided the first models for such studies and have until now given the bulk of useful insights ³⁾. It is well known that phenomenological Lagrangians provide a simple mnemonic device to enforce low energy theorems on physical coupling constants given arbitrary current field commutation rules ⁴⁾. Indeed, for the complicated commutation rules of $SU(3) \times SU(3)$, the Lagrangian methods have made the involved consistency equations of Ward identities ⁵⁾ an easy $SU(3)$ coupling exercise.

For evaluating consequences of dimensional assumptions, however, Lagrangians probably provide somewhat too big an apparatus. The reason is that the commutator carrying all information on the dimension of a field is extremely simple :

$$i [D_\mu(x), \phi(y)]_{x=y_0} = (x\partial + d_\phi) \phi(x) \delta^3(x-y) + \text{Schwinger term} \quad (1)$$

where $D_\mu(x)$ is the current density of dilatations, defined in terms of the symmetric local energy momentum tensor $\theta_{\mu\nu}(x)$ ⁶⁾ by

$$D_\mu(x) \equiv x^\nu \theta_{\mu\nu}(x) \quad (2)$$

such that the conservation condition reads

$$\partial^\mu D_\mu(x) = \theta^\mu_\mu(x) \equiv \theta(x) \quad (3)$$

The trivial algebraic structure of (1) allows one to solve many Ward identities at once. Thus for the three-point function $T(D_\mu(x)\phi(y)\phi(0))$ one finds immediately the low energy theorem ²⁾ :

$$\Gamma(0, p^2, p^2) = -2\Delta^{-1}(p^2) \left[(2-d_\phi) + p^2 \Delta^{-1}(p^2) \frac{\partial}{\partial p^2} \Delta(p^2) \right] \quad (4)$$

where $\Delta(p^2)$ is the propagator of the ϕ field

$$\Delta(p^2) = \int \frac{\rho_\phi(\mu^2) d\mu^2}{p^2 - \mu^2}$$

and Γ is the reduced vertex function defined by

$$-\Delta(p^2)\Delta((p-q)^2)\Gamma(q^2, p^2, (p-q)^2) = \int_{xy} e^{-i(qx-px)} \langle 0 | T(\phi(x)\phi(y)\phi(z)) | 0 \rangle \quad (5)$$

On the mass shell of a particle of mass μ this gives the universal result

$$\Gamma(0, \mu^2, \mu^2) = 2\mu^2 Z^{-1} \quad (6)$$

Here Z is the wave function normalization of the single particle state of mass μ . For convenience, we shall sometimes state the results in terms of the properly normalized vertex function $\hat{\Gamma} = Z\Gamma$. There is no dependence on the dimension of the interpolating field (as any on-mass shell result should be). The dimension enters, however, in the off-mass shell continuation, determining

$$\left. \frac{\partial}{\partial p^2} \hat{\Gamma}(0, p^2, \mu^2) \right|_{p^2=\mu^2} = d_\phi - 1 \quad (7)$$

One might suspect that a similar result could also be obtained for the derivative of $\Gamma(q^2, \mu^2, \mu^2)$ with respect to q^2 , in terms of the dimensional constant of the energy momentum tensor. Indeed, in a recent paper it was shown ⁷⁾ that using the additional assumptions of :

- 1) standard chiral symmetry properties of the Hamiltonian;
- 2) σ dominance of the $\Gamma(q^2, p^2, p'^2)$ vertex, (with $\phi = \delta A_\mu$),
- 3) smoothness of the vertex function,

one can in fact derive such a result for $\partial\Gamma/\partial q^2$, involving only the dimension d of the chiral symmetry breaking part in θ_{00} :

$$\frac{\partial \hat{\Gamma}}{\partial q^2}(0, m_\pi^2, m_\pi^2) = 1 + (d-2) \frac{m_\pi^2}{m_\sigma^2} \quad (8)$$

Obviously this equation implies, even for a once-subtracted dispersion relation for $\Gamma(q^2, \mu^2, \mu^2)$, that the $\sigma\pi\pi$ coupling constant ⁸⁾ (normalized by $\mathcal{L} = g_{\sigma\pi\pi} (m_\sigma/2) \sigma \pi^a \pi^a$) is

$$g_{\sigma\pi\pi} = \gamma \left(1 + [d-2] \frac{m_\pi^2}{m_\sigma^2} \right) \quad (9)$$

where $(m_\sigma^3 \gamma^{-1})$ is the coupling $\langle 0 | \theta | \sigma \rangle$ ⁹⁾.

A second very simple Ward identity can be derived for the two-point function $\langle 0 | T(D_\mu(x) \theta(0)) | 0 \rangle$. Taking its divergence leads directly to

$$\Delta_{\theta\theta}(0) = \langle 0 | i \left[\int d^3x D_0(x, 0), \theta(0) \right] | 0 \rangle \quad (10)$$

where $\Delta_{\theta\theta}$ is the propagator of the $\theta(x)$ field. Under the above assumption 2), the left-hand side becomes $-\gamma^{-2} m_\sigma^4$ such that

$$m_\sigma^4 \gamma^{-2} = -\langle 0 | i \left[\int d^3x D_0(x, 0), \theta(0) \right] | 0 \rangle \quad (11)$$

Equations (9) and (11) can be used to eliminate the unknown σ graviton coupling γ . The commutator in (11), on the other hand, can be evaluated by making dimensional assumptions on θ_{00} .

It is the purpose of this paper to discuss the consequences of (9) and (11), assuming only dimensional properties of the decomposition of θ_{00} , PCAC for π and K, and σ dominance of θ . In spite of the roughness of the information on possible $I=0$ s wave resonances ¹⁰⁾ the presently assumed principal candidate at $m_\sigma = 700$, $\Gamma_{\sigma\pi\pi} \approx 400$ helps considerably to distinguish between the possible dimensions of the different parts contained in θ_{00} .

2. DIMENSIONAL AND CHIRAL CONTENT OF $\theta_{00}(x)$

We adopt the assumptions on the $SU(3) \times SU(3)$ decomposition of the energy density conjectured a long time ago by Gell-Mann ¹¹⁾

$$\theta_{00} = \theta_{00}^* + \delta + u^0 + cu^8 \quad (12)$$

where θ_{00}^* and δ are $SU(3) \times SU(3)$ singlets. The terms u^0, u^8 are Lorentz scalars transforming like members of the $(3, \bar{3}) + (\bar{3}, 3)$ representation, which they form together with a nonet of pseudoscalar operators v^a ($a=0, \dots, 8$)

$$\begin{aligned} [A_0^a(x), u^b(0)]_{x_0=0} &= -i d^{abc} v^c(0) \delta^3(x) \\ [A_0^a(x), v^b(0)]_{x_0=0} &= i d^{abc} u^c(0) \delta^3(x) \end{aligned} \quad (13)$$

The term u can conveniently be split in a term conserving $SU(2) \times SU(2)$ S, and another term Σ , transforming like the fourth component of an $(\frac{1}{2}, \frac{1}{2})$ representation

$$u = u^0 + cu^8 = S + \Sigma$$

From this assumption one readily derives, using the equation of motion for A_0^i , $i=1, \dots, 8$:

$$\partial^\mu A_\mu^i = -(d^{i00} + cd^{i88})v^i - \delta^{i8}\sqrt{\frac{2}{3}}cv^0 \quad (14)$$

which in turn implies the so-called Σ commutator to be

$$\begin{aligned} \Sigma_{ij} \equiv i \left[\int d^3x A_0^i(x,0), \partial^\mu A_\mu^j(0) \right] &= \left(\frac{2}{3}\delta^{ij} + \sqrt{\frac{2}{3}}cd^{ij8} \right) u^0(0) \\ &+ \left(\sqrt{\frac{2}{3}}d^{ijk} + cd^{j8k}d^{i0k} + \frac{2}{3}c\delta^{j8}\delta^{i0k} \right) u^k(0) \end{aligned} \quad (15)$$

This equation can be used to derive the well-known low energy theorem for the propagator

$$\Delta^i(q) \equiv -i \int_x e^{-iqx} \langle 0 | T(\partial^\mu A_\mu^i(x) \partial^\nu A_\nu^j(0)) | 0 \rangle$$

$$\Delta^i(0) = \langle \Sigma^i \rangle = a^i \langle S \rangle + b^i \langle \Sigma \rangle \quad (16)$$

where

$$a^i = \begin{cases} 0 \\ \frac{r+1}{r+2} \\ \frac{1}{3} \end{cases}, \quad b^i = \begin{cases} 1 \\ \frac{r+1}{2r} \\ \frac{1}{3} \end{cases} \quad \text{for } i = \begin{cases} 1,2,3 \\ 4,5,6,7 \\ 8 \end{cases}$$

and

$$r = -\frac{2}{3} \frac{c+\sqrt{2}}{c} \quad (17)$$

From Eq. (16), we find for $i = \pi, K$ that the vacuum expectation values $\langle S \rangle$ and $\langle \Sigma \rangle$ can be expressed as

$$\langle \Sigma \rangle = \Delta^{\pi}(0) \quad (18)$$

$$\langle S \rangle = (r+2) \left(\frac{1}{r+1} \Delta^K(0) - \frac{1}{2r} \Delta^{\pi}(0) \right) \quad (19)$$

These two equations allow us to obtain for $u = S + \Sigma$

$$\langle u \rangle = \frac{r-2}{2r} \Delta^{\pi}(0) + \frac{r+2}{r+1} \Delta^K(0) \quad (20)$$

After these preliminaries, let us proceed to the evaluation of the commutator in Eq. (10)

$$i \left[\int D_\alpha(x,0) d^3x, \theta(0) \right]$$

To do so, we obviously need information on the dimensional content of $\theta(0)$. Let us assume that δ is a sum of scalars δ_n of definite

dimension d_n , while u has the dimension d ¹²⁾. Then it can be shown that θ has the form ²⁾

$$\theta = \sum_n (4 - d_n) \delta_n + (4 - d) u \quad (21)$$

As a consequence, Eq. (10) is immediately evaluated to give

$$\Delta_{\theta\theta}(0) = \sum_n (4 - d_n) d_n \langle \delta_n \rangle + (4 - d) d \langle u \rangle \quad (22)$$

Now we also have the following trivial identities

$$\begin{aligned} \langle \theta \rangle &= \sum_n (4 - d_n) \langle \delta_n \rangle + (4 - d) \langle u \rangle = 0 \\ \langle \theta_{00} \rangle &= \langle \theta_{00}^* \rangle + \sum_n \langle \delta_n \rangle + \langle u \rangle = 0 \end{aligned} \quad (23)$$

from which we immediately see that if among $\langle \theta_{00}^* \rangle$ and $\langle \delta_n \rangle$ there are less than two non-vanishing terms, then $\Delta_{\theta\theta}(0) = 0$. We reject this case for it would require the corresponding spectral function $\rho_{\theta}(\mu^2)$ to vanish everywhere. A corollary of this simple result is that there must be at least one δ_n with a non-vanishing vacuum expectation value (i.e., there must be a δ), although this can be a constant.

The important observation now is that if among θ_{00}^* and δ_n there are two and only two operators O' , O'' with non-vanishing vacuum expectation values ¹²⁾ then $\Delta_{\theta\theta}$ can be expressed completely in terms of $\langle u \rangle$ and the dimensions d' and d'' of O' and O'' respectively

$$\Delta_{\theta\theta}(0) = (d - d')(d'' - d) \langle u \rangle \quad (24)$$

Equation (24), together with the expression (20) for $\langle u \rangle$ becomes our central exact result

$$\Delta_{\theta\theta}(0) = (d-d')(d''-d) \left[\frac{r-2}{2r} \Delta^{\pi}(0) + \frac{r+2}{r+1} \Delta^K(0) \right] \quad (25)$$

The result can be restated in terms of a sum rule relating the spectral function of the scalar operator θ to those of the pseudoscalar operators $\partial^r A_r^{\pi}$ and $\partial^r A_r^K$:

$$\int \rho_{\theta}(p^2) \frac{d\mu^2}{\mu^2} = (d-d')(d''-d) \left[\frac{r-2}{2r} \int \rho_{\pi}(p^2) \frac{d\mu^2}{\mu^2} + \frac{r+2}{r+1} \int \rho_K(p^2) \frac{d\mu^2}{\mu^2} \right] \quad (26)$$

Let us discuss the consequences of this sum rule. We assume as usual that Δ^{π} , Δ^K are dominated by a π and K pole, respectively, i.e., the spectral functions are approximated by

$$\rho_{\pi}(p^2) = f_{\pi}^2 m_{\pi}^4 \delta(p^2 - m_{\pi}^2), \quad \rho_K(p^2) = f_K^2 m_K^4 \delta(p^2 - m_K^2)$$

In addition, σ dominance of $\Delta_{\theta\theta}$ gives

$$\rho_{\theta}(p^2) \equiv \frac{m_{\sigma}^6}{g^2} \delta(p^2 - m_{\sigma}^2) \quad (27)$$

Therefore we find in this approximation

$$\frac{m_{\sigma}^4}{g^2} = (d-d')(d''-d) \left[\frac{r-2}{2r} f_{\pi}^2 m_{\pi}^2 + \frac{r+2}{r+1} f_K^2 m_K^2 \right] \quad (28)$$

3. DISCUSSION OF THE RESULT

Two types of ideas have been forwarded as to what the mechanism of $SU(3) \times SU(3)$ breaking amounts to. In the old Gell-Mann scheme, recently revived by Brandt and Preparata ¹³⁾, θ_{00}^* of dimension four creates a massless world which is made massive by a term δ breaking dilatational symmetry. Then $SU(3) \times SU(3)$ is broken, conserving $SU(3)$, by u^0 , and finally a very small $SU(3)$ breaking cu^8 is introduced to cause the $SU(3)$ splitting of baryon and meson masses. In this scheme the smallness of the pion mass is an unexplained dynamical accident. The value r one finds from several considerations is $r \approx 3.3$ ¹³⁾. Thus, neglecting the small pion contribution in (28), we obtain

$$\frac{m_{\sigma^4}}{f_\pi^2} = (d-d')(d''-d)(1.2)f_K^2 m_K^2 \quad (29)$$

According to more recent ideas of Gell-Mann et al. ¹⁴⁾, however, the small masses of the pseudoscalar mesons and the quality of PCAC can most naturally be explained by assuming the vacuum to break the $U(3) \times U(3)$ symmetry of θ_{00}^* and creating a nonet of massless pseudoscalar Goldstone bosons. The term δ should commute with all $U(3) \times U(3)$ generators, except for Q_5^0 to lift the singlet particle η' to its high mass of 985 MeV. Finally, an approximately $SU(2) \times SU(2)$ invariant u term lifts K and η masses, leaving only the pion almost massless.

In this scheme r is estimated by the pseudoscalar masses as

$$r \approx \frac{m_\pi^2}{m_K^2 - m_\pi^2} \quad (30)$$

such that (29) becomes

$$\frac{m_{\sigma^4}}{f_\pi^2} = (d-d')(d''-d) \left[\frac{m_\pi^2}{2} (3f_K^2 - 2f_\pi^2) + m_K^2 (2f_K^2 - f_\pi^2) \right] \quad (31)$$

Notice that this result reduces to the one obtained by Lagrangian methods ^{3),4)} if one sets $f_K = f_\pi$, and $d' = 0$, $d'' = 4$ [Refs. 3) and 4) assume δ is a constant.]

In conjunction with Eq. (9) for $g_{\sigma\pi\pi}$, we can eliminate δ from (29) and (31) and we find, in terms of mass and width of the σ meson, our final result in the Brandt-Preparata scheme

$$\frac{m_\sigma}{\Gamma_\sigma} = X \frac{32\pi}{3m_\sigma^2} (1.2) f_K^2 m_K^2 \left(1 + [d-2] \frac{m_K^2}{m_\sigma^2}\right)^{-2} \quad (\text{BP})$$

while the Gell-Mann, Oakes, Renner scheme gives

$$\frac{m_\sigma}{\Gamma_\sigma} = X \frac{32\pi}{3m_\sigma^2} (2f_K^2 - f_\pi^2) m_K^2 \left(1 + [d-2] \frac{m_K^2}{m_\sigma^2}\right)^{-2} \quad (\text{GOR})$$

Here we have introduced for brevity the parameter $X = (d-d')(d''-d)$. Notice that in both schemes $X > 0$ and hence d necessarily lies between d' and d'' .

These equations can be used to get some limits as to the possible dimensions of the symmetry breakers δ and u . We shall assume that only integer dimensions occur in these local fields. As we have discussed in our last paper ⁷⁾, we believe that the naive dimensions of a field remain approximately a good concept in the presence of interactions ¹⁵⁾. This has been proved to any finite order in renormalizable field theory ¹⁶⁾. It has been pointed out by Wilson that in the Thirring model logarithmic factors arising in finite order perturbation theory pile up to change the dimension completely ¹⁷⁾. Also there is indication that the scale invariant $\lambda\phi^4$ theory has the same behaviour ¹⁸⁾. From investigations of the non-scale invariant ϕ^3 theory ¹⁶⁾ we know, however, that such a pile-up does not occur. It is our belief that any theory containing massive constants from the beginning will have approximately integer dimensions. These considerations severely limit the possible values d' , d'' and d can assume. From the Lehmann spectral representation we know that the dimension of a proper local operator has to be ≥ 1 .

Allowing also constants, we have $0 \leq d', d'', d \leq 4$. It has been argued ¹⁹⁾ that no dimension higher than four should play an important role in the Hamiltonian. If we accept this point of view, we obtain for every ordered triplet (d', d, d'') the following possible values of X :

$(d' \ d \ d'')$	X
4 3 2	1
4 3 1	2
4 3 0	3
4 2 1	2
4 2 0	4
4 1 0	3
3 2 1	1
3 2 0	2
3 1 0	2
2 1 0	1

Let us now assume that the σ particle under consideration is the experimentally inferred $\Sigma(700)$ of width $\Gamma_{\sigma\pi\pi} = 400$ ^{10), 20)}. Then we obtain the following values of X :

1) Brandt-Preparata scheme

- a) $f_K/f_\pi = 1$: $X = 4.7$ (≥ 3.3)
- b) $f_K/f_\pi = \frac{5}{4}$: $X = 3.0$ (≥ 2.2)

2) Gell-Mann, Oakes, Renner scheme

- a) $f_K/f_\pi = 1$: $X = 5.8$ (≥ 4.0)
- b) $f_K/f_\pi = \frac{5}{4}$: $X = 2.7$ (≥ 1.9)

The numbers in parentheses indicate the lower limits on X provided by the Adler-Weisberger relation ($g_{\sigma\pi\pi} \lesssim m_\sigma / (\sqrt{2} f_\pi)$ ²¹⁾).

We note that in both schemes the values of X are extremely sensitive to the value of f_K/f_π taken. We consider two cases:

a) For $f_K/f_\pi = 1$ we see that the experimental values for X do not appear in our table of possible X values; however, if $g_{\sigma\pi\pi}$ attains its Adler-Weisberger bound then $X \approx 3-4$. This high value for X would exclude all but three (d', d, d'') combinations, each of which have $d' = 4$, $d'' = 0$. If $d'' = 0$, however, $0''$ is a constant and hence in the Gell-Mann scheme [where θ_{00}^* commutes with the whole

$U(3) \times U(3)$ it is not capable of raising the η' mass (since it commutes with Q_5^0); this would have to be done by other terms δ_n with vanishing vacuum expectation value.

b) For $f_K/f_\pi = 5/4$ [which corresponds to $F_+(0)$ very close to one] we unfortunately find that our sum rule does not distinguish between the two proposed values for r ²²⁾, both schemes give $X=2-3$. This value for X allows us to reject many models, however, at the same time it is consistent with many (d', d, d'') combinations and hence the assumptions are consistent with no anomalously high dimensions entering the theory.

CONCLUSION

There has to be at least one operator δ in the Hamiltonian density with non-vanishing vacuum expectation value, although this operator can be a constant. When $F_+(0) \approx 1$ the σ dominance of $\Delta_{\theta\theta}$ is consistent with many dimensional structures of the Hamiltonian density in both the Gell-Mann, Oakes, Renner and the Brandt, Preparata symmetry breaking schemes (in particular it is consistent with having just one δ term of definite dimension). If f_K/f_π is very close to one, however, the high η' mass requires at least two terms of different dimensions in δ in the Gell-Mann, Oakes, Renner scheme if θ_{00}^* commutes with $U(3) \times U(3)$.

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REFERENCES AND FOOTNOTES

- 1) G. Mack, Nuclear Phys. B5, 499 (1968);
 G. Mack and A. Salam, Ann.Phys. 53, 174 (1969);
 K. Wilson, Phys.Rev. 179, 1499 (1969);
 M. Dal-Cin and H.A. Kastrup, Nuclear Phys. B15, 189 (1970).
- 2) M. Gell-Mann, Lecture Notes, Hawaii Summer School (1969).
- 3) A. Salam and J. Strathdee, Phys.Rev. 184, 174 (1969);
 C.J. Isham, A. Salam and J. Strathdee, Phys.Letters 31B, 300 (1970);
 S.P. DeAlwis and P.J. O'Donnell, Toronto preprint (1970);
 J. Ellis, DAMTP preprint (1970) - [there is a normalization error
 in this paper - see Eq. (14) - Ref. 4) agrees with us.];
 J. Wess and B. Zumino, to be published and Ref. 4).
- 4) For an elegant proof of the equivalence of Ward identities and
 low energy theorems extracted from phenomenological Lagrangians
 in the tree approximation see, for example,
 B. Zumino, Brandeis Lectures (1970).
- 5) See, for example, I. Gerstein and H.J. Schnitzer, Phys.Rev. 175,
 1876 (1968).
- 6) E. Huggins, Ph.D. thesis, Caltech (1962);
 F. Gürsey, Ann.Phys. 24, 211 (1963);
 C. Callan, S. Coleman and R. Jackiw, Ann.Phys. - to be published.
- 7) H. Kleinert and P.H. Weisz, CERN preprint TH.1234 (1970).
- 8) Notice that our result holds up to order $O(m_\pi^2/M^2)$ and
 $O(m_\sigma^2/M'^2)$ where M, M' are the next significant mass contri-
 bution in $\Delta\gamma_{A_1} \gamma_{A_1}$ and $\Delta_{\theta\theta_4}$. They are, in principle,
 valid for any m_π^2/m_σ^2 . We are aware that such a term is physically
 as small as the neglected terms. However, we shall carry it along
 for the sake of allowing for a comparison with Lagrangian results.

- 9) Notice that this is the same result as was obtained by Salam et al., Ref. 3), from phenomenological Lagrangians for the special value $\delta = m_\sigma / f_\pi$. But as it was pointed out by Ellis, Ref. 3), the conformal covariance properties assumed by Salam et al. do not determine δ . However, the physical significance of the arbitrary constant δ appearing in the construction of the chiral Lagrangian is perhaps not as manifest as here.
- 10) Particle Data Group, Revs. Modern Phys. 42, 87 (1970).
- 11) M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
- 12) Notice, however, that many popular models have a more complicated vacuum expectation value structure. For example, in the $SU(2)$ π - σ model, the term δ consists of the terms $1/4(3m_\pi^2 - m_\sigma^2) \times (\sigma^2 + \pi^2)$ and a constant $= f_\pi^2 / 8(3m_\pi^2 + m_\sigma^2) > 0$ (necessary to make the vacuum expectation value of the Lagrangian vanish). The first term has dimension two, the second dimension zero. The vacuum expectation value of θ_{00}^* is $f_\pi^2 / 8(m_\sigma^2 - m_\pi^2)$. Thus in the σ model there are just two terms among θ_{00}^* and δ with non-vanishing vacuum expectation value only when $m_\sigma^2 = m_\pi^2$ or $m_\sigma^2 = 3m_\pi^2$.
- 13) R.A. Brandt and G. Preparata, to be published in Nuclear Phys.
- 14) M. Gell-Mann, R. Oakes and B. Renner, Phys. Rev. 175, 2195 (1968).
- 15) If a local field $\phi(x)$ has a dimension d , then the operator product $\phi(x)\phi(0)$ diverges for $x \rightarrow 0$, like $1/(x^2)^d$ [see K. Wilson, Ref. 1]. It has been verified to finite order in perturbation theory that this singularity is modified only by logarithmic factors in the presence of interactions, (see next Ref.).
- 16) K. Symanzik, DESY preprint (1970).
- 17) K. Wilson, Stanford preprint (1970).
- 18) R. Gatto, private communication.

- 19) See, K. Wilson, Ref. 1).
- 20) J. Engels and G. Höhler, Karlsruhe preprint (1970).
- 21) In the σ model, the $g_{\sigma\pi\pi}$ coupling constant can be obtained as

$$g_{\sigma\pi\pi} \approx \frac{m_\sigma}{m_\pi} g_{\pi NN} = \frac{m_\sigma}{m_N} g_{\sigma NN}$$

The right part of this relation is very well satisfied [see Ref. 20] where $(g_{\sigma NN}^2)/4\pi \approx 18 \pm 2$. The left-hand side yields, however, upon using the Goldberg-Treiman relation ($g_{\pi NN} = (m_N/f_\pi)g_A$), the value $g_{\sigma\pi\pi} = (m_\sigma/f_\pi)g_A$. This result obviously violates the Adler-Weisberger limit $g_{\sigma\pi\pi} = (m_\sigma/\sqrt{2} f_\pi)$ giving a width of ≈ 1200 MeV. The reason is, clearly that in the σ model σ exchange alone makes up all the correctly normalized $\pi\pi$ scattering. There is no ρ meson to fill the missing half of the sum rule. Notice that $g_{\sigma\pi\pi} = (m_\sigma/m_N)g_{\sigma NN}$ can also be derived assuming an unsubtracted dispersion relation for the $\langle N|\theta|N \rangle$ vertex. Thus its failure indicates the necessity for a subtraction just like in the $\langle \sigma|\theta|\sigma \rangle$ vertex.

- 22) This is, however, a familiar situation, for in the pole approximation for the sum rule

$$\Delta^x(0) = \frac{1}{1+r} \Delta^\pi(0) - \frac{1}{r} \Delta^K(0)$$

with $f_K/f_\pi = 5/4$, one again obtains very similar estimates for m_x in the two schemes.

$$m_x = \begin{cases} .75 \left| \frac{f_\pi}{f_x} \right| m_K & \text{GOR.} \\ .6 \left| \frac{f_\pi}{f_x} \right| m_K & \text{B.P.} \end{cases}$$

i.e., both schemes require a small $|f_x/f_\pi|$ for a large m_x . [Recall with $f_K=f_\pi$ the two values for r , $r=r(\text{GOR})$ and $r=\infty$ give exactly the same estimates for $\langle u \rangle$ and $f_x^2 m_x^2 = 0$.]