ON THE VALUE OF \( \int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx \)

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ON THE VALUE OF $\int_0^{\pi/2} \log^a \cos x \log^p \sin x \, dx$ *)

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ABSTRACT

A closed expression is derived for the integral given in the title, where $n$ and $p$ are non-negative integers. As already remarked by Nielsen in a monograph on the generalized polylogarithms published early in this century, this integral is equal to $\pi$ times a homogeneous polynomial in $\zeta(q)$ (the Riemann zeta function for integer arguments) and $\log 2$, with rational coefficients. Explicit expressions for the integral are given for $0 < n \leq 5$, $0 \leq p \leq 5$, most of which have been found from the general formula by means of a computer.

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1. **INTRODUCTION**

In a recent paper [1] a closed expression was given for the integral

$$s_{np} = \frac{(-1)^{n+p+1}}{(n-1)! \cdot p!} \int_0^1 \frac{\log^{n-1} t \log^p (1-t)}{t} \, dt \quad (n,p > 0, \text{ integer}) \quad (1)$$

This integral is a special case of the so-called generalized polylogarithms, which were discussed early in this century by Nielsen [2]. These functions are of considerable interest for certain problems in quantum electrodynamics.

In his monograph, Nielsen [2] also remarked that the integral

$$r_{np} = \int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx$$

is

$$r_{np} = \frac{1}{2^{n+p+1}} \int_0^1 \frac{\log^n t \log^p (1-t)}{\sqrt{t(1-t)}} \, dt \quad (n \geq 0, \ p \geq 0, \ n + p \neq 0),$$

which is similar to (1), can be expressed as \(\pi\) times a polynomial in \(\eta(q) \ (1 \leq q \leq n + p)\), with rational coefficients, where

$$\eta(q) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^q} = \begin{cases} \log 2 & (q = 1) \\ (1 - 2^{1-q}) \zeta(q) & (q > 1) \end{cases} \quad (3)$$

This polynomial is homogeneous of degree \(n + p\) if one considers \(\eta(q)\) to be of degree \(q\), and if the degree of a product is the sum of the degrees of its factors. \(\zeta(q)\) is the Riemann zeta function for integer arguments.

The aim of this note is to give a closed formula for the integral, from which, at least for small \(n\) and \(p\), the expressions for \(r_{np}\) can be found with the help of a computer.

2. **THE EVALUATION OF THE INTEGRAL**

For the evaluation of (2) we use some of Nielsen's ideas and follow closely the method used for the integral (1). Since \(r_{np} = r_{pn'}\) it is sufficient to

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* Nielsen [2] actually writes \(\pi\) instead of \(\pi/2\) in the upper limit of (2) and \(n + p + 1\) instead of \(n + p\). From the context, however, it is clear that these discrepancies are due to misprints.
consider \( n \geq p \geq 0 \). The case \( n > 0, p = 0 \) has been treated by Bowman [3], who found a simple determinant for \( r_{np} \). We express \( r_{np} \) as a derivative of Euler's beta function [4] and obtain

\[
\begin{align*}
\frac{r_{np}}{2^{n+p+1}} & = \frac{\partial^{n+p}}{\partial \beta \partial \alpha^p} \left. \Gamma^{-1/2} (1 - t)^{a-1/2} \right|_{t=0} \\
& = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta \partial \alpha^p} \left[ \frac{\Gamma(1 + \alpha) \Gamma(1/2 + \beta)}{\Gamma(1 + \alpha + \beta) \Gamma(1/2)} \right]_{\alpha=\beta=0}
\end{align*}
\]

\( (4) \)

The appearance of the factor \( \pi \) is expected, and its early separation simplifies somewhat the following calculations. With the help of the power series [4]

\[
\log \Gamma(1 + z) = -\gamma z + \sum_{m=2}^{\infty} \frac{\zeta(m)}{m} z^m \quad (|z| < 1) \tag{5}
\]

where \( \gamma \) is Euler's constant, one finds

\[
\frac{r_{np}}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta \partial \alpha^p} \left. e^{\varphi(\alpha, \beta)} \right|_{\alpha=\beta=0} \tag{6}
\]

where

\[
\varphi(\alpha, \beta) = -\sum_{m=2}^{\infty} \frac{\zeta(m)}{m} \left[ (\alpha + \beta)^m - (\alpha - 1/2)^m - (\beta - 1/2)^m + 2(-1/2)^m \right] \tag{7}
\]

Expanding the exponential function into its power series, we have

\[
\begin{align*}
\frac{r_{np}}{2^{n+p+1}} & = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial \beta \partial \alpha^p} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{i=1}^{\infty} \frac{(-1)^{m_i} \zeta(m_i)}{m_i} \right. \\
& \times \left[ (\alpha + \beta)^{m_i} - (\alpha - 1/2)^{m_i} - (\beta - 1/2)^{m_i} + 2(-1/2)^{m_i} \right]_{\alpha=\beta=0}
\end{align*}
\]

\( (8) \)
We apply the Leibniz formula for the differentiation of products in its multinomial form, and, after carrying out the differentiations with respect to $\alpha$ and setting $\alpha = 0$, find the following expression

\[
\frac{\sigma_{np}}{2^{n+p+1}} \frac{d^n}{d\beta^n} \left\{ \delta_{op} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{\sum p_i = p} \right. \\
\left. \left\{ \left(1 - \delta_{p_i} \right) \sum_{m_i = p_i + 1}^{\infty} \frac{(-1)^{m_i}}{m_i} \frac{\zeta(m_i)}{m_i} \left( \beta^{m_i - p_i} - \left(\frac{1}{2}\right)^{m_i - p_i} \right) \right. \\
\left. - \delta_{p_i} \sum_{m_i = 2}^{\infty} \frac{(-1)^{m_i}}{m_i} \frac{\zeta(m_i)}{m_i} \left( \left( \beta - \frac{1}{2} \right)^{m_i} - \beta^{m_i} - \left(\frac{1}{2}\right)^{m_i} \right) \right\} \right. \\
\left. \right\}
\]

where $\delta_{ij}$ is the Kronecker symbol. Applying again the Leibniz formula for the differentiations with respect to $\beta$ and setting $\beta = 0$, we have

\[
\frac{\sigma_{np}}{2^{n+p+1}} \sum_{k=1}^{n+p} \frac{(-1)^{n+p+1-k}}{k!} \sum_{\sum p_i = p} \sum_{\sum n_i = n} \prod_{i=1}^{k} f(p_i, n_i)
\]

with

\[
f(p_i, n_i) = \left(1 - \delta_{p_i} \right) \left(1 - \delta_{n_i} \right) \frac{p_i + n_i - 1}{p_i + n_i} \left( \frac{p_i + n_i}{p_i} \right) \zeta(p_i + n_i)
\]

\[
+ \left( \delta_{p_i} - \delta_{n_i} \right) \zeta(p_i + n_i)
\]

and where, for $j \geq 1$,

*) A $\Sigma$ without limits indicates a sum over $i$ from 1 to $k$. 
\[ \zeta(j) = \sum_{m=j+1}^{\infty} (-1)^m \frac{\zeta(m)}{m} \left( \frac{m}{j} \right) \left( -\frac{1}{2} \right)^{m-j} \]

\[ = (-1)^j \sum_{m=j+1}^{\infty} \frac{1}{m} \left( \frac{m}{j} \right) \left( -\frac{1}{2} \right)^{m-j} \sum_{\ell=1}^{\infty} \frac{1}{\ell^j} \]

\[ = (-1)^j \sum_{\ell=1}^{\infty} \frac{1}{\ell^j} \sum_{q=0}^{\infty} \left( \frac{-1}{2\ell} \right)^q \left( \frac{1}{q+1} \right) \]

\[ = (-1)^j \sum_{\ell=1}^{\infty} \frac{1}{\ell^j} \left[ \left( 1 - \frac{1}{2\ell} \right)^{-j} - 1 \right] , \quad (12) \]

and hence

\[ \zeta(j) = \begin{cases} 
-2 \log 2 & \quad (j = 1) \\
(-1)^j \frac{2^j - 2}{j} \zeta(j) & \quad (j > 1) 
\end{cases} \quad (13) \]

or, using (3)

\[ \zeta(j) = (-1)^j \frac{2^j}{j} \eta(j) . \quad (14) \]

In the case \( n > 0, p = 0 \), Eq. (10) can be reduced to

\[ r_{n_0} = (-1)^n \frac{\pi}{2} n! \sum_{k=1}^{n} \frac{1}{k!} \sum_{n_1 \geq 1} \prod_{i=1}^{k} \frac{\eta(n_i)}{n_i} , \quad (15) \]

which is another form of Bowmar's determinant. This formula was already found by Nielsen [2]. Lewin [5] discussed the integral

\[ \]

*) Note that in [2] §22(6) the factor \( n! \) is missing.
\[ r_n^* = - \int_0^\pi \log^n \left( 2 \sin \frac{1}{2} x \right) \, dx \]
\[ = -2 \sum_{k=0}^{n} \binom{n}{k} r_{k0} \log^{n-k} 2 \]  \hspace{1cm} (16)

and develops the recurrence relation

\[ r_{n+1}^* = (-1)^n n! \left\{ \pi (1 - 2^{-n}) \zeta(n + 1) - \sum_{k=2}^{n-1} (-1)^k \frac{1 - 2^{-k-n}}{k!} \zeta(n - k + 1) r_k^* \right\} \]  \hspace{1cm} (17)

He also indicates that the idea of representing integrals by derivatives of gamma functions can be applied to several definite integrals. In most cases, however, he did not carry out the differentiations explicitly.

3. **Explicit Expressions for \( r_{np} \)**

For increasing values of \( n \) and \( p \), a straightforward evaluation of formula (10) becomes, even with the help of a fast computer, virtually impossible for reason of time. This is already the case for \( n = p = 5 \). The number of possibilities for \( p_i \) and \( n_i \) in formula (10) for a fixed \( k \) is

\[ [\min(p + 1, n + p + 2 - k) \min(n + 1, n + p + 2 - k)]^k \]  \hspace{1cm} (18)

where most of them are, of course, invalid. Thus for \( n = p = 5 \) (disregarding \( k = 1 \) and \( k = n + p \), where the sums can be found in a simple way as given below), the number of terms which must be examined is

\[ 6^4 + 6^6 + 6^8 + 6^{10} + 6^{12} + 5^{14} + 4^{16} + 3^{18} = 13024879492. \]  \hspace{1cm} (19)

Therefore, one has to try to reduce the number of terms in formula (10) in such a way that the remaining calculations can be done, at least for small \( n \) and \( p \), in a reasonable amount of time. By the following considerations, which were suggested in part by Rühl [6], the calculation time is considerably reduced.

For a given \( k \), we order the sets \( p_i \) and \( n_i \) in formula (10) which satisfy \( \Sigma p_i = p \) and \( \Sigma n_i = n \) in such a way that \( p_1 \leq p_2 \leq \ldots \leq p_k \) and \( n_1 \leq n_2 \leq \ldots \leq n_k \). Then we define an expression
\[
\text{per}_k(f) = \sum_{\alpha, \beta, \ldots} f(p_1, n_{\alpha}) f(p_2, n_{\beta}) \cdots f(p_k, n_\omega),
\]

(20)

where the \(\alpha, \beta, \ldots, \omega\) run over all \(k!\) possible permutations of the numbers \(1, 2, \ldots, k\), and where \(f(p_i, n_i)\) is defined by Eq. (11). The quantity \(\text{per}_k(f)\) is sometimes called a permanent (e.g. [7]) and differs from the usual determinant of a \(k \times k\) matrix only by the permanently positive sign of its terms. We consider then a fixed decomposition \(\{\ell_j\}\) of \(p\) into \(\ell_0\) times 0, \(\ell_1\) times 1, \ldots, \(\ell_p\) times \(p\), and a fixed decomposition \(\{m_j\}\) of \(n\) into \(m_0\) times 0, \(m_1\) times 1, \ldots, \(m_n\) times \(n\), which correspond to particular ordered sets \(p_i\) and \(n_i\), respectively, so that

\[
\sum_{j=0}^{p} \ell_j = k, \quad \sum_{j=0}^{p} j\ell_j = p
\]

\[
\sum_{j=0}^{n} m_j = k, \quad \sum_{j=0}^{n} jm_j = n.
\]

It is then clear that such a decomposition of \(p\) and \(n\) would produce

\[
\frac{(k!)^2}{\ell_0! \ell_1! \cdots \ell_p! m_0! m_1! \cdots m_n!}
\]

terms in the sums over \(p_i\) and \(n_i\) in Eq. (10). Using the definition (20), one can deduce that these terms can be replaced by

\[
\frac{k! \text{per}_k(f)}{\ell_0! \ell_1! \cdots \ell_p! m_0! m_1! \cdots m_n!},
\]

which gives, using the fact that \(f(0, 0) = 0\),

\[
r_{np} = \frac{m_{\lambda}! p!}{2^{n+p+1}} \sum_{k=1}^{n+p} \sum_{\lambda} \sum_{\mu} \frac{\Theta(k - \ell_0^{(\lambda)} - m_0^{(\mu)}) \text{per}_k(f)}{\ell_0^{(\lambda)} \ell_1^{(\lambda)} \cdots \ell_p^{(\lambda)} m_0^{(\mu)} m_1^{(\mu)} \cdots m_n^{(\mu)}},
\]

(22)

where the sums now run only over all possible sets \(\{\ell_j^{(\lambda)}\}\) (\(\lambda = 1, 2, \ldots, L\)) and all possible sets \(\{m_j^{(\mu)}\}\) (\(\mu = 1, 2, \ldots, M\)), and where
\[ \theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \]

As already remarked, we see from Eqs. (10) and (11) that the sums over \( p_i \) and \( n_i \) can be carried out directly for \( k = 1 \) and \( k = n + p \), so that finally

\[
\begin{align*}
\mathcal{R}_{np} &= \frac{n}{2^{n+p+1}} \left\{ (-1)^{n+p-1} (n+p-1)! \left[ 1 + \delta_{op}(1-2^n) \right] \zeta(n+p) \right. \\
+ n! p! & \sum_{k=2}^{n+p-1} \sum_{\{\ell_{\lambda}^{(\lambda)}\}} \sum_{\{\mu\}} \frac{\theta(k - \ell_{\lambda}^{(\lambda)} - m_{\mu}) \text{ perm}_{k}(f)}{\ell_{o}^{(\lambda)}; \ell_{1}^{(\lambda)}; \ldots; \ell_{p}^{(\lambda)}; m_{o}^{(\mu)}; m_{1}^{(\mu)}; \ldots; m_{n}^{(\mu)}; } \\
& \left. + (-2 \log 2)^{n+p} \right\} (n+p \geq 2),
\end{align*}
\]

the sum over \( k \) being zero for \( n + p = 2 \).

In the particular case \( n = p \), we can simplify further by using the fact that the sets \( \{\ell_{j}^{(\lambda)}\}, \{m_{j}^{(\mu)}\} (\lambda, \mu = 1, 2, \ldots, L) \) are identical and write

\[
\begin{align*}
\mathcal{R}_{nn} &= \frac{n}{2^{2n+1}} \left\{ - (2n-1)! \zeta(2n) + (2 \log 2)^{2n} \right. \\
+ (n!)^2 & \sum_{k=2}^{2n-1} \sum_{\{\ell_{j}^{(\lambda)}\}} \sum_{\{\ell_{j}^{(\mu)}\}} \frac{\theta(k - \ell_{o}^{(\lambda)} - \ell_{o}^{(\mu)}) \theta(\mu - \lambda) \text{ perm}_{k}(f)}{\ell_{o}^{(\lambda)}; \ell_{1}^{(\lambda)}; \ldots; \ell_{n}^{(\lambda)}; \ell_{o}^{(\mu)}; \ell_{1}^{(\mu)}; \ldots; \ell_{n}^{(\mu)}; } \\
& \left. (n \geq 1) \right),
\end{align*}
\]

the sum over \( k \) being zero for \( n = 1 \).

When we consider \( n = p = 5 \) and construct the appropriate sets \( \{\ell_{j}^{(\lambda)}\} \), we find that the remaining permanents in Eq. (24) involve

\[
6 \times 2! + 14 \times 3! + 18 \times 4! + 20 \times 5! + 15 \times 6! + 9 \times 7! + 5 \times 8! + 2 \times 9! = 986 \, 448
\]

terms, which is related to the number of terms in (19) by about 1 to 13200.
Using the last formulae (23) (24), a FORTRAN program has been written which computes the permanents \( \text{per}_k(r) \) by means of Eq. (20). The necessary permutations were generated by means of a library program [8]. The main program constructs a label corresponding to the product \( \zeta(q_1) \zeta(q_2) \cdots \zeta(q_k) \) \( (q_1 \leq q_2 \leq \cdots \leq q_k) \), where by definition \( \zeta(1) = \log 2 \), and calculates the appropriate rational coefficient from Eqs. (11) and (13). Then it finds the final coefficients of the polynomial by adding the coefficients of equal products.

In the following, the expressions obtained for \( r_{np} \) are listed for \( 0 < n \leq 5 \), \( 0 \leq p \leq 5 \). They were computed on a CDC 6600 computer at CERN, and were checked against their numerical values obtained for \( p > 0 \) by numerical integration, and for \( p = 0 \) by Bowman's determinant. The time needed to compute \( r_{55} \) was 205 seconds, of which 93 per cent was spent in the evaluation of the three coefficients for \( k = 8 \) and \( k = 9 \).

The integrals \( r_{11}, r_{20} \) can also be found in the integral tables. The value for \( r_{11} \) has been published by Bromwich [9].

\[
\begin{align*}
    r_{10} &= -\frac{3}{2} \log 2 \\
    r_{11} &= \frac{\pi}{8} \left(-\zeta(2) + 4 \log^2 2\right) \\
    r_{20} &= \frac{\pi}{4} \left(\zeta(2) + 2 \log^2 2\right) \\
    r_{21} &= \frac{\pi}{8} \left(\zeta(3) - 4 \log^3 2\right) \\
    r_{22} &= \frac{\pi}{16} \left(-3\zeta(4) - 8\zeta(3) \log 2 + 3\zeta^2(2) + 8 \log^4 2\right) \\
    r_{30} &= -\frac{\pi}{4} \left(3\zeta(3) + 3\zeta(2) \log 2 + 2 \log^3 2\right) \\
    r_{31} &= \frac{\pi}{16} \left(-3\zeta(4) + 6\zeta(3) \log 2 - 3\zeta^2(2) + 6\zeta(2) \log^2 2 + 8 \log^4 2\right) \\
    r_{32} &= \frac{\pi}{16} \left(6\zeta(5) + 15\zeta(4) \log 2 - 6\zeta(2)\zeta(3) + 6\zeta(3) \log^2 2 - 3\zeta^2(2) \log 2 \\
    &\quad\quad\quad - 4\zeta(2) \log^3 2 - 8 \log^5 2\right) \\
    r_{33} &= \frac{\pi}{64} \left(-60\zeta(6) - 144\zeta(5) \log 2 - 9\zeta(2)\zeta(4) + 90\zeta^2(3) - 180\zeta(4) \log^2 2 \\
    &\quad\quad\quad + 144\zeta(2)\zeta(3) \log 2 - 21\zeta^3(2) - 48\zeta(3) \log^3 2 + 36\zeta^2(2) \log^2 2 \\
    &\quad\quad\quad + 24\zeta(2) \log^4 2 + 32 \log^6 2\right) \\
    r_{40} &= \frac{\pi}{8} \left(21\zeta(4) + 24\zeta(3) \log 2 + 3\zeta^2(2) + 12\zeta(2) \log^2 2 + 4 \log^4 2\right)
\end{align*}
\]
\[\begin{align*}
\gamma_{41} &= \frac{\pi}{8} \left( 3\zeta(5) - 15\zeta(4) \log 2 + 9\zeta(2)\zeta(3) - 18\zeta(3) \log^2 2 + 3\zeta^2(2) \log 2 - 8\zeta(2) \log^3 2 - 4 \log^5 2 \right) \\
\gamma_{42} &= \frac{\pi}{16} \left( -15\zeta(6) - 36\zeta(5) \log 2 + 18\zeta(2)\zeta(4) - 9\zeta^2(3) - 12\zeta(2)\zeta(3) \log 2 + 6\zeta^3(2) + 16\zeta(3) \log^2 2 + 12\zeta(2) \log^4 2 + 8 \log^6 2 \right) \\
\gamma_{43} &= \frac{\pi}{32} \left( 90\zeta(7) + 210\zeta(6) \log 2 + 18\zeta(2)\zeta(5) - 135\zeta(3)\zeta(4) + 252\zeta(5) \log^2 2 - 90\zeta(2)\zeta(4) \log 2 - 126\zeta^2(3) \log 2 + 27\zeta^2(2)\zeta(3) + 120\zeta(4) \log^3 2 - 108\zeta(2)\zeta(3) \log^2 2 + 6\zeta^3(2) \log 2 - 24\zeta^2(2) \log^3 2 - 24\zeta(2) \log^5 2 - 16 \log^7 2 \right) \\
\gamma_{44} &= \frac{\pi}{64} \left( -630\zeta(8) - 1440\zeta(7) \log 2 - 120\zeta(2)\zeta(6) + 1035\zeta^2(4) - 1680\zeta(6) \log^2 2 - 288\zeta(2)\zeta(5) \log 2 + 2160\zeta(3)\zeta(4) \log 2 + 234\zeta^2(2)\zeta(4) - 576\zeta(2)\zeta^2(3) - 1344\zeta(5) \log^3 2 + 720\zeta(2)\zeta(4) \log^2 2 + 1008\zeta^2(3) \log^2 2 - 432\zeta^2(2)\zeta(3) \log 2 + 576\zeta(2) \log^4 2 + 576\zeta(2)\zeta(3) \log^3 2 - 48\zeta^3(2) \log^2 2 + 96\zeta^2(2) \log^4 2 + 64\zeta(2) \log^6 2 + 32 \log^8 2 \right) \\
\gamma_{50} &= -\frac{\pi}{8} \left( 90\zeta(5) + 105\zeta(4) \log 2 + 30\zeta(2)\zeta(3) + 60\zeta(3) \log^2 2 + 15\zeta^2(2) \log 2 + 20\zeta(2) \log^3 2 + 4 \log^5 2 \right) \\
\gamma_{51} &= \frac{\pi}{32} \left( -30\zeta(6) + 300\zeta(5) \log 2 - 135\zeta(2)\zeta(4) - 60\zeta^2(3) + 360\zeta(4) \log^2 2 - 60\zeta(2)\zeta(3) \log 2 - 15\zeta^3(2) + 200\zeta(3) \log^3 2 + 60\zeta(2) \log^4 2 + 16 \log^6 2 \right) \\
\gamma_{52} &= \frac{\pi}{32} \left( 90\zeta(7) + 210\zeta(6) \log 2 - 150\zeta(2)\zeta(5) + 165\zeta(3)\zeta(4) - 120\zeta(5) \log^2 2 + 90\zeta(2)\zeta(4) \log 2 + 210\zeta^2(3) \log 2 - 105\zeta^2(2)\zeta(3) - 240\zeta(4) \log^3 2 + 120\zeta(2)\zeta(3) \log^2 2 - 30\zeta^3(2) \log 2 - 140\zeta(3) \log^4 2 - 48\zeta(2) \log^5 2 - 16 \log^7 2 \right) \\
\gamma_{53} &= -\frac{\pi}{64} \left( -630\zeta(8) - 1440\zeta(7) \log 2 - 165\zeta(2)\zeta(6) + 990\zeta(3)\zeta(5) - 180\zeta^2(4) - 1680\zeta(6) \log^2 2 + 720\zeta(2)\zeta(5) \log 2 + 360\zeta(3)\zeta(4) \log 2 - 360\zeta^2(2)\zeta(4) + 405\zeta(2)\zeta^2(3) - 600\zeta(5) \log^3 2 + 180\zeta(2)\zeta(4) \log^2 2 + 360\zeta^2(2)\zeta(3) \log 2 - 60\zeta^4(2) + 60\zeta(4) \log^4 2 + 120\zeta(2)\zeta(3) \log^3 2 + 60\zeta^3(2) \log^2 2 + 168\zeta(3) \log^5 2 + 60\zeta^2(2) \log^4 2 + 88\zeta(2) \log^6 2 + 32 \log^8 2 \right) 
\end{align*}\]
\[ r_{54} = \frac{D}{64} \left( 2520\xi(9) + 5670\xi(8) \log 2 + 540\xi(2)\xi(7) + 210\xi(3)\xi(6) - 4230\xi(4)\xi(5) \\
+ 6480\xi(7) \log^2 2 + 1260\xi(2)\xi(6) \log 2 - 3960\xi(3)\xi(5) \log 2 - 4455\xi^2(4) \log 2 \\
- 450\xi^2(2)\xi(5) + 990\xi(2)\xi(3)\xi(4) + 630\xi^3(3) + 5040\xi(6) \log^3 2 \\
- 720\xi(2)\xi(5) \log^2 2 - 6120\xi(3)\xi(4) \log^2 2 + 270\xi^2(2)\xi(4) \log 2 \\
+ 1260\xi^2(2)\xi(3) \log 2 - 210\xi^3(2)\xi(3) + 2280\xi(5) \log^4 2 - 1440\xi(2)\xi(4) \log^3 2 \\
- 1680\xi^2(3) \log^3 2 + 360\xi^2(2)\xi(3) \log^2 2 - 45\xi^4(2) \log 2 + 432\xi(4) \log^5 2 \\
- 840\xi(2)\xi(3) \log^4 2 - 112\xi(3) \log^6 2 - 144\xi^2(2) \log^5 2 - 96\xi(2) \log^7 2 \\
- 32 \log^9 2 \right) \]

\[ r_{55} = \frac{D}{256} \left( -45360\xi(10) - 100800\xi(9) \log 2 - 9450\xi(2)\xi(8) - 3600\xi(3)\xi(7) + 5400\xi(4)\xi(6) \\
+ 73800\xi^2(5) - 113400\xi(8) \log 2 - 21600\xi(2)\xi(7) \log 2 - 84000\xi(3)\xi(6) \log 2 \\
+ 169200\xi(4)\xi(5) \log 2 - 18000\xi^2(2)\xi(6) + 39600\xi(2)\xi(3)\xi(5) - 33075\xi(2)\xi^2(4) \\
- 33300\xi^2(3)\xi(4) - 86400\xi(7) \log^3 2 - 252000\xi(2)\xi(6) \log^2 2 \\
+ 79200\xi(3)\xi(5) \log^2 2 + 89100\xi^2(2) \log^2 2 + 18000\xi^2(2)\xi(5) \log 2 \\
- 39600\xi(2)\xi(3)\xi(4) \log 2 - 25200\xi^3(3) \log 2 - 6750\xi^3(2)\xi(4) \\
+ 15300\xi^2(2)\xi^2(3) - 50400\xi(6) \log^4 2 + 96000\xi(2)\xi(5) \log^3 2 \\
+ 81600\xi(3)\xi(4) \log^3 2 - 54000\xi^2(2)\xi(4) \log^2 2 - 252000\xi(2)\xi^2(3) \log^2 2 \\
+ 8400\xi^3(2)\xi(3) \log 2 - 765\xi^5(2) - 18240\xi(5) \log^5 2 + 14400\xi(2)\xi(4) \log^4 2 \\
+ 16800\xi^2(3) \log^4 2 - 480\xi^2(2)\xi(3) \log^3 2 + 900\xi^4(2) \log^2 2 \\
- 2880\xi(4) \log^6 2 + 6720\xi(2)\xi(3) \log^5 2 + 640\xi(3) \log^7 2 + 960\xi^2(2) \log^6 2 \\
+ 480\xi(2) \log^8 2 + 128 \log^9 2 \right) \]

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REFERENCES


3. F. Bowman, Note on the integral $\int_0^{\pi/2} (\log \sin \theta)^n d\theta$, J. London Math. Soc. 22 (1947), 172-173.


