INCLUSIVE REACTIONS IN DUAL MODELS WITH
SECOND SHEET RESONANCES

P. Olesen ^{*)}
CERN - Geneva

A B S T R A C T

We investigate the effects of unitarity-like features in inclusive reactions by means of two dual resonance models with second sheet resonance poles and Mandelstam analyticity. The first model has recently been proposed by Cohen-Tannoudji, Henyey, Kane and Zakrzewski (CHKZ). The second model (a modification of the CHKZ model) has the additional feature that it approaches the Veneziano model in the zero-width limit. It turns out that the unitarity-like features contained in both models influence the transverse momentum spectrum strongly in the central region. The second model gives a transverse momentum cut-off of the form $\exp(-a_1 p_{\perp} - a_2 p_{\perp}^q)$, where $q > 1$ and where a_1 and a_2 are (not necessarily universal) constants. In the zero-width limit the spectrum decreases much faster [like $\exp(-a p_{\perp}^2)$]. In the fragmentation region the unitarity-like features of the models are of less importance.

*) On leave of absence from the Niels Bohr Institute, Copenhagen.

1. INTRODUCTION

During the last few years there has been much experimental and theoretical interest in inclusive reactions in which single particle distributions are observed at high energies. Of particular interest is the discovery of a scaling property of inclusive cross-sections. This property has been found in the multiperipheral model ¹⁾ and in the parton model ^{2),3)}.

More recently Mueller ⁴⁾ has discovered a generalized optical theorem which allows one to discuss single particle production at high energies in terms of Regge phenomenology. Mueller's idea is of particular importance for dual resonance models, since it allows one to calculate the single particle spectrum in terms of a discontinuity of the six-point function. The contribution from the relevant tree-diagrams has been evaluated by several authors ^{5),6),7)}. It turns out that dual resonance models exhibit very neat scaling behaviour. This approach has, however, certain disadvantages. Since unitarity is not satisfied diffraction cannot really be taken into account, and consequently the Pomeron enters in a rather artificial way (or alternatively, one has to restrict oneself to combinations of reactions where the Pomeron does not enter). Furthermore, if one compares the spectra calculated from the tree diagrams with experiments it turns out ⁸⁾ that, in general, the agreement is bad in the pionization region (and also in the central region). However, the general physical structure of the results obtained in Refs. 5),6),7) makes it rather likely that scaling would be a feature of more realistic models which have certain features associated with unitarity.

One way to investigate the effects of unitarity in inclusive reactions is to include dual (factorizable) loops. Such a program is certainly complicated since it involves the calculation of discontinuities of loops. At present it is not even known whether loops have Regge behaviour in the physical region. It is also questionable how relevant perturbative unitarity is in strong interaction physics, and it might be that one has to sum over an infinite number of loops in order to get results of physical interest.

In the present paper we shall therefore attempt a much less ambitious approach to study within a dual framework the effects of unitarity in inclusive reactions. We investigate single particle distributions in a model recently proposed by Cohen-Tannoudji, Henyey, Kane and Zakrzewski ⁹⁾ (CHKZ), as well as in a modified version of this model. The CHKZ model is dual and crossing symmetric, it has Regge asymptotic behaviour and second-sheet poles for resonances in all channels. The main point of the model is that in addition to the properties mentioned before it also satisfies Mandelstam analyticity. The CHKZ amplitude thus has many features usually associated with unitarity. The modified CHKZ model has the same features, but, in addition, it contains the Veneziano model as a limiting case when the resonances have zero width. We mention the main features of the CHKZ model in Section 2.

Although the CHKZ model (as well as the modified CHKZ model discussed in Section 2) does not satisfy unitarity exactly we feel that the model has so many unitarity-like features that it makes sense to ask (at least qualitatively) the question concerning the main effects of unitarity in inclusive reactions. It turns out that unitarity changes the behaviour of the single particle spectrum drastically in the pionization limit (and the "central" region), whereas it gives rather unimportant effects in the fragmentation region. In particular the transverse momentum distribution changes drastically when unitarity-like corrections are included. We also find that the approach to Feynman's scaling limit, in the fragmentation region ^{*)}, is given by

$$d\sigma = \frac{d^3 p_c}{p_{0c}} F(x, p_T) \left(1 + \frac{C}{\sqrt{s}}\right) \quad (1.1)$$

for the reaction $a + b \rightarrow c + x$. The function $F(x, p_T)$ depends on the reaction and C is a constant (i.e., C is independent of x and p_T) which depends on the coupling of the secondary trajectories. If the $(ab\bar{c})$ channel is exotic the constant C is very small ^{10), 11)} in the modified CHKZ model.

^{*)} In the pionization limit the correction is of order $1/\sqrt[4]{s}$.

The paper is organised the following way: in Section 2 we briefly discuss the CHKZ model as well as the modified model. In Section 3 we find the general scaling limit of inclusive reactions. In Section 4 we discuss in more detail the various special limits (in particular, limited fragmentation and pionization). The transverse momentum cut-off is studied in detail.

2. DESCRIPTION OF THE MODELS

2.1 The CHKZ model

For the reader's convenience we shall give a brief discussion of the CHKZ model ⁹⁾. The four-point amplitude is given by

$$M(s,t) = \int_0^1 dx \, x^{-\alpha(t(1-x))-1} (1-x)^{-\alpha(sx)-1} f(sx) f(t(1-x)),$$

(2.1)

where α represents the complex trajectory, $f(z)$ and $\alpha(z)$ are real ^{*)} for $z < 4m^2$ and have threshold branch points for $z = 4m^2$. In Eq. (2.1) we have assumed an exotic u channel, but otherwise one just adds $M(s,u)$ and $M(u,t)$ to $M(s,t)$ as usual. The Regge pole asymptotic behaviour is obtained by changing variables $a = -sx$, if $\text{Re } s \rightarrow -\infty$. This gives

$$M(s,t) = (-s)^{\alpha(t)} f(t) \int_0^\infty da \, a^{-\alpha(t)-1} f(-a).$$

(2.2)

This formula is actually valid for $|s| \rightarrow \infty$ in any direction of the complex plane. If we had performed the limit $\text{Re } s \rightarrow +\infty$ we would obtain, instead of Eq. (2.2)

*) In order that $M(s,t)$ exists we must assume that $f(z)$ vanishes faster than any inverse power of z when z becomes infinite anywhere in the physical sheet.

$$M(s,t) = s^{\alpha(t)} f(t) \int_0^{\infty} da a^{-\alpha(t)-1} f(a). \quad (2.3)$$

The equivalence of Eqs. (2.2) and (2.3) can be established by the following calculations. First we notice that

$$\int_{-\infty}^{+\infty} da a^{-\alpha(t)-1} f(a) = 0, [\operatorname{Re} \alpha(t) < 0], \quad (2.4)$$

since $f(a)$ vanishes faster than any power when $|a| \rightarrow \infty$. Integrating over the contour shown in Fig. 1 then immediately yields Eq. (2.4), since the integrand does not contain any singularities inside the contour. Using Eq. (2.4) we then have

$$\begin{aligned} & (-s)^{\alpha(t)} f(t) \int_0^{\infty} da a^{-\alpha(t)-1} f(-a) \\ &= -s^{\alpha(t)} f(t) \int_{-\infty}^0 da a^{-\alpha(t)-1} f(a) \\ &= s^{\alpha(t)} f(t) \int_0^{\infty} da a^{-\alpha(t)-1} f(a) \end{aligned} \quad (2.5)$$

when Eq. (2.4) was used in the last step. Equation (2.5) shows that Eqs. (2.2) and (2.3) are equivalent.

Returning to Eq. (2.1) we see that crossing symmetry is obvious, and the second sheet resonance poles arise from the end-point singularity near $x=0$ and are of the form $1/(n-\alpha)$ with $\operatorname{Im} \alpha \neq 0$. The pole residues have no "ancestors". The correct threshold behaviour can be explicitly included in the choice of α and f . Mandelstam analyticity can be shown by calculating the double discontinuity arising from $\operatorname{Im} \alpha$ and $\operatorname{Im} f$. Amplitudes with Regge pole asymptotic behaviour in one channel and Regge cut asymptotic behaviour in another, or cut behaviour in both, can easily be constructed

$$M(s,t) = \int_0^1 dx \int_{-\infty}^{J_{\max}} dj \int_{-\infty}^{J_{\max}} dj' x^{-j-1} (1-x)^{-j'-1} \times h(j, sx) h(j', t(1-x)) \quad (2.6)$$

with h being of the type

$$h(j, sx) = f_1(s(1-x)) \delta(j - \alpha(s(1-x))) + f_2(j, s(1-x)) \times \theta(\alpha_c(s(1-x)) - j). \quad (2.7)$$

The first term in Eq. (2.7) corresponds to the Regge pole contribution whereas the second term gives the cut contribution. Equation (2.6) has Mandelstam analyticity. Finally, the amplitude (2.1) can easily be generalized to N particle problems,

$$M_N = \int d\mu \prod_u u^{-\alpha(s_u(1-u))-1} f(s_u(1-u)), \quad (2.8)$$

where the product is over all planar channels and $d\mu$ is the usual Veneziano measure. The pole + cut model (2.6) can also easily be generalized to N particle reactions.

For our purpose the most relevant features of the CHKZ model are the following.

- a) With appropriate choices of the trajectories and of $f(z)$ the CHKZ model has reasonable high and low energy behaviours. In fact since the model incorporates second-sheet resonance poles it can be made to have a more realistic low energy behaviour than the Veneziano model. At high energies strong absorptive cuts are probably needed to describe the data, so in this respect the pole + cut version of the CHKZ model is also more realistic. Due to the flexibility of the model [the Regge trajectories and $f(z)$ are essentially arbitrary ^{*)} although these functions are

*) This represents a phenomenological input, analogous to the input trajectories used in Veneziano phenomenology. The advantage of the CHKZ model is that no "ancestors" are introduced for complex trajectories.

in principle fixed by unitarity] one can be reasonably justified in hoping that it can describe phenomenologically a large amount of experimental data.

- b) One can have the Harari-Freund model for the Pomeron satisfied in the pole+cut version of the CHKZ model. To be explicit, the amplitude

$$M(s,t) = \int_0^1 dx \int_{-\infty}^{J_{\max}} dj x^{-j-1} (1-x)^{-\alpha_P(tx)-1} \times h(j, s(1-x)) f(tx) \quad (2.9)$$

has t channel resonances but no s channel resonances (in the s channel, the poles coming from $x \approx 0$ are integrated over j , so that M itself has no poles). The asymptotic behaviour is

$$M \rightarrow s^{\alpha_P(t)} f(t) \int_0^1 da \int_{-\infty}^{J_{\max}} dj a^{-\alpha(t)-1} h(j,a), \quad s \rightarrow \infty, \quad (2.10)$$

and

$$M \rightarrow t^{\alpha_c(s)} \log^{\gamma} t G(t), \quad t \rightarrow \infty, \quad (2.11)$$

where $G(t)$ depends on the nature of the cut. The dual properties of the Pomeron (and hence of diffraction) are therefore not unreasonable. Another way to introduce s channel background is to have a "background-trajectory" $\alpha_B(s)$ in the s channel such that $\text{Re } \alpha_B(s)$ never crosses the real axis ($\alpha_B < 0$ for all s).

- c) The price paid for the nice features of the CHKZ model is its lack of nice factorization properties ⁹⁾ [at least if one does not make rather special choices of $f(z)$]. It is perhaps also a bad feature that double-poles enter ⁹⁾ [coming from the expansion of e.g., $x^{-\alpha(s(1-x))}$ in powers of x]. Such double poles are also present in dual loops, but there they presumably add up to give a renormalization of the trajectory.

2.2 A modified version of the CHKZ model

One could argue that the Veneziano amplitude is fundamental in the sense that it is a "Born approximation" relevant in the zero-width approximation. The CHKZ amplitude (2.1) does not, however, approach the Veneziano amplitude in the zero-width limit. It is possible to modify the CHKZ amplitude such that the zero-width limit reduces to the Veneziano amplitude ^{*}). Let us write the (complex) Regge trajectory as

$$\alpha(t) = \alpha_0(t) + \sigma(t), \quad (2.12)$$

with

$$\alpha_0(t) = \alpha' t + \alpha(0), \quad \sigma(0) \equiv 0. \quad (2.13)$$

Thus, $\alpha_0(t)$ is the straight-line part of the complex trajectory $\alpha(t)$. If the trajectory $\alpha(t)$ turns over at high t values, the term $\alpha' t$ is cancelled by a similar term in $\sigma(t)$.

The four-point amplitude

$$M(s, t) = \int_0^1 dx \, x^{-\alpha_0(t) - \sigma(t(1-x)) - 1} (1-x)^{-\alpha_0(s) - \sigma(sx) - 1} \times f(sx) f(t(1-x)) \quad (2.14)$$

with $f(z)$ vanishing faster than $e^{-\alpha'|z|}$ for $|z| \rightarrow \infty$,

$$f(z) e^{\alpha'|z|} \rightarrow 0 \quad (|z| \rightarrow \infty), \quad (2.15)$$

^{*}) After completion of the work reported in this paper the author received a preprint by M.G. Schmidt (University of Heidelberg preprint) where the same modification of the CHKZ model has been proposed.

has the same features as the CHKZ amplitude. Since the non-linear part σ of α is always associated with the argument $t(1-x)$ (or sx) the amplitude (2.14) has no "ancestors". The Regge limit is given by

$$M(s,t) = (-s)^{\alpha(t)} f(t) \int_0^{\infty} da a^{-\alpha(t)-1} e^{-\alpha' a} f(-a). \quad (2.16)$$

Again duality, Mandelstam analyticity, etc., can be shown as for the CHKZ model ⁹⁾.

In addition to the features of the CHKZ model the amplitude (2.16) approaches the Veneziano amplitude in the zero-width limit, provided we define this limit as ^{*)}

$$\sigma \rightarrow 0, \quad f(z) \rightarrow 1. \quad (2.17)$$

The approach to the zero-width limit is not "smooth". This follows from the fact that the amplitude (2.14) is not meromorphic when (2.15) is satisfied. The meromorphic Veneziano amplitude is then only obtained at the cost of giving up condition (2.15). Obviously Eq. (2.14) is then as usual only valid in the domain where $\alpha(s)$ and $\alpha(t)$ are negative. If condition (2.15) is satisfied Eq. (2.14) is valid everywhere.

The use of the modified amplitude (2.14) has the advantage that we can compare the results obtained by use of (2.14) with those obtained by use of the Veneziano amplitude. Deviations in these results can then be said to be due to the unitarity-like features (second sheet resonance poles and Mandelstam analyticity) of the amplitude (2.14) that are not shared by the Veneziano amplitude.

In the model (2.14) background can be simulated by absence of the linear term $\alpha_0(t)$ in Eq. (2.12) and by requiring that $\sigma(t)$ always remains negative.

^{*)} The limit $f \rightarrow 1$ can be considered as a weak coupling limit [i.e., $f = 1 + O(g^2)$ for $g \rightarrow 0$]. The zero-width approximation is then to be viewed as a weak coupling limit.

There is no difficulty in generalizing Eq. (2.14) to the N particle amplitude

$$M_N = \int d\mu \prod_u u^{-\alpha_0(s_u) - \sigma(s_u(1-u)) - 1} f(s_u(1-u)), \quad (2.18)$$

where the notation is similar to Eq. (2.8).

3. THE FEYNMAN SCALING LIMIT (GENERAL RESULTS)

We now consider the inclusive reaction

$$a + b \rightarrow c + X, \quad (3.1)$$

where X stands for "anything". The kinematical variables appropriate for the reaction (3.1) are

$$s = (p_a + p_b)^2, \quad (\text{total energy})$$

$$M^2 = (p_a + p_b - p_x)^2, \quad (\text{missing mass})$$

$$t = (p_b - p_c)^2,$$

$$u = (p_a - p_c)^2, \quad (3.2)$$

with the relation

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + M^2. \quad (3.3)$$

As has been much discussed in the literature ²⁾⁻⁷⁾ one can consider various limits. In the present section we shall be interested only in general results and we, therefore, consider the limit

$$s \rightarrow \infty, \quad M^2 \rightarrow \infty. \quad (3.4)$$

The basic approach to the calculation of the single particle distribution has been given by Mueller ⁴⁾, and we have illustrated in Fig. 2 that the cross-section (with appropriate phase space factors) can be computed in terms of the discontinuity of the six-point function with respect to M^2 . Since the six-point function contains many variables besides M^2 there arises the question as to which prescription one should use to obtain the correct discontinuity. We have not studied this question in a general field theoretic framework. Instead we have used the Veneziano model as a guide. The motivation for this procedure is the work of Di Giacomo et al. ¹²⁾ who consider the zero-width approximation as the limiting case of narrow-width resonances that are unitary in the Breit-Wigner sense. Since the model discussed in Section 2.2 is especially appropriate for the incorporation of narrow-width resonances ^{*}) we feel that it is reasonable to accept the results of Di Giacomo et al. ¹²⁾ as a guide for finding the correct prescription for the discontinuity. As has been discussed in particular by Gordon and Veneziano ⁵⁾ the prescription for evaluating the discontinuity is then

$$d\sigma = \frac{d^3 p_c}{p_{oc}} \frac{1}{\pi^2 s} \frac{\text{disc}_{M^2}}{2i\pi} M_6(s+i\epsilon, t, M^2, s-i\epsilon). \quad (3.5)$$

Here M_6 is the six-point function, and the variables are defined in Fig. 2. Although we cannot prove it, we feel that the prescription defined in Fig. 2 should also be valid when the assumptions used by Di Giacomo et al. ¹²⁾ and by Gordon and Veneziano ⁵⁾ are no longer valid, since the correct discontinuity presumably is independent of whether resonances are narrow or not. In the following we shall adopt Eq. (3.5) as our definition of $d\sigma$, but we emphasize again that this has only been shown to be correct if the resonances are sufficiently narrow, although it is likely that Eq. (3.5) is valid more generally.

^{*}) but not necessarily zero-width resonances.

There are several six-point functions that can a priori contribute to $d\sigma$. To illustrate the general technique we consider explicitly the contribution shown in Fig. 3. This contribution is given by

$$\begin{aligned}
 M_6 = & f_p(0)^2 \int_0^1 \int_0^1 \int_0^1 dx dy dz \frac{1}{(1-xyz)^2} x^{-\alpha(s(1-x))-1} z^{-\alpha(\bar{s}(1-z))-1} \\
 & y^{-\alpha(M^2(1-y))-1} \left(\frac{1-x}{1-yx} \right)^{-\alpha_{bc}(tx \frac{1-y}{1-xy})-1} \left[\frac{(1-y)(1-xyz)}{(1-xy)(1-zx)} \right]^{-\alpha_p(0)-1} \\
 & \left(\frac{1-z}{1-zy} \right)^{-\alpha_{bc}(tz \frac{1-y}{1-zy})-1} (1-xyz)^{-\alpha_p(0)-1} \\
 & \left(\frac{1-xy}{1-xyz} \right)^{-\alpha_b(m^2 xy \frac{1-z}{1-xyz})-2} \left(\frac{1-zx}{1-xyz} \right)^{-\alpha_b(m^2 zy \frac{1-x}{1-xyz})-2} \\
 & f_{ab}(s(1-x)) f_{ab}(\bar{s}(1-z)) f_{ab\bar{c}}(M^2(1-y)) f_{bc}(tx \frac{1-y}{1-xy}) \\
 & f_{bc}(tz \frac{1-y}{1-zy}) f_b(m^2 xy \frac{1-z}{1-xyz}) f_b(m^2 zy \frac{1-x}{1-xyz})
 \end{aligned}
 \tag{3.6}$$

for the CHKZ model. For the model discussed in Section 2.2 a very similar expression is valid, where the trajectories in (3.6) are replaced according to Eq. (2.18). For example,

$$\alpha_{bc}(tz \frac{1-y}{1-zy}) \rightarrow \alpha_0^{bc}(t) + \sigma_{bc}(tz \frac{1-y}{1-zy}),
 \tag{3.7}$$

etc. Since the details are so similar we shall not write down the expression for the model discussed in Section 2.2 explicitly. In Eq. (3.6) we have introduced the quantum numbers of the channels in the notation for α and f . Thus $f_p(0)$ is the residue which couples to the Pomeron α_p , f_b is the residue with the quantum

numbers of particle b , etc. The representation (3.6) is valid also for $\text{Re } M^2$ and $\text{Re } s$ positive, and we shall study the limits $\text{Re } s \rightarrow \infty$ and $\text{Re } M^2 \rightarrow \infty$. We, therefore, introduce the variables

$$a = s(1-x), \quad b = \bar{s}(1-z), \quad c = M^2(1-y). \quad (3.8)$$

Using that the functions $f(a)$, $f(b)$, $f(c)$ are strongly damped for large values of a , b , c we can extend the range of integrations to infinity. We therefore obtain after a few rearrangements

$$\begin{aligned} M_6 &\simeq (s\bar{s})^{\frac{1}{2}\alpha(0)} \int_P(0)^2 \int_0^\infty da db dc \\ &\quad S^{\alpha_{bc}(t \frac{c}{c+ap}) - \frac{1}{2}\alpha_P(0)} \bar{S}^{\alpha_{bc}(t \frac{c}{c+b\bar{p}}) - \frac{1}{2}\alpha_P(0)} \\ &\quad (M^2)^{-\alpha_{bc}(t \frac{c}{c+ap}) - \alpha_{bc}(t \frac{c}{c+b\bar{p}}) + \alpha_P(0)} a^{-\alpha_{bc}(t \frac{c}{c+ap}) - 1} \\ &\quad b^{-\alpha(t \frac{c}{c+b\bar{p}}) - 1} c^{-\alpha_P(0) - 1} (c+ap)^{\alpha_{bc}(t \frac{c}{c+ap}) + \alpha_P(0) - \alpha_b(M^2 \frac{b\bar{p}}{ap+b\bar{p}+c})} \\ &\quad (c+b\bar{p})^{\alpha_{bc}(t \frac{c}{c+b\bar{p}}) + \alpha_P(0) - \alpha_b(M^2 \frac{ap}{ap+b\bar{p}+c})} \\ &\quad (c+ap+b\bar{p})^{\alpha_b(M^2 \frac{b\bar{p}}{ap+b\bar{p}+c}) + \alpha_b(M^2 \frac{ap}{ap+b\bar{p}+c}) - 2\alpha_P(0)} \\ &\quad f_{ab}(a) \bar{f}_{ab}(b) f_{ab\bar{c}}(c) f_{bc}(t \frac{c}{c+b\bar{p}}) f_{bc}(t \frac{c}{c+ap}) \\ &\quad f_b(M^2 \frac{b\bar{p}}{b\bar{p}+ap+c}) f_{\bar{b}}(M^2 \frac{ap}{ap+b\bar{p}+c}), \end{aligned} \quad (3.9)$$

where

$$\rho = M^2/s, \quad \bar{\rho} = M^2/\bar{s}. \quad (3.10)$$

Note that $\bar{\rho}$ is not the complex conjugate of ρ . In Eq. (3.9) $\bar{f}_{ab}(b)$ enters because the path of integration for b is slightly below the real axis [see Eq. (3.8)], so by Schwartz's reflection principle $\bar{f}_{ab}(b)$ enters when $\epsilon \rightarrow 0$ (in $s - i\epsilon$). Introducing the new variables z, u and u' ,

$$a = \frac{zu}{\rho}, \quad b = \frac{zu'}{\bar{\rho}}, \quad c = z(1-u-u'), \quad (3.11)$$

we can express Eq. (3.9) in the slightly simpler form

$$\begin{aligned} M_6 \simeq & f_p(0)^2 (s\bar{s})^{\frac{1}{2}\alpha_p(0)} (\rho\bar{\rho})^{\frac{1}{2}\alpha_p(0)} \int_0^1 du \int_0^{1-u} du' \\ & u^{-\alpha_{bc}(t \frac{1-u-u'}{1-u})-1} (u')^{-\alpha_{bc}(t \frac{1-u-u'}{1-u'})-1} (1-u-u')^{-\alpha_p(0)-1} \\ & (1-u')^{\alpha_{bc}(t \frac{1-u-u'}{1-u'}) + \alpha_p(0) - \alpha_b(m^2 u')} \\ & (1-u)^{\alpha_{bc}(t \frac{1-u-u'}{1-u}) + \alpha_p(0) - \alpha_b(m^2 u)} \\ & f_{bc}(t \frac{1-u-u'}{1-u'}) f_{bc}(t \frac{1-u-u'}{1-u}) f_b(m^2 u) f_b(m^2 u') \\ & \int_0^\infty dz z^{-\alpha_p(0)-1} f_{ab}(\frac{zu}{\rho}) \bar{f}_{ab}(\frac{zu'}{\bar{\rho}}) f_{abz}(z(1-u-u')). \end{aligned} \quad (3.12)$$

In a similar way one can investigate the contributions from all other diagrams. The net result is that when one calculates the discontinuity according to Eq. (3.5) (or Fig. 2) one finds that the differential cross-section, in the fragmentation region ^{*}, is given by

$$d\sigma \simeq \frac{d^3 p_c}{p_{0c}} s^{\alpha_p(0)-1} G(p, t) [1 + c s^{\alpha_M(0) - \alpha_p(0)}], \quad (3.13)$$

^{*}) In the central region only the u channel contributes, and the corrections are of order $(\sqrt{s})^{\alpha_M(0) - \alpha_p(0)}$ (Section 4).

where C is a constant,

$$C = t_M(0)^2 / t_P(0)^2 \quad (3.14)$$

and $\alpha_M(0)$ is the intercept for the secondary trajectories $[\alpha_M(0) \approx \frac{1}{2}]$. The result (3.13) is also valid in the model discussed in Section 2.2.

Equation (3.13) shows that the models discussed in Section 2 are in accordance with Feynman's scaling hypothesis. We can express our variables in terms of Feynman's variables ³⁾,

$$|P_T^c|, \quad x = 2 \frac{p_L^c}{\sqrt{s}} \quad (3.15)$$

where p_T^c and p_L^c are the centre-of-mass transverse and parallel momenta of the detected particle, respectively. For $x > 0$ and x not wee (i.e., $x > m_c/\sqrt{s}$) we have

$$t = \frac{1-x}{x} (m_b^2 x - m_c^2) - \frac{(P_T^c)^2}{x},$$

$$p = \frac{M^2}{s} = 1 - \sqrt{x^2 + 4 \frac{(P_T^c)^2 + m_c^2}{s}} + O\left(\frac{1}{s}\right). \quad (3.16)$$

Equation (3.13) can therefore also be written

$$d\sigma \approx \frac{d^3 p_c}{p_{0c}} s^{\alpha_P(0)-1} F(x, p_T^c) \{1 + C s^{\alpha_M(0)-\alpha_P(0)}\}, \quad (3.17)$$

in accordance with Feynman ³⁾, Benecke, Chou, Yang and Yen ²⁾. We emphasize that Eqs. (3.13) and (3.17) are general results. The modified model (see Section 2.2) yields very similar results. Equations (3.13) and (3.17) are still valid. However, in this model C is very small if the channel $(ab\bar{c})$ is exotic, since in that case the zero-width approximation (the Veneziano model) gives a vanishing $d\sigma$.

4. THE LIMITED FRAGMENTATION AND THE CENTRAL REGION

In this Section we shall consider in more detail the limited fragmentation region and the central region. We first discuss these limits in terms of kinematics.

4.1 Kinematics

The general kinematical variables are given by Eq. (3.2). The limits are defined in the following.

4.1.A The fragmentation region of b

Here $s \rightarrow \infty$ and $p = M^2/s$ is fixed. Therefore, with t fixed, $u \rightarrow -\infty$. In particular, we shall consider the phase space boundary of the fragmentation region, $p \rightarrow 0$.

4.1.B The central region

Here $s \rightarrow \infty$, $t \rightarrow -\infty$, $u \rightarrow -\infty$ with

$$\frac{tu}{s} = (p_T^c)^2 + m_c^2. \quad (4.1)$$

4.2 The CHKZ model

We first consider the CHKZ model, discussed in Ref. 9) and in Section 2.1.

4.2.A The fragmentation region of b

To get an idea of what happens in the fragmentation region we study the limit where the mass M^2 of the undetected cluster of secondaries is small relative to s (the phase space boundary of the fragmentation region), i.e., $p \rightarrow 0$. In this limit Eq. (3.9) simplifies considerably. For example,

$$\alpha_{bc}(t \frac{c}{c+a_p}) \rightarrow \alpha_{bc}(t) \quad \text{for } p \rightarrow 0, \quad (4.2)$$

etc. As far as the contribution from Eq. (3.9) is concerned the discontinuity (3.5) becomes [with $\alpha_p(0) = 1$]

$$d\sigma^{(3.9)} \simeq \frac{d^3 p_c}{p_{0c}} \frac{f_p(0)^2}{\pi^2} f_b(0)^2 \left(\frac{s}{M^2}\right)^{2\alpha_{bc}(t) - \alpha_p(0)} f_{bc}(t)^2 \\ \left| \int_0^\infty da a^{-\alpha(t)-1} f_{ab}(a) \right|^2 \int_0^\infty dc c^{2\alpha_{bc}(t) - \alpha_p(0) - 1} \text{Im} f_{ab\bar{c}}(c). \quad (4.3)$$

From the sum rule (2.4) we find that Eq. (4.3) can be written

$$d\sigma^{(3.9)} \simeq \frac{d^3 p_c}{p_{0c}} \frac{f_p(0)^2}{\pi^2} f_b(0)^2 \left(\frac{s}{M^2}\right)^{2\alpha_{bc}(t) - \alpha_p(0)} f_{bc}(t)^2 \\ \left[\int_0^\infty da a^{-\alpha(t)-1} f_{ab}(-a) \right]^2 \int_0^\infty dc c^{2\alpha_{bc}(t) - \alpha_p(0) - 1} \text{Im} f_{ab\bar{c}}(c). \quad (4.4)$$

In addition to the contribution given by Eqs. (4.3) or (4.4) we find that there are three other contributions which do not vanish exponentially for $s \rightarrow \infty$. The total contribution is

$$d\sigma \simeq \frac{d^3 p_c}{p_{0c}} \frac{f_p(0)^2}{\pi^2} f_b(0)^2 \left(\frac{s}{M^2}\right)^{2\alpha_{bc}(t) - \alpha_p(0)} f_{bc}(t)^2 \\ \left| \int_0^\infty da a^{-\alpha_{bc}(t)-1} [f_{ab}(a) + f_{ac}(-a)] \right|^2 \\ \int_0^\infty dc c^{2\alpha_{bc}(t) - \alpha_p(0) - 1} \text{Im} f_{ab\bar{c}}(c). \quad (4.5)$$

If $f_{ac}(-a) = f_{ab}(-a)$ ($a > 0$) we can use the sum rule (2.4) to rewrite Eq. (4.5) in the form

$$d\omega \approx \frac{d^3 p_c}{p_{0c}} \frac{1}{\pi^2} f_P(0)^2 \left(\frac{s}{M^2}\right)^{2\alpha_{bc}(t) - \alpha_P(0)} f_{bc}(t)^2$$

$$4 \cos^2 \frac{\pi\alpha(t)}{2} \left[\int_0^\infty da a^{-\alpha(t)-1} f_{ab}(-a) \right]^2$$

$$\int_0^\infty dc c^{2\alpha_{bc}(t) - \alpha_P(0) - 1} \text{Im } f_{abc}(c). \quad (4.6)$$

The result (4.6) looks somewhat similar to the corresponding result for the Veneziano amplitude ⁵⁾⁻⁷⁾.

4.2.B The central region

In the central region we find that only one contribution, which is shown in Fig. 4, is of importance ^{*}). This contribution can be obtained from Eq. (3.12) by replacing s and $\bar{s}$ by u . One then finds

$$M_6^{\text{Fig. 4}} \approx f_P(0)^2 (M^2)^{\alpha_P(0)} \int_0^1 dv \int_0^{1-v} dv' v^{-\alpha_{bc}(t \frac{1-v-v'}{1-v})-1}$$

$$(v')^{-\alpha_{bc}(t \frac{1-v-v'}{1-v'})-1} (1-v-v')^{-\alpha_P(0)-1}$$

$$(1-v')^{\alpha_{bc}(t \frac{1-v-v'}{1-v'}) + \alpha_P(0) - \alpha_b(m^2 v')}$$

$$(1-v)^{\alpha_{bc}(t \frac{1-v-v'}{1-v}) - \alpha_b(m^2 v) + \alpha_P(0)}$$

$$f_{bc}(t \frac{1-v-v'}{1-v'}) f_{bc}(t \frac{1-v-v'}{1-v}) f_b(m^2 v) f_b(m^2 v')$$

$$\int_0^\infty dz z^{-\alpha_P(0)-1} f_{ac}(-xzv) f_{ac}(-xzv') f_{abc}(z(1-v-v')), \quad (4.7)$$

where we have assumed that x is small (but not "wee", $x > m_c/\sqrt{s}$), so that $\rho \approx 1$ and

$$\frac{u}{M^2} \approx -x \quad (4.8)$$

^{*}) In this region the cut in s is therefore unimportant. Thus, in order that Regge corrections are absent it is enough to require that the $(ab\bar{c})$ channel is exotic ^{10), 11)}.

In this region $|t|$ is very large. The functions f_{bc} in Eq. (4.7) are, therefore, very small $[f(z)$ decreases exponentially when $|z| \rightarrow \infty]$ unless $1 - v - v'$ is very small. Since we have to compute the discontinuity in M^2 we have to take the discontinuity of $f_{abc}(z(1 - v - v'))$ which starts at some threshold mass, M_T say. Therefore,

$$z \geq M_T / (1 - v - v'), \quad (4.9)$$

and since $1 - v - v'$ is small, z is very large. The main contribution to the z integral therefore comes when z is not too far from the end-point, i.e. $z \simeq M_T / (1 - v - v')$. The integrand therefore behaves essentially like

$$f_{bc}(t \frac{1-v-v'}{1-v'}) f_{bc}(t \frac{1-v-v'}{1-v}) f_{ac}(-x \frac{v M_T}{1-v-v'}) f_{ac}(-x \frac{v' M_T}{1-v-v'}) \quad (4.10)$$

Introducing a new variable

$$\xi = x / (1 - v - v')$$

we see that the factor (4.10) becomes

$$f_{bc}(tx/\xi(1-v')) f_{bc}(tx/\xi(1-v)) f_{ac}(-\xi v M_T) f_{ac}(-\xi v' M_T). \quad (4.11)$$

However, for x very small [see Eq. (3.16)]

$$tx \simeq -[(p_T^c)^2 + m_c^2], \quad (4.12)$$

so the factor (4.10) essentially corresponds to a decrease with p_T^c which is faster than any inverse power of p_T^c for $p_T^c \rightarrow \infty$.

To substantiate the above discussion we have discussed the example where

$$f(-z) = z^\delta e^{-cz^q}, \quad 0 < q < 1, \quad z \text{ large}. \quad (4.13)$$

For the details see Appendix A. The result is that

$$\frac{1}{s} \frac{d \text{disc}_{M_0^2}}{2i\pi} M_6 \underset{x \text{ small}}{\simeq} \text{const.} [(p_T^c)^2 + m_c^2]^{2\delta - \frac{1}{2}q} K_{-\frac{1}{q}}(2c \{M_0^2(p_T^{c2} + m_c^2)\}^{\frac{q}{2}}) \quad (4.14)$$

where $K_{-1/q}$ is a Bessel function. M_0 is given by the maximum with respect to z of $\text{Im} f_{abc}(z)$. Thus M_0 is certainly larger than the threshold mass M_T for the $(ab\bar{c})$ channel. For $p_T \rightarrow \infty$ we find from Eq. (4.14) ^{*)}

$$d\sigma \simeq \frac{d^3 p_c}{p_{0c}} \text{const.} p_T^{4\delta - \frac{3}{2}q} e^{-2c M_0^q p_T^q} \quad (4.15)$$

Since q cannot be equal to 1 the transverse momentum cut-off given by Eq. (4.15) is too slow to be in agreement with the data. It is therefore tempting to argue that the CHKZ model does not provide a good description of the data in the pionization region where x is small.

4.3 The modified CHKZ model

We shall now consider the modified CHKZ model introduced in Section 2.2. It is of importance that this model has the Veneziano model as a limiting case, since we can then compare our results with those obtained from the Veneziano model ⁵⁾⁻⁸⁾ in order to see what are the effects of unitarity.

4.3.A The fragmentation region of b

The calculations are very similar to the CHKZ model. We only mention the result corresponding to Eq. (4.6)

^{*)} The correction terms are of order $O(1/\sqrt{|u|})$. Since $u \sim O(\sqrt{s})$ for x small, the correction terms are of order $O(1/\sqrt[4]{s})$.

$$d\sigma \simeq \frac{d^3 p_c}{p_{0c}} \frac{f_p(0)^2}{\pi^2} f_b(0)^2 \left(\frac{s}{M^2}\right)^{2\alpha_{bc}(t) - \alpha_p(0)} \\ f_{bc}(t) 4 \cos^2 \frac{\pi\alpha(t)}{2} \left[\int_0^\infty da a^{-\alpha_{bc}(t)-1} e^{-\alpha' a} f_{ab}(-a) \right]^2 \\ \int_0^\infty dc c^{2\alpha_{bc}(t) - \alpha_p(0) - 1} e^{\alpha' c} \operatorname{Im} f_{ab\bar{c}}(c), \quad (4.16)$$

where $\alpha_{bc}(t)$ is the total trajectory. The functions $f(z)$ all decrease faster than $e^{-\alpha'|z|}$ for $|z| \rightarrow \infty$ as assumed in (2.15). For small t Eq. (4.16) is practically identical to the result obtained from the Veneziano model. For larger values of t Eq. (4.16) in general gives a more rapid decrease than the Veneziano model. However, the net effect is not very significant unless one has very precise experimental data. It therefore appears as if unitarity does not have much quantitative effect in the limit of low missing mass, $M^2/s \rightarrow 0$. The same is presumably true in the fragmentation region where M^2/s is fixed.

4.3.B The central region

Again only the diagram shown in Fig. 4 contributes, while all other diagrams vanish rapidly as a function of x for x small. The contribution from Fig. 4 is ^{*)}

$$\frac{\text{disc}_{M^2}}{2i\pi} M_6^{\text{Fig. 4}} \simeq f_p(0)^2 (M^2)^{\alpha_p(0)} \int_0^1 dv \int_0^{1-v} dv' \\ v^{-\alpha_{bc}^0(t) - \alpha_{bc}(t \frac{1-v-v'}{1-v}) - 1} (v')^{-\alpha_{bc}^0(t) - \alpha_{bc}(t \frac{1-v-v'}{1-v'}) - 1}$$

(continued)

*) With correction terms of order $O(1/\sqrt[4]{s})$, since in the central region $\sqrt{|u|} \sim O(\sqrt[4]{s})$, since

$$\begin{aligned}
 & (1-v-v')^{-\alpha_p(0)-1} (1-v')^{\alpha_{bc}^0(t) + \delta_{bc}(t \frac{1-v-v'}{1-v'}) + \alpha_p(0) - \alpha_b(0) - \delta_b(m^2 v')} \\
 & (1-v)^{\alpha_{bc}^0(t) + \delta_{bc}(t \frac{1-v-v'}{1-v}) - \alpha_b(0) - \delta_b(m^2 v) + \alpha_p(0)} \\
 & f_{bc}(t \frac{1-v-v'}{1-v'}) f_{bc}(t \frac{1-v-v'}{1-v}) f_b(m^2 v) f_b(m^2 v') \\
 & \int_0^\infty dz z^{-\alpha_p(0)-1} e^{-xz(v+v')} f_{ac}(-xz v) f_{ac}(-xz v') \\
 & e^{\alpha' z(1-v-v')} \text{Im} f_{abc}(z(1-v-v')). \quad (4.17)
 \end{aligned}$$

Here all functions $f(z)$ go to zero faster than $e^{-\alpha'|z|}$ for $|z| \rightarrow \infty$. The presence of the factors $\exp(-xz(v+v'))$ as well as the linear parts $\alpha_{bc}^0(t) = \alpha' t + \alpha_{bc}(0)$ changes the behaviour for large transverse momenta rather drastically. Using again the example

$$f(-z) = z^\delta e^{-cz^q}, \quad q > 1, \quad (4.18)$$

we have found in Appendix B that for $|p_T| \rightarrow \infty$ (up to a polynomial in p_T) ^{*}

$$\begin{aligned}
 d\sigma & \underset{x \text{ small}}{\sim} \frac{d^3 p_c}{p_{0c}} \text{const.} \exp \left\{ -\alpha' M_0 \left(4\eta + \frac{1}{\eta} \right) p_T \right. \\
 & \left. - c M_0^q (4\eta^q + 2 \cdot 2^{-q} \eta^{-q}) p_T^q \right\}, \\
 \eta & = 2^{-(q+1)/2q}, \quad (4.19)
 \end{aligned}$$

where it should be remembered that $q > 1$. M_0 is the (first) maximum of $e^{\alpha' z} \text{Im} f_{abc}(z)$, so M_0 is larger than the threshold mass of the channel $(ab\bar{c})$.

We have the following comments to the result (4.19):

^{*}) One cannot perform the weak coupling limit directly in Eq. (4.19), since the considerations in Appendix B are no longer valid when $g \rightarrow 0$. In Eq. (4.17) one can let $g^2 \rightarrow 0$ provided $M^2 < 0$, and then continue to $M^2 > 0$ [see Ref. 6)].

- a) In the Veneziano model $d\sigma$ decreases like $6), 7)$
 $\exp(-\text{const. } p_T^2)$, (which is too rapid to represent the data). It is interesting that the inclusion of the non-zero width implies a less rapid decrease with p_T . We therefore conjecture that unitarity has the effect that it modifies the transverse momentum cut-off given by the Veneziano model such that the decrease is less rapid.
- b) It is tempting to compare Eq. (4.19) with Hagedorn's model ¹³⁾. Disregarding for the moment the second term in the exponent in Eq. (4.19) one would have agreement with Hagedorn's distribution if M_0 is universal. Since M_0 is the (first) maximum of $\text{Im } f_{abc}(z)$ above threshold, our model does not contain intrinsically the requirement that M_0 is universal. On the other hand, one can certainly construct $\text{Im } f_{abc}(z)$ such that M_0 does indeed become universal. Thus we have found the interesting result that a model, which takes into account the finite width of the resonances is able to incorporate Hagedorn's transverse momentum distribution. The second term in the exponent (4.19) becomes important when $p_T \gg 1/M_0 (2\alpha'/c)^{1/(q-1)}$. Since M_0 is of the order of a few GeV the second term presumably first becomes of importance for p_T of the order a few GeV.

5. CONCLUDING REMARKS

Our main motivation for the research reported in the present paper was to see if unitarity had any important effects in inclusive reactions. The result is that the model which approaches the Veneziano model in the limit of zero-width resonances modifies the transverse momentum spectrum in a way which is at least qualitatively in good agreement with experiments.

We mention that it is also possible to study inclusive reactions in the Regge cut + pole model discussed in Section 2. The technique is a straightforward generalization of the work in Section 3. The result confirms Mueller's observation that the presence of Regge cuts introduces logarithmic factors. Furthermore, we find that if the Pomeron is a Regge cut, scaling is violated by a logarithmic factor.

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A P P E N D I X A

Here we shall investigate the asymptotic behaviour of Eq. (4.7) when $f(z)$ is given by Eq. (4.13). We first consider the z integral

$$I_2 = \int_{M_T^2/(1-v-v')}^{\infty} dz \ z^{-\alpha(0)-1} e^{-c z^q (v^q + v'^q) x^q} \operatorname{Im} f(z(1-v-v')) z^{2\delta} x^{2\delta} v^{\delta} v'^{\delta}. \quad (\text{A.1})$$

Here and in the following we drop unessential subscripts. M_T^2 is the threshold for the M_T^2 cut. The function $\operatorname{Im} f(z(1-v-v'))$ is strongly damped for large values of the argument. Since $1-v-v'$ is small (as explained in Section 4) the lower limit in (A.1) is quite large. We therefore replace $\operatorname{Im} f(z(1-v-v'))$ by its value at the first maximum. We obtain, when M_0 is the value of z where $\operatorname{Im} f(z)$ is maximal

$$I_2 = \frac{\operatorname{Im} f(M_0)}{v^q + v'^q} \frac{1}{q} x^{2\delta} v^{\delta} v'^{\delta} \left(\frac{M_0^2}{1-v-v'} \right)^{-\alpha(0)+2\delta-q} \exp \left\{ -c \left(\frac{M_0^2 x}{1-v-v'} \right)^q (v^q + v'^q) \right\} [1 + O(1-v-v')]. \quad (\text{A.2})$$

This asymptotic form is valid if $x/(1-v-v')$ is large. The integrand now contains the factor

$$\exp \left[-c (|t|(1-v-v'))^q \left(\frac{1}{(1-v)^q} + \frac{1}{(1-v')^q} \right) - c \left(\frac{M_0^2 x}{1-v-v'} \right)^q (v^q + v'^q) \right]. \quad (\text{A.3})$$

Since $1-v-v'$ is small both terms in the exponent contribute. The maximal contribution comes when the exponent is as small as possible. Since $1-v-v' \sim 0$ we can write $v' \simeq 1-v$ whenever no singularities are introduced. One then finds that the maximal contribution comes when

$$1 - v - v' = \sqrt{\frac{M_0^2 x}{|t|}} \left(\frac{v^q + (1-v)^q}{\frac{1}{(1-v)^q} + \frac{1}{v^q}} \right)^{\frac{1}{2q}} \quad (\text{A.4})$$

The value of the exponent (A.3) then is

$$\exp \left[-2(M_0^2 x |t|)^{\frac{q}{2}} \sqrt{2 + \frac{v^q}{(1-v)^q} + \frac{(1-v)^q}{v^q}} \right] \quad (\text{A.5})$$

This is as large as possible when the square root is as small as possible which implies

$$v = v' = \frac{1}{2} \quad (\text{A.6})$$

up to terms of order x . Thus, the exponent becomes

$$\exp \left[-4c (M_0^2 x |t|)^{q/2} \right]. \quad (\text{A.7})$$

The trajectory dependent factors in Eq. (4.7) then approach 1 for $x \rightarrow 0$. Introducing the new variable

$$2c |t|^p 2^p (1-v-v')^p \quad (\text{A.8})$$

one finds the result (4.14). The asymptotic result (4.15) can be obtained directly by use of the maximum exponent (A.7) and standard methods for obtaining the asymptotic values of an integral (Laplace's method) containing a large parameter (p_T).

A P P E N D I X B

In this Appendix we shall estimate the behaviour of the integral (4.17) with the function (4.18) for $p_T \rightarrow \infty$. Again we treat the z integral as in Appendix B to obtain

$$\begin{aligned} I_z &= \int_0^{\infty} dz z^{-\alpha(0)-1} e^{-\alpha' z x(v+v')} \Gamma(-xzv) \Gamma(-xzv') \\ &\quad \text{Im} \Gamma(z(1-v-v')) e^{\alpha' z(1-v-v')} \\ &\approx e^{\alpha' M_0} \text{Im} \Gamma(M_0) \frac{1}{\alpha'} \frac{v \delta_{v'} \delta x^{2\delta-1}}{v+v'} \left(\frac{M_0^2}{1-v-v'} \right)^{2\delta-\alpha(0)-1} \\ &\quad \exp \left\{ -\alpha' \frac{M_0^2 x(v+v')}{1-v-v'} - c \left(\frac{M_0^2 x}{1-v-v'} \right)^q (v^q + v'^q) \right\} \\ &\quad \left[1 + O \left(x \left(\frac{x}{1-v-v'} \right)^{q-1} \right) \right]. \end{aligned}$$

(B.1)

The integrand now contains the following exponent

$$\begin{aligned} \exp \left[-c |t|^q (1-v-v')^q \left(\frac{1}{1-v} + \frac{1}{1-v'} \right) - \alpha' \frac{M_0^2 x}{1-v-v'} (v+v') \right. \\ \left. - c \left(\frac{M_0^2 x}{1-v-v'} \right)^q (v^q + v'^q) \right]. \end{aligned}$$

(B.2)

The trajectory dependent factors can be written

$$\begin{aligned} \exp \left[-(\alpha_0(t)+1) \left(\ln \frac{1+2u}{1-2u} + \ln \frac{1-2u'}{1+2u'} \right) \right. \\ \left. - \delta(2t(u'-u)) \left(\ln \frac{1+2u}{1-2u} + \ln \frac{1+2u'}{1-2u'} \right) \right], \end{aligned}$$

(B.3)

with

$$v = \frac{1}{2} + u, \quad v' = \frac{1}{2} - u'$$

(B.4)

The σ part can be seen to be unimportant relative to the linear part $\alpha_0(t)$. Since $1 - v - v' = u' - u$, and since $1 - v - v'$ must be small, i.e., $u' \simeq u$, it follows that the maximum contribution from the exponent in Eq. (B.3) comes when $u \simeq u' \simeq 0$. Then the exponent (B.3) behaves like

$$e^{4\alpha' t (u' - u)} \quad (B.5)$$

The exponent (B.2) has maximum for

$$u' - u = \sqrt{\frac{M_0^2 x}{|t|}} \eta \quad (B.6)$$

where η satisfies

$$16\eta^2 + \frac{2}{A} \eta^{q+1} - \frac{2^{-q}}{A} \eta^{1-q} - 1 = 0, \quad (B.7)$$

$$A = \frac{\alpha'}{8c q} (M_0^2 x |t|)^{\frac{1-q}{2}}. \quad (B.8)$$

Since $q > 1$, $A \rightarrow 0$ for $|p_T| \rightarrow \infty$. Thus

$$\eta = 2^{-\frac{q+1}{2q}} (|p_T| \rightarrow \infty). \quad (B.9)$$

Using Eq. (B.6) the combined exponents (B.2) and (B.5) become

$$\exp \left\{ -\alpha' M_0 \left(4\eta + \frac{1}{\eta} \right) p_T - c M_0^q (4\eta^q + 2 \cdot 2^{-q} \eta^{-q}) p_T^q \right\}. \quad (B.10)$$

Using again Laplace's method one sees that the integral (4.17) exists for $x \rightarrow 0$ and the remaining p_T dependence is a polynomial in p_T .

R E F E R E N C E S

- 1) D. Amati, A. Stanghellini and S. Fubini, Nuovo Cimento 26, 6 (1962);
K. Wilson, Acta Phys.Austr. 17, 31 (1963);
D. Silverman and C.I. Tan, Phys.Rev. D3, 991 (1971);
N.F. Bali, A. Pignotti and D. Steele, University of Washington preprint RLO-1388-585 (1970);
C.E. De Tar, Phys.Rev. D3, 128 (1971).
- 2) J. Benecke, T.T. Chou, C.N. Yang and E. Yen, Phys.Rev. 188, 2159 (1969).
- 3) R.P. Feynman, Phys.Rev.Letters 23, 1415 (1969).
- 4) A.H. Mueller, Phys.Rev. D2, 2963 (1970).
- 5) D. Gordon and G. Veneziano, MIT preprint (1971).
- 6) M.A. Virasoro, Berkeley preprint (1971).
- 7) C.E. De Tar, K. Kang, C.I. Tan and J.H. Weis, MIT preprint (1971).
- 8) R.A. Brower and R.E. Waltz, CERN preprint (in preparation) (1971).
- 9) G. Cohen-Tannoudji, F. Henyey, G.L. Kane and W.J. Zakrzewski, Phys.Rev.Letters 26, 112 (1971).
- 10) Chan Hong-Mo, C.S. Hsue, C. Quigg and J.M. Wang, Phys.Rev.Letters 26, 672 (1971).
- 11) J. Ellis, J. Finkelstein, P.H. Frampton and M. Jacob, CERN preprint TH.1316 (1971).
- 12) A. Di Giacomo, S. Fubini, L. Sertorio and G. Veneziano, Phys. Letters 33B, 171 (1970).
- 13) R. Hagedorn, Nuovo Cimento 56A, 1027 (1968).

FIGURE CAPTIONS

- Figure 1 : Contour of integration.
- Figure 2 : Illustration of Mueller's optical theorem.
- Figure 3 : Dual six-point amplitude investigated in Section 3.
- Figure 4 : Dual six-point amplitude which is the only contribution in the central region.

FIG.1

$$d\sigma = \frac{d^3 p_c}{p_{oc}} \frac{1}{\pi^2 s}$$

FIG.2

FIG. 3

FIG. 4