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THE DUAL PARTON PICTURE

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A B S T R A C T

We make more quantitative the recent speculations that the N point Veneziano amplitude can be viewed as the limit of very high order unrenormalized Feynman diagrams. A general statistical formalism for investigating the behaviour of very high order Feynman diagrams is developed. Within the framework of the electric circuit analogy the concept of the mean resistance of a propagator is introduced. It is shown that the combined effect of the fluctuations around the mean resistances is smaller than one would expect. It then follows that a family of n dimensional integrals over Feynman parameters can be evaluated in the limit $n \rightarrow \infty$. This implies that the Feynman amplitude with soft ("wee") virtual particles (partons) can be replaced by a simple Gaussian expression (with no loop integrations) to a good approximation in the limit of very high order, provided the parton Lagrangian contains a super-renormalizable ϕ^3 part. Having obtained this result Nielsen's variational principle follows if the discrete Feynman net is replaced by a continuous distribution of propagators. The Regge slope is expressed in terms of the parton mass.

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1. - INTRODUCTION

In the present paper we propose a general statistical formalism for very high order Feynman diagrams. We shall show that this approach allows one to evaluate some integrals that are rather similar to ordinary Feynman amplitudes. The key technical ingredient in the new formalism is a statistical definition of the "most important" region of integration in the space of Feynman parameters.

The investigation of the statistical properties of very high order Feynman diagrams reported in this paper is motivated by an attempt to make more quantitative the recent speculations ^{1),2)} that the N point Veneziano amplitude can be viewed as the limit of very high order unrenormalized Feynman diagrams. The advantages of this dynamical picture (to be referred to as "the dual parton picture" in the following) of dual resonance models have already been much discussed in the literature ^{1),2),3)}. Here we only mention that duality is easily understood ^{1),2)} and the large degeneracy of the levels in the N point Veneziano amplitude is due to the multiparticle intermediate states ³⁾. From the point of view of dynamical structure the dual parton picture is a bootstrap in the sense that all observable hadrons are composed of an infinite sea of partons (i.e., the particles that propagate in the higher order Feynman net). In addition to the features mentioned above the dual parton picture also has some potential possibilities, e.g., with respect to hints as to how spin should be introduced in dual resonance models.

The main physical assumption concerning the nature of the partons is that the external partons (i.e., those partons out of which the external hadrons are composed) are soft ("wee" in the terminology of Feynman ⁴⁾). The physical reason for this assumption is that the exchange of hard partons in a hadron-hadron collision is in general much more difficult than the exchange of soft partons. If the hadron energies are large compared to the parton mass (which gives the scale of energy) the exchange of hard partons is very much suppressed by the fact that the hard partons are likely to retain their movements inside the hadrons practically unaffected by the scattering process. This is

not the case for the soft partons which can easily be exchanged. Self-consistency of the approach requires the parton mass to be of order a few hundred MeV, so the difficulty in exchanging hard partons sets in already at hadron energies of a few GeV. Since we want a picture which can be applied uniformly for all hadron energies, it is natural as a first approximation to assume that the partons are soft.

To see the consequences of this assumption, let us consider the regular Feynman net shown in Figs. 1a and 1b. The net is composed of cells which have the basic structure shown in Fig. 1b, corresponding to ϕ^3 and ϕ^4 vertices. If the total number of propagators is n , then the number of partons in an external hadron is of order $\sqrt{n}$ (as one can see from the geometry of the net - each "side" contains $\sqrt{n}$ propagators). Consider an external hadron H with momentum p_H . The assumption that the external partons contained in H are soft then requires that the momentum carried by an individual parton is of order $p_H/\sqrt{n}$.

This gives rise to a difficulty since we can also obtain observable hadrons by factorizing the net. Evidently the number of partons contained in the excited hadron is again of order $\sqrt{n}$ from the geometry of the diagram, but the momenta of the virtual partons are no longer necessarily of order $p_H/\sqrt{n}$, since the virtual momenta are integration variables which can take any value from $-\infty$ to $+\infty$. To avoid this difficulty we have to parametrize the Feynman integral in such a way that the virtual momenta are bounded and cannot exceed $\sim p_H/\sqrt{n}$. This requires that we do the integrations over the (arbitrary) "circulating loop momenta". This can be done in the standard way by introducing Feynman parameters, and the virtual momenta are then restricted in magnitude by Kirchhoff's laws. Therefore, in order to work with the assumption that the partons are soft we have to introduce Feynman parameters. Having done this all observable hadrons ("external" states as well as "excited" states obtained by factorization) have the same structure, and the dual parton picture becomes a bootstrap structure.

It is amusing to notice that the above picture is rather similar to a potential theory model with indefinitely rising Regge trajectories discussed by Mandelstam ⁵⁾. In Mandelstam's heuristic arguments the trajectory $\alpha(s)$ increases indefinitely when one considers a dynamical model where only n particle states are allowed and let $n \rightarrow \infty$ (provided the potential is strong enough). The particles in the n body state are then the elementary particles, i.e., the partons. The observable hadrons are bound states of the partons. In order that the trajectory increases indefinitely the limit $n \rightarrow \infty$ must be performed in such a way that the energy E is restricted by $E \ll nm$ (m is the parton mass) since the trajectory $\alpha(s)$ turns over at an energy $\sim nm$. This is similar to the assumption that the partons are soft in the dual parton picture, since this requires that $p_H/\sqrt{n} \ll m$, i.e., the limit $n \rightarrow \infty$ must be performed such that $p_H \ll \sqrt{nm}$.

We present our material in the following order. In Section 2 we discuss the case of a pure ϕ^4 model for the parton interaction (i.e., "fish-net" diagrams). It is shown that this type of parton Lagrangian is not able to produce bound states, and the Regge slope vanishes, $\alpha'(0) = 0$.

In Section 3 we discuss the much more interesting case where the parton Lagrangian is given by $\gamma_3 \phi^3 + \gamma_4 \phi^4$ (the most general renormalizable Lagrangian for scalar partons). An approximation where the Kirchhoff momenta are replaced by average values is introduced. Having made this approximation, no further approximations are needed.

In Section 4, the new statistical formalism for treating Feynman diagrams is proposed. The main ingredient in this approach is the formal definition of the most important region of integration in the Feynman amplitude (in the circuit formalism). In x space this region corresponds to an infinitely long segment of the light-cone with finite width into the timelike region.

In Section 5 we investigate the statistical fluctuations around the important region of integration. It is shown that the combined effect of the fluctuations is smaller than one would expect. By means of the results obtained for the fluctuations an asymptotic ($n \rightarrow \infty$) result is found for the approximate Feynman integral introduced in Section 3.

In Section 6 we introduce the continuous approximation in which the complicated structure of the Feynman net is replaced by a continuous distribution of propagators. The main new result is the following formula for the slope of the Regge trajectory,

$$\alpha'(0) = \frac{g}{\pi m^2} \frac{N_3}{3N_3 + 4N_4}, \quad (1.1)$$

where g is a simple factor depending on the geometry of the diagram (e.g., $g = \sqrt{3}$ for "bee-cells", $g = 1$ for a "fish-net"), m is the parton mass and N_x is the number of vertices of the ϕ^x type ($x = 3$ and/or 4). For $N_3 = 0$ the slope vanishes, in accordance with the result in Section 2. Only if the net contains ϕ^3 vertices do we get a non-trivial result. This feature is entirely due to the super-renormalizability of a ϕ^3 model.

2. - LARGE n BEHAVIOUR OF THE FEYNMAN AMPLITUDE IN A ϕ^4 MODEL

We start our investigation of the large n behaviour of the Feynman amplitude by considering a ϕ^4 model for the parton Lagrangian. This turns out to be the simplest case.

The Feynman amplitude is given by the well-known expression ^{6),7)}

$$F_n(p_i, p_k, m^2) = \int_0^1 \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} \frac{\delta(1 - \sum_{i=1}^n \alpha_i)}{\left[\sum_{j=1}^n \alpha_j (k_j(\alpha)^2 - m^2) \right]^{n-2k}}, \quad (2.1)$$

where we have disregarded a trivial constant. The $k_j(\alpha)$ are the Kirchhoff momenta ^{6),7)} which satisfy Kirchhoff's equations, i.e., energy-momentum conservation at each vertex and

$$\sum_{j=1}^n k_j \alpha_j \eta_{jr} = 0 \quad \text{for each loop, } r=1, \dots, k, \quad (2.2)$$

where η_{jr} is a sign factor defined by Bjorken and Drell ⁶⁾. The α 's are the standard Feynman parameters, n is the number of propagators and k is the number of loop momenta to be integrated. The quantity $\Delta_n(\alpha)$ is a determinant ^{6),7)}. By means of Kirchhoff's equations we can express $k_j(\alpha)$ in terms of the external parton momenta p_i and the α 's. Since Kirchhoff's equations are linear it follows from the softness of the external parton momenta, $p_i = O(p_H/\sqrt{n})$, that ^{*})

$$k_i(\alpha) = O(p_H/\sqrt{n}). \quad (2.3)$$

^{*}) Here and in the following p_H^2 stands for the maximum centre-of-mass energy in all possible hadron channels.

The circuit analogy ^{6),7)} also tells us that $k_i(\alpha)$ cannot be larger than something of order p_i . From Eq. (2.3) it follows that

$$\sum_{j=1}^n \alpha_j k_j(\alpha)^2 = O(p_H^2/n) \sum_{j=1}^n \alpha_j = O(p_H^2/n). \quad (2.4)$$

With $p_H^2 \ll nm$ it follows that the total heat generation (in the circuit analogy) (2.4) is of order $1/n$. Therefore, in Eq. (2.1), the term in the denominator proportional to m^2 is much larger than the heat generation. Therefore

$$\begin{aligned} \left[\sum_{j=1}^n \alpha_j (k_j^2 - m^2) \right]^{2k-n} &= (-m^2)^{2k-n} \exp \left\{ (2k-n) \ln \left[1 - \sum_{j=1}^n \alpha_j k_j^2 / m^2 \right] \right\} \\ &= (-m^2)^{2k-n} \exp \left[(n-2k) \sum_{j=1}^n \alpha_j k_j^2 / m^2 + O\left(\frac{n-2k}{n^2}\right) \right]. \end{aligned} \quad (2.5)$$

To investigate the behaviour of the exponent we shall calculate $n-2k$. Let us denote the number of external parton lines by e , and the number of vertices by N . In a ϕ^x model we then have

$$e + 2n = x N. \quad (2.6)$$

The number of loop momenta to be integrated is

$$k = n - N + 1. \quad (2.7)$$

Eliminating N we have

$$\frac{k}{n} = \frac{x-2}{x} - \frac{e-x}{xn} = \frac{x-2}{x} + O\left(\frac{e}{n}\right). \quad (2.8)$$

In a ϕ^4 model we thus have with $e \sim \sqrt{n}$ and $n \rightarrow \infty$,

$$n - 2k = \frac{e}{2} = O(\sqrt{n}) \quad (2.9)$$

Equation (2.5) therefore becomes in the limit $n \rightarrow \infty$

$$\begin{aligned} \left[\sum_{j=1}^m \alpha_j (k_j^2 - m^2) \right]^{2k \cdot n} &\rightarrow (-m^2)^{2k \cdot n} \exp O(p_H^2 / \sqrt{n}) \\ &\rightarrow (-m^2)^{2k \cdot n}. \end{aligned} \quad (2.10)$$

Hence, from Eqs. (2.1) and (2.10) we obtain

$$F_n(p_i, p_k, m^2) \rightarrow F_n(0, m^2), \quad n \rightarrow \infty. \quad (2.11)$$

It therefore follows that when the partons are soft the Feynman amplitude is given by the Feynman amplitude with zero external momenta in the limit $n \rightarrow \infty$.

Introducing the continuous approximation where the discrete net of Feynman propagators is replaced by a continuous distribution of propagators ^{1),2)} one can easily see ^{*)} that Eq. (2.11) corresponds to a Veneziano amplitude with vanishing Regge slope,

$$\alpha'_{\phi^4}(0) = 0. \quad (2.12)$$

The conclusion of this Section is therefore that a ϕ^4 model for the (soft) parton interaction is not strong enough to produce bound states lying on an indefinitely rising Regge trajectory. To produce an interesting model we have to include a ϕ^3 interaction in the Feynman net.

*) See Section 6 for further details.

3. - LARGE n BEHAVIOUR OF THE FEYNMAN AMPLITUDE IN A ϕ^3 MODEL

The reason that a ϕ^4 model cannot produce an interesting dual model is that the quantity $n - 2k$ is of order $\sqrt{n}$. In a ϕ^3 model the disappointing argument (2.10) is not valid, since there one has from Eq. (2.8) that for $n \rightarrow \infty$

$$n - 2k = n/3. \quad (3.1)$$

The exponent in Eq. (2.5), therefore, does not vanish when $n \rightarrow \infty$, that approaches a well defined limit.

To study this limit we shall consider the Feynman amplitude in a region without dynamical singularities. It is then well known that in such a region the expression (2.1) for F_n can be scaled and re-expressed in the exponentiated form ⁷⁾

$$F_n(p_i p_k, m^2) = \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} e^{\sum_{i=1}^n \alpha_i (k_i(\alpha)^2 - m^2)} \quad (3.2)$$

where we have left out various constants. The exponent in Eq. (3.2) is negative, and we shall restrict ourselves to a kinematical domain where

$$\sum_{i=1}^n \alpha_i k_i(\alpha)^2 \leq 0. \quad (3.3)$$

As before the partons are assumed to be soft, and hence the Kirchhoff momenta are of order $p_H/\sqrt{n}$ as in Eq. (2.3). It therefore follows that the exponent in Eq. (3.2)

$$\exp \sum_{i=1}^n \alpha_i (k_i(\alpha)^2 - m^2)$$

is dominated by the mass term, whereas $k_i(\alpha)^2$ is only a small perturbation. It is therefore reasonable to expect that for soft partons the important region of integration is the same when the external partons carry zero momenta [i.e., when $k_i(\alpha) = 0$ for all i] as when they carry soft momenta. Evidently this is an approximation, but the approximation becomes better when $n \rightarrow \infty$. The problem of finding the important region of integration is therefore reduced to the problem of finding the important region of integration in the amplitude

$$F_n(0, m^2) = \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} e^{-m^2 \sum_{i=1}^n \alpha_i}. \quad (3.4)$$

In the next Section we introduce a formalism which measures the important region of integration in Eq. (3.4) in a statistical sense in the limit $n \rightarrow \infty$. It is therefore a reasonable approximation (for $n \rightarrow \infty$) to replace the variable $k_i(\alpha)^2$ by $k_i(\bar{\alpha})^2$, where $\bar{\alpha}$ indicates (in a manner which is defined precisely in the next Section) the most important region of integration.

With this motivation we therefore introduce the approximate Feynman amplitude,

$$\tilde{F}_n(p_i p_k, m^2) = \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} e^{\sum_{i=1}^n \alpha_i (k_i(\bar{\alpha})^2 - m^2)}. \quad (3.5)$$

It should be noticed that the approximation (3.5) keeps some of the important properties of Eq. (3.2). To give some examples let us mention that :

- (a) $\tilde{F}_n$ depends on the details of the dynamics through the complicated determinant $\Delta_n(\alpha)$;
- (b) the approximate total heat $\sum \alpha_i k_i(\bar{\alpha})^2$ vanishes for $\alpha_i = 0$ (all i) ;

- (c) for large values of the Feynman parameters that part of the integrand which depends on the external momenta vanishes exponentially [in the kinematical region (3.3)] ;
- (d) the approximation (3.5) satisfies the boundary condition $\tilde{F}_n(p_i p_k) \rightarrow F_n(0)$ for $p_i \rightarrow 0$;
- (e) the exponent in Eq. (3.5) satisfies Kirchhoff's laws in the most important region of integration where $\alpha_i \approx \bar{\alpha}_i$.

Of course, the approximate amplitude (3.5) also differs from the exact amplitude (3.2). The difference becomes more important as one moves more outside of the important region of integration. It is not difficult to show that $\tilde{F}_n$ is an upper limit for F_n (this is shown in Section 6). Since the two integrands in Eqs. (3.2) and (3.5) agree in the most important region of integration, one can, however, expect that Eq. (3.5) is a reasonable first approximation to Eq. (3.2).

4. - MEAN RESISTANCES AND OTHER STATISTICAL CONCEPTS

Our purpose is to investigate very high order Feynman diagrams. Intuitively one would expect that such diagrams have some statistical properties, and we shall therefore borrow some concepts from statistics. In the previous Section we argued that for soft partons the problem consists in finding the most important region of integration in the integral (3.4). We shall therefore take the point of view that the statistical concepts should be defined in such a way that they refer to the integral (3.4).

First, let us normalize the Feynman amplitude [the exact amplitude (3.2) or the approximate amplitude (3.5)],

$$F_n(p_i p_k, m^2) = \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2 F_n(0, m^2)} e^{H(\alpha)} e^{-m^2 \sum_{i=1}^n \alpha_i} \quad (4.1)$$

where $H(\alpha)$ is the (exact or approximate) total heat generation. We restrict ourselves to renormalizable theories (i.e., with $n - 2k \geq 0$), since otherwise the integration over the circulating loop momenta cannot be performed without subtractions. Therefore, $F_n(0, m^2)$ is at most logarithmically divergent, but for regular nets without self-energy and tadpole insertions (see Figs. 1a and 1b) F_n is in fact convergent. We shall therefore not bother too much about cut-offs. However, in logarithmically divergent theories our approach is still valid as far as the most singular terms are concerned.

We can now think of F_n as a kind of expectation value of the exponentiated heat generation $\exp H(\alpha)$ with respect to the normalized distribution function

$$d_n(\alpha) = e^{-m^2 \sum_{i=1}^n \alpha_i} / F_n(0, m^2) \Delta_n(\alpha)^2. \quad (4.2)$$

It is then natural to introduce the mean resistance (in the electric circuit analogy ^{6),7)}) of the j th propagator by the definition

$$\overline{\alpha}_j^{(n)} = \int_0^\infty \prod_{i=1}^n d\alpha_i d_n(\alpha) \alpha_j. \quad (4.3)$$

We assume that the limit $n \rightarrow \infty$ has a meaning, and we denote the limiting value $\overline{\alpha}_j^{(\infty)}$ by $\overline{\alpha}_j$. We therefore have

$$\overline{\alpha}_j = \overline{\alpha}_j^{(n)} [1 + O(n^{-\epsilon})], \quad (4.4)$$

for some $\epsilon > 0$, and n sufficiently large. To give the limit $n \rightarrow \infty$ a meaning we can imagine that the finite net is embodied in the infinite net where the propagators $n+1, n+2, \dots$ are dummy.

Similarly we can introduce higher moments $\overline{\alpha_i \alpha_j^{(n)}} \rightarrow \overline{\alpha_i \alpha_j}$, etc. We specifically assume the existence of these moments for $n \rightarrow \infty$,

$$\overline{\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_N}^{(n)}} \rightarrow \overline{\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_N}} < \infty, \quad (4.5)$$

for $n \rightarrow \infty$ with N fixed ($N < \infty$).

It is of importance to realize that the "statistical variables" α_i in the distribution function (4.2) are not independent variables in the statistical sense. This is because these variables are correlated by the determinant $\Delta_n(\alpha)$, which depends in a very complicated manner on the α 's. The presence of correlations is easily shown for simple (low order) diagrams by calculating $\overline{\alpha_i \alpha_j}$ with $i \neq j$. One then sees that $\overline{\alpha_i \alpha_j} \neq \overline{\alpha_i} \overline{\alpha_j}$ ($i \neq j$), whereas a theory without correlations would give $\overline{\alpha_i \alpha_j} = \overline{\alpha_i} \overline{\alpha_j}$ ($i \neq j$). The reason why we have to do so much work in order to obtain an asymptotic expansion is mainly due to the statistical correlations in the distribution function $d_n(\alpha)$.

We also mention that if the α_j integration diverges in Eq. (4.1) (as is the case, e.g., for self-energy insertions), then it follows from Eq. (4.3) that $\overline{\alpha}_j = 0$. Similarly all higher moments vanish, $\overline{\alpha_i \alpha_j} = 0$, etc., when the cut-off goes to infinity. This only reflects the fact that the most important region of integration is $\alpha_j \approx 0$ because of the divergence. If the α_j integration is convergent, the most important region of integration is $\alpha_j \lesssim 1/m^2$ because of the exponential factor in the distribution (4.2). We therefore expect that in this case $\overline{\alpha}_j \lesssim 1/m^2$.

There is no reason to expect that the mean resistances and the higher moments diverge, since the integral (4.3) is strongly damped at large values of the integration variables. The only exception to this is if $m \rightarrow 0$. In this case infra-red divergencies show up and $\overline{\alpha}_i \rightarrow \infty$. We therefore assume that the parton mass does not vanish.

The mean values $\overline{\alpha}_i$ have a simple intuitive meaning. In α space $\overline{\alpha}_i$ measures the important region of integration. In x space this corresponds to an infinitely long segment of the light cone of finite invariant width $\overline{\alpha} \sim 1/m^2$ into the timelike region. Thus $\overline{\alpha}_i$ measures roughly how far one has to go inside the light cone for processes involving soft partons. In particular, in a finite model (e.g., ϕ^3) the light cone singularities do not in general dominate the dynamics.

From a statistical point of view it is natural to introduce the average of the mean resistances $\overline{\alpha}$ as a statistical measure of the important region of integration. For $\overline{\alpha}$ we can show a simple sum rule. From Eq. (4.3), we have

$$\begin{aligned}
 \overline{\alpha}^{(n)} &= \frac{1}{n} \sum_{i=1}^n \overline{\alpha}_i = \frac{1}{n} \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2 F_n(0, m^2)} \sum_{i=1}^n \alpha_i e^{-m^2 \sum_{i=1}^n \alpha_i} \\
 &= \frac{1}{n} \int_0^\infty dx e^{-m^2 x} \times \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{F_n(0, m^2) \Delta_n(\alpha)^2} \delta\left(\sum_{i=1}^n \alpha_i - x\right) \\
 &= \frac{1}{n} \int_0^\infty dx e^{-m^2 x} x^{n-2k} \int_0^1 \prod_{i=1}^n d\alpha_i \delta\left(\sum_{i=1}^n \alpha_i - 1\right) / F_n(0, m^2) \Delta_n(\alpha)^2
 \end{aligned}
 \tag{4.6}$$

where we have introduced the variable $x = \sum_{i=1}^n \alpha_i$ and performed a uniform scaling by use of the important equation (6), 7)

$$\Delta_n(\lambda \alpha) = \lambda^k \Delta_n(\alpha). \quad (4.7)$$

By the same method we have

$$F_n(0, m^2) = \int_0^\infty dx e^{-m^2 x} x^{n-2k-1} \int_0^1 \frac{\prod_{i=1}^n d\alpha_i \delta(\sum_{i=1}^n \alpha_i - 1)}{F_n(0, m^2) \Delta_n(\alpha)^2} \quad (4.8)$$

and hence

$$\overline{\alpha}^{(n)} = \frac{1}{n} \sum_{i=1}^n \overline{\alpha}_i = \frac{1}{m^2} (1 - \frac{2k}{n}). \quad (4.9)$$

Equation (4.9) is valid for renormalizable theories as far as the most singular terms are concerned ^{*)}. For a finite model like ϕ^3 , Eq. (4.9) is exact.

^{*)} Equation (4.9) can also be derived by means of a differentiation technique. We have

$$\overline{\alpha}^{(n)} = - \frac{1}{n} \left[\frac{\partial}{\partial m^2} F_n(0, m^2) \right] / F_n(0, m^2).$$

By a uniform scaling one can easily see that (Λ is the cut-off)

$$F_n(0, m^2) = (m^2)^{2k \cdot n} A_n \ln^B \frac{\Lambda^2}{m^2} + \text{less singular terms.}$$

where A_n is independent of m . Performing the differentiation, one obtains Eq. (4.9).

It is of interest to verify directly that Eq. (4.9) represents the most important region of integration in α space. From well known properties of the Γ function it follows that

$$(m^2)^{n-2k} \Gamma(n-2k) = \int_0^\infty dx e^{-m^2 x} x^{n-2k-1} \quad (4.10)$$

receives its most important contribution from

$$x = \sum_{i=1}^n \alpha_i \approx (n-2k) \frac{1}{m^2} , \quad (4.11)$$

provided $n-2k \rightarrow \infty$. Thus Eq. (4.9) represents the important region of integration.

By use of Eq. (2.8) we can write Eq. (4.9) in the limit $n \rightarrow \infty$ as

$$\bar{\alpha}^{(n)} = \frac{1}{m^2} \left(\frac{4-x}{x} + \frac{2(e-x)}{xn} \right) \rightarrow \frac{1}{m^2} \frac{4-x}{x} , \quad (4.12)$$

for a ϕ^x model. This result is easily generalized to a mixture of ϕ^3 and ϕ^4 models,

$$\bar{\alpha} = \frac{1}{m^2} \frac{N_3}{3N_3 + 4N_4} (\alpha_{\text{parton}} = \gamma_3 \phi^3 + \gamma_4 \phi^4), \quad (4.13)$$

where N_x is the number of vertices of the ϕ^x type.

The quantity $\overline{\alpha}^{(n)}$ has thus been shown to approach a constant in the limit $n \rightarrow \infty$, and $\overline{\alpha}^{(n)}$ depends only on the ratio $k/n = \text{constant}$ for large n . Thus, in the case of $\overline{\alpha}^{(n)}$ we have actually shown that it makes sense to perform the limit $n \rightarrow \infty$.

For a pure ϕ^4 model $\overline{\alpha} = 0$ (for $n \rightarrow \infty$). Since $\overline{\alpha}_k^{(n)} \geq 0$ for all k it follows that $\overline{\alpha}_k = 0$ (for $n \rightarrow \infty$). This is reasonable in view of the fact that a pure ϕ^4 model is rather singular.

For a ϕ^3 model a non-trivial result $\overline{\alpha} = 1/3m^2$ is obtained. This corresponds to the fact that because of the exponential damping $\exp(-m^2 \alpha_i)$ in $d_n(\alpha)$ and because of the non-singular character of a ϕ^3 model the most important region of integration for the variable α_i is from 0 to $\sim 1/m^2$. The reason why $\overline{\alpha} \neq 0$ is entirely the super-renormalizability of a ϕ^3 model, since this ensures that $n - 2k$ increases linearly with n . In the next Section we shall see that a similar pattern is valid for the higher order fluctuations.

5. - PROPERTIES OF HIGHER ORDER FLUCTUATIONS

The sum rule derived in the previous Section shows that the average of the mean resistances have a very simple and neat dependence on the dynamics. By use of the uniform scaling (4.7) of $\Delta_n(\alpha)$ we have circumvented a detailed investigation of the dynamics. In the present Section we shall utilize the same technique and derive sum rules for the higher order fluctuations. In particular we shall show that the combined effect of the fluctuations is smaller than one would expect. This enables us to evaluate an integral of the same type as the approximate Feynman amplitude (3.5).

We start by investigating the second order fluctuation,

$$\sum_{k=1}^n \overline{\alpha_i \alpha_k}^{(n)} = \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{F_n(0, m^2) \Delta_n(\alpha)^2} \sum_{k=1}^n \alpha_i \alpha_k e^{-m^2 \sum_{i=1}^n \alpha_i} \quad (5.1)$$

where we sum over k whereas i is a free index. By use of the same transformations as in Eq. (4.6) we obtain ^{*})

$$\begin{aligned} \sum_{k=1}^n \overline{\alpha_i \alpha_k}^{(n)} &= \int_0^\infty dx e^{-m^2 x} x^{n-2k+1} \int_0^1 \frac{\prod_{i=1}^n d\alpha_i \delta(\sum_{i=1}^n \alpha_i - 1)}{F_n(0, m^2) \Delta_n(\alpha)^2} \alpha_i \sum_{k=1}^n \alpha_k \\ &= \frac{1}{m^2} (n-2k+1) \overline{\alpha_i}^{(n)}, \end{aligned} \quad (5.2)$$

where Eq. (4.8) has been used. Using the sum rule (4.9) we can rewrite Eq. (5.2) as

$$\sum_{k=1}^n (\overline{\alpha_i \alpha_k}^{(n)} - \overline{\alpha_i}^{(n)} \overline{\alpha_k}^{(n)}) = \frac{\overline{\alpha_i}^{(n)}}{m^2}. \quad (5.3)$$

^{*}) Equation (5.2) can also be derived by differentiation and uniform scaling, using

$$\sum_{k=1}^n \overline{\alpha_i \alpha_k}^{(n)} = \frac{1}{F_n(0, m^2)} \frac{\partial}{\partial m^2} \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} \alpha_i \exp(-m^2 \sum_{i=1}^n \alpha_i).$$

The sum rule (5.3) is rather restrictive since it shows that the n fluctuations on the left-hand side add up to give a constant. This certainly implies that only a few of the fluctuations can be relatively large (of order $1/m^4$) and of the same sign, whereas the majority of the fluctuations are either very small or are large but cancel out in the sum. We shall soon see that the latter possibility is not realized.

Physically it is not difficult to understand this result. The diagonal fluctuation $\overline{\alpha_i^2} - \alpha_i^2$ is generally of order $1/m^4$. The fluctuations $\overline{\alpha_i \alpha_k} - \alpha_i \alpha_k$ where the propagator i is close to the propagator k are presumably also of order $1/m^4$. However, if the propagator k is far removed from the propagator i it is natural to think that these two propagators cannot "feel" each other, and hence $\overline{\alpha_i \alpha_k} - \alpha_i \alpha_k$ is very small, corresponding to almost no correlation. The number of diagonal and almost diagonal fluctuations is much smaller than n which explains the result (5.3).

Summing Eq. (5.3) over i and using Eq. (4.8) we obtain

$$\sum_{i,k=1}^n (\overline{\alpha_i \alpha_k}^{(n)} - \alpha_i^{(n)} \alpha_k^{(n)}) = \frac{1}{m^4} (n-2k). \quad (5.4)$$

This result can of course also be obtained directly by use of the same technique as in Eq. (5.2). Again, although there are n^2 fluctuations on the left-hand side of Eq. (5.4) the value of the sum is only of order n .

In general we are interested in more complicated sums than (5.4). In Appendix A we have shown the formulas

$$\begin{aligned} \sum_{j=1}^n d_{ij} \overline{\alpha_i \alpha_j}^{(n)} &= \sum_{j=1}^n d_{ij} \alpha_i^{(n)} \alpha_j^{(n)} \\ &+ O(\ln n \text{ Max } |d_{ij}|), \end{aligned} \quad (5.5)$$

$$\sum_{i,j=1}^n d_{ij} \overline{\alpha_i \alpha_j^{(n)}} = \sum_{i,j=1}^n d_{ij} \bar{\alpha}_i^{(n)} \bar{\alpha}_j^{(n)} + O(n \ln n \text{Max} |d_{ij}|). \quad (5.6)$$

Note that Eq. (5.6) follows from Eq. (5.5). The derivation of Eq. (5.5) is much more complicated than the derivation of the sum rule (5.3).

Equations (5.5) and (5.6) have the same general feature as the sum rules (5.3) and (5.4), namely that the correction terms are less than what one would expect. In Eq. (5.6) the number of terms is n^2 in the two sums, and in general the correction term would be of order $n^2 \text{Max} |d_{ij}|$. However, due to the smallness of some of the fluctuations the correction term is only of order $n \log \text{Max} |d_{ij}|$. If the d_{ij} are smooth, Eqs. (5.5) and (5.6) are asymptotic equations. This is true in particular if the d_{ij} have the same sign.

From Eqs. (A.3) and (A.4) in Appendix A we have the following generalization of Eq. (5.4)

$$\sum_{l=1}^i \sum_{m=1}^j (\overline{\alpha_l \alpha_m^{(n)}} - \bar{\alpha}_l^{(n)} \bar{\alpha}_m^{(n)}) = \frac{ij}{n} \frac{\bar{\alpha}}{m^2}, \quad (5.7)$$

where i and j are both large, $i \gg \sqrt{n}, j \gg \sqrt{n}$. Equation (5.7) shows that it is not possible for all the fluctuations in Eq. (5.4) to be large (of order $1/m^4$) and cancelling. If this was the case, Eq. (5.7) would be contradicted since i and j can be varied so as to exclude some of the large terms, thereby avoiding the cancellations. Equation (5.7) clearly indicates that many of the fluctuations are small, as is expected physically for correlations between propagators that are far away from each other.

By means of Eq. (5.6) one can show the following asymptotic theorem for a class of integrals.

Let $G(x)$ be a function and let

$$0 \leq |G''(x)| \leq M, \quad x = \sum_{i=1}^n \alpha_i \phi_i, \quad 0 \leq \alpha_i \leq \infty, \quad (5.8)$$

$$\phi_i = O(1/n),$$

then *)

$$F(\rho) \equiv \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} G\left(\sum_{i=1}^n \alpha_i \phi_i\right) e^{-m^2 \sum_{i=1}^n \alpha_i} \quad (5.9)$$

$$\rightarrow G\left(\sum_{i=1}^n \bar{\alpha}_i \phi_i\right) F(0), \quad n \rightarrow \infty.$$

To see this, let us expand G in a Taylor series with remainder ($0 < \theta < 1$),

$$G\left(\sum \alpha_i \phi_i\right) = G\left(\sum \bar{\alpha}_i \phi_i\right) + \sum (\alpha_i - \bar{\alpha}_i) \phi_i G'(\sum \bar{\alpha}_i \phi_i) \\ + \frac{1}{2} \left[\sum_{i=1}^n (\alpha_i - \bar{\alpha}_i) \phi_i \right]^2 G''\left(\sum \bar{\alpha}_i \phi_i + \theta \sum (\alpha_i - \bar{\alpha}_i) \phi_i\right). \quad (5.10)$$

Inserting the expansion (5.10) in the definition of $F(\rho)$ the second term on the right-hand side of Eq. (5.10) drops out because of the definition of $\bar{\alpha}_i$. Thus

$$F(\rho) / F(0) = G\left(\sum \bar{\alpha}_i \phi_i\right) + R \quad (5.11)$$

$$R = \frac{1}{2} \int_0^\infty \prod_{i=1}^n d\alpha_i d_n(\alpha) \left[\sum_{i=1}^n (\alpha_i - \bar{\alpha}_i) \phi_i \right]^2 G''\left(\sum \bar{\alpha}_i \phi_i + \theta \sum (\alpha_i - \bar{\alpha}_i) \phi_i\right). \quad (5.12)$$

*) The correction terms are of order

$$O(Mn \ln n (\max \phi_i)^2).$$

Using the uniform upper bound on $|G''|$ in (5.8) we have

$$\begin{aligned} |R| &\leq \frac{M}{2} \sum_{i,j=1}^n (\overline{\alpha_i \alpha_j} - \bar{\alpha}_i \bar{\alpha}_j) \phi_i \phi_j \\ &= \frac{M}{2} O\left(\frac{\ln n}{n}\right) \rightarrow 0, \quad n \rightarrow \infty, \end{aligned} \quad (5.13)$$

where we have used Eq. (5.6) as well as $\phi_i = O(1/n)$. Equations (5.11) and (5.13) verify the theorem (5.9).

Equation (5.9) shows that the distribution function $d_n(\alpha)$ has a kind of "functional invariance" since by integrating $G(\sum \alpha_i \phi_i)$ over the distribution $d_n(\alpha)$ we obtain the function G evaluated at the mean value of its argument.

It is clear that the function $G(x) = e^x$, $x = \sum \alpha_i \phi_i$, satisfies the conditions (5.8) for $\sum \alpha_i \phi_i < 0$. Hence we have the asymptotic result

$$\begin{aligned} F_n(\phi) &= \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} e^{\sum_{i=1}^n \alpha_i \phi_i} e^{-n^2 \sum_{i=1}^n \alpha_i} \\ &\rightarrow \exp\left(\sum_{i=1}^n \bar{\alpha}_i \phi_i\right) F_n(0), \quad n \rightarrow \infty \end{aligned} \quad (5.14)$$

provided $\sum \alpha_i \phi_i < 0$ in the whole range of integration *).

The asymptotic result (5.14) confirms that the mean values $\bar{\alpha}_i$ measure the most important region of integration so that $\exp \sum \alpha_i \phi_i$ receives its main contribution from $\alpha_i = \bar{\alpha}_i$ in such a way that $\exp \sum \alpha_i \phi_i$ is effectively replaced by $\exp \sum \bar{\alpha}_i \phi_i$. This result does not remain valid if the ϕ_i are numerically larger than $O(1/n)$.

*) The correction term in Eq. (5.14) is of order $O(n \log n (\text{Max } \phi_i)^2)$.

For the higher fluctuations one can show rather similar results (see Appendix B). For the quantity

$$\Sigma^{(N)} = \overline{\left[\sum_{i=1}^n (\alpha_i - \bar{\alpha}_i) \right]^N} \quad (5.15)$$

one can show the recursion relation (see Appendix B)

$$\Sigma^{(N+1)} = \frac{N}{m^2} \Sigma^{(N)} + \frac{N(N-2k)}{m^4} \Sigma^{(N-1)}. \quad (5.16)$$

This is a direct generalization of the sum rule (5.4). Equation (5.16) shows that $\Sigma^{(N)}$ is a polynomial in $n-2k$ of degree $N/2$ (N even) or $\frac{1}{2}N - \frac{1}{2}$ (N odd). In the Table we have given the first few total fluctuations calculated by means of Eq. (5.16). Various other results can be obtained for the higher moments. Some of these results are given in Appendix B. The asymptotic theorem (5.9) can also be generalized [by means of Eq. (B.7)]. Without giving the details we mention the result. Let

$$\phi_{i_1 \dots i_N} = O(1/n^N), \phi_{i_1 \dots i_N} \leq O, (\text{all } N) \quad (5.17)$$

then

$$\begin{aligned} F(\phi) &\equiv \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{\Delta_n(\alpha)^2} \exp \left[\sum_{N=0}^{N_0} \sum_{i,j=1}^n \alpha_{i_1} \dots \alpha_{i_N} \phi_{i_1 \dots i_N} \right] \\ &\quad \times \exp \left(-m^2 \sum_{i=1}^n \alpha_i \right) \\ &\rightarrow \exp \left[\sum_{N=0}^{N_0} \sum_{i,j=1}^n \frac{\alpha_{i_1} \dots \alpha_{i_N} \phi_{i_1 \dots i_N}}{m^2} \right] F(0) \\ &\rightarrow \exp \left[\sum_{N=0}^{N_0} \sum_{i,j=1}^n \bar{\alpha}_{i_1} \dots \bar{\alpha}_{i_N} \phi_{i_1 \dots i_N} \right] F(0), n \rightarrow \infty, N_0 \ll n. \end{aligned} \quad (5.18)$$

Equation (5.18) is shown by means of a straightforward generalization of the arguments (5.10) - (5.14) by use of Eq. (B.7).

6. - THE CONTINUOUS APPROXIMATION

We now return to the approximate expression (3.5) for the Feynman amplitude,

$$\tilde{F}_n(p_i p_k, m^2) = \int_0^\infty \prod_{i=1}^n d\alpha_i e^{\sum_{i=1}^n \alpha_i k_i(\bar{\alpha})^2} d_n(\alpha) F_n(0, m^2). \quad (6.1)$$

In view of the results obtained in Section 4 we now identify $\bar{\alpha}$ with the average of the mean resistances, since in a statistical sense this represents the most important region of integration for $n \rightarrow \infty$. Using the asymptotic theorem (5.14) and remembering that $k_1 = O(p_H/\sqrt{n})$ we obtain immediately

$$\begin{aligned} \tilde{F}_n(p_i p_k, m^2) &\rightarrow \exp \left[\sum_{i=1}^n \bar{\alpha}_i k_i(\bar{\alpha})^2 \right] F_n(0, m^2) \times \\ &\times \left[1 + O\left(\frac{P_H^4}{m^4 n} \ln n\right) \right] \end{aligned} \quad (6.2)$$

for $n \rightarrow \infty$. In Appendix C we have shown that the sum rule (4.9) implies that Eq. (6.2) can be rewritten as

$$\begin{aligned} \tilde{F}_n(p_i p_k, m^2) &\rightarrow \exp \left[\bar{\alpha} \sum_{i=1}^n k_i(\bar{\alpha})^2 \right] F_n(0, m^2) \times \\ &\times \left[1 + O\left(\frac{P_H^2}{\sqrt{n} m^2}\right) \right]. \end{aligned} \quad (6.3)$$

We emphasize that Eq. (6.3) is an exact consequence of the approximate Eq. (6.1).

For ϕ^x models with $x \geq 5$ the result (6.3) becomes meaningless ($\bar{\alpha}$ is formally negative although $\bar{\alpha} \geq 0$ by definition). This is of course due to the fact that $n - 2k < 0$ for such models, and the loop integrations cannot be performed without subtractions. In our case such a procedure is meaningless, since as $n \rightarrow \infty$ an infinite number of subtractions are needed due to the lack of renormalizability of a ϕ^x model with $x \geq 5$.

For ϕ^3 and ϕ^4 interactions Eq. (6.3) gives the asymptotic behaviour for $n \rightarrow \infty$ for soft partons with $p_H^2 \ll m^2 \sqrt{n}$, which is similar to the restriction $E \ll mn$ in Mandelstam's model ⁵⁾. In a pure ϕ^4 model $\bar{\alpha} = 0$, and hence

$$\begin{aligned} \tilde{F}_n(p_i p_k, m^2) &\rightarrow F_n(0, m^2), \quad n \rightarrow \infty, \\ \mathcal{L}_{\text{parton}} &= \gamma_4 \phi^4. \end{aligned} \quad (6.4)$$

In Section 2 we have shown that this result is valid for the exact Feynman amplitude. Thus Eq. (6.4) shows that the approximation (6.1) leads to the exact answer for a ϕ^4 model. This is not merely a random coincidence, since in both cases the result is due to the fact that $n - 2k = O(\sqrt{n})$ in a ϕ^4 model.

The only case where the above formalism leads to a non-trivial result is for a parton Lagrangian

$$\mathcal{L}_{\text{parton}} = \gamma_3 \phi^3 + \gamma_4 \phi^4, \quad \gamma_3 \neq 0. \quad (6.5)$$

It is interesting that this result is due to the linear decrease with n of the superficial degree of divergence $D = 4k - 2n$. This leads us to expect that also in a more realistic world where the partons have spin there must exist a scalar ϕ^3 parton to provide a super-renormalizable piece in the Lagrangian.

We mention that it is not difficult to show that the approximate Feynman amplitude (6.1) is an upper limit for the exact amplitude (for $n \rightarrow \infty$). In fact, one can show ^{*)} that for $n \rightarrow \infty$

*) Using a slight generalization of the statistical formalism which allows one to show that

$$F_n(p_i p_k, m^2) \rightarrow \int_0^\infty \frac{\pi d\alpha_i}{\Delta_n(\alpha)^2} e^{\bar{\alpha} \sum k_i(\alpha)^2} e^{-m^2 \sum \alpha_i}$$

from which the inequalities (6.6) follow immediately.

$$e^{\bar{\alpha} \text{Min} \sum k_i(\alpha)^2} F_n(0, m^2) \leq F_n(p_i p_k, m^2) \leq e^{\bar{\alpha} \sum k_i(\bar{\alpha})^2} F_n(0, m^2), \quad (6.6)$$

with correction terms of relative order $p_H^2/\sqrt{n}$. This shows that if the Kirchhoff momenta are slowly varying such that $\sum k_i(\alpha)^2$ is approximately a constant then the approximate Feynman amplitude is practically exact. However, constancy of $\sum k_i(\alpha)^2$ is by no means necessary in order that the approximate amplitude (6.1) becomes exact (or almost exact). The lower limit (6.6) is certainly rather crude, since it does not take into account that $\bar{\alpha}$ represents the most important region of integration. However, for the following, let us assume that the Kirchhoff momenta vary so slowly that Eq. (6.1) is a good approximation.

We now introduce the continuous approximation ^{1),2)} where the large Feynman net is (in some sense) replaced by a continuous distribution of propagators. As is well known ^{1),2),3)} this approximation in connection with the result (6.3) for soft partons leads to the Veneziano N point function with trajectory intercept $\alpha(0) = 1$. The continuous approximation is analogous to the approximation made by everybody who describes, e.g., a table consisting of atoms as a continuous mass distribution. One would expect an approximation of this type to be valid at least as far as the gross features of the discrete Feynman net are concerned.

We do not have much new to say about the continuous approximation and refer the reader to Refs. 1) and 2). The Kirchhoff momenta $k_i(\bar{\alpha})$ in Eq. (6.3) are replaced by a continuous current distribution, and $\bar{\alpha}$ becomes proportional to the specific resistance of the continuous distribution. Using that Kirchhoff's equations extremize the total heat generation, Eq. (6.3) can be rewritten as

$$F_n(p_i p_k, m^2) \sim \text{extr. val.}_j \exp \left[g \bar{\alpha} \int \sum_{\mu} j_{\mu}(x, y)^2 dx dy \right] \quad (6.7)$$

in the continuous approximation, where the variation is with respect to the two-dimensional current distribution $\underline{j}_\mu(x,y)$. Here g is a factor which only depends on the geometry of the net. For the net in Fig. 1a $g = \sqrt{3}/2$. To see this, let σ denote the specific resistance. If we consider a rectangle (Fig. 1a) containing ξ propagators (in Fig. 1a $\xi = 11$) in parallel connection the rectangle gives a resistance $\sigma / (\xi \sqrt{3}/2)$, since the distance between the opposite resistances (vertical sides in Fig. 1b) is $\sqrt{3}/2$ times the length of the resistance. On the other hand, the resistance of the rectangle is $\bar{\alpha}/\xi$, since $\bar{\alpha}$ is the resistance in each propagator. Identifying the two numbers we obtain $\sigma = \sqrt{3} \bar{\alpha} / 2$. For a fish-net $g = 1$. Clearly g is always a simple factor.

At the points where the external particles enter the diagram we have the boundary condition ^{*}) (we take the boundary to be a circle with radius r_0)

$$\underline{j}_\mu(r_0, \theta) \underline{n} = \sum_{i=1}^N p_i^H \delta(\theta - \theta_i) \quad (6.8)$$

where $\underline{n}$ is the normal to the boundary. The component of $\underline{j}$ perpendicular to $\underline{n}$ vanishes. This is a direct transcription of the boundary condition for the discrete net. Equations (6.7) and (6.8) constitute Nielsen's variational principle ⁸⁾. For a disk the solution is ⁸⁾

$$F_n^{\text{disk}}(p_i p_k, m^2) \underset{n \rightarrow \infty}{\sim} F_\infty^{\text{disk}}(0, m^2) \prod_{\substack{i,j \\ i \neq j}} |z_i - z_j|^{-\frac{g\bar{\alpha}}{\pi} p_i^H p_j^H} \quad (6.9)$$

where z_i and z_j are the points where the external hadron momenta p_i^H and p_j^H enter the boundary, respectively.

^{*}) We are here assuming that the external currents can be approximated by point sources. Extended sources lead to excited Reggeons as external particles ¹⁾.

To find the total contribution we have to sum over all possible graphs of order n ($n \rightarrow \infty$) that are topologically different (this is what the ordinary Feynman rules instruct us to do). Thus we have to sum over graphs where the external particles are hooked on the net in different positions. In the continuous approximation this is equivalent to integrating over the points z_i and z_j and summing over topologically different boundaries. We obtain

$$\begin{aligned} & \sum_{\text{all graphs of very high order}} F_n(p_i p_k, m^2) \\ & \rightarrow F_{\infty}^{\text{disk}}(0, m^2) \int \prod_{j=1}^N d\theta_j \prod_{i < j} |z_i - z_j|^{-2\alpha' p_i^H p_j^H} \\ & \quad + \text{contribution from annulus} + \dots, \end{aligned} \quad (6.10)$$

where

$$\alpha' = \frac{g}{\pi m^2} \frac{N_3}{3N_3 + 4N_4}. \quad (6.11)$$

For the net in Fig. 1a, $N_3/N_4 = 2$, $g = \sqrt{3}/2$, and hence

$$\alpha' = \sqrt{3} \alpha / 2\pi = \sqrt{3} / 10\pi m^2. \quad (6.12)$$

In Eq. (6.10) $z_i = \exp(i\theta_i)$. In Eq. (6.10) we have written explicitly only the contribution from the disk. However, since the Feynman rules instruct us to sum over all topologically distinct graphs it is clear that we have to include the contribution from the annulus⁹⁾, etc. This has the important consequence that the over-all coupling constant for the first term in Eq. (6.10) is restricted in order of magnitude relative to the over-all coupling constant in the second term. Since the first term is of order g^2 , where g is the "dual coupling constant", and the second term is of order g^4 , it follows that the strong coupling constant has to be restricted in order of magnitude if the parton picture

is the correct dynamical mechanism for dual amplitudes. This is not so surprising since the parton picture is actually a bootstrap scheme in the sense that all observable hadrons are composed of an infinite sea of partons.

From the known slope we find from Eq. (6.12) a parton mass $m = 230$ MeV. For a pure ϕ^3 model ("bee-cells") we obtain $m = 430$ MeV (this is actually an upper limit - more complicated nets than shown in Fig. 1a give $m < 230$ MeV).

The importance of the continuous approximation is that it allows an analytic continuation from the region without dynamical singularities [where Eq. (6.3) is valid] to a region with singularities. Regge behaviour and poles do not arise from the individual graphs, but enter via the sum over all topologically distinct graphs. This feature is understandable since in general a single Feynman graph does not have a physical meaning, only the sum over topologically distinct graphs gives something with a physical meaning.

7. - CONCLUSIONS

Let us sum up the main points of the present paper. We have obtained non-trivial results for a parton Lagrangian involving a ϕ^3 piece, and let us therefore concentrate on this case. We introduced a physical assumption, namely that the external partons are soft. This was justified by the fact that (in purely strong processes) the soft partons are overwhelmingly important already at a few GeV, and since we want a picture which is uniformly valid at all energies it is reasonable first to concentrate on the most important exchange mechanism. To give meaning to soft virtual partons we found it necessary to use the parametrization of the Feynman amplitude where the loop integrations have been done. We then introduced the approximation that the Kirchhoff momenta could be replaced (essentially) by their average values. The statistical formalism then provided the Gaussian form (6.3) of the Feynman amplitude, in a kinematical region without dynamical singularities. The analytic continuation into a region with dynamical singularities was then finally made by means of the continuous approximation.

On the technical side we are aware that the approximation of replacing the Kirchhoff momenta $k_i(\alpha)$ by $k_i(\bar{\alpha})$ needs further justification. One can expand $k_i(\alpha)$ in a power series, and then try to show that the higher order terms either contribute a negligible amount or, if this is not the case then, that they still lead to the form (6.3) with a modified $\bar{\alpha}$. The difficulty in this approach is that the derivatives of $k_i(\alpha)$ oscillate rapidly in sign and it is therefore difficult to extend the estimates in Section 5 of the limit $n \rightarrow \infty$ in a useful and reliable way. The fact that the result (6.3) is an upper bound makes us rather confident that this approximation is reasonable for soft partons.

However, for Feynman nets where the Kirchhoff momenta are slowly varying Eq. (6.3) is certainly a good approximation. The main thing one can learn from this result is probably that the parton Lagrangian has to have a ϕ^3 piece in order that a non-trivial result

emerges (this is true rigorously as the result in Section 2 shows). Ultimately one would like to consider the partons as fermions (quarks), and it is not likely that one can construct a super-renormalizable fermion model. At best one can construct a renormalizable model. This then suggests that in addition to quark-like partons, one also needs a scalar parton ("the ϕ^3 parton") which is super-renormalizable. The parton Lagrangian would then consist of two parts, namely an essentially scale invariant fermion Lagrangian (where scale invariance is broken only by mass terms) and a non-scale invariant ϕ^3 Lagrangian. In this respect the simple scalar Lagrangian $\gamma_3 \phi_3 + \gamma_4 \phi_4$ is already quite illustrative.

By analogy to Mandelstam's intuitive model ⁵⁾ based on potential theory one would expect that the parton coupling is strong. This is also natural since in the weak coupling limit the Born approximation is likely to dominate the dynamics at least at relatively high energies (a few GeV). We therefore believe that it is reasonable to assume that the parton picture only makes sense if the coupling is rather strong.

Finally, let us mention that if one allows some of the partons to be hard, one can show (by means of the statistical formalism) that the Gaussian term (6.3) of the Feynman amplitude does not emerge. Therefore we only obtain the N point Veneziano amplitude if the partons are soft. We do not like to interpret this result by saying that the Veneziano amplitude resulting from the soft parton picture corresponds to the Pomeron. Such an interpretation runs into severe difficulties with respect to the dual aspects of the whole approach (although one can use Feynman's arguments ⁴⁾ in favour of the Pomeron interpretation ¹⁰⁾). Instead we rather prefer to believe that the scalar pseudoworld will be modified in the right direction by the introduction of spinning partons.

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APPENDIX A

In this Appendix we shall show Eqs. (5.5) and (5.6). Since Eq. (5.6) is a simple consequence of Eq. (5.5) it is enough to verify Eq. (5.5).

For n sufficiently large we can write ($n \gg n_0 \gg 1$, n_0 fixed)

$$\overline{\alpha_i \alpha_k^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_k^{(n)}} = \overline{\alpha_i \alpha_k} - \overline{\alpha_i} \overline{\alpha_k} + \frac{K_{ik}}{n\epsilon} \quad (A.1)$$

where $\epsilon > 0$ indicates how rapidly the fluctuations in a net with n propagators approach the limiting value $\overline{\alpha_i \alpha_k} - \overline{\alpha_i} \overline{\alpha_k}$ in a net with an infinite number of propagators. Again we imagine that the finite net is embodied in the infinite net with the propagators $n+1, n+2, \dots$ dummy. Using Eqs. (5.3) and (A.1) we have (ignoring the second order effect coming from the difference between $\overline{\alpha_i}$ and $\overline{\alpha_i^{(n)}}$),

$$\begin{aligned} \sum_{j'=1}^j (\overline{\alpha_i \alpha_{j'}^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_{j'}^{(n)}}) &= \sum_{j'=1}^j (\overline{\alpha_i \alpha_{j'}} - \overline{\alpha_i} \overline{\alpha_{j'}}) \\ &+ \sum_{j'=1}^j K_{ij'} \left(\frac{1}{n\epsilon} - \frac{1}{j\epsilon} \right) \\ &= \frac{\overline{\alpha_i}}{m^2} + \sum_{j'=1}^j K_{ij'} \left(\frac{1}{n\epsilon} - \frac{1}{j\epsilon} \right), \end{aligned} \quad (A.2)$$

where j is sufficiently large, $j > n_0 \gg 1$. Similarly

$$\begin{aligned} \sum_{i'=1}^i (\overline{\alpha_{i'} \alpha_j^{(n)}} - \overline{\alpha_{i'}^{(n)}} \overline{\alpha_j^{(n)}}) \\ = \frac{\overline{\alpha_j}}{m^2} + \sum_{i'=1}^i K_{i'j} \left(\frac{1}{n\epsilon} - \frac{1}{j\epsilon} \right). \end{aligned} \quad (A.3)$$

Summing Eq. (A.2) over i and Eq. (A.3) over j and subtracting we obtain (with $j > i$, say) by use of Eq. (4.9)

$$\frac{\overline{\alpha_i}}{m^2} (j-i) = \frac{j\epsilon - i\epsilon}{j\epsilon i\epsilon} \sum_{i',j'=1}^{ij} K_{i'j'}. \quad (A.4)$$

By its definition (A.1) K_{ij} does not vanish for $n \rightarrow \infty$ (nor does it blow up, of course). Each K_{ij} is of order one (in suitable units), and the sum on the right-hand side of Eq. (A.4) can therefore maximally be of order ij . Therefore,

$$\frac{j-i}{j^\varepsilon - i^\varepsilon} \frac{j^i}{j^\varepsilon i^\varepsilon} = \frac{n^2}{\alpha} \frac{1}{ij} \sum_{i', j'=1}^{ij} K_{ij'} = O(1) \quad (A.5)$$

and thus ^{*)} $\varepsilon = 1$.

From Eq. (A.2) with $\varepsilon = 1$ we obtain by use of a simple identity (the "partial integration formula" for sums)

$$\begin{aligned} & \sum_{j=n_0}^n d_{ij} (\overline{\alpha_i \alpha_j^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}}) \\ &= \sum_{j=n_0}^n (d_{ij} - d_{i,j+1}) \sum_{j'=1}^j (\overline{\alpha_i \alpha_j^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}}) \\ &= d_{in_0} \frac{\overline{\alpha_i}}{n^2} + \sum_{j=n_0}^n (d_{ij} - d_{i,j+1}) \left(\frac{1}{n} - \frac{1}{j} \right) \sum_{j'=1}^j K_{ij'} , \end{aligned} \quad (A.6)$$

where $d_{in+1} = 0$ by definition. After some rearrangements Eq. (A.6) can be written

$$\begin{aligned} & \sum_{j=n_0}^n d_{ij} (\overline{\alpha_i \alpha_j^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}}) = \frac{\overline{\alpha_i}}{n^2} d_{in_0} \\ &+ \sum_{j=n_0}^n d_{ij} K_{ij} \left(\frac{1}{n} - \frac{1}{j} \right) + \sum_{j=n_0}^n d_{ij} \frac{1}{j^2} \sum_{j'=1}^{j-1} K_{ij'} \\ &+ d_{in_0} \left(\frac{1}{n} - \frac{1}{n_0-1} \right) \sum_{j'=1}^{n_0-1} K_{ij'} . \end{aligned} \quad (A.7)$$

^{*)} Under special circumstances [when the sum over K_{ij} is smaller than expected in Eq. (A.5)] ε can differ from one. This does not change the following developments in any essential manner.

Since the quantities K_{ij} by their definition (A.1) cannot blow up we obtain the inequality

$$\left| \sum_{j=n_0}^n d_{ij} (\overline{\alpha_i \alpha_j^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}}) \right| \leq \text{const.} \cdot \text{Max } |d_{ij}| \sum_{j=n_0}^n \frac{1}{j}. \quad (\text{A.8})$$

Now $n_0 \gg 1$, and therefore

$$\sum_{j=n_0}^n \frac{1}{j} \approx \int_{n_0}^n \frac{dx}{x} = \ln \frac{n}{n_0}. \quad (\text{A.9})$$

Thus (A.8) gives

$$\left| \sum_{j=n_0}^n d_{ij} (\overline{\alpha_i \alpha_j^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}}) \right| \leq \text{const.} \cdot \ln n \cdot \text{Max } |d_{ij}|. \quad (\text{A.10})$$

The merit of this inequality is that although the sum contains n terms, each of which are of order $|d_{ij}|$, the sum is bounded by $|d_{ij}| \log n$ and not by $|d_{ij}|n$. Since

$$\left| \sum_{j=1}^{n_0} d_{ij} (\overline{\alpha_i \alpha_j^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}}) \right| \leq \text{Max } |d_{ij}| n_0 = \text{const.} \cdot \text{Max } |d_{ij}| \quad (\text{A.11})$$

we finally obtain ^{*)}

$$\left| \sum_{j=1}^n d_{ij} (\overline{\alpha_i \alpha_j^{(n)}} - \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}}) \right| \leq \text{const.} \cdot \ln n \cdot \text{Max } |d_{ij}|. \quad (\text{A.12})$$

*) If, under special circumstances, $\epsilon \neq 1$, the factor $\log n$ should be replaced by $n^{1-\epsilon}$. This has no consequences for the applications since ϵ is certainly positive from its definition (A.1).

This inequality is equivalent to

$$\sum_{j=1}^n d_{ij} \overline{\alpha_i \alpha_j^{(n)}} = \sum_{j=1}^n d_{ij} \overline{\alpha_i^{(n)}} \overline{\alpha_j^{(n)}} + O(\ln n \max |d_{ij}|).$$

(A.13)

We have thus shown Eq. (5.5).

APPENDIX B

In this Appendix we shall investigate the higher moments.

Let us first consider the sum

$$\sum_{j_N=1}^n \overline{\alpha_{j_1} \dots \alpha_{j_N}}^{(n)} = \frac{-1}{F_n(0, m^2)} \frac{\partial}{\partial m^2} \int_0^\infty \frac{\pi d\alpha_i}{\Delta_n(\alpha)^2} \alpha_{j_1} \dots \alpha_{j_{N-1}} e^{-m^2 \sum \alpha_i}, \quad (B.1)$$

where we only sum over j_N . Using a uniform scaling we have

$$\begin{aligned} & \int_0^\infty \frac{\pi d\alpha_i}{\Delta_n(\alpha)^2} \alpha_{j_1} \dots \alpha_{j_{N-1}} e^{-m^2 \sum \alpha_i} \\ &= (m^2)^{2k-n-N+1} \int_0^\infty \frac{\pi d\alpha'_i}{\Delta_n(\alpha')^2} \alpha'_{j_1} \dots \alpha'_{j_{N-1}} e^{-\sum \alpha'_i}. \end{aligned} \quad (B.2)$$

The differentiation in Eq. (B.1) is therefore easily performed, and we obtain (it should be noticed that $n - 2k > 0$ in the renormalizable models, and the differentiation therefore always give a non-vanishing result)

$$\sum_{j_N=1}^n \overline{\alpha_{j_1} \dots \alpha_{j_N}}^{(n)} = \frac{1}{m^2} (N-1 + n\bar{\alpha}m^2) \overline{\alpha_{j_1} \dots \alpha_{j_{N-1}}}^{(n)}. \quad (B.3)$$

The sum over the N 'th order fluctuations is thus expressed entirely in terms of an $(N-1)$ 'th order fluctuation, analogously to Eq. (5.2).

Summing over all indices we obtain the relation

$$\sum_{j_1 \dots j_N=1}^n \overline{\alpha_{j_1} \dots \alpha_{j_N}}^{(n)} = \frac{1}{(m^2)^N} \frac{\Gamma(N + n\bar{\alpha}m^2)}{\Gamma(n\bar{\alpha}m^2)}. \quad (B.4)$$

For $n \gg N$ one can write this result in the form

$$\sum_{j_1 \dots j_N=1}^n [\overline{\alpha_{j_1} \dots \alpha_{j_N}}^{(n)} - \bar{\alpha}_{j_1}^{(n)} \dots \bar{\alpha}_{j_N}^{(n)}] = O(n^{N-1}). \quad (B.5)$$

Again there are cancellations among the N 'th order moments and the uncorrelated moments $\bar{\alpha}_{j_1} \dots \bar{\alpha}_{j_N}$. Using Eq. (B.4) one also has

$$\sum_{j_1 \dots j_N=1}^n \left[\overline{\alpha_{j_1} \dots \alpha_{j_N}}^{(n)} - \overline{\alpha_{j_1} \dots \alpha_{j_s}}^{(n)} \overline{\alpha_{j_{s+1}} \dots \alpha_{j_N}}^{(n)} \right] \quad (B.6)$$

$$= O(n^{N-1})$$

provided $n \gg N$. The case $n \lesssim N$ will not be discussed.

We mention that through a rather long chain of arguments one can show from Eq. (B.3) that Eq. (5.6) can be generalized to

$$\left| \sum_{j_1 \dots j_N=1}^n d_{j_1 \dots j_s} d_{j_{s+1} \dots j_N} \left[\overline{\alpha_{j_1} \dots \alpha_{j_N}}^{(n)} - \overline{\alpha_{j_1} \dots \alpha_{j_s}}^{(n)} \overline{\alpha_{j_{s+1}} \dots \alpha_{j_N}}^{(n)} \right] \right| \quad (B.7)$$

$$\leq \text{const. } n^{N-1} \ln n \text{ Max } |d_{j_1 \dots j_s} d_{j_{s+1} \dots j_N}| \quad (N \ll n)$$

We shall not give the proof since we do not need (B.7), but the arguments are rather straightforward generalizations of the arguments in Appendix A.

We shall also study fluctuations of the type

$$\Sigma^{(N)} = \sum_{i_1 \dots i_N=1}^n \overline{\prod_{j=1}^N (\alpha_{i_j} - \bar{\alpha}_{i_j})}^{(n)} \quad (B.8)$$

$$= \overline{\sum_{i=1}^n (\alpha_i - \bar{\alpha}_i)^N}^{(n)}$$

since these fluctuations have some nice properties. Defining

$$G_N(\lambda)$$

$$= \int_0^\infty \frac{\pi d\alpha_i}{F_N(0, m^2) \Delta_N(\alpha)^2} \left[\sum_{i=1}^n (\alpha_i - \bar{\alpha}_i) \right]^N e^{-\lambda \sum_{i=1}^n (\alpha_i - \bar{\alpha}_i)} \quad (B.9)$$

we have

$$\Sigma^{(N+1)} = -e^{-m^2 n \bar{a}} \left[\frac{\partial}{\partial \lambda} G(\lambda) \right]_{\lambda=m^2}. \quad (\text{B.10})$$

The differentiation in (B.10) can be carried out by using that

$$G_N(\lambda) = \lambda^{2k-n-N} \int_0^\infty \frac{\prod_{i=1}^n d\alpha_i}{F_n(0, m^2) \Delta_n(\alpha)^2} \left[\sum_{i=1}^n (\alpha_i - \lambda \bar{a}_i) \right]^N e^{-\sum_{i=1}^n (\alpha_i - \lambda \bar{a}_i)} \quad (\text{B.11})$$

After a few calculations the following recursion relation is obtained

$$\Sigma^{(N+1)} = \frac{N}{m^2} \Sigma^{(N)} + \frac{N(n-2k)}{m^4} \Sigma^{(N-1)}. \quad (\text{B.12})$$

Since we know $\Sigma^{(1)}$ and $\Sigma^{(2)}$ [Eqs. (4.9) and (5.4)],

$$\Sigma^{(1)} = 0, \quad \Sigma^{(2)} = \frac{n-2k}{m^4}, \quad (\text{B.13})$$

the recursion relation (B.12) can be used to calculate $\Sigma^{(N)}$ with $N \geq 3$. The result is given in the Table for the first few orders. It is obvious that $\Sigma^{(N)}$ is a polynomial in $n-2k$ of degree $N/2$ (N even) or $n/2 - \frac{1}{2}$ (N odd). We mention without proof that one can show that

$$\left| \sum_{\ell_1, \dots, \ell_N=1}^n d_{\ell_1, \dots, \ell_N} \overline{(\alpha_{\ell_1} - \bar{a}_{\ell_1}) \dots (\alpha_{\ell_N} - \bar{a}_{\ell_N})^{(n)}} \right| \leq \text{const. } n^{N-1} \ln n \text{ Max } |d_{\ell_1, \dots, \ell_N}| \quad (N \ll n). \quad (\text{B.14})$$

APPENDIX C

Here we shall study the sum

$$\sum_{i=1}^n \bar{\alpha}_i k_i(\bar{\alpha})^2. \quad (C.1)$$

We can always choose the way in which the propagators are ordered such that the Kirchhoff momenta are never increasing,

$$k_i(\bar{\alpha})^2 \geq k_{i+1}(\bar{\alpha})^2. \quad (C.2)$$

For a finite net this is obviously always possible. Consider the quantity

$$\begin{aligned} R_n &= \sum_{i=1}^n (\bar{\alpha}_i^{(n)} - \bar{\alpha}) k_i(\bar{\alpha})^2 \\ &= \sum_{i=1}^n [k_i(\bar{\alpha})^2 - k_{i+1}(\bar{\alpha})^2] \sum_{i'=1}^i (\bar{\alpha}_{i'}^{(n)} - \bar{\alpha}), \end{aligned} \quad (C.3)$$

where by definition $k_{n+1} = 0$. From the sum rule (4.13) we have

$$\bar{\alpha} = \frac{1}{n^2} \frac{N_3}{3N_3 + 4N_4} = \frac{1}{n} \sum_{i=1}^n \bar{\alpha}_i^{(n)} + O\left(\frac{e}{n}\right), \quad (C.4)$$

where e is the number of external partons, $e/n = O(1/\sqrt{n})$. Thus, with $\bar{\alpha}_i$ denoting $\bar{\alpha}_i^{(\infty)}$,

$$\sum_{i=1}^i (\bar{\alpha}_i^{(n)} - \bar{\alpha}) = \sum_{i=1}^i (\bar{\alpha}_i - \bar{\alpha}) + O(n^{1-\epsilon}), \quad (C.5)$$

where

$$\bar{\alpha}_i^{(n)} = \bar{\alpha}_i + O(n^{-\epsilon}). \quad (C.6)$$

From Eq. (C.4) with $O(e/n) = O(1/\sqrt{n})$ we see that $\epsilon = \frac{1}{2}$. Thus

$$\begin{aligned} \sum_{i=1}^i (\bar{\alpha}_i^{(n)} - \bar{\alpha}) &= \sum_{i=1}^i (\bar{\alpha}_i^{(i)} - \bar{\alpha}) + O(\sqrt{i}) + O(\sqrt{n}) \\ &= O(\sqrt{n}), \end{aligned} \quad (C.7)$$

since for i sufficiently large

$$\sum_{i=1}^i (\bar{\alpha}_i^{(i)} - \bar{\alpha}) = 0, \quad (C.8)$$

as one can see from the sum rule (C.4) with n replaced by i . With $n_0 \gg 1$ we thus have

$$\begin{aligned} |R_n| &\leq \sum_{i=n_0}^n [k_{i+1}(\bar{\alpha})^2 - k_i(\bar{\alpha})^2] O(\sqrt{n}) \\ &= -O(\sqrt{n}) k_{n_0}(\bar{\alpha})^2 = O(p_H^2/\sqrt{n}) \end{aligned} \quad (C.9)$$

where we have used that $|k_{n_0}(\bar{\alpha})^2| = O(1/n)$, and where the ordering (C.2) has been used ($k_i(\bar{\alpha})^2 < 0$). The sum from 1 to n_0 is of order n_0/n . Thus

$$R_n \rightarrow 0 \quad \text{for } n \rightarrow \infty \quad (C.10)$$

and we have the result

$$\sum_{i=1}^n \bar{\alpha}_i k_i(\bar{\alpha})^2 = \bar{\alpha} \sum_{i=1}^n k_i(\bar{\alpha})^2 + O(p_H^2/\sqrt{n}). \quad (C.11)$$

$\sum_{m=0}^{(1)}$	0
$\sum_{m=4}^{(2)}$	$n - 2k$
$\sum_{m=6}^{(3)}$	$2(n - 2k)$
$\sum_{m=8}^{(4)}$	$3(n - 2k)^2 + 6(n - 2k)$
$\sum_{m=10}^{(5)}$	$20(n - 2k)^2 + 24(n - 2k)$
$\sum_{m=12}^{(6)}$	$100(n - 2k)^3 + 220(n - 2k)^2 + 120(n - 2k)$
$\sum_{m=14}^{(7)}$	$720(n - 2k)^3 + 1464(n - 2k)^2 + 720(n - 2k)$
$\sum_{m=16}^{(8)}$	$700(n - 2k)^4 + 6580(n - 2k)^3 + 11088(n - 2k)^2 + 5040(n - 2k)$

- TABLE -

REFERENCES

- 1) B. Sakita and M.A. Virasoro - Phys.Rev.Letters 24, 1146 (1970).
- 2) H.B. Nielsen and P. Olesen - Phys.Letters 32B, 203 (1970).
- 3) G. Veneziano - Rapporteur talk at the XVth International Conference on High Energy Physics, Kiev (1970).
- 4) R.P. Feynman - Phys.Rev.Letters 23, 1415 (1969).
- 5) S. Mandelstam - Comments on Nuclear and Particle Physics 3, 65 (1969).
- 6) J.D. Bjorken and S.D. Drell - "Relativistic Quantum Fields", McGraw Hill, N.Y. (1965).
- 7) R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne - "The Analytic S Matrix", Cambridge University Press (1966).
- 8) H.B. Nielsen - NORDITA Preprint (1969) ;
D. Fairlie and H.B. Nielsen - Nuclear Phys. B20, 637 (1970).
- 9) D. Fairlie and H.B. Nielsen - Nuclear Phys. B20, 637 (1970) ;
C.S. Hsue, B. Sakita and M.A. Virasoro - Phys.Rev., to be published ;
G. Frye and L. Susskind - Phys.Letters 31B, 589 (1970) ;
C. Lovelace - Phys.Letters 32B, 703 (1970).
- 10) L. Susskind and H.B. Nielsen - CERN Preprint TH. 1230 (1970).

FIGURE CAPTION

Figure 1a Very high order Feynman net in a $\gamma_3 \phi^3 + \gamma_4 \phi^4$ model.

Figure 1b The basic structure of the cells.

FIG. 1