Thermal conduction before relaxation in slowly rotating fluids

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Abstract

For slowly rotating fluids, we establish the existence of a critical point similar to the one found for non-rotating systems. As the fluid approaches the critical point, the effective inertial mass of any fluid element decreases, vanishing at that point and changing of sign beyond it. This result implies that first order perturbative method is not always reliable to study dissipative processes occurring before relaxation. Physical consequences that might follow from this effect are commented.

04.25.-g, 05.70.Ln, 04.40.Dg, 97.60.-s

*On leave from Departamento de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela and Centro de Astrofísica Teórica, Mérida, Venezuela.
I. INTRODUCTION

An alternative path to the study of self-gravitating systems, which avoids the use of numerical procedures and/or the introduction of too restrictive simplifying assumptions, consists in perturbing the system, compelling it to withdraw from equilibrium state. Then, evaluating it after its departure from equilibrium, it is possible to study the tendency of the evolution of the object. This is usually done following a first order perturbative method which neglects quadratic and higher terms in the perturbed quantities. This applies whenever the relevant processes occurring in the self-gravitating object take place on time scales which are of the order of, or smaller than, hydrostatic time scale. In this case the quasistatic approximation fails [1] (e.g. during the quick collapse phase preceding neutron star formation) and the system is evaluated immediately after its departure from equilibrium, where immediately means on a time scale of the order of relaxation times.

Recently, it has been shown [2–5] that, for systems out of quasi-static approximation, a first order perturbative theory is not always satisfactory. In fact, there exist systems for which this method seems to be inadequate however small the perturbation is. These ones are those for which the parameter

\[ \alpha = \frac{\kappa T}{\tau (\rho + p)} \]

is close to, or beyond the so called critical point (\( \alpha = 1 \)). This combination of the temperature \( T \), the heat conduction coefficient \( \kappa \), the relaxation time \( \tau \), the energy density \( \rho \), and the pressure \( p \), has been found to be the same in spherically symmetric systems [2,3], and axially symmetric systems with reflection symmetry [5]. Also the viscous spherically symmetric case has been studied with similar results [4].

The astrophysical interest of the study of relativistic rotating fluids is past all doubt. Therefore, it seems important to establish, for such systems, the existence, or not, of a critical point as described above. With this aim, we assume that, initially, a non-viscous slowly rotating object is close to hydrostatic equilibrium (along the \( r \) coordinate) and nearly
to thermal adjustment (the so called *complete equilibrium* [1, p. 66]), as measured by a local Minkowskian observer. Therefore, the time derivatives of the radial velocity and heat flow can be neglected. At that time, we perturb the radial velocity and the heat flow, and we evaluate conservation equations and heat transport equation just after the perturbation takes place, neglecting quadratic and higher terms in the perturbed quantities. Here *just after the perturbation* means on a time scale which is of the order of the relaxation time. This is necessary if the relevant processes take place on time scales which are of the order, or smaller than, hydrostatic time scale. This meaning of *just after the perturbation* implies that physical quantities remain unchanged, but not the time derivatives of the perturbed quantities. These ones, are still small, but they cannot be neglected since the system is departing from the complete equilibrium.

As has been mentioned above, it is necessary to use a heat transport equation, together with the conservation equations, to find out the existence, or not, of the critical point. In order to keep clear of inconsistences, the heat transport equation cannot be the well-known Eckart one [6,7] because it assumes a vanishing relaxation time. Furthermore, this theory suffers from two important drawbacks: Non-causality (the thermal signals propagate at infinite speed), and unstability (all the predicted equilibrium states are unstable). Fortunately, there exist well physically founded thermodynamical theories that avoid these problems and that can deal with pre-relaxation processes [8–11]. In this work, we shall use the Israel-Stewart heat transport equation. Nevertheless, it is important to emphasize that, as in [2–5], the results found are also valid in the context of the *Extended Irreversible Thermodynamics* [10,11].

The paper is organized as follows. The next section is devoted to introduce the interior and exterior metrics used, and to construct the stress-energy tensor. Also the validity of the slow rotating limit is discussed. In section three, the conservation equations and heat transport equation are evaluated just after perturbation and we find the expression for the critical point. Finally, we discuss the results in the last section.

We adopt metric of signature $-2$ and geometrised units $c = G = 1$. The quantities
subscripted with \( r_1 \) denote that they are evaluated at the surface of the object, whereas a partial derivative with respect time is denoted by subscript \( ,0 \).

**II. ENERGY-MOMENTUM TENSOR**

We consider a nonstatic and axisymmetric distribution of matter and radiation. Let us assume that the interior metric is given by [12]

\[
\begin{align*}
 ds^2 &= Y^2 du^2 + 2 \frac{Y}{X} dudr + 2a \sin^2 \theta \left( \frac{Y}{X} - Y^2 \right) dud\phi - 2a \sin^2 \theta \left( \frac{Y}{X} \right) drd\phi \\
 &- R^2 d\theta^2 - \sin^2 \theta \left[ r^2 + a^2 \cos^2 \theta + 2a^2 \sin^2 \theta \left( \frac{Y}{X} - \frac{Y^2}{2} \right) \right] d\phi^2,
\end{align*}
\]

where \( u = x^0 \) is a timelike coordinate, \( r = x^1 \) is the null coordinate and \( \theta = x^2 \) and \( \phi = x^3 \) are the usual angle coordinates. Here \( R^2 = r^2 + a^2 \cos^2 \theta \), \( a \) is the angular momentum per unit mass in the weak field limit -the Kerr parameter-, and \( X \) and \( Y \) are arbitrary functions of \( u, r \) and \( \theta \). The \( u \)-coordinate is related to retarded time in a flat space-time and therefore, \( u \)-constant surfaces are null cones open to the future. In these coordinates \( r \)-constant surfaces are oblate spheroids.

The energy-momentum tensor may be expressed in the above coordinates (1). Nevertheless, the physical quantities appearing in it will be those measured by a local Minkowskian observer comoving with the fluid. Thus, it is necessary to introduce the local Minkowski coordinates \((t, x, y, z)\) related to these ones by

\[
\begin{align*}
 dt &= Y du + \frac{dr}{X} + a \sin^2 \theta \left( \frac{1}{X} - Y \right) d\phi \\
 dx &= \frac{dr}{X} + a \sin^2 \theta \frac{X}{d\phi} \\
 dy &= \left[ r^2 + a^2 \cos^2 \theta \right]^{1/2} d\theta \\
 dz &= \left[ r^2 + a^2 \cos^2 \theta \right]^{1/2} \sin \theta d\phi.
\end{align*}
\]

The radial velocity of matter is given by

\[
\frac{dr}{du} = \frac{XY R \omega_z}{R (1 - \omega_z) + Y a \omega_z \sin \theta}
\]
and the orbital velocity is

\[ \Omega = \frac{d\phi}{du} = \frac{Y \omega_z}{R \sin \theta (1 - \omega_x)} + Y a \omega_z \sin^2 \theta, \] (4)

where \( \omega_x \) and \( \omega_z \) are the corresponding components of the velocity of a fluid element as measured the locally Minkowski frame.

In the slow rotating limit \( a \ll 1 \), and consequently \( \Omega \ll 1 \). Thus, from (4)

\[ \omega_z = \frac{\Omega R \sin \theta (1 - \omega_x)}{Y (1 - \Omega a \sin^2 \theta)} = \frac{\Omega r \sin \theta (1 - \omega_x)}{Y} + \mathcal{O}(\Omega^2), \]

and \( \omega_z \) is also much less than unity. Note that in the static case (i.e. \( \omega_x = 0 \)) and for a local Minkowskian observer (i.e. \( Y = 1 \)), this means that every fluid element must move at non-relativistic velocity [13]. A simple calculus shows that this condition is accomplished by most of the known pulsars [14, p. 146].

The interior metric (1), can be matched to the Kerr-Vaidya exterior metric [15]

\[ ds^2 = \left( 1 - \frac{2mr}{\tilde{R}^2} \right) du^2 + 2dudr + \frac{4mr \tilde{a} \sin^2 \theta}{\tilde{R}^2} dud\phi - 2\tilde{a} \sin^2 \theta drd\phi - \tilde{R}^2 d\theta^2 - \sin^2 \theta \left[ r^2 + \tilde{a}^2 + 2mr \tilde{a}^2 \sin^2 \theta \right] d\phi, \] (5)

where \( \tilde{R} = r^2 + \tilde{a}^2 \cos^2 \theta \), \( \tilde{a} \) is the exterior Kerr parameter and \( m \) is the total mass. It is worth mentioning at this point that the metric above is not a pure radiation solution and may be interpreted as such only asymptotically [16]. A pure rotating radiation solution may be found in reference [17]. However, although the interpretation of the Carmeli-Kaye metric is not completely clear, the appearance of the critical point is independent of the shape and the intensity of the emission pulse, and will be put in evidence for small values of luminosity (see below).

A particular solution can be found in [12]. Nevertheless, in this work we shall not restrict ourselves to a particular solution, and we shall go on using the unknown functions \( X(u, r, \theta) \) and \( Y(u, r, \theta) \). These ones are constrained by the following junction conditions at the surface \( (r = r_1) \) [12]
\[
X_{r=r_1} = Y_{r=r_1} = \left( 1 - \frac{2mr}{R^2} \right)^{1/2},
\]
\[
\left( \frac{\partial X}{\partial \theta} \right)_{r=r_1} = \left( \frac{\partial Y}{\partial \theta} \right)_{r=r_1} = -\left( \frac{mra^2 \sin(2\theta)}{X (r^2 + a^2 \cos^2 \theta)^2} \right)_{r=r_1},
\]
\[
\left( \frac{\partial X}{\partial r} \right)_{r=r_1} = \left( \frac{\partial Y}{\partial r} \right)_{r=r_1} = -\left( \frac{m(r^4 - a^4 \cos^2 \theta + 2r^2a^2 \cos^2 \theta)}{X (r^2 - a^2 \cos^2 \theta)(r^2 + a^2 \cos^2 \theta)^2} \right)_{r=r_1}. \tag{6}
\]

Next, we assume that, for a local Minkowskian observer, comoving with the fluid, the space-time contains:

1. An anisotropic fluid of density \( \rho^{\text{mat}} \), radial pressure \( p^{\text{mat}} \) and tangential pressure \( p_\perp^{\text{mat}} \).

2. A radiation field of specific intensity \( I(x,t;\vec{n},\nu) \), radiation energy flow \( q \), radiation energy density \( \rho^{\text{rad}} \), and radiation pressure \( p^{\text{rad}} \).

The specific intensity of the radiation field \( I(x,t;\vec{n},\nu) \), is measured at the position \( x \) and time \( t \), traveling in the direction \( \vec{n} \) with a frequency \( \nu \). The moments of \( I(x,t;\vec{n},\nu) \) for a planar geometry can be written as [18]

\[
\rho^{\text{rad}} = \frac{1}{2} \int_0^\infty d\nu \int_{-1}^1 d\mu \ I(x,t;\vec{n},\nu) \ , \tag{7}
\]

\[
q = \frac{1}{2} \int_0^\infty d\nu \int_{-1}^1 d\mu \ \mu I(x,t;\vec{n},\nu) \tag{8}
\]

and

\[
p^{\text{rad}} = \frac{1}{2} \int_0^\infty d\nu \int_{-1}^1 d\mu \ \mu^2 I(x,t;\vec{n},\nu) \ . \tag{9}
\]

where \( \mu = \cos \theta \). In classical radiative transfer theory, the specific intensity of the radiation field, \( I(x,t;\vec{n},\nu) \) at the position \( x \) and time \( t \), traveling in the direction \( \vec{n} \) with a frequency \( \nu \), is defined so that, \( dE = I(x,t;\vec{n},\nu) \ dS \ \cos \alpha d\theta \ d\nu \ dt \), is the energy crossing a surface element \( dS \), into solid angle \( d\theta \) around \( \vec{n} \) (\( \alpha \) is the angle between \( \vec{n} \) and the normal to \( dS \)), transported by radiation of frequencies \( (\nu,\nu + d\nu) \), in time \( dt \) (see [18] for details).
For a nonrotating observer, the radiation portion of the stress-energy tensor reads [18,19]

\[
\hat{T}^R_{\mu\nu} = \begin{pmatrix}
\rho^{\text{rad}} & -q & 0 & 0 \\
-q & \rho^{\text{rad}} & 0 & 0 \\
0 & 0 & \frac{1}{2}(\rho^{\text{rad}} - p^{\text{rad}}) & 0 \\
0 & 0 & 0 & \frac{1}{2}(\rho^{\text{rad}} - p^{\text{rad}})
\end{pmatrix},
\]

(11)

The radiation part of the energy momentum tensor as seen by an observer comoving with the fluid can be found by means of a local rotation to (11)

\[
\hat{T}^R_{\mu\nu} = \begin{pmatrix}
\rho^{\text{rad}} + D^2 p^{\perp} & -q & 0 & DG \\
-q & \rho^{\text{rad}} & 0 & -Dq \\
0 & 0 & p^{\perp} & 0 \\
DG & -Dq & 0 & D^2 p^{\text{rad}} + p^{\perp}
\end{pmatrix},
\]

(12)

where \(D\) is an unknown function of \(u, r\) and \(\theta\) associated with the local dragging of inertial frames effect, \(p^{\perp} = \frac{1}{2}(\rho^{\text{rad}} - p^{\text{rad}})\) and \(G = \frac{1}{2}(3\rho^{\text{rad}} - p^{\text{rad}})\).

The material part of the energy-momentum tensor for this observer is given by

\[
\hat{T}^M_{\mu\nu} = (\rho_M + p_{\perp})\hat{U}_\mu \hat{U}_\nu - p_{\perp} \eta_{\mu\nu} + (p - p_{\perp})\hat{s}_\mu \hat{s}_\nu,
\]

(13)

where the Minkowski metric is denoted by \(\eta_{\mu\nu}\), \(\hat{s}_\mu = \delta^t_\mu\) and \(\hat{U}_\mu = \delta^t_\mu\). Thus, the energy-momentum tensor, as seen by a Minkowskian observer comoving with the fluid, can be written as

\[
\hat{T}_{\mu\nu} = \hat{T}^R_{\mu\nu} + \hat{T}^M_{\mu\nu}.
\]

(14)

In the slow rotation limit \(D\) is taken up to first order. Thus, in virtue of (12) and (13), (14) \(\hat{T}_{\mu\nu}\) can be expressed as

\[
\hat{T}_{\mu\nu} = (\rho + p_{\perp})\hat{U}_\mu \hat{U}_\nu - P_{\perp} \eta_{\mu\nu} + (p - p_{\perp})\hat{s}_\mu \hat{s}_\nu + 2\tilde{q}_{(\mu} \hat{U}_{\nu)} + 2\tilde{q}_{(\mu} \hat{D}_{\nu)} + 2G\hat{U}_{(\mu} \hat{D}_{\nu)},
\]

(15)

where \(\tilde{q}_\mu = -q\hat{s}_\mu\), \(\hat{D}_\mu = D\delta^t_\mu\), \(\rho = \rho^{\text{rad}} + \rho^{\text{mat}}\) is the total energy density, and \(p = p^{\text{mat}} + p^{\text{rad}}\) and \(p_{\perp} = p_{\perp}^{\text{rad}} + p_{\perp}^{\text{rad}}\) are the total radial pressure and the total tangential pressure respectively.
It remains to express the energy-momentum tensor in curvilinear coordinates (1), as seen by an observer at rest with respect to the Minkowskian coordinates given by (2). Thus, we apply a Lorentz boost and the coordinate transformation defined in (2). The boost velocity is, in the rotating case, \( \vec{\omega} = (\omega_x, 0, \omega_z) \)-see [20] for details. Assuming slow rotation limit, \( D, a \) and \( \omega_z \) are taken up to first order. Thus,

\[
T_{\mu\nu} = (p + p_\perp)U_\mu U_\nu - p_\perp g_{\mu\nu} + (p - p_\perp)s_\mu s_\nu + 2q_\mu U_\nu + 2q_\nu D_\nu + 2GU_\mu D_\nu,
\]

where, \( g_{\mu\nu} \) is given by (1),

\[
U_\mu = \gamma Y \delta_\mu^u + \frac{\gamma (1 - \omega_x)}{X} \delta_\mu^r + \gamma \left[ a \sin^2 \theta \left( \frac{1 - \omega_x}{X} - Y \right) - \omega_z r \sin \theta + \mathcal{O}(\omega_z^2) \right] \delta_\mu^\phi,
\]

\[
s_\mu = -\gamma \omega_x Y \delta_\mu^u + \left[ \frac{\gamma (1 - \omega_x)}{X} + \mathcal{O}(\omega_z^2) \right] \delta_\mu^r + \left[ r \sin \theta \frac{\omega_z}{\omega_x} (\gamma - 1) + \frac{\gamma a \sin^2 \theta}{X} [1 - \omega_x (1 - Y X)] + \mathcal{O}(\omega_z^2) \right] \delta_\mu^\phi
\]

\[
q_\mu = -qs_\mu,
\]

\[
D_\mu = \mathcal{O}(\omega_x^2) \delta_\mu^u + \mathcal{O}(\omega_z^2) \delta_\mu^r + \left[ Dr \sin \theta + \mathcal{O}(\omega_z^2) \right] \delta_\mu^\phi,
\]

\[
\omega = \sqrt{\omega_x^2 + \omega_z^2} = \omega_x + \mathcal{O}(\omega_z^2),
\]

and

\[
\gamma = \frac{1}{\sqrt{1 - \omega^2}} = \frac{1}{\sqrt{1 - \omega_x^2}} + \mathcal{O}(\omega_z^2).
\]

Here, \( \mathcal{O}(\omega_z^2) \) corresponds to quadratic and higher terms in \( \omega_z, D \) and \( a \).

**III. DEPARTURE FROM COMPLETE EQUILIBRIUM**

As has been mentioned in the Introduction, we assume that, before perturbation, the slowly rotating system is evolving along a sequence of states in which it is close complete
equilibrium. Therefore, \( u \)-derivatives of \( \omega_x \) and \( q \) can be neglected because it is close to hydrostatic equilibrium (along the \( r \) coordinate) and nearly thermally adjusted. A system which is thermally adjusted changes its properties considerably only within a time scale \( \tau_{cha} \) that is large as compared with the Kelvin-Helmholtz time scale \( \tau_{KH} \) [1, p. 66]. Thus, before perturbation we can assume that the \( u \)-derivatives of \( \rho, p \) and \( p_\perp \) are small, and consequently \( \omega_x \) too (\( i.e. \) we can neglect quadratic and higher terms in \( \omega_x \)). On the other hand, if the system is close to hydrostatic equilibrium, then the hydrostatic time scale \( \tau_{hyd} \sim \sqrt{r^3/m} \) is much shorter than the Kelvin-Helmholtz time scale \( \tau_{KH} \sim m^2/2rl \), and inertial terms in the equation of motion \( T^\mu_{\nu;\mu} = 0 \) can be ignored. This condition will be accomplished for small values of luminosity \( l \), and consequently for small values of \( q \). Thus, before perturbation \( q \sim O(\omega_x) \).

We shall evaluate the system just after perturbation. Where, as stated before, \textit{just after perturbation} means on a time scale of the order of the relaxation time. Physically, this implies that the perturbed quantities (\( \omega_x \) and \( q \)) are still much less than unity. Nevertheless, the system is departing from complete equilibrium and the \( u \)-derivatives of \( \omega_x \) and \( q \) must be small but different from zero (\( i.e. \) \( q,0 \sim \omega_x,0 \sim O(\omega_x) \)).

Thus, the system is characterized by:

- **Before perturbation**

\[
\rho,0 \sim p,0 \sim p_\perp,0 \sim \omega_x \sim q \sim O(\omega_x) \quad (22)
\]

\[
\omega_x,0 \sim q,0 \sim O(\omega_x^2). \quad (23)
\]

- **After perturbation**

\[
\rho,0 \sim p,0 \sim p_\perp,0 \sim \omega_x \sim q \sim \omega_x,0 \sim q,0 \sim O(\omega_x) \quad (24)
\]

In order to clarify the existence of a critical point in slowly rotating fluids, we shall use conservation equations \( T^\mu_{\nu;\mu} = 0 \).
A. Conservation equations

Before perturbation, conservation equations read

\[ R_\nu := T_{\nu,\mu}^\mu = 0. \]  \hspace{1cm} (25)

After perturbation, physical quantities contained in (16) remain unchanged since we are evaluating the system on a time scale of the order of the relaxation time. Therefore, the only new terms appearing in conservation equations are those containing \( u \)-derivatives of \( \omega_x \) and \( q \), and conservation equations can be written as

\[ \tilde{T}_{\nu,\mu} = \tilde{R}_\nu + \tilde{\omega}_{x,0} F_\nu + \tilde{q}_{0} G_\nu = 0, \]  \hspace{1cm} (26)

where tilde denotes that the quantity is evaluated after perturbation, and \( F_\nu \) and \( G_\nu \) do not depend on \( \omega_x \), \( q \) or \( u \)-derivatives of physical variables since we are applying first order perturbation theory. The only terms that can contain \( \tilde{\omega}_{x,0} \) and \( \tilde{q}_{0} \) in \( \tilde{T}_{\nu,\mu} = 0 \) are of the form \( \tilde{T}_{\nu,0}^0 \). By means of (1), (17-20) and (16), we find four equations of the form (26) -see appendix A for details-

\[ \tilde{R}_u = (\rho + p) \tilde{\omega}_{x,0} + \tilde{q}_{0}, \]  \hspace{1cm} (27)
\[ \tilde{R}_r = \frac{2}{XY} [(\rho + p) \tilde{\omega}_{x,0} + \tilde{q}_{0}], \]
\[ \tilde{R}_\theta = 0 \]
\[ \tilde{R}_\phi = 0. \]

Note that \( \tilde{R}_\phi \) is not the total meridional force acting on a given fluid element since it contains terms in \( \tilde{\omega}_{z,0} \). Nevertheless, \( \tilde{R}_r \) does not contain \( u \)-derivatives of physical variables. Thus, \( \tilde{R}_r > 0 \) is the total outward force (pressure gradient + gravitational) along the \( r \)-coordinate acting on a given fluid element after perturbation.

The \( u \)-derivative of the heat flow \( \tilde{q}_{0} \) can be connected with \( \tilde{\omega}_{x,0} \) by means of an adequate heat transport equation.
B. Heat transport equation

As it is well-known, Eckart-Landau transport equation [6,7] assumes a vanishing relaxation time. This fact leads to undesirable predictions: An infinite speed for the propagation of the thermal signals and unstable equilibrium states [21]. Thus, it is necessary to adopt a relativistic thermodynamic theory leading to a hyperbolic equation for the propagation of thermal signals. On the other hand, we are evaluating the system just after its departure from hydrostatic equilibrium and thermal adjustment (in the sense described above). Thus, to be consistent with this choice we must use a heat transport equation with non vanishing relaxation time.

We shall use the Israel-Stewart relativistic transport equation [8,9]. For viscous free fluid distributions, this one can be written as [22]

\[ \tau h^{\mu\nu} U_\alpha q_{\nu,\alpha} + q^\mu = \kappa h^{\mu\nu} (T_\nu - T U^{\alpha} U_{\nu;\alpha}) - \frac{1}{2} \kappa T^2 \left( \frac{T}{\kappa T^2} U^\beta \right)_\beta q^\mu + \tau \omega^{\mu\nu} q_{\nu}, \]

(28)

where \( \kappa, \tau \) and \( T \) denote thermal conductivity, thermal relaxation time and temperature respectively, \( h^{\mu\nu} = U^\mu U^\nu - g^{\mu\nu} \) is the projector onto the hypersurface orthogonal to \( U^\mu \) and \( \omega_{\mu\nu} = h^{\alpha}_{\mu} h^{\beta}_{\nu} U_{[\alpha;\beta]} \) is the vorticity.

Before perturbation, transport equations (28) can be symbolized as

\[ H^\mu = 0. \]

(29)

Just after perturbation (28) can be written as

\[ \tilde{H}^\mu + \tilde{\omega}_{x,0} I^\mu + \tilde{q}_{0,0} J^\mu = 0. \]

(30)

Nevertheless, physical quantities contained on \( H^\mu \) do not change just after perturbation, and \( \tilde{H}^\mu = H^\mu \). Thus, from (29), expression (30) takes the form

\[ \tilde{\omega}_{x,0} I^\mu + \tilde{q}_{0,0} J^\mu = 0. \]

(31)

Vectors \( I^\mu \) and \( J^\mu \), as \( F_\nu \) and \( G_\nu \) in the preceding section, do not depend on \( \omega_x \), \( q \) or \( \nu \)-derivatives of physical variables.
The components of (28) contain $u$-derivatives of $q$ and $\omega_x$ up to first order are

\[ \tau h^{\mu\nu} U^\alpha q_{\nu\alpha} \]  

and

\[ -\kappa T h^{\mu\nu} U^\alpha U_{\nu\alpha}. \]  

Therefore, heat transport equation (28) just after perturbation is given by - see appendix B for details -

\[ \tilde{q}_{\alpha,0} = -\frac{\kappa T}{\tau} \tilde{\omega}_{x,0}, \]  

for any value of $\mu$.

C. Equation of motion

We are now in position to find the equation of motion just after perturbation. From (27) and (34) we can write

\[ \tilde{R}_r = \frac{2 (\rho + p)}{XY} (1 - \alpha) \tilde{\omega}_{x,0}, \]  

where

\[ \alpha = \frac{\kappa T}{\tau (\rho + p)}. \]

As it has been noted in section III A, $\tilde{R}_r > 0$ is the total outward force along the $r$-coordinate acting on a given fluid element. Note that it vanishes for $\alpha = 1$ (the critical point). This fact has an important consequence: For $\alpha = 1$, $\tilde{R}_r$ vanishes even though the $u$-derivative of the radial velocity is different from zero. This method also predicts an anomalous behaviour beyond the critical point - equation (35). If $\alpha > 1$, then an outward force ($\tilde{R}_r > 0$) implies an inward acceleration ($\tilde{\omega}_{x,0} < 0$). Therefore, it seems that, for systems out of quasi-static approximation, first order perturbation theory can not be applied close to the critical point or beyond it.
On the other hand, (35) may be compared with the Newtonian form

$$Force = mass \times acceleration,$$

where here, the term

$$\frac{2(\rho + p)}{X Y} (1 - \alpha)$$

stands for the effective inertial mass. Below the critical point, this inertial mass decreases as $\alpha$ grows up. This seems to be connected with the dynamical stability of the system, leading to a minimum stability for $\alpha = 1$ [23].

IV. CONCLUSIONS

In this paper we have studied the departure, of a slowly rotating fluid distribution, from a state close to hydrostatic equilibrium (along the $r$ coordinate) and nearly thermally adjusted. Our aim has been to elucidate the existence of a critical point similar to the found for non-rotating systems [2–5]. The existence of this critical point implies that first order perturbative method is not always satisfactory to study pre-relaxation processes (i.e. processes that take place on time scales smaller than the hydrostatic time scale).

We have found that, also in this case, there exists such critical point. This one is given by condition

$$\alpha = \frac{\kappa T}{\tau (\rho + p)} = 1,$$

and it coincides with this one found in spherically simmetric case [2–4] and in axially simmetric case [5]. Therefore, condition $\alpha = 1$, establishes an upper limit for which pre-relaxation processes can be studied by means of a first order perturbative method. This result is also valid if the initial system configuration is strictly in complete equilibrium and radially static (i.e. $\omega_x = \omega_{x,0} = q = q_{,0} = 0$).

Note that this method predicts, for values of $\alpha$ less than unity, that the effective inertial mass decreases as $\alpha$ grows. Intuitively, this means that the departure from equilibrium
or quasi-equilibrium will be steeper for larger $\alpha$'s, or, in other words, that the smaller $\alpha$, the larger dynamical stability. This point, has been recently illustrated, by means of an expression for the active gravitational mass in terms of $\alpha$ [23]. It is interesting to emphasize that, at least in non-rotating configurations, causality and stability conditions [21] not always forbid the existence of the critical point [3,4].

Finally it is also worth noticing that the critical point and the inflationary equation of state for non-dissipative systems ($p = -\rho$) are similar in that they imply the vanishing of the inertial mass term. Therefore one might wonder about the plausibility of an inflationary scenario in a Universe at, or close to, the critical point.

**ACKNOWLEDGMENTS**

This work has been partially supported by the Spanish Ministry of Education under Grant No. PB94-0718

**APPENDIX A: CONSERVATION EQUATIONS JUST AFTER PERTURBATION**

In the slow rotating limit, $U^\mu$, $s^\mu$ and $D^\mu$ read

$$U^\mu = \left[ \frac{\gamma (1 - \omega_x)}{Y} + \mathcal{O} (\omega_x^2) \right] \delta_{u}^\mu + \left[ \omega_x X + \mathcal{O} (\omega_x^2) \right] \delta_{r}^\mu + \left[ \frac{\omega_z}{r \sin \theta} + \mathcal{O} (\omega_z^2) \right] \delta_{\phi}^\mu \tag{A1}$$

$$s^\mu = \left[ \frac{\gamma (1 - \omega_x)}{Y} + \mathcal{O} (\omega_x^2) \right] \delta_{u}^\mu + \left[ -\gamma X + \mathcal{O} (\omega_z^2) \right] \delta_{r}^\mu \tag{A2}$$

$$+ \left[ -\frac{\omega_z}{r \sin \theta} \left( \frac{\gamma - 1}{\omega_x} \right) + \mathcal{O} (\omega_z^2) \right] \delta_{\phi}^\mu$$

$$D^\mu = \mathcal{O} (\omega_x^2) \delta_{u}^\mu + \mathcal{O} (\omega_x^2) \delta_{r}^\mu + \left[ -\frac{D}{r \sin \theta} + \mathcal{O} (\omega_z^2) \right] \delta_{\phi}^\mu \tag{A3}$$

and $q^\mu = -qs^\mu$. From (16) the only terms that contain $u$-derivatives of $\omega_x$ and $q$ (up to first order in $\omega_x$ and $\omega_z$) in conservation equations $T^\mu_{\nu,\mu} = 0$ are of the form $T^0_{0,0}$. In particular
\[(\rho + p_\perp) (U^0 U_\nu) = (\rho + p_\perp) \left[ \frac{\gamma (1 - \omega_x)}{Y} U_\nu \right], (A4) \]

\[(p - p_\perp) (s^0 s_\nu) = (p - p_\perp) \left[ \frac{\gamma (1 - \omega_x)}{Y} s_\nu \right], \]

\[G \left( U^0 D_\nu + U_\nu D^0 \right) = G \left[ \frac{\gamma (1 - \omega_x)}{Y} D_\nu \right], \]

\[(q^0 U_\nu + q^0 D_\nu + q_\nu U^0 + q_\nu D^0) = -q_0 \left[ \frac{\gamma (1 - \omega_x)}{Y} (U_\nu + D_\nu + s_\nu) \right] - q \left[ \frac{\gamma (1 - \omega_x)}{Y} (U_\nu + D_\nu + s_\nu) \right]. \]

Thus, from (17-20), (21), (A4) and following the definition of \(F_\nu\) and \(G_\nu\) given in (26), we find up to first order in \(\omega_x\)

\[\omega_{x,0} F_u = (\rho + p_\perp) \left[ \frac{1}{1 + \omega_x} \right] - (p - p_\perp) \left[ \frac{\omega_x}{1 + \omega_x} \right] - q \left[ \frac{1 - \omega_x}{1 + \omega_x} \right], \]

\[= -\omega_{x,0} (\rho + p), \]

\[\omega_{x,0} F_r = \left( \frac{\rho + p - 2q}{X Y} \right) \left[ \frac{1 - \omega_x}{1 + \omega_x} \right] = -\omega_{x,0} \frac{2 (\rho + p)}{X Y}, \]

\[\omega_{x,0} F_\theta = 0, \]

\[\omega_{x,0} F_\phi = 0, \]

\[q_0 G_u = -q_0, \]

\[q_0 G_r = -q_0 \frac{2}{X Y}, \]

\[q_0 G_\theta = 0, \]

\[q_0 G_\phi = 0, \]

where we have neglected in \(\omega_{x,0} F_\phi\) terms of the form \(\omega_{x,0} D, \omega_{x,0} a\) and \(\omega_{x,0} \omega_z\), and in \(q_0 G_\phi\) terms of the form \(q_0 D, q_0 a\) and \(q_0 \omega_z\). Thus,

\[F_\mu = - (\rho + P) \left( 1, \frac{2}{X Y}, 0, 0 \right), \]

and

\[G_\mu = - \left( 1, \frac{2}{X Y}, 0, 0 \right). \]
The expression of conservation equations can be found by means of (26):

$$\tilde{R}_\nu = [(\rho + p) \tilde{\omega}_{x,0} + \tilde{q}_0] \left( 1, \frac{2}{XY}, 0, 0 \right).$$  \hspace{1cm} \text{(A9)}

**APPENDIX B: HEAT TRANSPORT EQUATION JUST AFTER PERTURBATION**

The right-hand terms in (28)

$$-\frac{1}{2} \kappa T^2 \left( \frac{\tau}{\kappa T^2} U^\beta \right) q^\alpha,$$

and

$$\tau \omega^{\mu \nu} q_\nu,$$

contain factors of the form $\omega_{x,0}q$, which are of second order. Therefore, the only terms in (28) that contain $u$-derivatives of $\omega_x$ and $q$ up to first order are of the form

$$\tau h^{\mu \nu} U^0 q_{\nu,0}$$ \hspace{1cm} \text{(B1)}

and

$$-\kappa T h^{\mu \nu} U^0 U_{\nu,0}.$$ \hspace{1cm} \text{(B2)}

Using (A1), (18) and (19) in (B1) and (B2)

$$\tau h^{\mu \nu} U^0 q_{\nu,0} = \frac{\gamma (1 - \omega_x)}{Y} \tau h^{\mu \nu} (-q_{0\nu} - q_{s\nu,0})$$ \hspace{1cm} \text{(B3)}

$$-\kappa T h^{\mu \nu} U^0 U_{\nu,0} = -\frac{\gamma (1 - \omega_x)}{Y} \kappa T h^{\mu \nu} U_{\nu,0}.$$

Vectors $I^\mu$ and $J^\mu$ (31) do not depend on $\omega_x$, $q$ and their $u$-derivatives because of we are using first order perturbation theory. Thus, from (B3)

$$\tilde{\omega}_{x,0} I^\mu = \tilde{\omega}_{x,0} \frac{\kappa T}{XY} h^{\mu \nu},$$ \hspace{1cm} \text{(B4)}

and
\[ \tilde{q}_0 \mathcal{J}^\mu = \tilde{q}_0 \frac{\tau}{XY} h^{\mu r}, \]  

(B5)

where we have neglected terms of the form \( \tilde{q}_0 a \). Therefore, from (B4) and (B5), expression (31) takes the form

\[ \tilde{q}_0 = -\frac{\kappa T}{\tau} \tilde{\omega}_{x,0}, \]  

(B6)

which is valid for any \( \mu \).
REFERENCES


