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MESON MESON SCATTERING.

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A B S T R A C T

A review is presented on
present theoretical and phenome-
nological knowledge of pseudo-
scalar meson meson interactions.

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1. - INTRODUCTION

In the present paper we shall review the theory and phenomenology of processes involving two pseudoscalar mesons in both the initial and the final states.

Within recent years a large number of works have been devoted to study this field of meson-meson scattering. The material presented is heavily biased in favour of works less than two years old.

On the theoretical side the interest arises from the fact that these processes are among the very simplest hadronic ones. External particles are spinless and masses are low so that crossing is a powerful constraint. No comprehensive theory exists, however, so that we shall give up any attempt of a treatment starting with first principles and gradually making predictions.

Nevertheless very interesting fragments of theories do exist and evidences are that they describe something close to reality. Therefore we shall emphasize what may now be considered the orthodox picture for which the main ingredients are : (1) current algebra (soft meson) theory of low energy amplitudes ; (2) SU_3 nonet patterns for natural spin parity mesons (those are the ones that couple to 0^-0^-) ; (3) absence of exotic resonances, and (4) linearity of Regge trajectories. All of these of course are summarized by the Veneziano model. At the present time it is unclear whether this is due to sheer luck or whether the model is a first step towards a near complete theory for hadrons. We shall mainly consider it a useful reference frame from which inevitable deviations can be conveniently measured.

As for these deviations, several attempts have been made to produce quantitative estimates : hard meson or effective Lagrangian techniques, breaking schemes for hadronic symmetries, unitarization procedures. In most cases, no general consensus for the importance or even direction of these corrections has been reached and we shall deal with those topics to a lesser extent.

In contrast to the model achievements, the massive efforts to "bootstrap" hadronic amplitudes out of the notions of analyticity, crossing and unitarity have so far had a very modest success. Only recently has it been possible by a somewhat drastic reformulation of the problem to obtain results of general physical interest. Only low energy $\pi\pi$ scattering has been treated that way to any noticeable extent, however.

On the experimental side studies of meson-meson scattering have been going on for more than a decade in so far as several natural spin parity boson resonances with strong decay into two pseudoscalars have been known that long. Values for masses and decay widths of these objects are constantly updated by the particle data group tables and we shall not be concerned with those to any large extent. Rather we shall define experimental meson-meson scattering as the stage at which bump-hunting changes into phase shift phenomenology.

That branch of high energy physics is peculiar in that due to the absence of a mesonic target, one tries to obtain amplitudes that are sufficiently simple for theoreticians to look at by studying as intermediate steps, reactions that are in fact exceedingly complicated. For high energy di-meson production processes, the Chew-Low-Goebel idea (Section 4) suggests that this can be done in a way which is independent of our ignorance, but due to lack of statistics anything near a model-independent analysis has so far been impossible. One has had to pretend that the complicated process takes place according to an extremely simple model (one-pion exchange) and whatever structures one has observed have been ascribed to subtleties in the meson-meson reaction.

This is clearly very unsatisfactory and as high statistics experiments now being processed become available, a considerable improvement on the level of sophistication of the data analysis will have to be urged. Already, many more suggestions for better treatments now exist that have so far been applied, but several more will have to be added to make the big experimental investments meaningful. We shall discuss a number of associated problems.

Notwithstanding the difficulties, in particular elastic $\pi\pi$ phase shifts now are available with believable details associated with them and various ambiguities are getting resolved. Also, brave attempts to extract $K\pi$ phase shifts are taking place although the difficulties are much greater and the field is much less matured. As for the $K\bar{K}$ system, couplings to inelastic channels only can as yet be studied and little is known outside the resonance bumps (f , f' , g , ϕ , A_2). One exception is the S^* enhancement which promises to play a major role in sorting out the combined $\pi\pi$ and $K\bar{K}$ behaviour of the s waves. For KK or for processes involving η 's, next to nothing that we shall be concerned with in this paper is known experimentally.

In Section 5, we describe a number of low energy processes that may provide various special results complementary to the ones obtained from peripheral high energy production.

Finally, in Section 6 conclusions are attempted in the phase shift language.

References 1)-6), 143), 145) contain reviews on a number of specialized topics.

2. - BASIC FORMALISM

In this Section we compile several formulas of general validity pertaining to meson-meson scattering.

2.1. - Amplitude normalizations. Cross-sections.

For the transition

$$|i\rangle \rightarrow |f\rangle$$

the S matrix element S_{fi} and the invariant Feynman amplitude $\mathcal{M}_{fi}$ are related thus

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^4(P_f - P_i) \frac{\mathcal{M}_{fi}}{\sqrt{1.T(2E_j)}} \quad (2.1)$$

We use units $\hbar = c = 1$ throughout. State vectors are non-invariantly normalized $\langle i | i \rangle = \langle f | f \rangle = 1$, specifically for one-particle states

$$\langle p' | p \rangle = (2\pi)^3 \delta^3(p' - p) \rightarrow V \delta_{pp'} \rightarrow \delta_{pp'}$$

where $V \equiv 1$ is the quantization volume, $P_i(P_f)$ is the sum of all four-momenta in the initial (final) state and E_j is the energy of the j'th particle, j running over both initial and final particles.

For the binary reaction

$$a + b \rightarrow c + d$$

we define standard invariants

$$s \equiv (p_a + p_b)^2, \quad t \equiv (p_d - p_b)^2, \quad u \equiv (p_c - p_b)^2 \\ s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \quad (2.2)$$

(metric : $p_\mu q_\mu \equiv p_0 q_0 - \vec{p} \cdot \vec{q}$).

Many different normalizations are in common use. We define for each N the invariant amplitude

$$^N A \equiv -\frac{N^{-1}}{16\pi} \mathcal{M} \quad (2.3)$$

Then the differential cross-section in the centre-of-mass system (c.m.) becomes

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{4}{s} N^2 \frac{q_f}{q_i} |^N A|^2 \\ \frac{d\sigma}{dt} &= -N^2 \frac{4\pi}{s q_i^2} |^N A|^2 = -\frac{\pi}{q_i q_f} \frac{d\sigma}{d\Omega} \end{aligned} \quad (2.4)$$

where $q_i(q_f)$ is the magnitude of the initial (final) c.m. three-momentum.

The total elastic cross-section is

$$\sigma_{el} \equiv \epsilon_f \int_{4\pi} d\Omega \frac{d\sigma}{d\Omega} \quad (2.5)$$

Here we have defined for later convenience a two-particle channel index

$$\epsilon_f = \begin{cases} 1 & \text{for two different particles in } |f\rangle \\ \frac{1}{2} & \text{for two identical particles in } |f\rangle \end{cases} \quad (2.6)$$

2.2. - Unitarity. Partial wave expansions.

The unitarity condition $SS^\dagger = \mathbb{1}$ takes the form

$$\begin{aligned} -2(2\pi)^4 \delta^4(P_i - P_f) \frac{\text{Im } \mathcal{M}_{fi}}{\sqrt{\prod_{j \in (f,i)} (2E_j)}} &= \\ (2\pi)^8 \sum_n \delta^4(P_f - P_n) \frac{\mathcal{M}_{fn}}{\sqrt{\prod_{j \in (f,n)} (2E_j)}} \frac{\mathcal{M}_{ni}^*}{\sqrt{\prod_{j \in (i,n)} (2E_j)}} \delta^4(P_i - P_n) & \end{aligned} \quad (2.7)$$

For $|i\rangle \equiv |f\rangle$ this gives the optical theorem

$$\text{Im } \tilde{A}_{i=f}^N = \frac{q\sqrt{s}}{8\pi N} \sigma_{tot} \quad (2.8)$$

We define partial wave amplitudes $\tilde{A}_{fi}^N(s)$ for spinless particles by

$$\tilde{A}_{fi}^N(s, \cos\theta) = \sum_{\ell} (2\ell+1) \tilde{A}_{fi}^{\ell}(s) P_{\ell}(\cos\theta) \quad (2.9)$$

where θ is the c.m. scattering angle.

For two-particle intermediate states only (ν)

$$\sum_n \rightarrow \sum_{(\nu)} \int \frac{d^3 p_1^{(\nu)}}{(2\pi)^3} \frac{d^3 p_2^{(\nu)}}{(2\pi)^3} \epsilon_{\nu}$$

so

$$\text{Im } \tilde{A}_{fi}^{\ell} = \frac{2N}{\sqrt{s}} \sum_{(\nu)} \theta(s-s_{\nu}) \epsilon_{\nu} q_{\nu} (\tilde{A}_{\nu f}^{\ell})^* (\tilde{A}_{\nu i}^{\ell}) \quad (2.10)$$

Here s_{ν} is the ν channel threshold, q_{ν} the magnitude of the ν channel c.m. three-momentum and ϵ_{ν} is defined in Eq. (2.6).

Equation (2.10) may be (under certain conditions ⁷⁾) continued below s_i, s_f to give

$$\begin{aligned} & \text{Im} \left[\tilde{A}_{fi}^{\ell} / (q_i q_f)^{\ell} \right] \\ &= \frac{2N}{\sqrt{s}} \sum_{(\nu)} \theta(s-s_{\nu}) \epsilon_{\nu} q_{\nu}^{2\ell+1} \left[\tilde{A}_{\nu f}^{\ell} / (q_{\nu} q_f)^{\ell} \right]^* \left[\tilde{A}_{\nu i}^{\ell} / (q_{\nu} q_i)^{\ell} \right] \end{aligned} \quad (2.11)$$

which is only equivalent to Eq. (2.10) for $s > s_i, s_f$.

For elastic scattering in a state of well-defined isospin I we get in particular below the first inelastic threshold

$$\text{Im } N_{A^{\ell,I}} = \frac{2N}{\sqrt{s}} \epsilon q |N_{A^{\ell,I}}|^2 \quad (2.12)$$

which may be solved to give

$$N_{A^{\ell,I}} = \frac{1}{N\epsilon} \frac{\sqrt{s}}{2} \frac{e^{i\delta_\ell^I} \sin \delta_\ell^I}{q} \quad (2.13)$$

where δ_ℓ^I is the real phase shift. Above the first inelastic threshold Eq. (2.13) is generalized as

$$N_{A^{\ell,I}} = \frac{1}{N\epsilon} \frac{\sqrt{s}}{2} \frac{\eta_\ell^I e^{2i\delta_\ell^I} - 1}{2iq} \quad (2.14)$$

where δ_ℓ^I is still the real phase shift (not the phase of $N_{A^{\ell,I}}$) and η_ℓ^I is the real inelasticity,

$$0 \leq \eta_\ell^I \leq 1. \quad (2.15)$$

Then we get for the total, the elastic and the inelastic cross-sections in a state with angular momentum ℓ and isospin I

$$\begin{aligned} \sigma_{\text{tot}}^{\ell,I} &= \frac{2\pi(2\ell+1)}{\epsilon q^2} (1 - \eta_\ell^I \cos 2\delta_\ell^I) \\ \sigma_{\text{el}}^{\ell,I} &= \frac{\pi(2\ell+1)}{\epsilon q^2} |\eta_\ell^I e^{2i\delta_\ell^I} - 1|^2 \\ \sigma_{\text{inel}}^{\ell,I} &= \frac{\pi(2\ell+1)}{\epsilon q^2} (1 - (\eta_\ell^I)^2) \end{aligned} \quad (2.16)$$

(symbols ϵ , q , etc., always pertain to initial state).

For two-particle channels only, S matrix elements may be defined thus

$$S_{if}^{\ell, I} = S_{fi}^{\ell, I} = 4i\sqrt{q_i q_f} N \sqrt{\epsilon_i \epsilon_f} \frac{N_A^{\ell, I}{}_{i \rightarrow f}}{\sqrt{s}} \quad \text{for } i \neq f \quad (2.17)$$

$$S_{if}^{\ell, I} = \eta_{\ell}^I e^{2i\delta_{\ell}^I}$$

for $i = f$

These are independent of the normalization, N. The unitarity condition $S^\dagger S = \mathbf{1}$ becomes a matrix equation, however, only open channels are summed over.

As an illustration we write down the solution for the $\pi\pi \rightarrow \pi\pi, K\bar{K} \rightarrow K\bar{K}$ and $\pi\pi \rightarrow K\bar{K}$ system assuming that no other channels couple. q and k are c.m. three-momenta of pions and kaons respectively and $\delta_{\ell}^I(\pi\pi \rightarrow K\bar{K})$ is defined to be the phase of the amplitude $N_A^{\ell, I}(\pi\pi \rightarrow K\bar{K})$. Then

$$\frac{q}{k} \sigma^{\ell, I}(\pi\pi \rightarrow K\bar{K}) = 2 \frac{k}{q} \sigma^{\ell, I}(K\bar{K} \rightarrow \pi\pi) \quad \eta_{\ell}^I(\pi\pi) = \eta_{\ell}^I(K\bar{K}) \quad (2.18)$$

$$\delta_{\ell}^I(\pi\pi \rightarrow k\bar{k}) \equiv \delta_{\ell}^I(\pi\pi) + \delta_{\ell}^I(K\bar{K}) \pmod{\pi}$$

2.3. - Isospin conventions ^{8),9),10)}

Let $|I, m, a\rangle$ denote a set of particle states transforming according to isospin I ($I_3 \equiv m$),

$$|I, m, a\rangle = \phi_{a, m}^\dagger |0\rangle,$$

and let $|(I, m)^*, a\rangle$ denote the corresponding set of antiparticle states :

$$|(I, m)^*, a\rangle = \phi_{(\bar{a}, \bar{m})}^\dagger |0\rangle = \phi_{a, m} |0\rangle.$$

Then the set $|I, m, \bar{a}\rangle$ defined by

$$|I, m, \bar{a}\rangle \equiv C_{m m'} |I, m'\rangle^*, a \rangle \quad (2.19)$$

will have the same isospin transformation properties as $|I, m, a\rangle$ provided ⁸⁾

$$C_{m m'} = \eta_{\bar{a}} \delta_{m', -m} (-)^{I+m} \quad (2.20)$$

$$|\eta_{\bar{a}}| = 1, \quad \text{independent of } m, m'$$

We shall take

$$\eta_{\bar{a}} \equiv (-)^{Y_{\bar{a}}/2 - I} \quad (2.21)$$

where $Y_{\bar{a}}$ is the hypercharge of $\bar{a}$. This choice is different from that of Carruthers and Krish ($\eta \equiv -1$) ¹⁰⁾ and of Neville ($\eta \equiv 1$) ⁸⁾ but conforms to De Swart's SU_3 convention ⁹⁾ and has the advantage that for the self-conjugate particles η and π , $|\bar{\pi}, m\rangle \equiv |\pi, m\rangle$, $|\bar{\eta}, 0\rangle = |\eta, 0\rangle$.

Thus our convention for pseudoscalar mesons becomes

$$\begin{aligned} |\pi, 1\rangle &= -|\pi^+\rangle, \quad |K, \frac{1}{2}\rangle = |K^+\rangle, \quad |\bar{K}, \frac{1}{2}\rangle = |\bar{K}^0\rangle \\ |\pi, 0\rangle &= |\pi^0\rangle, \quad |K, -\frac{1}{2}\rangle = |K^0\rangle, \quad |\bar{K}, -\frac{1}{2}\rangle = -|K^-\rangle \\ |\pi, -1\rangle &= |\pi^-\rangle, \quad |\eta, 0\rangle = |\eta\rangle, \quad (|\eta', 0\rangle = |\eta'\rangle) \end{aligned} \quad (2.22)$$

With these conventions we list the isospin decomposition for a few reactions

$$\begin{aligned} A(\pi^+\pi^+ \rightarrow \pi^+\pi^+) &= A(\pi^-\pi^- \rightarrow \pi^-\pi^-) = A^{(2)} \\ A(\pi^+\pi^0 \rightarrow \pi^+\pi^0) &= A(\pi^-\pi^0 \rightarrow \pi^-\pi^0) = \frac{1}{2}(A^{(2)} + A^{(1)}) \\ A(\pi^+\pi^- \rightarrow \pi^0\pi^0) &= \frac{1}{3}(A^{(0)} - A^{(2)}) \\ A(\pi^+\pi^- \rightarrow \pi^+\pi^-) &= \frac{1}{6}(2A^{(0)} + 3A^{(1)} + A^{(2)}) \end{aligned} \quad (2.23)$$

$$\begin{aligned}
 A(K^+\pi^+ \rightarrow K^+\pi^+) &= A(K^-\pi^- \rightarrow K^-\pi^-) = A^{(3/2)} \\
 A(K^+\pi^- \rightarrow K^+\pi^-) &= A(K^-\pi^+ \rightarrow K^-\pi^+) = \frac{1}{3}(A^{(3/2)} + 2A^{(1/2)}) \\
 A(K^+\pi^- \rightarrow K^0\pi^0) &= A(K^-\pi^+ \rightarrow \bar{K}^0\pi^0) = \frac{\sqrt{2}}{3}(A^{(3/2)} - A^{(1/2)})
 \end{aligned} \tag{2.24}$$

$$\left. \begin{aligned}
 A(\pi^+\pi^- \rightarrow K^+K^-) &= \frac{1}{2}A^{(1)} + \sqrt{\frac{1}{6}}A^{(0)} \\
 A(\pi^+\pi^- \rightarrow K^0\bar{K}^0) &= \frac{1}{2}A^{(1)} - \sqrt{\frac{1}{6}}A^{(0)}
 \end{aligned} \right\} \tag{2.25}$$

2.4. - Isospin crossing matrices

For pseudoscalar meson-meson scattering we shall define the s channel to describe scattering of $Y \geq 0$ particles only. Then the s, t and u channels are defined by

$$\begin{aligned}
 \text{s channel} &: a + b \rightarrow c + d \\
 \text{t channel} &: \bar{d} + b \rightarrow c + \bar{a} \quad (\text{note particle order}) \\
 \text{u channel} &: \bar{c} + b \rightarrow \bar{a} + d
 \end{aligned} \tag{2.26}$$

The s channel amplitude $A_s^I(ab \rightarrow cd)$ is supposed to be an analytic function of (s, t, u) having singularities given for example by the Landau rules ¹¹⁾. When continued to the t or u channel physical regions it describes a linear combination of t or u channel isospin amplitudes, the coefficients of which are given by the crossing matrices. For a general discussion of all the subtleties involved in finding that, see Neville ⁸⁾. Here we just summarize the rules.

Consider two channels 1 and 2 with isospins I_1, I_2 and where no attention is as yet paid to particle order :

$$\begin{aligned}
 \text{channel 1} &: (ax) \rightarrow (by), \quad I_1 \\
 \text{channel 2} &: (a\bar{y}) \rightarrow (b\bar{x}), \quad I_2
 \end{aligned}$$

Then

$$A_1^{I_1} = \sum_{I_2} \mathcal{C}_{12}(I_1, I_2) A_2^{I_2} \quad (2.27)$$

where the crossing matrix is given by

$$\eta_x \eta_y^* \mathcal{C}_{12}(I_1, I_2) = (-)^{I_1 + \bar{I}_2 + I_x - \bar{I}_y} (2I_2 + 1) \phi \times \begin{Bmatrix} I_a & I_x & I_1 \\ I_b & \bar{I}_y & I_2 \end{Bmatrix} \quad (2.28)$$

Here η_x, η_y are the phases [Eq. (2.21)] pertaining to the antiparticle (we call particles, states with $Y \geq 0$) of the pair $(x, \bar{x}), (y, \bar{y})$. The last term is a 6-j symbol ¹²⁾. ϕ is a phase factor constructed according to the rules of Table I. Then we have for example

$$\mathcal{C}_{st}(I_s, I_t) = (2I_t + 1) (-)^{2I_a + 2I_b + 2I_c - \frac{1}{2}(\bar{Y}_a + \bar{Y}_c) - I_s - I_t} \times \begin{Bmatrix} I_b & I_a & \bar{I}_s \\ I_c & I_d & I_t \end{Bmatrix} \quad (2.29)$$

$$\mathcal{C}_{su}(I_s, I_u) = (2I_u + 1) (-)^{I_a + 2I_b - I_c - \frac{1}{2}(\bar{Y}_a + \bar{Y}_c)} \begin{Bmatrix} I_b & I_a & \bar{I}_s \\ I_d & I_c & \bar{I}_u \end{Bmatrix}$$

Using similar expressions for $\mathcal{C}_{ut}, \mathcal{C}_{ts}, \mathcal{C}_{us}, \mathcal{C}_{tu}$, the consistency may be checked by the relations

$$\mathcal{C}_{ts} = \mathcal{C}_{st}^{-1} ; \mathcal{C}_{ut} = \mathcal{C}_{us} \cdot \mathcal{C}_{st} \text{ etc.} \quad (2.30)$$

In Eq. (2.29) the convention Eq. (2.21) has been used whereas the right-hand side of Eq. (2.28) is convention independent.

In addition to isospin amplitudes, covariant amplitudes are often used. These are defined in terms of the covariant states. Thus for pions, $|\pi_a\rangle$, $a = 1, 2, 3$, are defined by

$$|\pi^\pm\rangle = \frac{1}{\sqrt{2}} (|\pi_1\rangle \pm i|\pi_2\rangle), \quad |\pi^0\rangle = |\pi_3\rangle. \quad (2.31)$$

For kaons

$$\chi_K = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases} \quad \text{for } K = \begin{cases} K^+, K^- \\ K^0, \bar{K}^0 \end{cases}$$

$$\chi_K^C = i\tau^2 \chi_K = \begin{cases} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \quad \text{for } K = \begin{cases} K^+, K^- \\ K^0, \bar{K}^0 \end{cases} \quad (2.32)$$

χ^T denotes the transposed spinor (row) and

$$\vec{\tau} = (\tau^1, \tau^2, \tau^3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

are the Pauli matrices.

Below we list for each pseudoscalar binary reaction the crossing properties of the amplitudes in common use as well as symmetry properties following from Bose statistics and charge conjugation invariance.

The lacking crossing matrices may be obtained from Eq. (2.30).

$\pi\pi\pi\pi$

s channel: $\pi\pi \rightarrow \pi\pi$; t channel: $\pi\pi \rightarrow \pi\pi$;

u channel $\pi\pi \rightarrow \pi\pi$

$$\mathcal{C}_{st} = \frac{1}{6} \begin{pmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{pmatrix} \quad ; \quad \mathcal{C}_{su} = \frac{1}{6} \begin{pmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix} \quad (2.33)$$

$$\langle d, c | T | b, a \rangle = A \delta_{ab} \delta_{cd} + B \delta_{ac} \delta_{bd} + C \delta_{ad} \delta_{bc} \quad (2.34) \quad 13)$$

$$\begin{pmatrix} A_s^{(0)} \\ A_s^{(1)} \\ A_s^{(2)} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \quad (2.35)$$

$$A(s, t, u) = B(t, s, u) = C(u, t, s) \quad (2.36)$$

even under $u \leftrightarrow t$.

K π $\bar{K}\pi$

s channel : $K\pi \rightarrow K\pi$; t channel : $\pi\pi \rightarrow K\bar{K}$;
u channel : $\bar{K}\pi \rightarrow \bar{K}\pi$

$$\mathcal{C}_{st} = \begin{pmatrix} 1/\sqrt{6} & 1 \\ 1/\sqrt{6} & -1/2 \end{pmatrix} ; \quad \mathcal{C}_{su} = \frac{1}{3} \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix} \quad (2.37)$$

$$\begin{aligned} & \langle K'\pi^c | T | \pi^a K \rangle \\ &= \chi_{K'}^T \left\{ \delta_{ac} A^{(+)} + \frac{1}{2} [\tau^c, \tau^a] A^{(-)} \right\} \chi_K \end{aligned} \quad (2.38)$$

$$\begin{aligned} A^{(+)} &= \frac{1}{\sqrt{6}} A_t^{(0)} = \frac{1}{3} (2A_s^{(3/2)} + A_s^{(1/2)}) = \frac{1}{3} (2A_u^{(3/2)} + A_u^{(1/2)}) \\ A^{(-)} &= \frac{1}{2} A_t^{(1)} = \frac{1}{3} (A_s^{(1/2)} - A_s^{(3/2)}) = \frac{1}{3} (A_u^{(1/2)} - A_u^{(3/2)}) \end{aligned} \quad (2.39)$$

$$A^{(\pm)}(s, t, u) = \pm A^{(\pm)}(u, t, s) \quad (2.40)$$

KK $\bar{K}\bar{K}$

s channel : $KK \rightarrow KK$; t channel : $\bar{K}K \rightarrow K\bar{K}$;
u channel : $\bar{K}K \rightarrow \bar{K}K$

$$\mathcal{C}_{st} = -\frac{1}{2} \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} ; \quad \mathcal{C}_{su} = \frac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \quad (2.41)$$

$$\begin{aligned} \langle K^4 K^3 | T | K^2 K^1 \rangle &= \frac{1}{2} A^0 (\chi_3^T \chi_4^c) ((\chi_2^c)^T \chi_1) \\ &+ \frac{1}{2} A^1 (\chi_3^T \vec{\tau} \chi_4^c) ((\chi_2^c)^T \vec{\tau} \chi_1) \\ A^0 &\equiv A_s^{(0)} , \quad A^1 \equiv A_s^{(1)} \end{aligned} \quad (2.42)$$

$$A^{(i)}(s, t, u) = \mp A^{(i)}(s, u, t) \quad (2.43)$$

$\pi\eta\pi\eta$

s channel : $\pi\eta \rightarrow \pi\eta$; t channel : $\eta\eta \rightarrow \pi\pi$;
u channel : $\pi\eta \rightarrow \pi\eta$

$$\mathcal{G}_{st} = -\frac{1}{\sqrt{3}} \quad ; \quad \mathcal{G}_{su} = 1 \quad (2.44)$$

$$\langle \eta\pi^c | T | \eta\pi^a \rangle = \delta_{ca} A \quad (2.45)$$

$$A \equiv A_s^{(1)}$$

$$A(s, t, u) = A(u, t, s) \quad (2.46)$$

$K\pi\bar{K}\eta$

s channel : $K\pi \rightarrow K\eta$; t channel : $\eta\pi \rightarrow K\bar{K}$;
u channel : $\bar{K}\pi \rightarrow \bar{K}\eta$

$$\mathcal{G}_{st} = -\frac{\sqrt{6}}{2} \quad ; \quad \mathcal{G}_{su} = -1 \quad (2.47)$$

$$\langle \eta K' | T | \pi^a K \rangle = \chi_{K'}^T \tau^a \chi_K A \quad (2.48)$$

$$A \equiv \frac{1}{\sqrt{3}} A_s^{(1/2)}$$

$$A(s, t, u) = A(u, t, s) \quad (2.49)$$

$K \eta \bar{K} \eta$

s channel : $K \eta \rightarrow K \eta$; t channel : $\eta \eta \rightarrow K \bar{K}$;
u channel : $\bar{K} \eta \rightarrow \bar{K} \eta$

$$\mathcal{C}_{st} = -\frac{1}{\sqrt{2}} ; \quad \mathcal{C}_{su} = 1 \quad (2.50)$$

$$\langle \eta K' | T | \eta K \rangle = \chi_{K'}^T \chi_K A$$

$$A \equiv A_s^{(\frac{1}{2})} \quad (2.51)$$

$$A(s, t, u) = A(u, t, s) \quad (2.52)$$

$\eta \eta \eta \eta$

$$\mathcal{C}_{st} = \mathcal{C}_{su} = 1 \quad (2.53)$$

The (one) scattering amplitude is completely symmetrical in s, t, u.

Replacing one or more η particles with an η' leaves the crossing matrices unchanged, however, some of the symmetry properties get altered. Our conventions here differ slightly from those of Canning ¹⁴⁾ and Osborn ¹⁵⁾.

3. - THEORIES OF MESON-MESON SCATTERING

In this Section we shall try to set up a coherent picture of meson-meson scattering as it may be viewed at this time. We shall not claim that it describes the truth in every respect, but we shall suggest that it accommodates a sufficient number of attractive ideas that disproving it would be just as interesting as having it confirmed in detail. More likely, several essential parts of the picture will survive and several will need modifications. We shall make comments on which features appear to be more believable than others.

Those processes involving η particles will be first ignored and only in subsection 3.5 shall we comment on some of the difficulties that appear to be peculiar to that particle or else may indicate a more fundamental difficulty with the approach as such.

3.1. - Soft-meson theory of low energy parameters.

This particular treatment of low energy interactions due to Weinberg proceeds in several steps ^{16),15),17)}.

1) For the covariant amplitudes of subsection 2.4, one writes down the most general off-shell linear expansion in the variables s, t, u, q_i^2 (the virtual mass squared of the i 'th meson, $i = 1, \dots, 4$). Imposing several constraints such as $s + t + u = \sum_i q_i^2$ and symmetry properties coming from Bose statistics and C invariance, one reduces the number of independent variables and free parameters.

We make a few comments on the significance of this linear expansion.

- (i) - It is the simplest form compatible with the requirements of PCAC and current algebra. Deviations are certain to exist but their evaluation is much more model dependent.

- (ii) - It can only be interpreted as the truncation of a complete polynomial expansion of the amplitude in a singularity free region in the space of all the variables. On-shell, this means in the triangle

$$s \leq s_0, \quad t \leq t_0, \quad u \leq u_0 \quad (3.1)$$

where s_0 , t_0 and u_0 are the lowest thresholds (physical or pseudophysical) in the corresponding channels. The deviation from the true amplitude in the s channel physical region will be of the order

$$\begin{aligned} (a_0)^2 (s-s_0)^{1/2} & \quad \text{for } \text{Im } A \\ \frac{1}{2} (a_0)^3 (s-s_0) & \quad \text{for } \text{Re } A \end{aligned} \quad (3.2)$$

where a_0 is the s wave scattering length in the s channel (possibly zero by Bose statistics - then the accuracy is improved and determined by the p wave). Consequently for not too large scattering lengths the linear approximation may well be a sensible first approximation for the real part below the resonance region.

- (iii) - It enforces "smoothness" in the form of absence of d and higher waves. Then for the on-shell amplitude the condition $s + t + u = \sum_i m_i^2 = \text{constant}$ leads to a further reduction on the number of parameters thereby in general implying sum rules for the scattering lengths. These are at least easy and convenient checks on models ignoring higher waves.

2) - In the second step one employs the LSZ formalism to contract two mesons of the total four-point function, using for the interpolating field the divergence of the axial current having the right quantum numbers. This defines the one particular off-shell form for the amplitude which is relevant for imposing current algebra information. It further involves introducing a PCAC constant F_1 normalized by

$$(2\pi)^{3/2} \sqrt{2\rho_0} \langle p_\mu, i | \partial_\mu A_\mu^i | 0 \rangle = F_i m_i^2 \quad (3.3)$$

where i is an SU_3 index. For $i = \pi$ this may be related to the π - μ decay :

$$\Gamma_{\pi^+ \rightarrow \mu^+ \nu_\mu} = G^2 \cos^2 \theta_c (m_\mu^2 / \mu^3) (\mu^2 - m_\mu^2)^2 F_\pi^2 / 2\pi \quad (3.4)$$

where G is the weak interaction coupling constant, θ_c is the Cabibbo angle and m_μ and μ are muon and pion mass. Experimentally¹⁸⁾ this gives

$$F_\pi \approx 0.66\mu \approx 93 \text{ MeV} \quad (3.5)$$

The Goldberger-Treiman relation on the other hand says

$$F_\pi = \frac{M g_A}{G_r} \approx 0.61\mu \approx 85 \text{ MeV} \quad (3.6)$$

where M is the nucleon mass, $g_A \approx 1.23$ and $G_r^2/4\pi \approx 14.6$. When quoting theoretical predictions of low energy parameters we shall give a "theoretical uncertainty" corresponding to

$$F_\pi = (89 \pm 4) \text{ MeV} \quad (3.7)$$

3) - At the "Adler points" $q_i^\mu = 0$, $q_j^2 = m_j^2$ ($j \neq i$), the meson-meson amplitudes vanish when the particular off-shell continuation defined by PCAC is used. The Adler condition fixes the connection between the on-shell and the off-shell behaviour of the amplitude. However, taken alone it provides no information whatever on the on-shell behaviour.

4) - At the "current algebra points" (CA points), $q_i^\mu = q_{j \neq i}^\mu = 0$, $q_k^2 = m_k^2$, $k \neq i, j$ current algebras provide low energy theorems that may be interpreted on-shell using the Adler condition (3) and the smoothness assumption (here linearity). To order $q_i \cdot q_j$ the amplitude at these points contains two terms involving the commutators

$$q_\mu^i \delta(x_0 - y_0) [A_0^i(y), A_\mu^j(x)], \delta(x_0 - y_0) [A_0^i(x), \partial_\mu A_\mu^j(y)] \quad (3.8)$$

The first term is given by the usual $SU_3 \times SU_3$ Gell-Mann current algebra. It is related to the vector current

$$f_{ijk} V_\mu^k$$

Provided one can find an equal-mass channel for which this is one of the conserved currents, V^1, V^2, V^3, V^8 , a derivative of the amplitude is known exactly at this CA point. For processes involving the η , however, this is not possible and further assumptions on the nature of $SU_3 \times SU_3$ symmetry breaking must be made. The second term also requires knowledge on this breaking, however, when no η particle is present it proves sufficient to assume non-exoticity of the breaking to fix the (linear) amplitude completely.

Once a complete model for the off-shell amplitude (linear or not) is given, a number of matrix elements of commutators of the form (3.8) are fixed although they are unknown a priori from the Gell-Mann algebra. This information may be used to throw further light on the nature of symmetry breaking which the model implies. It has been fashionable to analyze this information according to the Gell-Mann - Oakes-Renner (GOR) scheme ¹⁹⁾ according to which the Hamiltonian density is written

$$\mathcal{H}(x) = \mathcal{H}_0(x) - u^0(x) - c u^8(x) \quad (3.9)$$

Here u^i , $i = 0, \dots, 8$ are scalar SU_3 singlet and octet members of an irreducible $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of $SU_3 \times SU_3$ and parity. In this model, matrix elements of σ terms [last term in (3.8)] and matrix elements of non-conserved vector currents are expressible as linear combinations of the quantities ¹⁵⁾

$$J_j^{ik}((p'-p)^2) \equiv \langle i, p' | u^j(0) | k, p \rangle$$

where i and k are SU_3 labels for the meson states. The coefficients of these depend on the parameter c of Eq. (3.9). Analyzing amplitudes for pions and kaons this way Osborn ¹⁵⁾ obtained a consistent value of

$$c = -2\sqrt{2} \frac{m_K^2 - \mu^2}{2m_K^2 + \mu^2} \quad (3.10)$$

which is the value also obtained by GOR ¹⁹⁾. The fact that $c \simeq -\sqrt{2}$ rather than $c \simeq 0$ may be interpreted as the statement that $SU_2 \times SU_2$ is a better symmetry than SU_3 (if off-shell behaviours are roughly linear). That point of view has recently been questioned on various grounds ^{20), 21)}, however, we shall not discuss this further here.

Below we shall give the results of the soft-meson treatment of binary processes for pions and kaons in what may be considered increasing order of model dependence :

- (i) on-shell consequences of linearity ;
- (ii) consequences of the Adler conditions (PCAC), i.e., no new on-shell information ;
- (iii) low-energy theorems depending only on the Gell-Mann algebra ;
- (iv) consequences of the non-exoticity assumption.

We take $N^{-1} = 16\pi$ in Eq. (2.3). Scattering lengths and effective ranges are defined by

$$q^{2\ell+1} \cot \delta_c^I = (a_c^I)^{-1} + \frac{1}{2} r_c^I q^2 + O(q^4) \quad (3.11)$$

The connection to the partial wave amplitudes of Eq. (2.14) is

$$\frac{\sqrt{s} q^{2l}}{2N\epsilon} (N A^{l,I})^{-1} = q^{2l+1} \cot \delta_e^I - i q^{2l+1} \quad (3.12)$$

where the first term on the right from unitarity is expected to be holomorphic in a region around the physical threshold.

$\pi\pi \rightarrow \pi\pi$

The most general linear form of A [Eq. (2.34)] has no explicit dependence on q_i^2 ($i = a, b, c, d$) :

$$A(s, t, u, q_i^2) = \alpha + \beta(t+u) + \gamma^2 s$$

(i) on-shell this implies the important rule

$$2a_0^0 - 5a_0^2 = 18\mu^2 a_1^1 \quad (3.13)$$

(ii) PCAC gives

$$A(\mu^2, \mu^2, \mu^2) = 0 \quad \text{i.e. } \alpha + 2\mu^2\beta + \gamma^2\mu^2 = 0$$

(iii) CA implies

$$\frac{\partial}{\partial x} A(\mu^2 + x, 0, \mu^2 - x) = \frac{1}{F_\pi^2} = \gamma^2 - \beta$$

or

$$L \equiv \frac{1}{6} (2a_0^0 - 5a_0^2) = 3\mu^2 a_1^1 = \frac{\mu}{8\pi F_\pi^2} = (0.10 \pm 0.01) \mu^{-1} \quad (3.14)$$

where L is Weinberg's universal length.

(iv) non-exoticity of σ term :

$$A(\mu^2, 0, \mu^2) = C(\mu^2, 0, \mu^2) = 0$$

or

$$A(s, t, u) = \frac{1}{F_\pi^2} (s - \mu^2) \quad (3.15)$$

$$a_0^0/a_0^2 = -\frac{7}{2}$$

$$\begin{aligned} A_0^0(s = \mu^2/2) &= 0; \quad a_0^0 = \frac{7}{4}L = (0.17 \pm 0.02)\mu^{-1} \\ A_0^2(s = 2\mu^2) &= 0; \quad a_0^2 = -\frac{1}{2}L = (-0.050 \pm 0.005)\mu^{-1} \\ r_0^0 &= (-7.3 \pm 0.7)\mu^{-1}; \quad r_0^2 = (6.0 \pm 0.6)\mu^{-1} \end{aligned}$$

We note that the well-known old s wave dominance theory of $\pi\pi$ scattering by Chew and Mandelstam^{13), 22)} had

$$a_0^0/a_0^2 \approx +5/2 \quad (3.16)$$

corresponding to an essentially constant value for A ($\beta \approx \gamma \approx 0$). This apparently simple behaviour, however, requires a very complicated off-shell extrapolation to cope with the current algebra information. Note in (3.15) that not only are s wave scattering lengths small, but also effective ranges are large.

$K\pi \rightarrow K\pi$

For the amplitudes of Eq. (2.38) one writes

$$\begin{aligned} A^{(+)} &= A + B(s+u) + Ct + D(p_1^2 + p_2^2) \\ A^{(-)} &= A'(s-u) \end{aligned} \quad (3.17)$$

where p_1, p_2 are kaon momenta.

(i) on-shell this gives the sum rule

$$a_0^{1/2} - a_0^{3/2} = 6m_K\mu(a_1^{1/2} - a_1^{3/2}) \quad (3.18)$$

(ii) PCAC implies

$$\begin{aligned} A + 2B m_K^2 + C \mu^2 + 2D m_K^2 &= 0 & (\text{soft } \pi) \\ A + 2B \mu^2 + (m_K^2 + D m_K^2) &= 0 & (\text{soft } K) \end{aligned} \quad (3.19)$$

(iii) PCAC + CA :

$$F_\pi^2 = F_K^2 \equiv F^2, \quad a_0^{1/2} - a_0^{3/2} = \frac{3 m_K}{m_K + \mu} L \quad (3.20)$$

(iv) PCAC + non-exoticity of σ terms (absence of $I = \frac{3}{2}$)

$$D = 0 \quad (\text{i.e., } P_1^2, P_2^2 \text{ dependence redundant})$$

$$A^{(+)} = \frac{1}{4 F^2} (s + u + 2t - 2m_K^2 - 2\mu^2) \quad (3.21)$$

$$A^{(-)} = \frac{1}{4 F^2} (s - u)$$

$$a_1^{3/2} = 0, \quad a_0^{1/2} / a_0^{3/2} = -2 \quad (\text{s-wave dominance : } a_0^{1/2} / a_0^{3/2} = 1)$$

$$a_0^{1/2} = 2 \frac{m_K}{m_K + \mu} L = (0.16 \pm 0.02) \mu^{-1}$$

$$a_0^{3/2} = -\frac{m_K}{m_K + \mu} L = (-0.078 \pm 0.008) \mu^{-1} \quad (3.22)$$

$$a_1^{1/2} = (0.011 \pm 0.001) \mu^{-3}$$

$$r_0^{1/2} = (-5.0 \pm 0.5) \mu^{-1}; \quad r_0^{3/2} = (17 \pm 2) \mu^{-1}$$

KK $\rightarrow$ KK [cf. Eq. (2.42)]

$$A^0 = A'(u-t), \quad A^1 = A + B(u+t) + Cs$$

(i) No sum rule involving only physical scattering lengths exists.

(ii) PCAC :

$$A + 2B m_K^2 + C m_K^2 = 0 \quad (3.23)$$

(iii) PCAC + CA :

$$A' = 0, \quad a_1^0 = 0, \quad B - C = \frac{1}{F^2} \quad (3.24)$$

(iv) PCAC + non-exoticity of σ terms

$$\begin{aligned} A^0 &\equiv 0 \\ A^1 &= \frac{1}{F_2} (t + u - 2m_K^2) \\ a_0^1 &= -\frac{m_K}{2\mu} L = (-0.18 \pm 0.02) \mu^{-1} \end{aligned} \quad (3.25)$$

We summarize this section by stressing the importance of the two different pieces of information provided by current algebra and the non-exoticity assumption (both interpreted by PCAC) :

- (i) Gell-Mann current algebra determines a derivative of the amplitude and fixes low energy p waves uniquely ;
- (ii) the structure of the σ term (i.e., the amount of exotics) determines the zero pattern below threshold - in particular ratios of s wave scattering lengths.

This information is not obtained by the Adler condition alone, in fact, several authors have considered the effect of allowing exotic contributions to the σ term while of course keeping the Adler zero ²³⁾⁻²⁷⁾.

These two types of information are crucial for fixing the amplitude and any theory must provide for these two basic types of input. In the following, we shall consider examples on how this comes about.

3.2. - Resonance couplings and KSFR relations.

Moving away from the low energy region we start this section by summarizing for easy reference the vector and tensor meson couplings to a pair of pseudoscalars. Then as a first step towards an understanding of the relation between the low energy and the resonance regions we comment on the dynamics behind the often discussed KSFR relation.

For the vector-pseudoscalar-pseudoscalar (VPP) coupling,
we write

$$\begin{aligned} \mathcal{L}_{VPP} = & i g_{\rho KK} \vec{P}_\mu (\bar{K} \frac{\vec{\tau}}{2} \delta_\mu K) + g_{\rho \pi \pi} \vec{P}_\mu (\vec{\pi} \times \partial_\mu \vec{\pi}) \\ & + i g_{K^* K \pi} (\bar{K}_\mu^* \vec{\tau} [K \delta_\mu \vec{\pi}] + [\vec{\pi} \delta_\mu \bar{K}] \vec{\tau} K_\mu^*) \\ & + i g_{K^* K \eta} (-\bar{K}_\mu^* [K \delta_\mu \eta] + [\bar{K} \delta_\mu \eta] K_\mu^*) \\ & + \frac{i}{2} (g_{\omega KK} \omega_\mu [\bar{K} \delta_\mu K] + g_{\phi KK} \phi_\mu [\bar{K} \delta_\mu K]) \end{aligned} \quad (3.26)$$

Here

$$K \equiv \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \bar{K} = (K^- \quad \bar{K}^0), \quad \pi^\pm = \frac{1}{\sqrt{2}} (\pi_1 \mp i \pi_2), \quad \pi^0 = \pi_3 \quad (3.27)$$

and similarly for ρ and K^* . The name of the field indicates the particle which is destroyed. Note that despite the apparent difference the pion convention is consistent with that of Eq. (2.31).

Within an SU_3 framework an ω - ϕ mixing angle occurs. In general, we define for physical fields X, X'

$$\left. \begin{aligned} X &= X_8 \sin \theta + X_1 \cos \theta \\ X' &= X_8 \cos \theta - X_1 \sin \theta \end{aligned} \right\} m(X') > m(X) \quad (3.28)$$

Then in a broken SU_3 scheme ^{(28), (18)}, we have

$$\begin{aligned} g_{\rho \pi \pi} = g_{\rho KK} &= 2g(1 + \alpha_1); \quad g_{K^* K \pi} = -g_{K^* K \eta} \sqrt{3} \\ &= g(1 - \alpha_1/2) \end{aligned} \quad (3.29)$$

From the experimental ρ and K^* widths

$$\alpha_1 \approx -0.14 \quad (3.30)$$

whereas in the limit of SU_3 symmetry

$$\alpha_1 = 0, \quad g_{\omega KK} = \sqrt{3} g \sin \theta; \quad g_{\phi KK} = \sqrt{3} g \cos \theta \quad (3.31)$$

Experimentally the $\omega - \phi$ mixing angle is

$$\theta_{\omega\phi} \approx 40^\circ \quad (3.32)$$

whether a linear or a quadratic mass formula is used. This is close to the ideal nonet value (SU_6 or quark model) ^{29), 30), 31)}

$$\tan^2 \theta = \frac{1}{2} \quad (3.33)$$

With this value the ideal nonet coupling limit is elegantly written for PVV and TVV (T : tensor mesons, $J^{PC} = 2^{++}$) ³¹⁾.

$$\begin{aligned} \mathcal{L} = & g_V^2 \text{Tr} \{ (P^\mu \overleftrightarrow{\partial}_\mu P^\nu) V_\mu^\nu \} \\ & + g_T^2 [\text{Tr} \{ (P^\mu \overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}_\nu P^\rho) T_{\mu\nu}^\rho \} \\ & + 2x \text{Tr} \{ (P^\mu \overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}_\nu P^\rho) T_{\mu\nu}^\rho \} + y \text{Tr} \{ (P'^\mu \overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}_\nu P'^\rho) T_{\mu\nu}^\rho \}] \end{aligned} \quad (3.34)$$

with

$$P^\mu = \begin{bmatrix} \frac{\eta_8}{\sqrt{6}} + \frac{\pi_0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_8}{\sqrt{6}} - \frac{\pi_0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{bmatrix}, \quad P' = \sqrt{\frac{1}{3}} \eta_0 \mathbb{1} \quad (3.35)$$

and similarly for T^9 with $(\omega_0, \omega_8, \vec{\rho}, K^*) \rightarrow (f_0, f_8, A_2, K^{**})$.

Conservation of C implies that the VPP coupling is pure F and the TPP coupling is pure D . x and y are related to the (η, η') mixing. Thus for ideal mixing ($\sin \theta = \sqrt{\frac{1}{3}}$), $x = y = 1$.

$$\left\{ \begin{array}{ccc} \frac{\sqrt{3}}{2} + \frac{\eta}{\sqrt{6}} - & K^* & - K^* \\ K^* & \frac{\sqrt{3}}{2} + \frac{\eta}{\sqrt{6}} - & \rho \\ K^* & \rho & \frac{\sqrt{3}}{2} + \frac{\eta}{\sqrt{6}} + \end{array} \right\} = \sqrt{3}$$

We now consider the KSFR relation ^{32),33)}

$$\frac{1}{F_\pi^2} = \frac{2 g_\rho^2 \pi \pi}{m_\rho^2} \approx \frac{96 \pi T_\rho}{m_\rho^3} \quad (3.36)$$

where terms of the order $(\mu/m_\rho)^2$ have been ignored. Experimentally we have for the left and the right-hand sides respectively

$$2.5 \pm 0.3 \quad \text{and} \quad 1.6 \pm 0.3$$

As pointed out by several people this relation has a simple interpretation in terms of $\pi\pi$ scattering ³⁴⁾⁻³⁷⁾.

Parametrizing δ_1^1 in the neighbourhood of the ρ resonance by a p wave Breit-Wigner form

$$\frac{e^{i\delta} \sin \delta}{q} = - \frac{g^2 q^2}{s - m_\rho^2 + i g^2 q^3}, \quad g^2 = m_\rho T_\rho / q^3 = \frac{2}{3} \frac{g_\rho^2 \pi \pi}{4\pi} \frac{1}{m_\rho} \quad (3.37)$$

and extrapolating to the threshold region ignoring finite width effects we get ($N^{-1} = 16\pi$)

$$A^{(1)} = \frac{2 g_\rho^2 \pi \pi}{m_\rho^2} (t-u) = \frac{96 \pi T_\rho}{m_\rho^3} (t-u).$$

Comparing with the current algebra prediction Eq. (3.14), the KSFR relation (3.36) results.

The validity of this relation has been interpreted as a success for ρ dominance and finite width corrections have been considered to improve agreement with experiment ^{37),36)}. As discussed by Kamal ³⁵⁾, however, the dynamics behind appears to be nowhere near ρ dominance in a straightforward meaning of the word. The point is that crossed channel ρ exchange gives a contribution to a_1^1 at threshold about the same magnitude as the direct channel resonance. Interestingly enough, however, the kind of E (or σ) particle predicted by Veneziano theory (see next subsection) or the algebraic

realization of chiral symmetry ^{38),39)} always "conspires" with the ρ to give back precisely Eq. (3.36) in the absence of other contributions.

Consider for example an $I_s = 1, s = 0$ dispersion relation (cf. subsection 5.5). Saturating with narrow width ρ and ϵ mesons gives

$$A^{(1)}(s=0, t, u) = g_{\rho\pi\pi}^2 \frac{t-s}{m_\rho^2 - u} + g_{\rho\pi\pi}^2 \frac{s-u}{m_\rho^2 - t} + 4g_{\epsilon\pi\pi}^2 \frac{1}{m_\epsilon^2 - t} - 4g_{\epsilon\pi\pi}^2 \frac{1}{m_\epsilon^2 - u} \quad (3.38)$$

where

$$T_\epsilon = 3 \frac{g_{\epsilon\pi\pi}^2}{4\pi} \frac{g_\epsilon}{m_\epsilon^2}$$

and the Lagrangian $\mathcal{L} = g_{\epsilon\pi\pi} \epsilon \cdot \vec{\pi} \cdot \vec{\pi}$ have been used. Expanding near $t = u = 2\mu^2$ and comparing with the current algebra prediction, we get

$$\frac{1}{F_\pi^2} = \frac{g_{\rho\pi\pi}^2}{m_\rho^2} + \frac{4g_{\epsilon\pi\pi}^2}{m_\epsilon^4} \quad (3.39)$$

This reproduces Eq. (3.36) precisely if

$$m_\epsilon = m_\rho, \quad T_\epsilon / T_\rho = 9/2 \quad (3.40)$$

This kind of ϵ state further has the effect of exactly cancelling the crossed ρ contribution to $I_s = 1, t = 0$ dispersion relation, thereby partly justifying the above derivation. However, although we thus have a suggestive link between Eqs. (3.36) and (3.40) none of the two can be derived at this level. Note that inconsistencies occur if one tries to account for crossed channel contributions as in perturbation theory ³⁵⁾, rather than by using dispersion relations to avoid double counting.

Equation (3.39) is the Adler sum rule in its most simple form ⁴⁰⁾ [cf. Eqs. (3.14), (5.40)]. In a phenomenological application of course the high energy contribution (Reggeized ρ exchange) should not be ignored (indications are that it is about 25% ¹⁰⁸⁾).

At this point we comment on an often cited paper by Bowcock and John ⁴¹⁾. The idea of these authors is to consider the $I = 1$ s channel forward amplitude supposed to be given by the p wave for $4\mu^2 \leq s \lesssim 1 \text{ GeV}^2$. In principle one might perform an analytic continuation from that region onto the left-hand cut thereby obtaining in particular the imaginary part of the combined u channel $I = 0, 1, 2$ contributions [Eq. (2.33)]. Ignoring $I = 2$ and taking the $I = 1$ part from the input, information on $I = 0$ results. In Ref. 41), a simple parametrization of the ρ bump in terms of known mass and width values is used. Using an ingenious prescription for carrying out the analytic continuation, the left-hand cut is reached although it is realized that due to the presence of experimental errors the result only makes sense on the average (see also subsections 4.4 and 5.2). However, the analytic form chosen did have an essentially zero imaginary part on the left-hand cut, so from the above we know that something like Eq. (3.40) must emerge. We conclude that to prove or disprove the existence of a strongly coupled $\mathcal{E}$ state by this kind of analytic continuation it is insufficient to exploit the presence of a ρ bump. In fact the object to be continued is the experimental deviation of the shape from a pure Breit-Wigner form (i.e., something with zero left-hand cut).

We finish this subsection by noticing that relations similar to Eq. (3.36) may be derived in other reactions. Thus, assuming K^* dominance of the $I = \frac{1}{2}$, $K\pi$ p wave gives in a similar way

$$\left(\frac{g_{\rho\pi\pi}}{g_{K^*K\pi}} \right)^2 = 4 \frac{m_\rho^2 - 4\mu^2}{m_{K^*}^2 - (m_K + \mu)^2} \quad (3.41)$$

In the SU_3 limit, $m_{K^*} = m_\rho$, $m_K = \mu$, this is consistent with Eq. (3.29) with $\alpha_1 = 0$. However, the correction to SU_3 represented by Eq. (3.41) is +30% whereas the phenomenological value of α_1 Eq. (3.30) gives a -35% correction. Therefore it does not seem particularly useful to pursue this game much further.

3.3. - Veneziano theory

In this subsection we summarize and discuss properties of conventional Veneziano models for binary processes involving pions and kaons and postpone to subsection 3.5 any remarks concerning the troubles introduced by going outside this framework. Lovelace⁴²⁾ in his fundamental paper pointed to a possible deep connection between the dual theory of strong interactions and current algebras or other symmetry schemes. At this moment it appears quite unclear whether this dream will be realized in any complete sense. However, it has been demonstrated that the most simple treatment in a very convenient way provides model amplitudes that accommodate rather consistently :

- (i) the low energy amplitudes of subsection 3.1 ;
- (ii) natural spin parity mesons in exchange degenerate SU_3 nonet patterns in reasonable agreement with experiment.

This achievement in itself is impressive, but recently the discussion has centred rather on the existing limitations preventing further progress :

- (iii) unrealistic lower daughters, i.e., presence of ghosts and lack of general factorization ;
- (iv) unrealistic off-shell extrapolations to safely impose current algebra constraints or study symmetry patterns implied by the model ;
- (v) lack of unitarity and diffractive contributions.

The new predictions of the model are :

- (vi) relations between masses and couplings of particles with different (arbitrary) spin at the parent level ;

- (vii) existence of daughter states, the coupling of which one should not take too seriously ;
- (viii) specific behaviours of low partial waves in exotic channels.

We now list the Lovelace-Veneziano type models for $\pi\pi$, πK and KK scattering. No comprehensive derivation can be given, but we shall make brief comments on several points and refer to the original literature ^{42)-46), 14)} for more detail as well as to existing reviews ^{6), 47)-50)}.

Defining

$$V_{xy} = - \frac{T(1-\alpha_x(s)) T(1-\alpha_y(t))}{T(1-\alpha_x(s) - \alpha_y(t))} \quad (3.42)$$

$$= -(1-\alpha_x(s) - \alpha_y(t)) B(1-\alpha_x(s), 1-\alpha_y(t))$$

we write ($N^{-1} = 16\pi$)

for $\pi\pi \rightarrow \pi\pi$

$$A(s, t, u) = f_{\pi\pi}^2 (V_{\rho\rho}(s, t) + V_{\rho\rho}(s, u) - V_{\rho\rho}(t, u)) \quad (3.43)$$

for $K\pi \rightarrow K\pi$

$$A^{(\pm)}(s, t, u) = \frac{1}{2} f_{K\pi}^2 (V_{K^*\rho}(s, t) \pm V_{K^*\rho}(u, t)) \quad (3.44)$$

for $KK \rightarrow KK$

$$A^0(s, t, u) = f_{KK}^2 (V_{\phi\rho}(t, u) - V_{\phi\rho}(u, t))$$

$$A^1(s, t, u) = f_{KK}^2 (V_{\phi\rho}(t, u) + V_{\phi\rho}(u, t)) \quad (3.45)$$

Several types of input have been provided at this level. Thus

α_ρ is the degenerate ρ -f-w- A_2 trajectory,
 α_K^* is the degenerate $K^*(890)$ - $K^{**}(1420)$ trajectory,
 α_ϕ is the degenerate $\phi(1019)$ - $f'(1500)$ trajectory.

These must be linear functions with equal slopes ⁴⁸⁾ from very general duality requirements. The degeneracy stems from the following SU_3 or quark model type information (supported by experiment) : absence of $I = 2$ $\pi\pi$ resonances implies ρ -f degeneracy, and presence of only one ρ state implies decoupling of f' from $\pi\pi$. Absence of $I = \frac{3}{2}$ $K\pi$ resonances implies $K^* - K^{**}$ degeneracy. Absence of $Y = 2$ KK resonances implies separate degeneracy of ρ - A_2 , ω -f and ψ - f' trajectories.

Figure 1 shows the Chew-Frautschi plot for the natural spin parity mesons indicating that exchange degeneracy works well ; also shown are daughter states [names from Ref. 47], some of which have experimental support (black circles) and most of which have not (open circles). Equations (3.43)-(3.45) exhibit correct spin structures of the (real) resonance poles and the normalization constants may be determined from the observed ρ and K^* widths. Also factorization of parent states is satisfied provided

$$f_{K\pi}^4 = f_{\pi\pi}^2 f_{K^*K}^2 \quad (3.46)$$

Lovelace further assumed that the forms (3.43)-(3.45) not only provided good approximations to the true amplitude on-shell, but also that going off-shell they choose to approximate well the particular form defined by PCAC. With this bold assumption the Adler condition gets predictive power and a number of surprising results follow.

- 1) - Trajectory intercepts become quantized and

$$\alpha_\rho(s) = \frac{1}{2} + \frac{s - \mu^2}{2(m_\rho^2 - \mu^2)}$$

$$\alpha_{K^*}(s) = \alpha_\rho(s) - \Delta$$

$$\alpha_\phi(s) = \alpha_{K^*}(s) - \Delta \quad (3.47)$$

with

$$\Delta \equiv \frac{m_K^2 - \mu^2}{2(m_\rho^2 - \mu^2)}$$

Those are the ones plotted in Fig. 1 directly. Equation (3.47) follows only if each term in Eqs. (3.43)-(3.45) is required to satisfy the Adler condition separately, otherwise terms with $V_{\rho\rho}$ and $V_{\eta\eta}$ in Eq. (3.45) could occur¹⁴⁾. These and satellite terms are discarded ad hoc⁴²⁾.

From Eq. (3.47) SU_3 nonet mass formulae with ideal mixing angle [Eq. (3.33)] follow. Thus for the $J^{PC} = 1^{--}$ and 2^{++} states^{14),43),44)}

$$\begin{aligned} m_{\phi}^2 - m_{K^*}^2 &= m_{K^*}^2 - m_{\rho}^2, \quad m_{\rho}^2 = m_{\omega}^2 \\ m_{f'}^2 - m_{K^{*+}}^2 &= m_{K^{*+}}^2 - m_{A_2}^2, \quad m_{A_2}^2 = m_f^2 \end{aligned} \quad (3.48)$$

in fair agreement with experiment^{30),31),51)}. In addition of course the linearity gives rise to an infinity of further mass relations not contained in simple symmetry schemes, such as

$$\begin{aligned} m_f^2 - \mu^2 &= 3(m_{\rho}^2 - \mu^2) \\ m_{\eta}^2 - \mu^2 &= 5(m_{\rho}^2 - \mu^2) \end{aligned} \quad (3.49)$$

The degeneracy of trajectories and the ideal singlet octet mixing follows from duality and absence of exotics alone without using the specific forms (3.43)-(3.45)⁵²⁾.

Ademollo et al.⁵³⁾ have used forms like Eq. (3.42) in conjunction with pion PCAC only, to suggest an even more general quantized structure of intercepts.

2) - The Euler function $B(1-\alpha_x, 1-\alpha_y)$ in Eq. (3.42) is much more slowly varying than V_{xy} itself within the triangle (3.1). Replacing it by a constant, all the low energy amplitudes of subsection 3.1. are precisely reproduced up to normalization. The normalization is given by the low energy theorems, provided^{45),46)}

$$f_{\pi\pi}^2 = f_{K\pi}^2 = f_{KK}^2 = g_{\rho\pi\pi}^2 \quad (3.50)$$

and

$$\frac{1}{F_{\pi}^2} = \pi \frac{g_{\rho\pi\pi}^2}{m_{\rho}^2}$$

or, allowing for the variation of the beta function,

$$\frac{1}{F_{\pi}^2} = 3.10 \frac{g_{\rho\pi\pi}^2}{m_{\rho}^2} \quad (3.51)$$

This may be considered a modified KSFR relation, however, in noticeably better agreement with experiment than Eq. (3.36). Also the ρ vs. $\mathcal{E}$ contribution ambiguity described in subsection 3.2 is here fixed and a model for the remaining contributions available.

3) - Parent states factorize in general and the condition that first daughters factorize is precisely the Adler condition Eq. (3.47)⁵⁴. Lower daughters do not factorize, however, for finite Δ [Eq. (3.47)] and their predicted widths are fairly meaningless. Examples on parent and daughter width relations are (elastic widths always)

$$\begin{aligned} T_{\mathcal{E}}/T_{\rho} &= 9/2 \text{ (cf. Eq. (3.40))}, & T_f/T_{\rho} &= 9\sqrt{3}/20 \\ T_g/T_{\rho} &= \frac{15\sqrt{5}}{112}, & \bar{T}_{\rho'}/T_{\rho} &= \frac{\sqrt{3}}{2} \\ \frac{T_K}{T_{K^*}} &= 6.6, & \frac{T_{K^{**}}}{T_{K^*}} &= 1.0, & \frac{T_{K^{**}'}}{T_{K^*}} &= 2.7 \end{aligned} \quad (3.52)$$

For further details and comparison with experiment, see Table II below.

4) - Definite multiplet coupling patterns arise. Thus for the signaturized amplitudes

$$V_{xy}^{\pm}(s, t, u) \equiv V_{xy}(s, t) \pm V_{xy}(s, u) \quad (3.53)$$

the pole residue at $\alpha_x(s) = J$ has the following spin structure ¹⁴⁾ :

(i) at right signature points

$$\begin{aligned} & \frac{-2(2\alpha' q_i q_f)^J}{\alpha'(J-1)!(2J+1)} C_J \quad \text{for (spin } J) \\ & \frac{-2(2\alpha' q_i q_f)^{J-1} \cdot J}{\alpha'(J-1)!(2J-1)} B C_{J-1} \quad \text{for (spin } J-1) \end{aligned} \quad (3.54)$$

(ii) at wrong signature points

$$\frac{-2(2\alpha' q_i q_f)^{J-1} \cdot J}{\alpha'(J-1)!(2J-1)} A C_{J-1} \quad \text{for (spin } J-1)$$

Here $q_i(q_f)$ is initial (final) c.m. three-momentum evaluated at the pole and

$$P_J(z) = \frac{z^J}{C_J} + \dots$$

$$A \equiv \frac{1}{2} [\alpha' \sum m_i^2 + \alpha_x(0) + 2\alpha_y(0) - 1] \quad (3.55)$$

$$B \equiv \frac{1}{2} [(m_a^2 - m_b^2)(m_c^2 - m_d^2)(\alpha')^2] / (J - \alpha_x(0))$$

For elastic scattering the partial wave amplitude $(\eta_J^I e^{2i\delta_J^I} - 1)/2iq$ has a residue in the narrow resonance approximation

$$R_J^I = T' m_J / q_J \quad (T' \text{ partial width}) \quad (3.56)$$

In general it is related to the product of resonance coupling constants to the initial and final states ⁵⁵⁾

$$R_J^I (i \rightarrow f) \propto g_i(J) g_f(J) (q_i q_f)^J / m_J \epsilon_f \quad (3.57)$$

Comparing Eqs. (2.14), (3.54) and (3.57) it is seen that the coupling constants for the leading trajectory states are independent of intercepts and masses and only depend on J and the quantum number contents. As mentioned above they then just become the SU_3 couplings Eq. (3.34) with ideal mixing. This result is certainly interesting although it is

clear that a fair amount of SU_3 and quark model information has been conveyed to the formulae. A convenient way of visualizing the way this comes about is given by the Harari-Rosner diagrams ⁵⁶⁾. At the first daughter level mass splittings explicitly enter the couplings via A and B Eq. (3.55) and SU_3 is broken for these couplings in a particular way. Since still lower daughters have completely unrealistic couplings, it is unclear what the significance of that is, however.

Having established the principal successes of the theory, we now comment on some of the troubles that have recently been much discussed.

1) - Since no Pomeron or diffractive contributions are present in the theory it is not clear whether the agreement of the low energy amplitudes with the soft meson ones (or with experiment for that matter) should be considered a success or a difficulty. Brooker and Taylor ⁵⁷⁾ tried (for $\pi\pi$) to add an ad hoc Pomeron form (fixed at high energies by factorization in πP , PP and $\bar{P}P$) and estimated that corrections as large as 50% might occur. However, in a Regge absorption model, diffractive effects are introduced by multiplying partial wave amplitudes by a slowly varying factor giving a negligible low energy effect ⁴⁷⁾. Also Kugler ⁵³⁾ noticed that (in $\pi\pi$) the amplitudes $A_x^{I, x=1}$ ($x = s, t, u$) at low energies satisfy unsubtracted fixed x dispersion relations with no high energy Pomeron contribution (cf. also subsection 5.5). Since then all amplitudes can be expressed as a linear combination of these, the form (3.43) is justified and adding something would be double counting. The argument of course only works at low energies and only when exotic channels are present.

2) - A large amount of work has been devoted to studies of what the underlying breaking mechanism for hadron symmetries implied by the model is. It seems clear by now that the model is not sufficiently well defined to make such a study meaningful. We shall briefly see at which level the breakdown takes place.

In $\pi\pi$, the σ term is given by

$$A(0, \mu^2, \mu^2) = -2g_{\pi\pi\pi}^2 \frac{T(1-\alpha_p(0)) T'(1-\alpha_p(\mu^2))}{T(1-\alpha_p(0) - \alpha_p(\mu^2))} = O(\mu^2 \alpha') \quad (3.58)$$

when α_p is given by Eq. (3.47). This led Lovelace⁴²⁾ and Yellin⁵⁹⁾ to speculate that the deviation from $SU_2 \times SU_2$ chiral symmetry was small and determined by [Eq. (3.47)]

$$\alpha_p(0) - \frac{1}{2} = O(\mu^2 \alpha') \quad (3.59)$$

(in itself an impressive prediction) so that an analysis according to the GOR scheme¹⁹⁾ is suggested. Cronin and Kang⁶⁰⁾, however, noticed that in $K\pi \rightarrow K\pi$ this is impossible with the form Eq. (3.44). The point is that in addition to the two usual low energy theorems obtained by taking either the two pions or the two kaons off-shell, GOR predict that the two functions obtained by restricting a particular $K\pi$ amplitude to the two lines corresponding to the one-pion soft and one-kaon soft respectively, have similar s dependences. That is impossible for Eq. (3.44) with $m_K^2 - \mu^2 \neq 0$. At first sight this may be cured simply by multiplying Eqs. (3.43)-(3.45) with off-shell form factors, one for each particle^{61), 46)}, $X_i(q_i^2)$ $i = a, b, c, d$ with $X_i(m_i^2) = 1$. Interestingly, these are then required to have poles at the positions expected from Ref. 53). However, Fayyazuddin⁶²⁾ and Ellis and Osborn⁶²⁾ showed that whichever the form factors chosen an inconsistency persists already with the Gell-Mann algebra. One considers the difference between the values of the amplitudes at the two off-shell points

$$(s, t, u) = (0, \mu^2, m_K^2) \quad ; \quad (s, t, u) = (0, m_K^2, \mu^2).$$

By PCAC this is related to the matrix element of a difference between two σ terms (3.8). Using the Heisenberg equations of motion and the Jacobi identity, this is given by the Gell-Mann charge algebra. Using the low energy theorems, F_π , F_K and all form factors may be eliminated to give

$$(m_K^2 - \mu^2) f_+(0) = [V_{K^*p}(0, m_K^2) - V_{K^*p}(0, \mu^2)] \\ \times \left\{ \frac{\partial}{\partial s} V_{K^*p}(s = m_K^2, 0) \frac{\partial}{\partial s} V_{K^*p}(s = \mu^2, 0) \right\}^{-1/2} \quad (3.60)$$

where $f_+(0)$ is the well-known $K_{\ell 3}$ form factor [see, e.g., Ref. 63) and subsection 5.4]. This is an explicit function of the masses

$$x \equiv (m_K^2 - \mu^2) \alpha' \propto SU_3 - \text{breaking} \\ y \equiv \alpha'^2 \propto SU_2 \times SU_2 - \text{breaking}$$

and so

$$f_+(0) = 1 + O(xy^2)$$

in contradiction with the Ademollo-Gatto theorem ⁶⁴⁾

$$f_+(0) = 1 + O(x^2)$$

In $KK \rightarrow KK$ the inconsistency is simpler. With Eq. (3.45) the low energy theorems read

$$\frac{1}{2F_K^2} = \frac{\partial}{\partial t} V_{\phi p}(t = m_K^2, 0) f_{KK}^2 \\ \frac{1}{2F_K^2} = \frac{\partial}{\partial u} V_{\phi p}(0, u = m_K^2) f_{KK}^2 \quad (3.61)$$

however, the two right-hand sides are in the ratio 1.8 : 1 and this will continue to be true for any form factor modification : both get multiplied by $x_K(0) x_K(m_K^2)$ ⁴⁶⁾.

What is the lesson to be learned from these troubles ?

As we have noted, present day Veneziano theory even on-shell and in the tree-approximation is unable in a comprehensive form ⁶⁵⁾ to accommodate particles with realistic quantum numbers and trajectory intercepts. Further, the off-shell extrapolations defined by Eqs. (3.43)-(3.45) whether with form factors or not have unrealistic singularity structures ⁶⁶⁾ and a fundamentally new approach is needed to incorporate

currents in the theory. In view of that it is not surprising that inconsistencies occur. Perhaps it is more surprising that one has effectively to go to second order derivatives (as above) before they become obvious. By patching with satellites, the troubles may be moved a bit further away ⁶⁷⁾⁻⁶⁹⁾, but nothing very useful seems to be learned thereby.

3) - The model's lack of unitarity was an obvious trouble from the very beginning and when comparison with experiment is wanted at the phase shift level, something has to be done about it. The dual multi-loop program has recently had very impressive formal progress but since it so far only applies to a very unrealistic model world, it has little practical value as yet [see for example Ref. 70], and references contained therein]. Since unitarity requires Regge trajectories to become complex above threshold, the simplest phenomenological prescription has been to add to the α 's of Eq. (3.42), imaginary parts above thresholds and leave them unchanged below. However, not only does that introduce ancestors, violate crossing and analyticity, but it still violates unitarity badly in that resonances like ρ and ϵ get equal widths although they have very different couplings. Lovelace's K matrix device overcomes partly these troubles. For an elastic partial wave amplitude, F_ℓ , normalized as $(e^{2i\delta_\ell} - 1)/2iq$ and for the corresponding partial wave projection of the Veneziano formula, V_ℓ , one writes ⁴⁷⁾

$$F_\ell(s) = V_\ell(s) / (1 + \rho(s) V_\ell(s)) \quad (3.62)$$

This form satisfies elastic unitarity exactly provided $\rho(s)$ has an imaginary part

$$\text{Im} \rho(s) = -q$$

above threshold. Then infinities associated with real axis poles have been removed and sensible phase shifts may be drawn (Figs. 2 and 3). Since no genuine new information has been conveyed to the theory it is not to be expected that parameters like resonance masses and widths are

improved over the ones in the simple version. In particular, low energy crossing is not well satisfied since $\rho(s)$ has been taken independent of ℓ . Therefore the fairly large predicted corrections to scattering lengths need not be taken seriously. In fact several people have considered the effect of partly restoring crossing by more complicated $\rho_\ell(s)$ functions thereby giving evidence that unitarity corrections to scattering lengths are probably not much more than 20%⁷¹⁾⁻⁷⁵⁾. Also (in $\pi\pi$) the Adler constraint may readily be re-introduced by putting $\rho_\ell(\mu^2) = 0$, if desired.

The K matrix method is easily extended to take into account several two-body channels. Lovelace thus considered the combined $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow K\bar{K}$, $K\bar{K} \rightarrow K\bar{K}$ system [cf. Eqs. (2.17), (2.18)]. This has no noticeable effect on the $(\pi\pi)$ phase shifts below ~ 850 MeV, however, the presence of the S^* (Fig. 1, Table II) and the $K\bar{K}$ threshold at ~ 995 MeV crucially influences the behaviour of δ_0^0 above that energy (see also subsections 4.6, 4.7).

Table II is a (slightly up-dated) reproduction of Lovelace's⁴⁷⁾ with parent state parameters listed first. These may serve as useful reference guides. As for daughter parameters the important claim of the theory is that such states should exist. However, masses and couplings are likely to be severely distorted. In particular phase shifts need not resemble Breit-Wigner's at all. In Figs. 2 and 3, resonating s waves are thus likely to be strongly influenced by (unrealistic) lower daughter states above the resonance. The form of exotic s waves and p waves is less model dependent although the above-mentioned violation of low energy crossing would mean that (absolute) values are too large, may be by as much as 30%. For further details on the experimental situation see sections 4 and 5 as well as the Particle Data Group listings.

3.4. - General principle theories on low energy $\pi\pi$ interaction.

Well over half the total theoretical activity on meson-meson scattering continues to be devoted to considerations on low energy $\pi\pi$ scattering in which the notions of crossing, analyticity and unitarity constitute (or are claimed by the authors to constitute the most important part of the input. Roughly half of this activity takes the form of model building employing the above constraints in an approximate way, whereas the remaining half is devoted to working out rigorous consequences of these.

On the model side we further divide the discussion according to the following two somewhat reciprocal problems :

- (i) what is the minimum input needed in addition to the above general principles to completely determine the low energy amplitude ?
- (ii) are current algebra (or Veneziano) type amplitudes consistent with unitarity at all, and if so, what is the extrapolation to the resonance region (<1 GeV) ?

An enormous number of attempts has been made to solve these problems, and the spread in suggested answers has not been less enormous. Recently a considerable clarification has taken place, however, and it seems that results of general physical interest not contained in the previous subsections have emerged. This is due in part to the equally large effort made on the rigorous side of working out constraints according to the Martin program which we discuss first [for reviews, see Refs. 76)-79)].

One considers the following problem. Find and describe in the most convenient way the class of functions $A(s,t,u)$ satisfying the conditions of crossing, analyticity and positivity pertaining to a $\pi\pi$ amplitude [Eq. (2.34)].

The crossing properties were described in subsection 2.4.

The analyticity properties required are a subset of those following from the Mandelstam representation [for an elaborate list of results proved in axiomatic field theory, see, e.g., Ref. 78] :

- (i) for $0 < s < 4\mu^2$ the amplitude is real analytic in the t plane cut along $(-\infty, -s)$ and $(4\mu^2, \infty)$, and similarly (by crossing) for other pairs of (s, t, u) ; in this interval the amplitude satisfies a fixed s dispersion relation with at most two subtractions ;
- (ii) for $s > 4\mu^2$ the amplitude is analytic in such a way that partial wave expansions converge within the Lehmann ellipse : foci at $t = 0$ and $t = -(s - 4\mu^2)$ and semi-major axis $a(s) = \frac{1}{2}(s + 4\mu^2)$.

Most of this is conveniently summarized in the well-known Mandelstam diagram for $\pi\pi$ scattering (Fig. 4). An s channel partial wave amplitude is defined by an integration of the invariant amplitude for fixed s between the lines $x_s = -1$ and $x_s = +1$ ($x_s \equiv \cos\theta_s$), i.e., $t = -(s - 4\mu^2)$ and $t = 0$. For fixed $s \geq -4\mu^2$ the nearest singularities in $\cos\theta_s$ are the t and u channel normal thresholds, $t = 4\mu^2$ ($u = 4\mu^2$). Very many properties of partial wave amplitudes (pwa's) become obvious when looked at this way. For example s channel pwa's are real analytic functions in the s plane cut along $(-\infty, 0)$ and $(4\mu^2, \infty)$. Near $s = 4\mu^2$

$$\text{Im} f_\ell(s) \propto q^{4\ell+1} \quad (3.63)$$

and near $s = 0$

$$\text{Im} f_\ell(s) \propto (-s)^{3/2}$$

the coefficient of the last being easily related to the crossed channel scattering lengths.

By positivity is meant that for pwa's with definite isospin one has ($N > 0$)

$$\text{Im} A^{\ell, I}(s) > 0, \quad s \geq 4\mu^2 \quad (3.64)$$

which is a considerably weaker condition than the full unitarity requirement [cf. Eq. (2.14)]

$$\text{Im } A^{I,l}(s) = \frac{2\epsilon N_q}{\sqrt{s}} R_l^I |A^{I,l}|^2 \quad (3.65)$$

where $\epsilon = \frac{1}{2}$ for $\pi\pi \rightarrow \pi\pi$ and

$$R_l^I \equiv \sigma_{\text{tot}}^{I,l} / \sigma_{\text{el}}^{I,l} \equiv 1 \quad , \quad s \leq 16\mu^2 \quad (3.66)$$

That amplitudes do exist which satisfy these requirements exactly even with full Mandelstam analyticity and with Eqs. (3.65), (3.66) was shown by Atkinson [Ref. 80] and earlier references cited there, see also Kupsch, 81]. However, little is known about the low energy parameters corresponding to these models and to what extent they exhaust the possibilities.

By far the majority of the work on the Martin program has taken the form of working out constraints among pwa's in the unphysical region $0 \leq s \leq 4\mu^2$ [for exceptions, see Roskies, 82) and Wanders, 83]. These take on all sorts of complicated forms and the significance of many is not properly understood. Nevertheless, we shall try to comment on the different kinds of information obtained.

A - Constraints from crossing symmetry alone ⁸⁴⁾⁻⁸⁷⁾

Necessary and sufficient conditions for a candidate set of partial wave amplitudes to belong to a crossing symmetric amplitude were obtained by Balachandran and Nuyts ⁸⁴⁾ and given a very simple interpretation by Roskies ⁸⁵⁾ and Basdevant et al. ⁸⁶⁾. For the completely symmetric amplitude $F(s,t,u)$ for $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ for example, one has with integer N and P

$$\int \int ds (s-u)^{2N+1} F(s,t,u) t^P = 0 \quad (3.67)$$

where the integration is over the Mandelstam triangle $s, t, u > 0$. Performing the integrations over $x_s \equiv \cos\theta_s = 1 + 2t/(s-4\mu^2)$ and s , relations between polynomial moments of the pwa's $f_l^{00}(s)$ with $l \leq 2N + 1 + P$ result. The completeness of these follow from the completeness of the polynomials (in two variables) used over the Mandelstam triangle. The relations involving s waves only, are

$$\begin{aligned} \int_0^{4\mu^2} (s-4\mu^2)(2f_0^0(s) - 5f_0^2(s)) ds &= 0 \\ \int_0^{4\mu^2} (s-4\mu^2)(f_0^0(s) + 2f_0^2(s)) ds &= 0 \end{aligned} \quad (3.68)$$

For finite L the number of relations involving f_l^I with $l \leq L$ only, increases with L (for $L = 1$ there are five). These "Roskies constraints" have been proved particularly useful for building and testing low energy $\pi\pi$ models.

Another interesting consequence of crossing alone is that for given s waves, d waves (and higher) follow surprisingly uniquely⁸⁷⁾. Of the many complicated relations ensuring that we quote as examples (for $\pi^0\pi^0 \rightarrow \pi^0\pi^0$)

$$\begin{aligned} (f_0(0.572) - f_0(1.294))/2.050 &< f_2(0.572) < (f_0(0.572) - f_0(1.087))/1.498 \\ (f_0(2.288) - f_0(1.091))/1.929 &< f_2(2.288) < (f_0(2.288) - f_0(1.244))/1.620 \end{aligned} \quad (3.69)$$

Qualitatively, the point is that a big d wave arises from a big curvature of F in the $t u$ direction which by crossing implies a big curvature in the s direction, to show up in the s wave since that is just the average of F (for fixed s).

B - Conditions following from positivity alone ^{88),89),16)}

Also here necessary and sufficient conditions have been obtained^{88),89)}, however, the constraints have a much more complicated form in that either infinitely many pwa's are involved or else only inequalities can be obtained. Basically two types of information arise :

- (i) signs on derivatives are fixed, in fact the shape of the s wave $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ amplitude $f_0^{00}(s)$ is determined in $0 < s < 4\mu^2$ almost uniquely up to transformations

$$f_0^{00} \rightarrow a f_0^{00} + b, \quad a > 0, b \text{ real} \quad (3.70)$$

and the curvature is positive ⁷⁶⁾ ;

- (ii) positivity improves the smoothness information of analyticity ; thus ^{76), 88)}

$$a_{\ell+2}^{00} < \frac{1}{16} [((\ell+2)(\ell+1))/((\ell+\frac{5}{2})(\ell+\frac{3}{2}))] a_{\ell}^{00}, \quad (3.71)$$

for $\ell \geq 2$

where a_{ℓ}^{00} are the $\pi^0 \pi^0$ elastic scattering lengths ; from Fig. 4, it is almost trivial that (3.71) holds for sufficiently large ℓ , however, analyticity alone is unable to say for how large ℓ . To prove relation (3.71), one uses the Froissart-Gribov formula for $\ell \geq 2$

$$f_{\ell}^{00}(s) = \frac{4}{\pi(s-4\mu^2)} \int_{4\mu^2}^{\infty} Q_{\ell}\left(\frac{2t}{4\mu^2-s} - 1\right) F_{\ell}(s,t) dt \quad (3.72)$$

with

$$F_t = \sum_{\ell} (2\ell+1) \text{Im} f_{\ell}^{00} P_{\ell}(\cos \theta_t)$$

the validity of which follows from analyticity (i), (ii) above.

The basic new information yielding (3.71) is the positivity of F_t [that follows since $\text{Im} f_{\ell}^I \geq 0$ for $I = 0, 1, 2$ implies $\text{Im} f_{\ell}^{00} \geq 0$; also for $0 \leq s \leq 4\mu^2$ and $t \geq 4\mu^2$, $\cos \theta_t \geq 1$ so that $P_{\ell}(\cos \theta_t) \geq 1 > 0$].

Similar relations for any I may be derived.

C - Conditions following from positivity and crossing 90)-94)

By now an infinity of such relations are known. However, unlike in the above two cases no sufficient set of constraints have been obtained. The known relations in general restrict further the amplitude but often have a very complicated and intransparent form.

In neither of the above cases A, B, C does one know the most "efficient" set of constraints. This means that for finite L, the complete set of relations between the f_{ℓ}^I 's with $\ell \leq L$ only, is unknown.

It is possible that given the needed effort a more complete solution may be obtained. As far as the constraints of crossing, analyticity and positivity go, however, it is important to realize that these are invariant under the transformations

$$A(s, t, u) \rightarrow A'(s, t, u) = a + bs + cA(s, t, u), \quad c > 0 \quad (3.73)$$

This means that they admit low energy s and p waves with virtually any scattering lengths and effective ranges, so that taken alone these constraints would seem to be insufficient to resolve phase shift ambiguities for example. Once s and p waves are given the constraints are able to fix almost uniquely the complete amplitude, however.

To get absolute bounds on the lower partial waves, it is therefore necessary to exploit at least the non-linear content of unitarity. Bounds can then be obtained (however, only because the additional phenomenological assumption of no stable $\pi\pi$ state is made) 95)-100). Thus for the Chew-Mandelstam coupling constant

$$\lambda \equiv -2A(4\mu^2/3, 4\mu^2/3, 4\mu^2/3)$$

and for the scattering lengths, one obtains

$$\begin{aligned} -2.6 < \lambda < 17 \\ a_0^2 \geq -7.1\mu^{-1}, \quad a_0^0 \geq -7.5\mu^{-1}, \quad a_0^{\circ} > -2.30\mu^{-1} \end{aligned} \quad (3.74)$$

When compared with the current algebra predictions Eq. (3.15) and

$$\lambda = -\frac{\mu^2}{96\pi F_\pi^2} = -(0.0083 \pm 0.0008), \quad a_0^{00} = \frac{2}{3}a_0^2 + \frac{1}{3}a_0^0 = (0.025 \pm 0.003)\mu^{-1} \quad (3.75)$$

they appear to have mainly academic interest, although they do represent a considerable improvement^{96),100)} over the original version⁹⁵⁾.

We now discuss problem (ii) above : are current algebra type amplitudes consistent with unitarity ?

The question is very relevant. We saw in subsections 3.2 and 3.3 that an $\mathcal{E}$ state with $m_{\mathcal{E}} = m_\rho$, $\Gamma_{\mathcal{E}} = 9/2\Gamma_\rho$ was either indicated or predicted. If now the same simple-minded reasoning which led to the KSFR relation is applied, we get

$$a_0^0 \approx \frac{\Gamma_{\mathcal{E}}}{m_{\mathcal{E}}} \mu^{-1} \approx \frac{9}{2} \frac{\Gamma_\rho}{m_\rho} \mu^{-1} \approx 0.7 \mu^{-1} \quad (3.76)$$

which is four times the current algebra value so that a major inconsistency is conceivable.

Several authors have claimed that in fact very large unitarity corrections to Weinberg's scattering lengths were necessary and/or that $\delta_0^0 \leq 30^\circ$ below 1 GeV is required^{101),34),102),103)}. Several of these treatments ignore partly or completely crossing constraints and some have been shown⁸⁵⁾ to be explicitly inconsistent. One is left with the impression that the more carefully crossing is treated the smaller are the unitarity corrections and the larger is the $\mathcal{E}$ width so that in fact $\Gamma_{\mathcal{E}}/\Gamma_\rho \simeq 9/2$ may well be consistent^{104),105)}. At any rate it can now be said with considerable confidence that models do exist that approximate closely Weinberg's amplitude, show a broad $\mathcal{E}$ resonance and satisfy unitarity and crossing to a high degree of approximation¹⁰⁶⁾⁻¹⁰⁸⁾.

Perhaps the most elegant approach is that based on a Padé approximant treatment of effective Lagrangians satisfying current algebra with non-exotic σ terms explicitly^{106),107)}. One exploits the analytic properties of the perturbation series expansion terms not only as functions of s and t but also of the coupling constant. The philosophy is that this information is enough to calculate the approximate amplitude from a few low order Feynman graphs even if the coupling constant is far too large for the perturbation sum to make sense (for recent reviews, see Zinn-Justin^{109),110)}). Whether the philosophy holds and the method can be used in a fundamental way to build a hadronic theory is not clear. Existing models have some essential drawbacks : the ratio of the ρ width and the f width is not right, trajectories behave in an unsatisfactory way and exotic resonances occur at higher energies. However, as far as the low energy region goes it has been shown that basically the same unitary crossing symmetric current algebra-like amplitude emerges whether one starts with the σ model¹⁰⁶⁾ and predicts the ρ (σ curves on Fig. 2) or one starts with an elementary Yang-Mills ρ field coupled to a conserved current¹⁰⁷⁾, and predicts a σ . Note that the ρ dominance dynamics only makes sense in the weak coupling limit (cf. subsection 3.2), but consistently renormalizes as the coupling is increased.

We finally discuss the first question raised in this subsection : uniqueness of the low energy $\pi\pi$ amplitude.

It is an important fact that Weinberg's amplitude is consistent with low energy unitarity. However, on-shell Weinberg predicts two parameters [say, a_1^1 (or L) and a_0^0/a_0^2] and we would like to know whether arbitrary values of these are equally consistent with unitarity so that a two-dimensional ambiguity prevails. From the Chew-Mandelstam treatment²²⁾ it is immediately clear that the freedom is at least one-dimensional since for a large range of λ 's solutions were found (with $a_0^0/a_0^2 \simeq 5/2$). After almost a decade of confusion there now appears to be strong hints that in a certain sense the ambiguity is essentially only one-dimensional and there are even interesting claims as to the nature of it.

One tries to build the amplitude from pwa's parametrized in some way or other to satisfy unitarity. Realizing that some information from outside is needed, the most common thing to do has been to assume the p wave to be given by experiment which at low energies essentially amounts to input $a_1^{(108), (111), (112)}$, however, other (roughly equivalent) approaches are equally possible $^{(113), (114)}$. Then the important crossing information is imposed in some way or other. Whether one prefers to use variations over the N/D iteration theme $^{(102), (115)}$ or to impose "Roskies-constraints" and others of the above kind seems largely to be a matter of taste. In the latter case, part of the positivity information is satisfied automatically by the chosen unitary parametrization. A careful treatment allowing considerable flexibility is that of Le Guillou et al. $^{(108)}$. Left-hand cuts of the pwa's (d waves are neglected in most treatments) are elaborately treated, thereby eliminating some of the parameters. More are fixed by imposing rigorous constraints in the Mandelstam triangle. The results are shown on Fig. 2 (GMN) and the range of allowed solutions corresponds to $\pm 15^\circ$ at the upper energy for δ_0^0 and $\pm 5^\circ$ for δ_0^2 . For the solution of Fig. 2, the scattering lengths are $a_0^0 = 0.21 \mu^{-1}$, $a_0^2 = -0.073 \mu^{-1}$, $a_1^1 = 0.045 \mu^{-3}$ in good agreement with Eq. (3.15).

Solutions with much larger deviations were obtained in Ref. 111). However, these are in obvious disagreement with the rule Eq. (3.13) indicating that d waves are very important $^{(116)}$:

$$2a_0^0 - 5a_0^2 = 18\mu^2 [a_1^1 - 5\mu^2(2a_2^0 - 3a_2^2) + \dots] \quad (3.77)$$

in contradiction with the assumptions. In fact by treating d waves in a more consistent way these solutions were recently eliminated by Bonnier $^{(117)}$.

The solutions of Le Guillou et al. $^{(108)}$ (Fig. 2) are noteworthy in connection with the possible existence of an $\mathcal{E}$ meson. Although δ_0^0 barely passes through 90° and if so at a very high energy, a second sheet $\mathcal{E}$ pole is found, having a very stable mass of

$m_\epsilon \simeq 420$ MeV with a very large but less stable width of $\Gamma_\epsilon = 270-520$ MeV. Also, corresponding to the solution of Fig. 2, the ϵ pole residue [of the partial wave normalized as $\sqrt{s}/q (e^{i\delta_0^0} \sin \delta_0^0)$ is ¹¹⁸⁾

$$R_{pole} = 14.7 e^{i56^\circ} \mu^2 \quad \text{for } (m_\epsilon, \Gamma_\epsilon) = (420 \text{ MeV}, 380 \text{ MeV}) \quad (3.78)$$

whereas a narrow resonance approximation would give

$$R_{N.R.} = -2 \Gamma_\epsilon m_\epsilon \sqrt{m_\epsilon^2 - 4\mu^2} = -22.0 \mu^2 \quad (3.79)$$

This is a show-case for demonstrating how little a resonating phase shift need resemble a Breit-Wigner one, and how large the penalties are likely to be if narrow resonance formulae are used in phenomenological work in the ϵ . Below threshold the $\pi\pi$ amplitude of Ref. 117) can no doubt be fitted very well by real axis poles (it is close to either Weinberg's or Lovelace-Veneziano's forms) but obtained parameters need have little to do with the dynamically significant residue Eq. (3.78). This is also part of the reason why the estimate (3.76) is fairly meaningless, the other part of the reason being that the derivation ignores the presence of a below threshold zero [Eqs. (3.15)] that strongly influences the shape of the $I = 0$ s wave.

Results of the above models are further illuminated by the work of Dilley ¹¹⁹⁾. He considers the class of amplitudes $A(s, t, u)$ (first used by Iliopoulos ¹²⁰⁾) that can be represented by polynomials in the variables

$$q_s \equiv \frac{1}{2} \sqrt{4\mu^2 - s}, \quad q_t \equiv \frac{1}{2} \sqrt{4\mu^2 - t}, \quad q_u \equiv \frac{1}{2} \sqrt{4\mu^2 - u}. \quad (3.80)$$

Crossing symmetry and correct threshold behaviour is readily imposed and elastic unitarity can be approximately enforced by a numerical minimisation procedure. Figure 5 shows what parts of the (a_0^0, a_0^2) plot are populated by solutions. Near zero they can be described in a first approximation as two (one-dimensional) lines having $a_0^0/a_0^2 \simeq 5/2$ (I) (the s wave dominant solutions of Chew et al. ²²) and $a_0^0/a_0^2 \approx -7/2$ (II) (Weinberg's non-exotic σ term value). If a_1^1 is assumed from the ρ and d waves are ignored, the line $2a_0^0 - 5a_0^2 = 18\mu^2 a_1^1$ [Eq. (3.13)] intersects branch II close to the CA point. This is completely consistent with the aforementioned models. If it is true it suggests the following exceedingly interesting "theorem" (ignoring the bend-away of branch II) :

For not too large values of the scattering lengths, the requirements of crossing, analyticity and unitarity imply either

- (1) s wave dominance, $a_0^0/a_0^2 \simeq 5/2$, or
- (2) non-exoticity, $a_0^0/a_0^2 = -7/2$.

Thus, if the Veneziano model had had a_0^0/a_0^2 appreciably different from $-7/2$ it would have been inconsistent with low energy unitarity. Now, however, it is so consistent that unitarity (at least in its low energy elastic form) is unable to fix the over-all constant (this statement is consistent with, but does not follow from, the multiloop perturbative point of view).

There are considerable reasons to believe that this theorem is not literally true. Existing work, however, would suggest that some penalties like large d waves or unrealistic high energy contributions would be associated with models corresponding to points outside the regions of Fig. 5. What is next needed is a way of describing and measuring what "large" and "unrealistic" means (see also subsection 5.5).

The result, as it stands, may partly explain why so many works in the past assuming so much about the singularity structure (even resonances masses) have been so unsuccessful. The prospects for a combined low energy and high energy bootstrap is harder to assess. The latest strip approximation treatments based on P, P' and ρ trajectories gave low energy amplitudes similar to Fig. 2, but several other features are unsatisfactory or not clearly understood ¹²¹).

3.5. - The η problem.

The simplest Veneziano formula for $\pi\eta \rightarrow \pi\eta$ scattering has the form ^{14),54),122),123)} [cf. Eq. (3.42)]

$$A(s,t,u) \propto V_{A_2 f}(s,t) + V_{A_2 A_2}(s,u) + V_{f A_2}(t,u) \quad (3.81)$$

which ensures correct signatures for A_2 and f (degenerate with ρ and ω - see subsection 3.3). Imposing the Adler zero on each term in the usual way gives [whether in the soft pion limit ($s = m_\eta^2$, $t = \mu^2$, $u = m_\eta^2$) or in the soft η limit ($s = \mu^2$, $t = m_\eta^2$, $u = m_\eta^2$)]

$$1 - \alpha_\rho(\mu^2) - \alpha_\rho(m_\eta^2) = 0, \text{ i.e. [Eq. (3.47)]} \quad (3.82)$$

$$m_\eta = \mu$$

Much discussion has been devoted to this miserable prediction and many cures have been proposed, but nothing very satisfactory has so far emerged.

Baacke et al. ¹²²⁾ point out that when applied to the reaction $\pi\eta' \rightarrow \pi\eta$ (relevant to $\eta' \rightarrow 2\pi\eta$ decay - one of the rare cases where experimental information on a $0^-0^- \rightarrow 0^-0^-$ amplitude involving the η is conceivable), the form (3.81) is anyway inconsistent with phenomenology in that a large δ coupling is predicted in disagreement with the observed small δ width. If one then feels forced to modify Eq. (3.81) by adding satellites it is easy enough to achieve an Adler zero without Eq. (3.82) via a cancellation mechanism. That, however, is against the rules which gave predictive power to the Adler constraint in subsection 3.3.

Osborn notices that already at the soft meson level amplitudes involving η 's have a qualitatively different off-shell extrapolation from the ones of subsection 3.2 ¹⁵⁾. Thus in the GOR model, the $\pi\eta \rightarrow \pi\eta$ amplitude is

$$A(s, t, u, q_1^2, k_1^2, q_2^2, k_2^2) = \frac{1}{3F_\pi^2} \left\{ s + t + u - 3\mu^2 - \frac{m_\eta^2 - \mu^2}{m_\eta} (k_1^2 + k_2^2) \right\} \quad (3.83)$$

where q 's and k 's are pion and η momenta. Thus explicit dependence on the k 's are indicated and if this is replaced by the on-shell value the Adler zero is not present (i.e., this new off-shell form is irrelevant for CA). Maybe therefore the off-shell extrapolation of Eq. (3.81) is qualitatively different from that for Eqs. (3.43)-(3.45). Similar points have been made ^{124), 125), 126)} arguing from simple Lagrangian models satisfying the GOR scheme ¹⁹⁾. The form Eq. (3.83) gives on-shell a constant amplitude corresponding to a scattering length

$$a_0^1 \approx 0.008 \mu^{-1} \quad (F_\eta = F_\pi) \quad (3.84)$$

The treatment is unrealistic, however, in that the coupling of η' to the divergence $\partial_\mu A_8^\mu$ has to be ignored.

Brout et al. ¹²⁷⁾ using various types of chiral and dual input achieve the Adler condition through a special cancellation scheme (not using Veneziano forms), thereby obtaining the attractive rules (together with many more)

$$\alpha_\eta(0) = -\frac{1}{4}, \quad \alpha_{\eta'}(0) = -\frac{3}{4} \quad (3.85)$$

However, even though one may be able to get around the Adler trouble at this level, the prediction (3.82) appears to be a more fundamental feature of the whole dual scheme. We have seen in $0^-0^- \rightarrow 0^-0^-$ that SU_3 quantum numbers of external and internal states fixes masses and couplings of the internal (V, T, etc.) states at the quark model nonet values; for example $m_\omega = m_\rho$ and $m_F = m_{A_2}$ is predicted. Similarly by considering binary processes with external vector mesons so that pseudoscalars can couple internally, the result (3.82) is extremely hard to avoid ^{128), 129), 130)}. More recently, Finkelstein ¹³¹⁾

has given a very simple but general argument for believing that dual meson meson scattering amplitudes saturated with narrow resonances on linearly rising trajectories with universal slopes will either have exotic states or else contain parity doublets [Eq. (3.82)].

Maybe, then, the title of this subsection should have been the pseudoscalar meson problem. We shall not try to assess the future prospects of dual theories, but take the pragmatic point of view that no strong suggestion for amplitudes pertaining to processes with the η is at hand. However, essentially no experimental information is available, anyway.

3.6. - Closing remarks.

Several fields on the fringe of meson meson physics have not been treated. These include hard meson or effective Lagrangian treatments ^{132),133),134)}, calculations of $\delta_0^0(\pi\pi)$ by consideration of the form factor $\langle\pi|\sigma|\pi\rangle$ ¹³⁵⁾⁻¹³⁸⁾ and theories on scale symmetry breaking ^{139),140)}. While these have bearings on several aspects of high energy physics they seem to provide little information on meson meson amplitudes, not already contained in the previous subsections.

Before we go to the experimental part we stress the most important theoretical predictions.

- (i) Low energies (subsection 3.1) : scattering lengths are small and effective ranges are large ; signs favour resonances in non-exotic channels and disfavour resonances in exotic channels.
- (ii) Medium high energies : phase shifts (Figs. 2, 3) are suggested for $\pi\pi$ and $K\pi$. Much interest concerns $\delta_0^0(\pi\pi)$. Whereas theories reasonably agree below the ρ [$\delta_0^0(500 \text{ MeV}) \simeq 45^\circ \pm 10^\circ$] Veneziano theory prefers a value considerably higher than 90° above the ρ while the remaining curves are at or below 90° in Fig. 2. However, the first is biased in favour of large values by the

presence of high energy, low daughter S state with possibly unrealistic coupling, whereas the latter are biased against large values by the parametrization chosen and neglect of complications for the $K\bar{K}$ threshold S^* region. As for $\delta_{\frac{1}{2},0}(K\pi)$ (Fig.3) a κ state (Table II) is strongly indicated but the phase shift could differ very much more from a Breit-Wigner. Exotic phases are rather uniquely predicted [in particular the near zero value of $\delta_{\frac{3}{2},1}(K\pi)$].

- (iii) For higher energy resonances see Table II and comments in subsection 3.3. Huntings for traces of lower daughters and exotics will be followed with particular interest.

4. - INFORMATION FROM PERIPHERAL DI-MESON PRODUCTION

The absence, for the time being, of mesonic targets implies that experimental information on meson meson scattering is possible only in highly indirect ways. By far the majority of our knowledge derives from high energy peripheral meson production. New high statistics experiments now being planned or analyzed ensure that this will continue to be so for a number of years. Reviews : Refs. 4),5),141)-145).

4.1. - Phenomenological framework.

Consider a process (Fig. 6) for which pion exchange is allowed. Here a 0^- meson (four-momentum q , mass m) scatters on a target nucleon (four-momentum p , mass M) to produce a three-body final state of a baryon and two 0^- mesons with four-momenta p' , q' and q'' and masses M' , m' and m'' respectively. The final baryon will be either a nucleon or a $\Delta_{33}(1236)$ resonance.

The process is described by five invariant variables :

$$\begin{aligned} W^2 &= (q+p)^2 && \text{total c.m. energy squared} \\ S &= (q'+q'')^2 && \text{effective di-meson mass squared} \\ \Delta^2 &= -(p-p')^2 && \text{invariant squared momentum transfer to} \\ &&& \text{baryon } (>0 \text{ in physical region}) \\ t &= (q-q')^2, \quad u = (q-q'')^2 && (4.1) \\ M_{M'm'}^2 &= (q'+p')^2, \quad M_{M'm''}^2 = (q'+p'')^2 \end{aligned}$$

Momentum conservation and mass-shell conditions imply

$$\begin{aligned} S + M_{M'm'}^2 + M_{M'm''}^2 &= W^2 + M'^2 + m'^2 + m''^2 \quad (\text{Dalitz plot}) \\ S + t + u &= m^2 + m'^2 + m''^2 - \Delta^2 \end{aligned} \quad (4.2)$$

For experimental studies, the following variables are preferred :

$$\begin{aligned} W^2 & \quad (\text{fixed for given beam momentum}) \\ (S, \Delta^2) & \quad (\text{Chew-Low plot}) \\ (\theta, \phi) & \quad \text{polar angles of the outgoing meson } m' \text{ in the di-meson c.m.} \end{aligned} \quad (4.3)$$

An infinity of co-ordinate systems can define (θ, φ) . In most studies either the (GJ) ¹⁴⁶⁾ or the helicity frame (H) is used. In both the y axis is taken orthogonal to the production plane (containing the beam and the outgoing baryon). Further in the GJ, $\hat{z}$ is the direction of the incoming 0^- meson (i.e., $\hat{q}$ as seen in the di-meson centre-of-mass) and in the H frame, $\hat{z}$ is opposite to the direction of the outgoing baryon (i.e., $-\hat{p}'$ as seen in the di-meson centre-of-mass). In every case $\hat{x}$ is specified by the right-hand rule.

Goebel ¹⁴⁷⁾ and Chew and Low ¹⁴⁸⁾ suggested to analyze data according to the formulae ¹⁴⁹⁾

$$\frac{d\sigma(W^2, s, \Delta^2, \theta, \varphi)}{ds d\Delta^2 d\Omega} = \frac{\sqrt{s}}{8\pi p^2 W^2} \tilde{q}(-\Delta^2) \frac{\Delta^2}{(\Delta^2 + \mu^2)^2} \frac{G^2}{4\pi} \left(\frac{d\sigma(s, t)}{d\Omega} \right)_{\text{meson}} \quad (4.4.a)$$

for final baryon = nucleon, and

$$\frac{d\sigma(W^2, s, \Delta^2, \theta, \varphi)}{ds d\Delta^2 d\Omega dM'} = \frac{\sqrt{s}}{8\pi^3 p^2 W^2} \tilde{q}(-\Delta^2) \frac{1}{(\Delta^2 + \mu^2)^2} Q(-\Delta^2) M'^2 \sigma(M') \left(\frac{d\sigma}{d\Omega} \right)_{\text{me}} \quad (4.4.b)$$

for final baryon = $\Delta_{33}(1236)$.

p is the initial centre-of-mass three-momentum, μ is the pion mass and

$$\begin{aligned} (\tilde{q}(-\Delta^2))^2 &= \frac{1}{4s} [s - (m + \sqrt{-\Delta^2})^2] [s - (m - \sqrt{-\Delta^2})^2] \\ (Q(-\Delta^2))^2 &= \frac{1}{4M'^2} [M'^2 - (M + \sqrt{-\Delta^2})^2] [M'^2 - (M - \sqrt{-\Delta^2})^2] \end{aligned} \quad (4.5)$$

M is the proton mass, $G^2/4\pi = 2 \times 14.6$ for $\pi^- p \rightarrow \pi^- \pi^+ n$, $\sigma(M')$ is the πN cross-section in the Δ_{33} region and the above cross-sections have been summed over final and averaged over initial baryon polarizations. $(d\sigma/d\Omega)_{\text{meson}}$ is the on-shell meson meson scattering cross-section corresponding to the upper vertex of Fig. 6.

The vanishing of Eq. (4.4a) at $\Delta^2 = 0$ is a peculiar feature of pseudoscalar coupling to the $\bar{N}N$ vertex (evasion).

The point behind Eq. (4.4) is that according to general belief (studies of Feynman graphs) the amplitude of Fig. 6 has a pole at $\Delta^2 = -\mu^2$, and the cross-section has a double pole, the residue of which is given exactly by Eq. (4.4). Therefore, one attempts to suitably extrapolate the left-hand sides to the unphysical region, $\Delta^2 = -\mu^2$, thereby obtaining $(d\sigma/d\Omega)$ meson. Then the problem of finding meson meson phase shifts is just as easy (and just as difficult) as for "ordinary" processes. However, it turns out that Eq. (4.4) is satisfied so poorly in the physical region that modifications are needed.

The most general form for the invariant amplitude for the process $O^-(m) \frac{1}{2}^+(M) \rightarrow O^-(m') O^-(m'') \frac{1}{2}^+(M)$ is ¹⁵⁰⁾

$$\mathcal{M}(W^2, s, \Delta^2, \theta, \phi) = \bar{u}(p') \{ A_1 \gamma_5 (q \cdot \gamma) + A_2 \gamma_5 (q' \cdot \gamma) + A_3 \gamma_5 (q'' \cdot \gamma) + A_4 \epsilon_{\lambda\mu\nu\sigma} \gamma^\lambda q^\mu q'^\nu q''^\sigma \} u(p) \quad (4.6)$$

The first Eq. (4.4) corresponds to

$$\mathcal{M} = \bar{u}(p') \gamma_5 u(p) \frac{A(s, t)}{\Delta^2 + \mu^2} G \quad (4.7)$$

where $A(s, t)$ is the invariant meson meson scattering amplitude. Evidently the dynamical sophistication of that model is exceedingly low compared with the most general possibility.

For phenomenological studies, one normally employs helicity amplitudes rather than invariant ones ¹⁵¹⁾. These may be analyzed in terms of di-meson angular momentum,

$$\mathcal{M}_{\lambda' \mu; \lambda}(W^2, s, \Delta^2, \theta^H, \phi^H) = \sum_{\ell} f_{\lambda' \mu; \lambda}^{\ell}(W^2, s, \Delta^2) d_{\mu 0}^{\ell}(\theta^H) e^{i\mu\phi^H} \quad (4.8)$$

Here $\lambda (\lambda')$ is the initial (final) baryon helicity in the over-all centre-of-mass, $f_{\lambda' \mu; \lambda}^{\ell}$ is the amplitude for producing the dipion system in a state of angular momentum ℓ and helicity μ . Equation (4.8) applies directly to the helicity frame. The differential cross-section is $(p'$ is the final baryon centre-of-mass momentum)

$$\frac{d\sigma}{ds d\Delta^2 d\Omega^H} = \frac{1}{2} \frac{\rho'}{s\rho} \sum_{\lambda\lambda'\mu} |M_{\lambda'\mu;\lambda}|^2$$

$$= \left[\sum_{\ell\ell'\mu\mu'} (\rho_{\mu\mu'}^{\ell\ell'})^H d_{\mu 0}^{\ell}(\theta^H) d_{\mu' 0}^{\ell'}(\theta^H) e^{i(\mu-\mu')\varphi^H} \right] \frac{d\sigma}{ds d\Delta^2}$$

(4.9)

with

$$\frac{d\sigma}{ds d\Delta^2} (\rho_{\mu\mu'}^{\ell\ell'})^H \equiv \sum_{\lambda\lambda'} f_{\lambda'\mu;\lambda}^{\ell} (f_{\lambda'\mu;\lambda}^{\ell'})^* \frac{\rho'}{2s\rho}$$

The helicity amplitudes satisfy

$$f_{\lambda',-\mu;-\lambda}^{\ell} = \zeta(-)^{\mu} f_{\lambda',\mu;\lambda}^{\ell} \quad (4.10)$$

where ζ is a notation dependent phase independent of μ (but dependent on parities and λ, λ')^{151), 152)}. Equations (4.9) and (4.10) imply

$$\left. \begin{aligned} (\rho_{-\mu,-\mu'}^{\ell\ell'})^H &= (-)^{\mu-\mu'} (\rho_{\mu,\mu'}^{\ell\ell'})^H \\ (\rho_{\mu\mu'}^{\ell\ell'})^{H*} &= (\rho_{\mu'\mu}^{\ell'\ell})^H \\ \sum_{\ell\ell'\mu\mu'} (\rho_{\mu\mu'}^{\ell\ell'})^H &\equiv \text{Tr}(\rho^H) = 1 \end{aligned} \right\} \quad (4.11)$$

As pointed out by Gottfried and Jackson¹⁴⁶⁾, Eqs. (4.8)-(4.11) hold in the GJ frame as well provided the following changes are made

$$\begin{aligned} (\theta, \varphi)^H &\rightarrow (\theta, \varphi)^{GJ}; \quad M_{\lambda'\mu;\lambda} \rightarrow M_{\bar{\lambda}'\lambda;m}^{(c)} \\ f_{\lambda'\mu;\lambda}^{\ell} &\rightarrow f_{\bar{\lambda}'\lambda;m}^{\ell(c)}; \quad (\rho_{\mu\mu'}^{\ell\ell'})^H \rightarrow (\rho_{mm'}^{\ell\ell'})^{GJ} \end{aligned} \quad (4.12)$$

Here $M_{\bar{\lambda}'\lambda;m}^{(c)}$ and $f_{\bar{\lambda}'\lambda;m}^{\ell(c)}$ are helicity amplitudes for the cross channel process

$$B(M) \bar{B}(M') \rightarrow m, m', m''$$

analytically continued on to the physical region for Fig. 6. The density matrices in the two frames are connected by

$$(\rho_{\mu\mu'}^{ll'})^{GJ} = (-)^{\mu-\mu'} \sum_{m m'} (\rho_{m m'}^{ll'})^H d^l(\chi)^m_{\mu} d^{l'}(\chi)^{m'}_{\mu'} \quad (4.13)$$

where the crossing angle χ is given by ¹⁵²⁾

$$\cos \chi = \left\{ (-\Delta^2 + s - m^2)(W^2 + s - M'^2) - 2s\delta \right\} / \left\{ [-\Delta^2 - (m + \sqrt{s})^2] [-\Delta^2 - (m - \sqrt{s})^2] [W^2 - (M' + \sqrt{s})^2] [W^2 - (M' - \sqrt{s})^2] \right\}^{1/2} \quad (4.14)$$

with

$$\delta \equiv s + M^2 - M'^2 - m^2$$

Given the density matrix elements, the angular distribution is fixed, however, the converse is not true. Writing in either frame

$$W(\theta, \varphi) = \sum_{LM} \langle Y_L^M \rangle Y_L^M(\theta, \varphi)$$

with

(4.15)

$$W(\theta, \varphi) \equiv \frac{d\sigma}{ds d\Delta^2 d\Omega} / \frac{d\sigma}{ds d\Delta^2}$$

the moments (or statistical tensors) have been defined ($\langle Y_0^0 \rangle \equiv 1/\sqrt{4\pi}$). These are the directly measurable quantities. One has [see, for example, Rose ¹⁵³⁾, Eq. (4.34)]

$$\begin{aligned} \langle \text{Re} Y_L^M \rangle &= \frac{1}{2\sqrt{2L+1}} \sum_{ll'm} \sqrt{(2l+1)(2l'+1)} (-)^m \\ &\times \langle l, l', M+m, -m | l l' L M \rangle \\ &\times \langle ll' 00 | l l' L 0 \rangle (1 + (-)^{l+l'-L}) \text{Re} \rho_{M+m, m}^{ll'} \end{aligned} \quad (4.16)$$

For fixed beam momentum (i.e., W^2) they depend on the point (s, Δ^2) in the "Chew-Low plane" bounded by the "Chew-Low boundary" :

- 1) the line $s = (m' + m'')^2$
- 2) the curve
$$(\Delta^2)_{\min}^{\max} = \frac{1}{2W^2} [W^4 - W^2(m^2 + M^2 + M'^2 + s) + (M^2 - m^2)(M'^2 - s) \pm 4W^2 p p'] \quad (4.17)$$

To perform a "Chew-Low extrapolation" from this physical region to the pion pole, some knowledge on the singularity structure of the extrapolated object is necessary.

1) - The invariant amplitudes $A_1, \dots, A_4$ Eq. (4.6) have dynamical singularities only ¹⁵⁰⁾. For each of the ten two-particle channels, below threshold poles (corresponding to the existence of a stable particle with the quantum numbers of the channel) and normal threshold branch points occur in the invariant associated with the channel. Thus for fixed physical values of W^2 , s , t and $M_{M', m'}^2$, the A_i 's are analytic in the Δ^2 plane cut along $(-\infty, \Delta_1^2)$ and (Δ_2^2, ∞) possibly with a pole at $\Delta^2 = -\mu^2$. Here

$$\begin{aligned} \Delta_1^2 &= -9\mu^2 \text{ for } \pi N \rightarrow 2\pi N' & \text{and} \\ \Delta_1^2 &= -4\mu^2 \text{ for } KN \rightarrow K\pi N' \end{aligned}$$

whereas Δ_2^2 depends on W^2 , s , t , $M_{M', m'}^2$ and is associated with a threshold in one of the remaining nine channels. Little is known about higher order singularities as given by the Landau rules ¹¹⁾. Generally they are supposed to be further away from the physical region. However, for $B' = \Delta_{33}(1236)$ the diagrams of Fig. 7 give rise to anomalous thresholds

$$\begin{aligned} \Delta_a^2 &\approx -5\mu^2(1+i) + 2X^2(\mu/M + i) \approx (-3.6 + i4.6)\mu^2 \quad (a) \\ \Delta_a^2 &\approx -2\mu^2(1+i) + \frac{1}{2}X^2(\mu/M + i) \approx (-1.6 + i2.8)\mu^2 \quad (b) \end{aligned} \quad (4.18)$$

with

$$X^2 = M_{\Delta_{33}}^2 - M^2 - \mu^2$$

and associated cuts extending from Δ_1^2 . In the absence of any information about the discontinuity across these cuts, one must suspect that the one-pion exchange (OPE) pole is considerably more difficult to separate from background singularities than for processes with $B' = N$ (see, however, the remark at the end of subsection 4.3).

2) - The partial wave helicity amplitudes $f_{\lambda', \mu; \lambda}^\ell$ (DCHA) and $f_{\bar{\lambda}' \lambda; m}^{\ell(c)}$ (CCHA) have two types of additional singularities.

- (i) Being pwa's they are obtained from the invariant amplitudes by integrations over t and $M_{M'm'}^2$. That produces in general complex singularities of complicated forms ¹⁵⁴⁾, which gives a smaller effective value for Δ_2^2 .
- (ii) Being helicity amplitudes they contain kinematical singularities which are exceedingly complicated to describe for a five-point function ^{155), 156)}. A simplified idea about them can be obtained by viewing the outgoing di-meson system as an effective particle $[\sqrt{s}, \ell]$ (which is reasonable at least at a resonance) and resort to the extensive treatments for four-point functions ^{152), 157)-159)}. Then :

- (a) - on the boundary of the physical region [in particular (4.17)] square root singularities may be present via the half-angle factors :

$$f_{\lambda', \mu; \lambda}^\ell = \left[\sin \frac{\Theta}{2} \right]^{| \lambda - \mu + \lambda' |} \left[\cos \frac{\Theta}{2} \right]^{| \lambda + \mu - \lambda' |} \tilde{f}_{\lambda', \mu; \lambda}^\ell$$

$$f_{\bar{\lambda}' \lambda; m}^{\ell(c)} = \left[\sin \frac{\Theta_c}{2} \right]^{| m - \bar{\lambda}' + \lambda |} \left[\cos \frac{\Theta_c}{2} \right]^{| m + \bar{\lambda}' - \lambda |} \tilde{f}_{\bar{\lambda}' \lambda; m}^{\ell(c)} \quad (4.19)$$

where the reduced amplitudes $\tilde{f}_{\lambda', \mu; \lambda}^\ell$, $\tilde{f}_{\bar{\lambda}' \lambda; m}^{\ell(c)}$ are regular across the boundary. Θ (Θ_c) is the direct channel (crossed channel) centre-of-mass scattering angle of the outgoing baryon ; across the boundary $\sin \Theta/2$ vanishes as a square root of the invariants ($\cos \Theta/2 \approx 1$) ; this is true for Θ_c as well

$$\varphi(W^2, \Delta^2) = -4\Delta^2 p_c^2 p_c'^2 \sin^2 \Theta_c \quad (4.20)$$

where p_c (p_c') is the initial (final) centre-of-mass three-momentum for the process $B\bar{B}' \rightarrow O^-(m) [\sqrt{s}, l]$, and $\varphi(W^2, \Delta^2)$ is the Kibble function ¹⁶⁰⁾, a polynomial in the invariants with first order zeros at the kinematical boundaries, positive in physical regions and negative outside ;

- (b) - at thresholds and pseudothresholds $\bar{W}^2 = (M \pm m)^2$, $W^2 = (M' \pm \sqrt{s})^2$ for DCHA's and $\Delta^2 = -(M \pm M')^2$, $\Delta^2 = -(m \pm \sqrt{s})^2$ for CCHA's] and at $W^2 = 0$ for DCHA's, $\Delta^2 = 0$ for CCHA's, square root singularities in positive and negative powers may be present. These are irrelevant for DCHA's as far as Chew-Low extrapolation goes but could be of major importance for CCHA's, in particular for $M = M'$ but see below.

3) - Density matrix elements and moments have :

- (a) - a subset of the singularities of the f^{ℓ} 's ; several singularities between the physical region and the OPE pole drop out, which is extremely fortunate. Thus, from Eqs. (4.9) and (4.16) we see that $(\rho_{mm'}^{\ell\ell'})$'s with $m = m'$ and $\langle Y_L^M \rangle$'s with $M = 0$ (those are the ones containing the OPE double pole) have no kinematical boundary singularities [a) above]. Also, the alarming pseudothreshold singularity at $\Delta^2 = 0$ (for $M = M'$) disappears from $(\rho_{mm'}^{\ell\ell'})^{GJ}$ due to constraints among the $f_{\lambda', \lambda; m}^{\ell(c)}$'s as is evident from Eqs. (4.13), (4.14) ;
- (b) - additional poles are present whenever $(d\sigma/ds d\Delta^2)$ is zero ; the positions completely depend on dynamical details and are unknown a priori ; as we shall see in subsection 4.3, one is almost certainly present between $\Delta^2 = 0$ and $\Delta^2 = -\mu^2$, which constitutes a major worry for Chew-Low extrapolations of normalized quantities.

The GJ frame ¹⁴⁶⁾ was introduced as a means of discriminating between various exchange mechanisms. Thus

$$\begin{aligned} \text{for normal parity exchange } (\rho_{m, -m'}^{\ell\ell'})^{GJ} &= -(-)^{m'} (\rho_{m, m'}^{\ell\ell'})^{GJ} \\ \text{for abnormal parity exchange } (\rho_{m, -m'}^{\ell\ell'})^{GJ} &= (-)^{m'} (\rho_{m, m'}^{\ell\ell'})^{GJ} \end{aligned} \quad (4.21)$$

More specifically, for pion-exchange only

$$(\rho_{m m'}^{\ell\ell'})^{GJ} = \delta_{m0} \delta_{m'0} f(\ell, \ell') \quad (4.22)$$

in particular

$$\langle Y_L^M \rangle = 0, \quad M \neq 0 \quad (4.23)$$

which is just the well known Treiman-Yang test for spin zero exchange : the dipion angular distribution is independent of the Treiman-Yang angle, φ^{GJ} .

The rules (4.21), however, are approximately valid in the helicity frame as well, the correction being of order Δ^2/W^2 ¹⁶¹⁾.

Equations (4.21)-(4.23) constitute tests for OPE. Others are specific predictions of Eq. (4.4) : vanishing of (a) at $\Delta^2 = 0$, specific Δ^2 dependence and specific W^2 dependence (p_{lab}^{-2}). In the following subsections we shall discuss how to deal with deviations from these as suggested by theory or experiment.

4.2. - Off-shell effects.

One very popular way of viewing di-meson production is to consider a modified OPE mechanism by which the graph in Fig. 6 is supposed to represent scattering of a meson on a virtual pion. In such a picture the Lorentz structure is still that of Eq. (4.7) (for $B' = N$) but one allows for the generalization

$$M = \bar{u}(p') g_5^2 u(p) \frac{A(s, t, \Delta^2)}{\Delta^2 + \mu^2} G \quad (4.24)$$

where $A(s, t, \Delta^2)$ is called an off-shell meson meson scattering amplitude.

Two modifications have been much considered :

- (i) meson meson pwa's, a_ℓ , are modified off-shell by real multiplicative form factors depending on ℓ and Δ^2 only ;
- (ii) the variation of a_ℓ with s is allowed to be correlated to the variation with Δ^2 in a particular way suggested by PCAC.

Form factors

In the partial wave projection performed to obtain $f_\ell^\ell(\nu)$ [a common name for $(f_{\lambda, \mu; \lambda}^\ell)^H$ and $(f_{\lambda, \lambda; m}^\ell)^{GJ}$] the integration limits depend on Δ^2 . This effect introduces a dependence $(\tilde{q}(-\Delta^2))^\ell$ in $f_\ell^\ell(\nu)$. If that was the only non-pole Δ^2 dependence we would have

$$(\Delta^2 + \mu^2) f_{(\nu)}^\ell(s, \Delta^2) = \left(\frac{\tilde{q}(-\Delta^2)}{\tilde{q}(\mu^2)} \right)^\ell [f_{(\nu)}^\ell(s, \Delta^2) (\Delta^2 + \mu^2)] \Big|_{\Delta^2 = -\mu^2} \quad (4.25)$$

where $[f_{(\nu)}^\ell(s, \Delta^2) (\Delta^2 + \mu^2)] \Big|_{\Delta^2 = -\mu^2} = a_\ell(s) \beta(G, (\nu))$

and the quantities $\beta(G, (\nu))$ may be evaluated exactly [see, for example, Ref. 145]. If $a_\ell(s)$ is dominated by a narrow resonance this is just what covariant Feynman rules would give, however, the Δ^2 behaviour is in gross disagreement with experiment.

Dürr and Pilkuhn (DP) ¹⁶²⁾ and Benecke and Dürr (BD) ¹⁶³⁾ showed that the replacement

$$\begin{aligned}
 & (\tilde{q}(-\Delta^2)/\tilde{q}(\mu^2))^L \rightarrow F_\ell(-\Delta^2) \\
 \text{with } & F_\ell(-\Delta^2) \propto (\tilde{q}(-\Delta^2)/\tilde{q}(\mu^2))^L \text{ for } -\Delta^2 \approx (\sqrt{s} \pm m)^2 \\
 & F_\ell(\mu^2) = 1 \\
 & F_\ell(-\Delta^2) \rightarrow \begin{cases} \text{constant (DP)} \\ 1/\Delta^2 - 1 \quad \text{(BD)} \end{cases} \text{ for } \Delta^2 \rightarrow \infty
 \end{aligned} \tag{4.26}$$

arose naturally in non-relativistic models with force range R (DP), or in ladder approximation treatments of Bethe-Salpeter equation solutions for exchange forces from a particle with mass $= R^{-1}$ (BD). $F_\ell(-\Delta^2)$ is a "pion form factor" for the vertex

$$\pi(-\Delta^2) \rightarrow O^-(m) + [\sqrt{s}; L]$$

expressible in terms of spherical Bessel and Neumann functions (DP) or Legendre functions of the second kind (BD). Specifically

$$F_0^{DP}(-\Delta^2) \equiv 1, \quad F_1^{DP}(-\Delta^2) = N_1 \frac{R \tilde{q}(-\Delta^2)}{\sqrt{1 + R^2 \tilde{q}^2(-\Delta^2)}} \tag{4.27}$$

where N_1 is determined by Eq. (4.26) and where R is a parameter. From the first Eq. (4.27), DP suggest the name, kinematical form factors.

Wolf ¹⁶⁴⁾ in a series of papers fitted a large number of differential cross-sections for reactions supposed to be dominated by OPE. In addition to the above form factors he also supplied the ones for the lower vertex and the propagator of Fig. 6. The relatively nice appearance of these fits made Ming Ma et al. ¹⁶⁵⁾ (see also Schlein ^{142), 143)}) and subsequently many other workers to consider form factor modified Chew-Low extrapolations. We make a few general comments on this approach.

- (i) - Modified with form factors, Eqs. (4.4) continue to decouple the W^2 and the Δ^2 dependence in apparent agreement with experiment ¹⁶⁴⁾. This is a success of OPE only if the pion trajectory has zero slope but indicates failure of OPE if α_π has universal slope.

- (ii) - The ability of form-factor-modified OPE forms to fit $(d\sigma/ds d\Delta^2)$ in cases where $\rho_{n,m}$ (n or $m \neq 0$) is known not to be small $[\rho_{10}^{11}]$ and ρ_{10}^{10} in $\pi^- p \rightarrow \pi^+ \pi^- n$, see for example Ref. 145] is a warning that these do not provide stringent tests for OPE;
- (iii) - In perturbation theories of weak and electromagnetic interactions, factorizable form factors may be defined. In hadronic processes this is only possible in an obvious way for pole residues. Therefore, although the idea of minimizing the number of free parameters in a Chew-Low extrapolation by implementing form factors obtained from fits to other reactions is attractive, it can hardly be justified. This is so in particular when the approach is modified to allow for non-evasive terms ¹⁶⁵⁾.
- (iv) - DP or BD form factors allow for additional singularities in the Δ^2 channel s wave. Since, however, the crossed channel partial wave expansion anyway diverges in the region of interest the significance of this is unclear. The divergence trouble is well known to be overcome by Reggeization. Then the Lorentz content is more complicated than in Eqs. (4.7), (4.24) (whether α_π is conspiring or evading - see next subsection). Further, the residue function β_π does factorize (being a j plane pole residue). Unfortunately the π trajectory is extremely poorly known at present.
- (v) - Form factors are clearly to be used where they belong but nowhere else. Below we shall consider cases where blind form factor fits to low statistics data will produce systematic errors.

Correlation effects

Models incorporating PCAC suggest a particular behaviour of the off-shell pwa's that cannot be described by factorizing the s and Δ^2 dependences like

$$a_\ell(s, \Delta^2) = a_\ell(s) F_\ell(\Delta^2) \quad (4.28)$$

Thus for $\pi\pi$ scattering [Eq. (3.15)] we have

$$A^{(0)} = \frac{1}{F_\pi^2} (2s - \Delta^2 - 2\mu^2), A^{(1)} = \frac{1}{F_\pi^2} (t - u), A^{(2)} = \frac{1}{F_\pi^2} (\mu^2 - \Delta^2 - s) \quad (4.29)$$

describing approximately pure s, p and s waves respectively (this is exact on-shell).

We see that no correlations are present for p waves so that form factor successes for the ρ [or $K^*(890)$] have no bearing on s wave treatments. This is a consequence of Bose statistics only.

On the other hand for $\pi^-\pi^+ \rightarrow \pi^-\pi^+$ (as obtained from $\pi^-p \rightarrow \pi^-\pi^+n$), Eq. (4.29) gives for the s wave

$$A_0^{+-}(s, \Delta^2) = \frac{1}{2F_\pi^2} (s - \Delta^2 - \mu^2) \quad (4.30)$$

Making an average fit of that of the form (4.28) over the interval $0 < \Delta^2 \lesssim 10\mu^2$ will give in an average sense

$$\left. \begin{aligned} \langle A_0^{+-}(s) \rangle_{\text{exp}} &\approx \frac{1}{2F_\pi^2} (s - 6\mu^2) \\ (A_0^{+-}(s))_{\text{th}} &= \frac{1}{2F_\pi^2} s \end{aligned} \right\} \quad (4.31)$$

Wagner¹⁶⁶⁾ analyzed this effect using the K matrix unitarized Lovelace-Veneziano model for the upper vertex Fig. 6. Instead of BD or DP form factors the $(\tilde{q}^2)^l$ trouble was dealt with by an overall multiplicative exponential $e^{-A\Delta^2}$ roughly analogous to Reggeization at one energy. The s wave was found to decrease for positive Δ^2 as Eq. (4.30) for low energies ($\lesssim 500$ MeV). For higher energies only, the p waves showed a slight ($\approx 20\%$) decrease (at $\Delta^2 = 9\mu^2$), whereas the s wave was lowered by as much as 75% to 60% for $\sqrt{s}$ between 500 and 900 MeV; δ_0^2 was affected by less than 30%.

Gutay et al. ¹⁶⁷⁾, Loos et al. ¹⁶⁸⁾ and Scharenguivel et al. ²⁷⁾ went the other way and attempted to determine $\pi\pi$ parameters from the observed correlations. In the linear approximation one has with PCAC imposed (subsection 3.1)

$$A_0^{+-}(s, \Delta^2) = \frac{1}{2} \{ (\alpha + 2\gamma)s - (5\alpha + 6\gamma) + (3\alpha + 2\gamma)\Delta^2 \} \quad (4.32)$$

and

$$a_0^0/a_0^2 = - \frac{4\gamma - 3\alpha}{8\gamma + 6\alpha} \quad (4.33)$$

It is argued that the most model-independent feature of the effect is the behaviour of the point, $\Delta^2 = \Delta_0^2(s)$ where the forward backward asymmetry vanishes, as a function of s . That is obtained by putting Eq. (4.32) zero, i.e.,

$$\Delta_0^2(s) = \frac{5\alpha + 6\gamma - (\alpha + 2\gamma)s}{3\alpha + 2\gamma} \quad (4.34)$$

Experimentally a straight line was indeed found for $\sqrt{s} \lesssim 600$ MeV, and γ/α was equal to the CA value, -1, to within 40%, giving by Eq. (4.33) the important result

$$a_0^0/a_0^2 = -3.3_{-1.1}^{+0.6} \quad (4.35)$$

in agreement with Weinberg's value of $-7/2$ [Eq. (3.15)]. Figure 8 shows $\Delta_0^2(s)$ vs. s [Ref. 168] for $\sqrt{s} \lesssim 1$ GeV. The low energy straight line behaviour seems to be modified in the right direction by the Veneziano ansatz.

It is not quite clear what the significance of these nice results is. When the lower vertex in Fig. 6 is weak (lepton target) the Adler condition is a consequence of the current current nature of the interaction. In the present hadronic context, however, there is

no reason to believe in its validity and without it, the result Eq. (4.35) does not follow. At least, one more charge model needs to be considered.

However, the important point to remember is that correlation effects of the above-mentioned order of magnitude follow just from symmetry considerations of the off-shell amplitude without imposing PCAC. One simple way of dealing with them is to make the replacement

$$\frac{A_{\pi\pi}(s,t)}{\Delta^2 + \mu^2} \rightarrow \frac{A_{\pi\pi}(s,t)}{\Delta^2 + \mu^2} + b \quad (4.36)$$

where $A_{\pi\pi}(s,t)$ is the on-shell $\pi\pi$ scattering amplitude and b is a relatively featureless background depending on (s,t,Δ^2) but constant in the linear approximation and depending on isospin channels in a known way (cf. subsection 2.4). Afterwards (4.36) may be further modified by form factors.

Correlation effects of a completely different origin have been discussed. Thus Bishop et al.¹⁶⁹⁾ suggested to use Coulomb interference to obtain low energy phase shifts. For scattering of charged pions, one-photon exchanges produce poles at $t = 0$ ($u = 0$) For $\Delta^2 = -\mu^2$ these occur at $\theta = 0$ (π) whereas for $\Delta^2 > 0$ they are outside the physical region and contribute a forward backward asymmetry depending on s and Δ^2 . Although that does not seem to be the most powerful method available for studying low energy phase shifts, it constitutes a calculable effect which should not be confused with other ones.

4.3. - Absorptive effects.

One of the most extensively studied reactions from the point of view of Chew-Low extrapolations is

$$\pi^- p \rightarrow \pi^+ \pi^- n \quad (4.37)$$

which would seem to be a particularly attractive pure OPE candidate.

However, (1) Marateck et al.¹⁷⁰⁾ found that extrapolating the total $\pi\pi$ cross-section using the evasive constraint

$$\frac{d\sigma}{dsd\Delta^2} = 0 \quad \text{at} \quad \Delta^2 = 0$$

[cf. Eq. (4.4.a)], the results were clearly below the p wave unitarity limit at the ρ mass. For a non-evasive extrapolation, data were insufficient to yield a useful accuracy. Also (2) if s and p waves only are considered, the isotropic term in the angular distribution (in the GJ frame) is given by [Eq. (9.9)]

$$\rho_{00}^{00} + (2\rho_{11}^{11} + \rho_{-1,-1}^{11}) \quad (4.38)$$

For OPE only, the last term is zero and the isotropic part is a measure of the s wave. However, doing again an evasive extrapolation, the result this time exceeds the unitary limit for s waves by a very large amount^{145), 171)}.

Both of these observations strongly suggest that some unnatural parity exchanges are present so that $(d\sigma/dsd\Delta^2) \neq 0$ for $\Delta^2 = 0$ and depolarized ρ 's dominate the isotropic distribution.

Below we briefly discuss predictions of these effects in various theoretical models [vector meson dominance (VMD), absorption, conspiracy) and try to estimate from the models what the cures will be in practical Chew-Low extrapolation studies.

Much interest has focused on the possible connection (via VMD) between transverse neutral ρ production and (inverse) isovector photoproduction

$$\begin{aligned}\pi^- p &\rightarrow \rho_{Tr}^0 n \\ \gamma^* n &\leftarrow p \pi^+\end{aligned}\quad (4.39)$$

For the first the GJ frame cross-section vanishes in a pure OPE model, whereas experimentally, the second shows a forward peak (within $\Delta^2 \approx \mu^2$)¹⁷²⁾.

The invariant amplitude for the reactions (4.39) may be decomposed as¹⁷³⁾

$$\begin{aligned}M = i \bar{u}(p') \gamma_5 \{ & B_1 \not{\epsilon} \not{k} + 2 B_2 \not{\epsilon} \not{P} + 2 B_3 (\epsilon \cdot q) + 2 B_4 \epsilon \cdot k \\ & - B_5 \not{\epsilon} + B_6 (\epsilon \cdot P) \not{k} + B_7 (\epsilon \cdot k) \not{k} + B_8 (\epsilon \cdot q) \not{k} \} u(p)\end{aligned}\quad (4.40)$$

Here $k \equiv q' + q''$ is the vector particle momentum. Also $P \equiv \frac{1}{2}(p + p')$. $B_{4,7}$ do not contribute to the physical cross-section but play a role in discussing possible kinematical singularities (B_i , $i = 1, \dots, 8$ do not have any¹⁷³⁾). The conditions that ρ be coupled to a conserved current are

$$\begin{aligned}m_\rho^2 B_1 + \frac{1}{2}(W^2 - \mathcal{U}) B_2 + (\Delta^2 + m_\rho^2 + \mu^2) B_3 + 2 m_\rho^2 B_4 &= 0 \\ -2 B_5 + \frac{1}{2}(W^2 - \mathcal{U}) B_6 + 2 m_\rho^2 B_7 + (\Delta^2 + m_\rho^2 + \mu^2) B_8 &= 0\end{aligned}\quad (4.41)$$

where $\mathcal{U} \equiv (q - p')^2$. For $m_\rho^2 = 0$ this is just the requirement of gauge invariance¹⁷⁴⁾⁻¹⁷⁶⁾.

For OPE only,

$$B_3 = \frac{\sqrt{2} G \cdot f}{\Delta^2 + \mu^2} = -2 B_4, \quad B_i = 0, \quad i \neq 3, 4 \quad (4.42)$$

where $G^2/4\pi = 14.6$ and $f = g_{\rho\pi\pi}$ or e for the two processes (4.39). Thus OPE is not a gauge invariant description for $m_\rho^2 = 0$ (Reggeized pion exchange is ¹⁷⁴). This is well known in perturbation theory in which in general all diagrams of given order must be taken together. Doing this in the electric Born model (no magnetic ρ NN coupling) one has from the nucleon pole terms $[1/(W^2 - M^2), 1/(U - M^2)]$ the further specific prediction for B_1 (Δ^2/W^2 small) ¹⁷⁷ (together with a similar one for B_2)

$$B_1 \sim 2\sqrt{2}Gf W^{-2} \quad \text{independent of} \quad \Delta^2 \quad (4.43)$$

A priori there is no reason whatever to expect that this model makes sense for high energy peripheral production, but as we shall see, that value of B_1 arises in more complicated models as well ¹⁷⁸.

Cross-sections for production of transverse and longitudinal ρ 's may be written down from the expressions for the helicity amplitudes. Thus in the direct channel one has for Δ^2/W^2 small ¹⁷⁶) [notation: $M\lambda_N$; $\lambda(\rho, \nu)$, λ_N , $M_i^{(\pm)}$ are parity conserving amplitudes; Eq. (4.41) has been used to replace B_2, B_6 by B_4, B_7]:

$$\begin{aligned} M_1 &= M_{+;0+} = (m_\rho W^2/2M)(B_7 + B_8) \\ M_2 &= M_{-;0+} = -(m_\rho \sqrt{\Delta^2}/M)(B_3 + B_4) \\ M_3^{(+)} &= M_{+;1+} + M_{+;-1+} = -(\sqrt{\Delta^2}/2M)B_5 \\ M_3^{(-)} &= M_{+;1+} - M_{+;-1+} = -(\sqrt{\Delta^2}/2M)W^2 B_8 \\ M_4^{(+)} &= M_{-;1+} + M_{-;-1+} = (-W^2 B_1 + 2MB_5) \frac{1}{\sqrt{2}M} \\ M_4^{(-)} &= M_{-;1+} - M_{-;-1+} = (-W^2 B_1 + 2\Delta^2 B_3 + 2MB_5) \frac{1}{\sqrt{2}M} \end{aligned} \quad (4.44)$$

Then in the electric Born model the helicity frame cross-sections are

$$\left(\frac{d\sigma_{\perp}}{d\Delta^2}\right)^H = \frac{f^2}{4\pi} \frac{G^2}{4\pi} \frac{\pi}{W^2 q^2} \frac{\mu^4 + \Delta^4}{(\Delta^2 + \mu^2)^2}$$

$$\left(\frac{d\sigma_{\parallel}}{d\Delta^2}\right)^H = \frac{f^2}{4\pi} \frac{G^2}{4\pi} \frac{\pi}{W^2 q^2} \frac{\frac{1}{2} m_\rho^2 \Delta^2}{(\Delta^2 + \mu^2)^2} \quad (4.45)$$

The first (transverse) cross-section exhibits the non-zero forward peak in (inverse) photoproduction.

Similar expressions may be written down for Δ^2 channel helicity amplitudes and polarized cross-sections in the GJ frame ¹⁷⁴⁾. Here the pion pole (B_3) does not contribute to non-zero helicity states [Eq. (4.22)]. Therefore $(d\sigma_{\perp}/d\Delta^2)^{GJ}$ exhibits no forward structure for ρ production. This has been used as an argument against VMD ¹⁷⁹⁾. However, kinematical singularities at $\Delta^2 = (m_\rho \pm \mu)^2$ (see subsection 4.1) introduce a strong m_ρ^2 dependence in the ρ_{nm}^{GJ} 's, whereas the corresponding singularities at $W^2 = (M \pm \mu)^2$, $(M \pm m_\rho^2)$ in ρ_{nm}^H has little effect on high energy forward structures. Therefore, VMD predictions could be straightforward at most in the helicity frame [Refs. 176), 177), 179)].

Cho and Sakurai ¹⁷⁷⁾ suggest to interpret VMD as independence of the Ball amplitudes B_i , $i = 1, \dots, 8$ of m_ρ^2 (this is unfortunately not an unambiguous assumption ^{174), 175), 176)}). Then the two Eqs. (4.41) are actually four equations and all eight (six) amplitudes for ρ production may be generally expressed in terms of the four independent ones for photoproduction. Using the pseudomodel of Jackson and Quigg ¹⁸⁰⁾ (a phenomenological model based on absorbed Reggeization and FESR's) longitudinal and transverse cross-sections for ρ production may be predicted in either frame (Figs. 9a, 9b, data points from Hyams et al. ¹⁸¹⁾). $(d\sigma/d\Delta^2)$ has a strong forward dip but is $\neq 0$ for $\Delta^2 = 0$ ($t = 0$ in the figures). However, $(d\sigma_{\perp}/d\Delta^2)^H$ has a forward peak whereas $(d\sigma_{\parallel}/d\Delta^2)^{GJ}$ is featureless. Particularly interesting are the very forward $\sqrt{\Delta^2}$ structures in ρ_{nm}^{GJ} (ρ^J in Fig. 9b). These predictions should soon be tested. The forms correspond to a value of B_1 much as in Eq. (4.43).

The connection between this value and absorption is particularly evident in the model of Williams¹⁸²⁾. For transverse ρ production in the helicity frame, the half-angle factors Eq. (4.19) suggest that only nucleon flip terms matter for Δ^2/W^2 small $[\bar{M}_4^{(\pm)}$ in Eq. (4.44)]. Pure OPE Eq. (4.42) gives

$$M_4^{(-)} \propto \frac{\Delta^2}{\Delta^2 + \mu^2} = 1 - \frac{\mu^2}{\Delta^2 + \mu^2} \quad (4.46)$$

Here the constant first term contributes an s wave that violates unitarity enormously at high energies. Williams argues that the main helicity dependent effects of absorption¹⁸³⁾ is to completely remove such terms. By Eq. (4.44) this device is seen to produce precisely Eq. (4.43)¹⁷⁸⁾.

Regge conspiracy fits have a similar important B_1 ^{184),185)} corresponding to an isoscalar j plane pole correlated to the pion at $\Delta^2 = 0$, which, however, leads to well-known troubles with factorization in other processes¹⁸⁶⁾. The relevant point here, however, is that several arguments exist for believing in a non-vanishing B_1 of the type Eq. (4.43).

The important consequence of this for

$$\frac{d\sigma}{d\Delta^2} = \frac{d\sigma_{\perp}}{d\Delta^2} + \frac{d\sigma_{\parallel}}{d\Delta^2}$$

is easily seen from Eq. (4.45). The OPE zero at $\Delta^2 = 0$ is now shifted to a value of Δ^2 between 0 and $-\mu^2$. However, neither $d\sigma_{\perp}/d\Delta^2$ nor $d\sigma_{\parallel}/d\Delta^2$ has a zero at that same place. Therefore the normalized density matrix elements and moments will have a pole at that point and possibly a zero nearby as emphasized by Kane and Ross^{144),187)}. Williams estimated the consequence of this for Chew-Low extrapolations in his particular model. Figure 10a shows that ignoring the effect can produce systematic errors of $\approx 20\%$ and suggests an explanation for why the unitary limit at the ρ was not reached in Ref. 170) (see above). Figure 10b indicates that the forward backward asymmetry may be affected to a lesser extent by the pole zero

anomaly [provided very peripheral events are ignored (!)]. Similar calculations were carried out by Kimel who also provided a scheme for doing absorption corrections at successive levels of accuracy ¹⁸⁸).

Due to the absence of OPE constraints for $B' = \Delta_{33}(1236)$ at $\Delta^2 = 0$ [Eq. (4.4.b)] we may expect that a corresponding effect is absent for these processes. This may partly compensate for the anomalous threshold trouble [Eqs. (4.18)].

In the following two subsections we discuss two different ways of dealing with all these effects in practical Chew-Low extrapolation work, which may be denoted (1) cross-section extrapolation, and (2) amplitude parametrization.

4.4. - Cross-section extrapolation,

This method furnishes in principle a completely model independent analysis. However, very high statistics is required and physical information difficult to implement.

To avoid kinematical singularities between the physical region and the pion pole, $\Delta^2 = -\mu^2$, we consider the $\langle \text{Re } Y_L^M \rangle^{GJ}$'s with $M = 0$. These quantities have a double pole at $\Delta^2 = -\mu^2$ but unlike $d\sigma/d\Delta^2 ds$ they have no single pole (subsection 4.1). To avoid the poles at places where $d\sigma/ds d\Delta^2 = 0$ (see preceding subsection) it is further advisable to consider the unnormalized objects

$$\frac{d\sigma}{d\Delta^2 ds} \langle \text{Re } Y_L^0 \rangle \quad (4.47)$$

In general the absolute normalization is experimentally poorly known but to the extent that only one over-all constant can allow for the ignorance, that may be determined by requiring extrapolated meson meson pwa's to be unitarity.

One then multiplies (4.47) by $(\Delta^2 + \mu^2)^2$ or some more suitable quantity (including if wanted the inverse of a DP or BD type form factor ¹⁶⁵⁾, however, care should be exercised that it contains no complex poles or other unwanted new singularities.; call the object thus obtained $f(\Delta^2)$ for each value of s . Fitting $f(\Delta^2)$ to a polynomial in Δ^2 the residue is obtained by evaluating the polynomial at $\Delta^2 = -\mu^2$.

Several people, however, (Cutkosky and Deo ¹⁸⁹⁾, Ciulli ¹⁹⁰⁾, see also Pišút ¹⁹¹⁾) have suggested that improved accuracy, better error estimates and more economical use of the data is achieved if these operations are performed in some conformally transformed variable, z , rather than in Δ^2 . Ideally the conformal transformation should map the part of the Δ^2 plane outside the cuts pertaining to (4.47), onto an ellipse in the new z plane such that the interval on the real positive Δ^2 axis on which data are to be used, is mapped onto the interval between the foci of the z plane ellipse ; i.e., without loss of generality, $-1 \leq z \leq 1$. In practice one may avoid the complicated complex cuts of (4.47) (cf. subsection 4.1) by cutting away a certain segment of the Δ^2 plane. Explicit expressions for the needed transformation are conveniently summarized in Ref. ¹⁸⁹⁾ and are very easy to incorporate in a computer program. The points about the conformal transformation are as follows.

- (i) Ignoring first the effect of experimental errors, least square polynomial fits to the data on $-1 \leq z \leq 1$ will converge throughout the z plane ellipse and therefore throughout the cut Δ^2 plane. This is to be compared with Δ^2 polynomial fits that only converge in a small finite part of the Δ^2 plane (an ellipse with foci at the end points of the interval containing used data, and bounded by Δ^2 plane singularities - not a circle as often stated). According to general theorems therefore ¹⁸⁹⁾⁻¹⁹¹⁾ fewer terms are needed for given approximation accuracy in the z variable than in the Δ^2 variable.

- (ii) Low order polynomials in z correspond to complicated "form factors" in Δ^2 . Also, the pole position is relatively nearer to the data interval in the z plane than in the Δ^2 plane. Therefore use of the z variable means that a smoother object has to be extrapolated a shorter distance.
- (iii) When experimental errors are present, only the lowest order polynomial giving fit to the data can be used in a justifiable way to perform the extrapolation. Both (i) and (ii) then argue in favour of z over Δ^2 . The extrapolation error is a sum of the usual statistical error and a truncation error associated with neglect of higher terms in the polynomial expansion. The latter can be estimated using the fact that at the pole position, $z = z_p$, the rate of convergence of the polynomial expansion is like that of a geometric series with quotient

$$r = \frac{r}{R} = \frac{\alpha + \beta}{a + b} .$$

Here a and b are the semi-axes of the z plane ellipse and α and β are the semi-axes of a new ellipse with foci at ± 1 and $\alpha = |z_p|$. The truncation error is often neglected in experimental Chew-Low extrapolation work which results in plots where the error bar on the extrapolated value is smaller than the error bars on the data points. That is highly unrealistic. Of course the trouble is only greater when one works in the Δ^2 variable.

Schemes that are technically more elegant have been worked out ¹⁹²⁾ and in particular the delicate error estimate has been given a more satisfactory treatment.

The use of quantities like (4.47) only is unsatisfactory in that a considerable amount of experimental information is not used ($\langle Y_L^{M \neq 0} \rangle$). Also it is not obvious that (4.47) is the most practical quantity to extrapolate. Chia Chang Shih ¹⁹³⁾ recently suggested to work rather with cross-sections with well-defined values of the invariants (4.1) instead of well-defined values of di-meson angular momentum. For fixed W^2 the quantity

$$F(s, M_{M'm'}^2, t, \Delta^2) \equiv (\Delta^2 + \mu^2) \langle |dM|^2 \rangle \quad (4.48)$$

is considered, where $\langle |dM|^2 \rangle$ is the spin-averaged invariant amplitude squared [Eq. (4.6)] which may be obtained from the data. Then F may be split as

$$F(s, M_{M'm'}^2, t, \Delta^2) = F_B(s, M_{M'm'}^2, t, \Delta^2) + \frac{R(s, t)}{\Delta^2 + \mu^2} \quad (4.49)$$

where the residue $R(s, t)$ is independent of Δ^2 and $M_{M'm'}^2$, and is related in a known way to the invariant meson meson scattering amplitude squared. For fixed values of s and $M_{M'm'}^2$, F_B and R are parametrized by suitable functions of (t, Δ^2) and t respectively (using polynomials in conformally transformed variables as above or something more or less equivalent^{192),193)}). An unbiased optimal estimate of $R(s, t)$ is obtained for the fit with the most slowly varying F_B (this can be precisely defined^{189),192)}). The parametrization of $F_B(t, \Delta^2)$ is complicated by the fact that singularities are present not only for fixed values of t and Δ^2 but also for fixed values of other invariants linearly dependent on t and Δ^2 (cf. subsection 4.1). The very noteworthy point about keeping $M_{M'm'}^2$ fixed rather than $\ell(m', m'')$ is technically that the horrible complex singularities and kinematical singularities of the quantities $f_{(\nu)}^\ell$ are avoided and fuller use of the analyticity information can be made. Physically it means that the data are analyzed for fixed points in the Dalitz plot and one is able to stay away from complicated N^* resonance regions. The most important drawback appears to be that the requirement of $R(s, t)$ being independent of $M_{M'm'}^2$ has not been built into the method. This would appear to be necessary with any foreseeable statistics.

4.5. - Amplitude parametrization.

Doing the parametrization at the amplitude level is in principle a much more powerful way of making use of the data, and also physical information can be conveyed to the analysis. To the extent that a single theoretical model such as a five-point Veneziano amplitude, was able to fit the data, one might feel satisfied that at least consistency with theory had been established. At present, however, B_5 phenomenology is far short of dealing with all the effects of the preceding subsections. Therefore one has to resort to compromises and build models with poorer predictive power but larger flexibility. If the latter is emphasized the scheme is likely to be technically quite complicated. So far no really satisfactory proposal has been given, however, we shall comment on some of the ones available.

Schlein ¹⁹⁴⁾ suggested to exploit the factorization assumption made for example in all absorption calculations

$$f_{(\nu)}^{\ell}(s, \Delta^2) = \tilde{f}_{(\nu)}^{\ell}(s, \Delta^2) a_{\ell}(s) \quad (4.50)$$

where $a_{\ell}(s)$ is the on-shell meson meson pwa and the reduced amplitudes $\tilde{f}_{(\nu)}^{\ell}$ are (relatively) real. To the extent that $\tilde{f}_{(\nu)}^{\ell}$ is slowly varying a rather efficient parametrization is achieved which is able to accommodate generalized absorption effects. Note that Eq. (4.50) is assumed also for those helicity amplitudes to which the pole does not contribute. This implies that the method relies on a certain final state interaction philosophy more than on dominance of the pion pole.

Froggat and Morgan ¹⁹⁵⁾ noticed that the reduced amplitudes $\tilde{f}_{(\nu)}^{\ell(c)}$ cannot be slowly varying with Δ^2 due to the presence of kinematical singularities. Instead (for $\pi N \rightarrow 2\pi N$) the following parametrization is suggested for $\ell = 0, 1$ (s and p waves)

$$\begin{aligned}
 f_{\frac{1}{2}, \frac{1}{2}; 0}^{l(c)}(s, \Delta^2) &= a_l(s) \left[-\frac{\sqrt{\Delta^2}}{\Delta^2 + \mu^2} + \frac{T_0^l}{\sqrt{\Delta^2}} \right] \\
 f_{\frac{1}{2}, -\frac{1}{2}; 0}^{l(c)}(s, \Delta^2) &= a_l(s) T_0^l \sqrt{\Delta^2 / \Delta_{\min}^2 - 1} / \sqrt{\Delta^2} \quad (l=0, 1) \\
 f_{\frac{1}{2}, \frac{1}{2}; \pm 1}^{1(c)}(s, \Delta^2) &= a_1(s) T_1^1 \sqrt{\Delta^2 / \Delta_{\min}^2 - 1} / \sqrt{\Delta^2} \\
 f_{\frac{1}{2}, -\frac{1}{2}; \pm 1}^{1(c)}(s, \Delta^2) &= a_1(s) T_1^1 [\sqrt{\Delta^2 / \Delta_{\min}^2} \pm 1] / \sqrt{\Delta^2}
 \end{aligned} \tag{4.51}$$

For $l = 1$ the expressions are obtained from formulae similar to Eq. (4.45), however, written down in the Δ^2 channel (174), (196). Then the factorization assumption Eq. (4.50) is made on the Ball amplitudes for each s .

$$B_i(w^2, \Delta^2) = a_i(s) \tilde{B}_i(w^2, \Delta^2) \tag{4.52}$$

For $i \neq 3$ (4), $\tilde{B}_i$ is assumed independent of Δ^2 and s (T_1^1 is related to B_1 and T_0^1 to B_8) and for $i = 3$ the pion pole is inserted. Also some of the B_i 's do not contribute to leading order in $\sqrt{\Delta^2}$ in ρ_{nm}^{GJ} and others are assumed small [near vanishing of $(\rho_{1, -1}^{11})^{GJ}$]. In a completely similar way for s wave production, $f_{\nu}^{0(c)}(s, \Delta^2)$ are parametrized in terms of (two) invariant amplitudes for the process $\pi N \rightarrow \sigma N$, where σ is a scalar particle (196).

In a fit to data from several collaborations on $\pi^- p \rightarrow \pi^+ \pi^- n$ Scharenguivel et al. (197) showed that with available statistics (which is poor in the crucial region $\Delta^2 \lesssim 2\mu^2$) this model adequately describes the data. For B_1 (i.e., T_1^1) a value quite similar to the theoretical estimate Eq. (4.43) results, whereas for B_8 no consistent value was obtained.

The model illustrates the basic advantage of amplitude parametrization. As we saw in subsection 4.3, the degree of non-evasiveness ($d\sigma/ds d\Delta^2 \neq 0$) determines the pole zero anomaly of normalized moments and prevent those from extrapolating smoothly. However, due to the large

steep dip in $d\sigma/dsd\Delta^2$, the value at $\Delta^2 = 0$ is hard to determine. In the model of Eq. (4.51) on the other hand, the crucial parameter (T_i^l) may be more readily measured by looking at those quantities ($\langle Y_L^l \neq 0 \rangle$ for example) not containing the pion pole and ignored in moment extrapolation (cf. preceding subsection). In fact in such schemes no extrapolation is done at all (computationally); the parameters are determined by fittings in the physical region and the pole residue simply obtained by evaluating the parametrized forms at the pion pole.

Equations (4.51) are complicated by the unpleasant singularity at $\Delta^2 = 0$. However, (subsection 4.1) this is a somewhat spurious trouble as far as observables go. It is avoided by working in the helicity frame and using the direct channel expressions Eqs. (4.44). This is in essence what Kane and Roos¹⁸⁷⁾ do, allowing slightly more general than constant behaviour for the $\tilde{B}_i$'s [Eq. (4.52)] but neglecting all but nucleon flip terms (as may be justified for transverse ρ production).

As they stand, these schemes ignore (s, Δ^2) correlations (subsection 4.2). This may be partly cured by the device Eq. (4.36). At higher energies where d waves and yet higher pwa's are present, analogous procedures may be quite laborious to work out. More important, however, is the complete neglect of competing processes such as

$$\pi N \rightarrow \pi N^*$$

or (related to that) diffractive dissociation. To allow in a consistent way for all possible effects it would seem to be inevitable to base the parametrization on the amplitudes A_i ($i = 1, \dots, 4$), Eq. (4.6).

4.6. - Experimental results on $\pi\pi$ scattering.

The reactions mainly used in peripheral pion production are (for a recent experimental review, see Schlein ¹⁴³) :

$$\begin{aligned}
 \pi^- p &\rightarrow \pi^- \pi^0 p & (a) \\
 \pi^- p &\rightarrow \pi^- \pi^+ n & (b) \\
 \pi^- p &\rightarrow \pi^0 \pi^0 n & (c) \\
 \pi^+ n &\rightarrow \pi^0 \pi^0 p & (d) \\
 \pi^+ n &\rightarrow \pi^+ \pi^- p & (e) \\
 \pi^+ p &\rightarrow \pi^+ \pi^+ n & (f) \\
 \pi^+ p &\rightarrow \pi^+ \pi^- \Delta^{++} & (g) \\
 \pi^- p &\rightarrow \pi^- \pi^- \Delta^{++} & (h)
 \end{aligned}
 \tag{4.53}$$

Apart from (a) they are all charge exchange reactions so that $I = 0$ exchange to the nucleon (in particular vacuum exchange) is forbidden. In (a) clear ω like contributions are seen (Treiman-Yang distribution ¹⁹⁸), $\sigma(\pi^- p \rightarrow \rho^0 n) / \sigma(\pi^- p \rightarrow \rho^- p) \neq 2$ ¹⁹⁹).

Consider first the region below the $K\bar{K}$ threshold region (≈ 1 GeV) where $\pi\pi$ scattering is overwhelmingly dominated by the well studied ρ meson. Important s waves are present in addition and most of recent activity has focused on δ_0^0 .

Figure 11 shows the new results of Baton et al. ²⁰⁰), using reaction (4.53.a) to first obtain δ_1^1 and δ_0^2 (P_1 and S_2). The advantage of that is that no $I = 0$ is present. Chew-Low extrapolation using conformal mapping techniques was applied (subsection 4.4) which is supposed to deal automatically with the ω . No violations of the evasive constraint was observed. We shall come back to other waves in Fig. 11.

Figure 12 shows results for δ_0^2 from Colton et al. ²⁰¹) using the modified non-evasive form factor method ¹⁶⁵) to extrapolate reaction (4.53h) to the pole. Several older results and two theoretical estimates are shown as well.

Clearly δ_0^2 is negative, slowly decreasing in the region ($600 \text{ MeV} \lesssim \sqrt{s} \lesssim 850 \text{ MeV}$). At the ρ mass a value of $\delta_0^2 = -15^\circ \pm 5^\circ$ is likely. Similarly there is not much discussion about δ_1^1 in this region (Fig. 11).

Most information on δ_0^0 in the ρ region at present comes from study of s-p interference in reaction (4.53.b). Figure 13 shows the results of Scharenguivel et al. ²⁰²⁾. More specifically these authors study the forward backward asymmetry and extrapolate to the pion pole. From the preceding sections we expect such a treatment to accommodate form factor dependences and correlation effects and to be only slightly affected by non-evasive contributions, although this last point relying on Fig. 10.b is model dependent. Figure 13 exhibits the well-known up-down ambiguity ^{203),204)}. This is inherent to the study of s-p interference only, in which (neglecting δ_0^2) one measures $\sin \delta_1^1 \sin \delta_0^0 \cos(\delta_1^1 - \delta_0^0)$, so that the ambiguity

$$\delta_0^0 \rightarrow \frac{\pi}{2} + (\delta_1^1 - \delta_0^0) \quad (4.54)$$

results. Different treatments (different values for δ_0^2 , use of forward backward asymmetry or $\langle Y_1^0 \rangle$, etc.) produce somewhat different "up" and "down" branches. Recent activity has centered on attempts to resolve this ambiguity.

Starting with the below ρ region, the discussion is somewhat confused by the fact that different people have very different up and down branches. There seem by now to be general agreement, however that, say at 500 MeV, δ_0^0 is probably neither $>70^\circ$ nor $<20^\circ$, and the fairly model independent theoretical predictions (Fig. 2) holds up.

Cline et al. ²⁰⁵⁾ proposed a simple way of measuring scattering lengths. Using data on reactions (4.53.b,d,f) they point out that in form-factor modified OPE models, the approximation $\cos(\delta_0^0 - \delta_0^2) \approx 1$ at low energies yields

$$R_1 \equiv R(\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^+)) = \frac{1}{9}[r^2 + 1 - 2r]$$

$$R_2 \equiv R(\sigma(\pi^0\pi^0)/\sigma(\pi^+\pi^-)) = \frac{1}{2}\left[\frac{r^2 + 1 - 2r}{r^2 + \frac{1}{9} + r}\right]$$

where $r \equiv \sin \delta_0^0 / \sin \delta_0^2$.

Consistency yields

$$r \approx a_0^0/a_0^2 = -(3.1 \pm 1.1)\mu^{-1} \quad (4.55)$$

in nice agreement with Weinberg. Possible non-OPE forces and neglect of effective ranges render the result uncertain, but the method illustrates the power of combining data from different interactions in the analysis.

Scharenguivel et al. ²⁷⁾ pursue the forward backward asymmetry method. They make a joint linear s and Δ^2 parametrization to allow for PCAC type correlations (cf. subsection 4.2). Knowledge on low energy δ_1^1 and δ_0^2 is obtained by extrapolating from higher energies by effective range formulae. Ambiguities in this extrapolation are mainly responsible for the width in their band of solutions (Fig. 14).

In reaction (4.53.b), the obvious way to resolve the ambiguity is to get good values for the extrapolated isotropic part of the $\pi^+\pi^-$ distribution since in that, only even ℓ waves contribute to the pole residue. As we have seen, however, depolarized ρ 's completely dominate the isotropic part in the physical region (subsection 4.3). Most recent attempts to circumvent this difficulty agree with Figs. 14, 11 and 2 ^{206),207),171),200)}.

The $\pi^0\pi^0$ channels (4.53.d,e) are free of depolarized ρ troubles but experimentally extremely difficult. High statistics experiments are nevertheless now in progress and are hoped to play an important role in finally settling the question. So far, however, no general agreement has been arrived at from these reactions neither below nor above the ρ . Figures 15 ²⁰⁸⁾ and 16 ²⁰⁹⁾ represent the two different schools of thought.

Coming to the above ρ region and in particular the region above 1 GeV, it is generally realized that allowance for inelasticities and higher waves are necessary. Baton et al. ²⁰⁰⁾, Fig. 17, find possible inelasticities above the 4π threshold, 560 MeV, but below 1 GeV, much larger than has been so far suspected.

Direct measurements of $\pi\pi$ inelasticities have so far been sparse but clearly are of key importance for higher energy phase shift analyses. Hyams et al. ²¹⁰⁾ analyze data on $\pi^-p \rightarrow K^+K^-n$ at 11 GeV/c, $\pi^-p \rightarrow K_1^0 K_1^0 n$ at 6 and 12 GeV/c and $\pi^-p \rightarrow \pi^+\pi^-\pi^+\pi^-n$ at 3.9 GeV/c using the non-evasive form factor modified OPE scheme ¹⁶⁵⁾ for di-meson mass below 1100 MeV. Huge inelasticities were observed. Within the first 30 MeV above $K\bar{K}$ threshold, $\pi^-\pi^+ \rightarrow K^-\bar{K}^+$ appear to reach the unitary limit. The following averages over the first 100 MeV above $K\bar{K}$ threshold summarize the findings.

$$\begin{aligned}\sigma(\pi^-\pi^+ \rightarrow K^-\bar{K}^+) &= (2.2 \pm 0.5) \text{ mb} \\ 2 \cdot \sigma(\pi^-\pi^+ \rightarrow K_1^0 \bar{K}_1^0) &= (1.4 \pm 0.6) \text{ mb} \\ \sigma(\pi^-\pi^+ \rightarrow \pi^-\pi^+\pi^-\pi^+) &= (0.7 \pm 0.2) \text{ mb} \\ \sigma(\pi^-\pi^+ \rightarrow \text{all inelastic}) &\gg (2.8 \pm 0.5) \text{ mb}\end{aligned}\tag{4.56}$$

or

$$\eta_0^0 \lesssim 0.45^{+0.13}_{-0.43}$$

The first two lines should agree for pure $I = 0$ s wave, however, important corrections could arise from neglected non-OPE forces. The strong onset of inelasticity at the $K\bar{K}$ threshold (≈ 995 MeV) is hard to explain away in terms of production mechanism, however, and is likely to be a crucial aspect of the meson meson scattering. Also, the complete lack of events below 995 MeV strongly suggests that δ_0^0 is nearly elastic in this region ^{211),212),197)} and suggests that Fig. 17 represents spurious effects.

Oh et al. ¹⁹⁷⁾ analyze reactions (4.53.b) using an absorption modified OPE formalism. Figure 18 shows their results again with important but less marked inelasticities.

Evidence that the $I = 0$ s wave is appreciable in the f^0 region is clear in the angular distribution of the decay $f^0 \rightarrow \pi^+\pi^-$ ^{213),214)} as shown [reaction (4.53.g)] in Fig. 19. A strong $\ell = 2$ component is evident, but at $\cos\theta = 0$ there is a depletion of events as compared to a pure $(P_2(\cos\theta))^2$ (dashed curve). The p wave,

however, must be small causing little asymmetry. Figure 20 shows the ABC collaboration results ²¹⁴⁾. The general agreement with Fig. 18 ¹⁹⁷⁾ appears to be satisfactory.

There is, however, considerable reasons to believe that the suggested constant behaviour of δ_0^0 (near 90°) from 800 MeV to about 1400 MeV is qualitatively incorrect. The point is that an s wave $K\bar{K}$ inelasticity will make η_0^0 decrease above $2m_K$ as $\sqrt{s-4m_K^2}$ [Eqs. (2.16)-(2.18)]. By analyticity this implies a large acceleration of δ_0^0 below 995 MeV which by unitarity must be in the positive direction. This effect (reminiscent of the Ball-Frazer mechanism) may turn out to be crucial for disclosing the behaviour of δ_0^0 above the ρ . It is clearly best analyzed in conjunction with a $K\bar{K}$ production analysis (see next subsection).

Fine structures around 1100 MeV have been observed by several people ^{215),216),200),197)}. Figures 11 and 17 show examples. The interpretation is unclear, and it is also not clear whether in fact it is the same effect. Baton et al. ²⁰⁰⁾ and Oh et al. ¹⁹⁷⁾ suggest a ρ' type interpretation (dotted curves in Fig. 18). However, Miller et al. ²¹⁶⁾ rule out $I = 1$ from lack of appearance in the $\pi^-\pi^0$ channel. $S^* \rightarrow 2\pi$ is generally discarded (the structure is too narrow for S^* and too strong for an s wave) whereas a $I = 2$ interpretation might work (full curve in Fig. 18). Split structures in the f^0 (an SU_3 companion of the A_2) has been looked for and both claimed ²¹⁷⁾ and disclaimed ²¹⁸⁾.

Phase shift analysis in the g meson region has so far not been possible and it is quite unclear whether one or more (possibly narrow) resonances are required [for a review, see Ref. 219]. However, the expected spin 3 assignment appears by now to have been proven [reaction (4.53.b)] ²²⁰⁾.

Finally, Fig. 21 shows suggestions for total and elastic $\pi^+\pi^-$ cross-sections by Caso et al. ²²¹⁾ from 1.5 to 3.0 GeV/c². The total cross-sections were obtained from forward elastic differential cross-sections assuming purely imaginary amplitudes and using the optical theorem. The results are compatible with asymptotic estimates from πN , NN and $\bar{N}N$ using earlier quark and Regge models : $\sigma_{\pi\pi}^{\text{tot}} = 14 \text{ mb}$ ²²²⁾, $\sigma_{\pi\pi}^{\text{tot}} = 16 \text{ mb}$ ²²³⁾.

4.7. - Experimental information from $K\bar{K}$ production.

The study of elastic $K\bar{K}$ scattering by means of a Chew-Low type technique is unlikely to become feasible in the near future : cross-sections for $\bar{K}N \rightarrow K\bar{K}Y$ are ≈ 100 times smaller than those for $\pi N \rightarrow 2\pi N$ and the K exchange pole is ≈ 12 times further away from $\Delta^2 = 0$ than the OPE pole.

Pion initiated $K\bar{K}$ production can provide information on $K\bar{K}$ couplings to various inelastic channels. For the most part the present status of experimental sophistication is far short of making a phase shift analysis meaningful. Available knowledge is most conveniently summarized in terms of partial decay modes for a variety of bumps

f, f', g, S^* (natural spin parity)

ϕ, A_2 (unnatural spin parity)

and updated constantly by Particle Data Group (see also Table II). For summaries on bump-hunting, see for example Lai ²²⁴⁾ and Astier ²²⁵⁾.

For a review on the possibly split A_2 see Barbaro-Galtieri ²²⁶⁾. At present the situation appears more confused than ever by the fact that recent high statistics experiments have given no evidence for splitting ²²⁷⁾.

Pion initiated S^* production with subsequent decay into $K\bar{K}$ is an exception to the above general situation and a reaction which is likely to play a crucial role (combined with analysis of $\pi N \rightarrow 2\pi N'$) in disentangling s waves in $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow K\bar{K}$ and $K\bar{K} \rightarrow K\bar{K}$. In the approximation that no other channels (4π) than $\pi\pi$ and $K\bar{K}$ couple, Eq. (2.17) gives for the S matrix for fixed I and l ²²⁸⁾

$$\begin{pmatrix} S(\pi\pi) & S(\pi\pi \rightarrow K\bar{K}) \\ S(\pi\pi \rightarrow K\bar{K}) & S(K\bar{K}) \end{pmatrix} = \begin{pmatrix} \eta e^{2i\delta(\pi\pi)} & i\sqrt{1-\eta^2} e^{i\delta(\pi\pi \rightarrow K\bar{K})} \\ i\sqrt{1-\eta^2} e^{i\delta(\pi\pi \rightarrow K\bar{K})} & \eta e^{2i\delta(K\bar{K})} \end{pmatrix} \quad (4.57)$$

with $\delta(\pi\pi \rightarrow K\bar{K}) = \delta(\pi\pi) + \delta(K\bar{K})$. Beusch ²²⁸⁾ fitted the $K_1^0 K_1^0$ mass spectrum ²¹⁰⁾ to Breit-Wigner's for S^* , f^0 and A_2 plus background to get estimates for the total cross-sections for S^* and f^0 production. These were assumed to be produced by pion exchange for which there is some evidence (whereas for A_2 of course OPE is forbidden and natural parity exchange appears to dominate). With these, the s-d interference term $\langle Y_2^0 \rangle$ may be interpreted to yield a value for $\cos(\delta_0^0(\pi\pi \rightarrow K\bar{K}) - \delta_2^0(\pi\pi \rightarrow K\bar{K}))$. Assuming the d wave to be given by an inelastic f^0 Breit-Wigner, $\delta_0^0(\pi\pi \rightarrow K\bar{K})$ is fixed mod π (see Fig. 22). Next, parametrizing $S(K\bar{K})$ by a complex effective range formula

$$S(K\bar{K}) = \frac{q \cot \Delta + i q}{q \cot \Delta - i q} \quad , \quad q \cot \Delta = \frac{1}{a + ib} - \frac{1}{2} r q^2 \quad (4.58)$$

or an inelastic resonance form, both $\delta_0^0(K\bar{K})$ and η_0^0 are parametrized and therefore [Eq. (4.57)] $|S(\pi\pi \rightarrow K\bar{K})|^2$. As is well known, the S^* may be fitted either way ^{229), 230)}. Parameters depend much on background assumptions. For the resonant solution Beusch gives

$$1640 \text{ MeV} \leq m_{S^*} \leq 1100 \text{ MeV} \quad ; \quad 150 \text{ MeV} \leq T_{S^*} \leq 300 \text{ MeV} \quad (4.59)$$

and for the form, Eq. (4.58)

$$a + ib = (1.3 + i 0.35) F \quad ; \quad r = 1.4 F \quad (4.60)$$

With either parametrization $\delta_0^0(K\bar{K})$ is fixed and $\delta_0^0(\pi\pi)$ may be obtained from the value of $\delta_0^0(\pi\pi \rightarrow K\bar{K})$. Figure 23 shows predicted $\pi\pi$ s wave Argand diagrams assuming $\delta_0^0(\pi\pi) = 90^\circ$ at the $K\bar{K}$ threshold ; rotated diagrams will result for other values.

As emphasized by Morgan ²³¹⁾ considerably more interesting results emerge if below threshold $\pi\pi$ information is conveyed to the analysis and if analyticity is properly implemented. Due then to the (infinite) acceleration of $\delta_0^0(\pi\pi)$ just below the $K\bar{K}$ threshold (see preceding subsection), the value of this phase at threshold is a very inconvenient parameter. Instead, Morgan [using a K matrix formalism starting from Eq. (4.57)] produces solutions for $\delta_0^0(\pi\pi)$ and η_0^0 that join to either the down or the up branch of Baton et al. (Fig. 11) at 800 MeV, and for which the amount of acceleration is determined by the Beusch s wave part of the $K_1^0 K_1^0$ production (neglecting 4π inelasticities). It is found in either case that the $I = 0$ s wave has a zero near below the $K\bar{K}$ threshold. This is in qualitative agreement with the prediction of Lovelace ⁴⁷⁾, Fig. 2, who has, however, the zero at a lower energy due to the lower mass of the S^* (it also appears to be on a different sheet). In either case, specific predictions for $\pi^+\pi^-$ production in reactions (4.53.b,g) follow. Thus for the solution joining onto the down branch at 800 MeV, the $\pi^+\pi^-$ mass spectrum will show a sharp drop just below the $K\bar{K}$ threshold (this is impossible for the up solution, since that stays near the bottom of the Argand diagram all the time ²¹²⁾). Also, $\langle Y_1^0 \rangle$ will drop at threshold in either case, however, for the down solution a moderate rise just below will show up. These statements are easily verified by looking at the behaviours of the s and p waves on the Argand diagram. Preliminary data ^{212),232)} do indeed seem to prefer the down branch solution below 850 MeV.

We finally emphasize again that with these kinds of ϵ and S^* resonances [note that Eq. (4.58) also has a pair of ($K\bar{K}$) second sheet poles, however, below threshold] conventional Breit-Wigner phenomenology is both insufficient and misleading.

4.8. - Experimental results on $K\pi$ scattering.

The following reactions have been mostly used

For a discussion of the different experimental peculiarities with these, see Schlein²³³), 143).

Experimentally, non-charge exchange reactions are dominated by natural spin parity exchange, which is less dominant (but never forbidden) in the charge exchange reactions (4.61)-(4.64). Compared with the analogous situation in $\pi N \rightarrow 2\pi N'$ almost everything acts against you. Data are fewer and the production mechanisms more complicated. Also Bose statistics does not help to cut down on the number of partial waves and no G parity selection rules help cut down on low energy inelasticity. For these reasons, more courage is needed to undertake a Chew-Low analysis and even more reservation for evaluating the results. One advantage, however, is that K^0 's are considerably easier to observe than π^0 's.

Trippe et al. ²³⁴⁾ used reaction (4.61.a) and found some very interesting evidence for a broad κ state (assuming OPE dominance) from the $K^{*0}(890)$ decay asymmetry. That reaction still appears to be one of the more attractive candidates for obtaining $K\pi$ phase shifts, but clear evidences for non-OPE contributions exist.

Figure 24 shows density matrix elements for K^{*0} production at 5.0 GeV/c ²³⁵⁾. Although the Treiman-Yang distribution is quite flat in this reaction, large deviations from OPE ($\rho_{00}^{GJ} \equiv 1$) are seen. Curves are based on absorption OPE calculations but in particular fail to reproduce the large value of $\text{Re } \rho_{10}$. Also the magnitude and shape of the differential cross-section is unsatisfactory. Reaction (4.64.b) should have differential cross-section equal to that of (4.61.a) if OPE dominates. Actually it is smaller by a factor 4 at $\cos\theta_{K^*} = 0.98$ whereas the two become equal at $\cos\theta_{K^*} = 0.87$. Werner et al. ²³⁶⁾ showed that this difference cannot be accounted for by absorbed OPE so clearly other natural parity exchanges with both charge conjugations must be present in these reactions (possibly $A_2 \pm \rho$; density matrix elements for the two are rather similar). A similar trouble is well-known in several other pairs of reactions (cf., for example, Kane ²³⁷⁾).

Fu et al. ²³⁸⁾ study in some detail reaction (4.61.a) in an attempt to isolate OPE signals. Of particular competing importance they consider a $\pi^- \Delta^{++}$ enhancement around 1.58 GeV which must have $I = \frac{1}{2}$ [absence of $\pi^0 \Delta^{++}$ enhancement in reaction (4.61.b)]. This effect contributes part of the $K\pi$ asymmetry and should not be confused with $K\pi$ scattering. It corresponds to events with K^+ very forward in the GJ frame and large $K^+ \pi^-$ effective mass. Figure 25 shows attempts to eliminate these events by the cuts $\sqrt{s} < 1.54$ GeV and $\cos\theta^{GJ} < 0.5$. These leave a very forward peak ($\Delta^2 \lesssim 3\mu^2$, $|t'|$ in Fig. 25.e) which is absent in (c) and (d) and is evidence for a pion pole effect. Similarly Fig. 26 shows decay angular distributions for $K^*(890)$ and $K^{**}(1420)$. For $\Delta^2 < 5\mu^2$ these are produced in a polarized fashion typical of OPE, distorted possibly by lower $K\pi$ waves (a), (c), whereas for $\Delta^2 > 5\mu^2$ a high degree of depolarization is evident (b), (d).

For the $\mathcal{K}$ interpretation (i.e., large $\delta_0^{\frac{1}{2}}$) of the $K^*(890)$ a symmetry it is perhaps fortunate that it seems to be a mainly peripheral phenomenon. Also in favour of $\mathcal{K}$ is the fact that the large number of events between $K^*(890)$ and $K^{**}(1420)$ in the $K\pi$ mass spectrum are mainly very peripheral ones.

A large difference between the forward backward a symmetries in reactions (4.61.a,b) has been known for some time to exist above $\simeq 1$ GeV ¹⁴¹⁾. If one refuses to interpret this in terms of a non-OPE $I = \frac{1}{2} \Delta\pi$ correlation ²³⁸⁾, a huge value for the exotic $I = \frac{3}{2} K\pi$ cross-section is obtained ²³⁹⁾. A similarly large value $[\sigma(\frac{3}{2}) \gtrsim 80 \text{ mb}]$, was obtained using reaction (4.61.d). Bakker et al. ²³⁹⁾, however, pointed out that this last reaction is overwhelmed by background from (4.61.a) and using reaction (4.64.a) they give

$$\sigma(\frac{3}{2}) \approx (2 \pm 1) \text{ mb} \quad (4.65)$$

over the region from threshold to 1.2 GeV in agreement with Cho et al. ²⁴⁰⁾ using the same reaction up to 2 GeV and with estimates from the current algebra scattering length (subsection 3.1).

Figure 27 shows the latest results for $\delta_0^{\frac{1}{2}}$ from the International K^+ collaboration ²⁴⁰⁾ [see also Ref. 241]. Reactions (4.61.a,b) were used and mainly the polynomially pole extrapolated s-p interference $\langle Y_1^0 \rangle$, used. The $K^*(890)$ was parametrized by an elastic Breit-Wigner (cf. Fig. 27) and the tail of the $K^{**}(1420)$ by a d wave Breit-Wigner with 50% inelasticity. The $\delta_0^{\frac{3}{2}}$ can be obtained from the different asymmetry in $K\pi$ elastic and charge exchange scattering. As noticed, however, this can be influenced heavily by correlations, but using in addition the smallness of the $K^-\pi^-$ cross-section (4.65) large values were excluded. Assuming $\delta_1^{\frac{3}{2}}$ to be negligible, $\delta_0^{\frac{3}{2}}$ comes out negative around -30° over the whole region up to 1 GeV but having several unexplained 2-3 standard deviation fluctuations of $\pm 20^\circ$. The near zero of $\delta_1^{\frac{3}{2}}$ is confirmed by studying the angular $K^-\pi^-$ distribution and may also be obtained in (4.61.a) from the obtained p wave deviation from a pure $K^*(890)$ (here taken as Breit-Wigner) which gives a negative value $> -10^\circ$. This is all in agreement with soft meson or Veneziano model expectations (Fig. 3).

The interesting $\delta_0^{\frac{1}{2}}(\kappa)$ is seen to exhibit an up-down ambiguity coming from measurement of s-p interference only. The down branch is rising considerably slower than in Fig. 3, apparently not reaching 90° below 1100 MeV. The up branch passes through 90° at 865 MeV like in Fig. 3, but corresponds to a much narrower resonance ($\Gamma < 40$ MeV) that can always be hidden under experimental resolution. Regarding the existence of the κ meson it seems that either the width has to be much smaller than in Table II (214 MeV) or else it has to be much larger. The down solution is sufficiently similar to what we have become used to call an ϵ type δ_0^0 in $\pi\pi$ that it cannot be considered evidence against κ , however, next to nothing could be said about the corresponding mass value. It might be very different from the point where $\delta_0^{\frac{1}{2}}$ passes through 90° (if it does - it does not have to, for the resonance pole to be there).

Inelasticities in $K\pi$ are not cut down by G parity as in $\pi\pi$, however, the lowest one must be p wave from Bose statistics. One study¹⁴¹⁾ seems to indicate that they are small below 1100 MeV. In the $K^{**}(1420)$ region the inelasticity is well known to be $\approx 50\%$ ²⁴²⁾. Whereas earlier measurements on the relative branching ratio

$$\Gamma(K^*(890)\pi)/\Gamma(K\pi)$$

were in agreement with the SU_3 value of 30% (50%) for an unsplit (split) A_2 , later measurements seemed to prefer a value close to 100%²⁴²⁾. A small value (44%) was recently reported again, however,²⁴³⁾.

Partial waves in the 1300-1500 MeV region were recently discussed by Firestone et al.²⁴⁴⁾ based on data on reaction (4.62.a) at 12 GeV/c. Figure 28 shows $K\pi$ decay angular distributions and $d\sigma/ds d\Delta^2$ in the $K\pi$ mass bands 1.3-1.4 GeV (a), (c) and 1.4-1.5 GeV (b), (d) respectively. Figure 28b shows the expected distribution for a 2^+ object $[K^{**}(1420)]$ produced by OPE. Evidence for appreciable s wave background (depletion of events near $\cos\theta = 0$; cf. also Fig. 19) found in reaction (4.61.a) at 9 GeV/c²³⁸⁾ (Fig. 26) is still clear whereas no appreciable p wave asymmetry is observed. However, the angular distribution in the lower mass band is markedly different even

though in the $K\pi$ mass spectrum the two together seem to contribute one peak. The authors interpret this as evidence for a new $K_N^*(1370)$ resonance with $\Gamma < 150$ MeV. The spin assignment is linked to the assumed production mechanism. Spin 2^+ with p wave background is possible only with appreciable vector exchange, whereas pion-exchange (experimental Treiman-Yang distribution is flat) suggests 0^+ with a d wave background consistent with the tail of a $K^{**}(1420)$ of width 100 MeV. That fits with an $I = \frac{1}{2}$ s wave passing 90° above 1100 MeV looping around to 270° at 1370 MeV and still staying away from the bottom of the Argand diagram at 1420 MeV. It will be interesting to see if the effect holds up both in connection with questions on a split versus unsplit 2^+ nonet [a previous search for a split $K^{**}(1420)$ has been negative ²⁴⁵⁾] and in connection with Veneziano-type second daughter structures.

5. - INFORMATION FROM OTHER SOURCES

5.1. - Low energy inelastic pion-nucleon interaction.

Single pion production in πN collisions in the region just above threshold and below the πN^* threshold ($\sqrt{s} \simeq 410$ MeV) can provide information on the lowest $\pi\pi$ scattering and thus supplement the high energy studies of the same process in a convenient way. The production mechanism and analysis philosophy, however, is completely different.

The di-pion mass distribution has been known for a long time to deviate from phase space in a way that has led to suggestions for a low energy narrow $I = 0$ $\pi\pi$ resonance²⁴⁶⁾. As emphasized by Roberts and Wagner²⁴⁷⁾ the observed effect is actually just what is expected from a $\pi\pi$ $I = J = 0$ wave with small positive scattering length and large negative effective range like predicted by Weinberg or Veneziano theory (Section 3).

From elastic πN phase shift analysis²⁴⁸⁾ one knows that the P_{11} wave is the only one showing important inelasticities in the energy region in question. It follows that the final di-pion state is either in an $I = J = 0$ relative state being s wave with respect to the nucleon or, in an $I = J = 1$ state being p wave with respect to the nucleon and thus being considerably damped by centrifugal barrier effects (or forbidden as in the new $\pi^0\pi^0$ data at 378 MeV²⁴⁹⁾).

In that picture the production amplitude takes the form

$$A(\pi N \rightarrow 2\pi N) = N \times (\text{phase-space}) \times f_0^0(s) \quad (5.1)$$

where $f_0^0(s)$ is the on-shell $\pi\pi$ $I = J = 0$ pwa and N is an energy dependent normalization constant that may be obtained either from the πN inelasticity $\eta_{P_{11}}$ or from the data (thus checking the elastic phase shift results).

In the single channel K matrix unitarized Veneziano model with

$$a_0^0 = 0.345 \mu^{-1} , r_0^0 = -4.15 \mu^{-1} \quad (5.2)$$

[compare with Eq. (3.15)] a good fit to available low energy $\pi^+ \pi^-$ data was obtained ²⁴⁷.

Botke in an analysis incorporating the $\pi^0 \pi^0$ data allowed a_0^0 and r_0^0 to be free parameters and also took into account small inelasticities in other πN waves (S_{11} and D_{11}). As far as the $\pi \pi$ aspect of the work goes, the following results were obtained ²⁵⁰

$$a_0^0 = (0.2^{+0.08}_{-0.1}) \mu^{-1} , a_0^0 = 0.2 \text{ implies } r_0^0 = (-5.2^{+0.2}_{-0.5}) \mu^{-1} \quad (5.3)$$

$$a_0^0 = 0.15 \text{ implies } r_0^0 = (-6.5 \pm 0.5) \mu^{-1}$$

In the linear model for the below threshold $\pi \pi$ amplitude, these may be translated into information on a_0^0 and a_0^2 :

$$a_0^2 = \frac{a_0^0}{10} [1 + 3a_0^0 r_0^0 \mu^2] \quad (5.4)$$

The results thus obtained are plotted in Fig. 29. Also shown are the (a_0^0, a_0^2) points corresponding to Eq. (3.15) and Ref. 247). The indicated results of Maung et al. ²⁴⁹ ($\pi^0 \pi^0$ experiment) were obtained by parametrizing $f_0^0(s)$ by the s wave dominant Chew-Mandelstam forms ²²), i.e., having $a_0^0/a_0^2 \approx 5/2$ or $a_0^0 r_0^0 \approx 1$. The authors notice that their a_0^0 agrees with Weinberg, but the relatively inconsistent nature of the fits (and large uncertainty) might be an indication that the parametrization in fact disagrees with Weinberg.

Nevertheless, the difference between the results obtained by the various groups (Fig. 29) together with the fact that they all were able to more or less fit the data, indicate that these experiments measure a certain combination of low energy parameters much better than a_0^0 and r_0^0 individually.

A similar phenomenon is known in the current algebra treatment of the same process. Here the soft pion techniques relate the amplitude for $\pi N \rightarrow 2\pi N$ to that for $\pi N \rightarrow \pi N$ at threshold in a way that depends on the same σ commutator that enters the $\pi\pi$ treatment, i.e., on a_0^0/a_0^2 . Olsson and Turner ²⁶⁾ thus obtained results that may be summarized as, Ref. 39)

$$|2a_0^0 + a_0^2| = (0.4 \pm 0.1)\mu^{-1} \quad (5.5)$$

and plotted in Fig. 29 as well. Analyzing the Dalitz distribution for $\pi^- p \rightarrow \pi^- \pi^+ n$ at 74 MeV by Gribov's formula (effective ranges ignored), Ref. 251), Blair et al. ²⁵²⁾ obtained

$$a_0^0 - a_0^2 = (0.42 \pm 0.10)\mu^{-1}.$$

5.2. - Low energy elastic pion-nucleon interaction.

Results of analysis of this process were among the very first evidences that the $I = J = 0$ $\pi\pi$ wave at low energies had a predominantly positive phase shift δ_0^0 of appreciable magnitude ²⁵³⁾.

The idea is to take the accurate information on πN scattering in the physical region $s \geq (M+\mu)^2$, $-4q^2 \leq t \leq 0$ and from that deduce properties on the amplitude in the pseudophysical range, $4\mu^2 \leq t \leq 4M^2$ for the process $\pi\pi \rightarrow \bar{N}N$. In this region the amplitude may be partial wave expanded and the pwa's for even (odd) angular momentum J and $I = 0$ (1) satisfy the generalized unitarity rule

$$\arg(f_{\pm}^J(t)) \equiv \delta_J^I(t) \pmod{\pi}, \quad 4\mu^2 \leq t \leq t_{in} \quad (5.6)$$

Here δ_J^I is the $\pi\pi$ phase shift for angular momentum J and isospin $I = 0, 1$, the $\pm$ subscript is a nucleon helicity label.

Among the several proposals for performing the above analytic extrapolation, the method using backward dispersion relations is the easiest to describe. Donnachie et al. ²⁵⁴⁾ used that method to find evidence for a $J = I = 0$ resonance. Details in the treatment have meanwhile been criticized on various points and the calculations

recently redone by Nielsen et al.²⁵⁵). We shall comment briefly on those points and estimate how the process can continue to supplement information from other sources.

The most interesting results come from study of $f_+^0(t)$. Consider the specific πN backward amplitude

$$\begin{aligned} F^{(+)}(\nu) &= \frac{4\pi\sqrt{s}}{E} (f_1^{(+)}(\nu, \cos\theta = -1) - f_2^{(+)}(\nu, \cos\theta = -1)) \\ &= \frac{1}{M} \frac{8\pi}{p_t^2} \sum_{J \text{ even}} (J + \frac{1}{2}) (p_t q_t)^J f_+^J(t) \end{aligned} \quad (5.7)$$

The (+) indicates that $I = 0$ in the t channel ($\pi\pi \rightarrow N\bar{N}$) has been selected; E is the s channel c.m. nucleon energy, $\nu \equiv q^2$ the s channel c.m. three-momentum squared and $\nu \equiv -4t$ at $\cos\theta = -1$, M is the nucleon mass and p_t and q_t are the t channel c.m. nucleon and pion three-momenta.

$F^{(+)}(\nu)$ is a real analytic function defined in the ν plane cut along $(0, \infty)$ (physical $\pi N \rightarrow \pi N$ region) and $(-\infty, -\mu^2)$ (the pseudophysical, $4\mu^2 \leq t \leq 4M^2$, and physical, $4M^2 \leq t \leq \infty$, $\pi\pi \rightarrow \bar{N}N$ regions) and having a Born pole at

$$\nu = -\mu^2 + \frac{\mu^4}{4M^2}$$

with known residue. Using the first line in (5.7) $F^{(+)}(\nu)$ may be calculated for $\nu \geq 0$ using known πN phase shifts [for $\nu \rightarrow \infty$ baryon Regge models may be (and have been) used, however, this region is irrelevant to the purpose at hand). Errors in the backward direction are considerable, but the no-flip combination chosen helps to reduce them. Provided $F^{(+)}(\nu)$ is known with sufficient accuracy on the right-hand cut, it may be continued onto the left-hand cut where the last line in Eq. (5.7) is valid. Truncating that expansion in some way or other and separating the first term $f_+^0(t)$, Eq. (5.6) can be used to obtain $\delta_+^0(t)$ for $t \leq t_{in}$. In all treatments t_{in} is taken of the order of 1 GeV^2 for which there is now good evidence (subsection 4.6). Above this value $f_+^0(t)$ even if it could be obtained is not related in any simple way to $\pi\pi$ scattering.

Comparing this method with the Chew-Low method (Section 4) we make the following comments : both require an extrapolation ; however, whereas in the latter we only need to extrapolate to an isolated point (the pion pole) in the holomorphy region, in the former extrapolations onto the boundary (the left-hand $\sqrt{}$ cut) is required ; the mathematical extrapolation theory ^{189),191),255)} tells us that the penalties for this are terrible. Thus, if ϵ is a measure of the experimental error, the extrapolation error in the Chew-Low case will be

$$\epsilon_{CL} \propto \epsilon^p \quad \text{with } p \text{ slightly smaller than } 1 \quad (5.8)$$

whereas in the present application it will be

$$\epsilon_{backw.} \propto |\log \epsilon|^{-1} \quad (5.9)$$

The effect of this is that in the πN backward case extremely poor resolution of structures in f_+^0 results and only averages over the whole region up to 1 GeV is obtainable with a useful accuracy ($\approx 50\%$). Improving πN phase shift by an order of magnitude at low energies, a non-trivial combination of a_0^0 and r_0^0 could be measured with good accuracy, but fine structures in the ρ region are unlikely to be obtainable. The main reason why this kind of analysis makes sense at all is that low energy elastic πN amplitudes are so fantastically much simpler to describe and better known than high energy inelastic ones.

The treatment has the almost unique advantage that $I = 0$ may be isolated and the sign of δ_0^0 is unambiguously obtained as positive, ruling out several suggestions for the opposite.

The methods of Donnachie et al. ²⁵⁴⁾ have been criticized on the following points.

- (i) To achieve the analytic continuation in a convenient way, a conformal transformation much like the one described in subsection 4.4 was applied, however, a circle rather than an ellipse was used

which is unjustifiable¹⁹⁰⁾. In practice the two differ by at most a few percent in this case and the effect can be ignored with present accuracy²⁵⁵⁾.

- (ii) In obtaining $f_{\pm}^0(t)$ from $F^{(+)}(\nu)$ for $t \geq 4\mu^2$ ($\nu \leq -\mu^2$), terms with $J \geq 2$ were effectively ignored. This has been shown to produce significant systematic errors²⁵³⁾ due to the peculiar behaviour of the s and u channel Born poles. The effect, however, is readily taken into account by explicit treatment of the Born pole for all J's in Eq. (5.7), and neglecting the non-pole part of $f_{\pm}^J(t)$ for all J's ≥ 2 can be justified.

Figure 30 shows the results for $f_{\pm}^0(t)$ ²⁵⁵⁾. Several ways of performing the extrapolation were found to agree within errors. Curve B in Fig. 30.b is the result of the unjustified truncation of Eq. (5.7) whereas the more correct curve B1 was obtained as described above.

The following conclusions can be made : δ_0^0 is predominantly positive below 1 GeV. Large local errors (factors of 2 or 3) in Fig. 30 can be expected but the average should be right to about 50%. Thus it appears impossible for $\text{Re } f_{\pm}^0$ to have a clear cut zero near the ρ and ending with a large negative value [this was entirely possible in Ref. 254), cf. curve B] and thus the down branch in the region above the ρ is preferred for δ_0^0 . If values for δ_0^0 are known from other sources, a similar analysis can provide more reliable values for the ϵ NN coupling²⁵⁶⁾. If, however, narrow resonance approximations are used²⁵⁷⁾ we expect from Eqs. (3.78), (3.79) that systematic errors of a factor 2-3 are likely to occur again.

5.3. - Experimental information from K_{e4} decay.

K_{e4} decay was shown by Pais and Treiman²⁵⁸⁾ to provide a reaction where low energy $\pi\pi$ scattering can be studied in an extremely clear way (no other hadrons in the final state) using only a minimum of assumptions. Further the validity of these may be tested on the data in a completely model independent way.

Unfortunately, the branching ratio for the process is only $\approx 3 \cdot 10^{-5}$ and four invariant amplitudes and five invariant variables are needed to a complete description. So far only 338 fully determined events ^{259),260)} have been published and a more model-dependent analysis must be made. However, the total number of available events is expected to be soon significantly increased.

Only the reaction

$$K^+ \rightarrow \pi^+ \pi^- e^+ \nu \quad (5.10)$$

has been studied. We shall assume time-reversal invariance and the $|4I| = \frac{1}{2}$ rule but these may be tested given the needed statistics. Then only $I = 0, 1$ $\pi\pi$ scattering contribute. The kinematical di-pion mass range is

$$2\mu \approx 280 \text{ MeV} \leq M_{\pi\pi} \leq m_K = 494 \text{ MeV}$$

However, experimentally virtually no events are present above 400 MeV. Therefore an effective range parametrization of s and p waves should suffice to describe the data.

The questions we should like to ask are :

- (1) can current-algebra-type $\pi\pi$ amplitudes describe the data ?
- (2) if so, to what extent can this be proven ?
- (3) if not so, what goes wrong ?

In addition, it is of great theoretical interest to consider several questions connected with the soft pion relations between K_{e4} ^{261),262),63)} and $K_{\ell 3}$.

As for answers to (1)-(3) the present status is : from the Veneziano model fit by Roberts and Wagner ²⁶³⁾ the answer to (1) is : definitely yes ; from the more phenomenological fits of Ely et al. ²⁶⁰⁾ and Berends et al. ²⁶⁴⁾, the answer to (2) is : not to any appreciable extent at all (see also Kalmus ²⁶⁵⁾).

To lowest order in weak interaction, the amplitude for (5.10) is

$$\mathcal{M} \propto \langle \pi^+(p_+) \pi^-(p_-) | J_\lambda^V + J_\lambda^A | K^+(k) \rangle \{ \bar{u}_\nu(p_\nu) (1 + \gamma_5) u_e(p_e) \}$$

with

$$\langle \pi^+(p_+) \pi^-(p_-) | J_\lambda^V + J_\lambda^A | K^+(k) \rangle = \frac{1}{m_K} [F P_\lambda + G Q_\lambda + R k_\lambda] \quad (5.11)$$

$$+ \frac{H}{m_K^3} \epsilon_{\lambda\mu\nu\sigma} k_\mu P_\nu Q_\sigma \quad (5.12)$$

and

$$P \equiv p_+ + p_- , \quad Q \equiv p_+ - p_-$$

The invariant axial form factors F, G, R and the vector form factor H depend on $M_{\pi\pi}$ = di-meson mass, $M_{\ell\ell}$ = di-lepton mass, θ_π = angle of π^+ in di-pion rest system w.r.t. the di-lepton line of flight in the K^+ rest system. The two additional variables are taken as

θ_ℓ = angle of e^+ in the di-lepton rest system w.r.t., the di-lepton line of flight in the K^+ rest system ;

φ = angle of the di-lepton plane w.r.t., the di-pion plane in the K^+ rest system.

The form factor R is of particular importance for the soft pion relations between $K_{\ell 3}$ and K_{e4} but gets multiplied by m_e^2 in the decay distribution and will be ignored for the rest.

The soft pion theorems give no statements for H . For the two remaining axial form factors, the simplest way to satisfy them is to have ²⁶¹⁾.

$$F \equiv G \equiv \text{constant} = f_+(0) m_K \sin \theta_c / F_\pi \sqrt{2} \approx 0.8 \quad (5.13)$$

This result neglects any final state $\pi\pi$ interaction which is just the effect we want to study. Therefore one writes ²⁶⁵⁾

$$\begin{aligned}
 F(M_{\pi\pi}, M_{\ell\nu}, \theta_{\pi}) &= f_s(M_{\pi\pi}, M_{\ell\nu}) e^{i\delta_0^0(M_{\pi\pi})} + f_p(M_{\pi\pi}, M_{\ell\nu}) e^{i\delta_1^1(M_{\pi\pi})} \\
 G(M_{\pi\pi}, M_{\ell\nu}, \theta_{\pi}) &= g_p(M_{\pi\pi}, M_{\ell\nu}) e^{i\delta_1^1(M_{\pi\pi})} \times \cos \theta_{\pi} \\
 H(M_{\pi\pi}, M_{\ell\nu}, \theta_{\pi}) &= h_p(M_{\pi\pi}, M_{\ell\nu}) e^{i\delta_1^1(M_{\pi\pi})}
 \end{aligned}
 \tag{5.14}$$

where (time reversal invariance) f_s, f_p, g_p, h_p are real functions in the physical region. Then Eq. (5.13) may be rephrased as

$$f_s \approx g_p \approx 0.8, \quad |f_p| \ll |f_s| \tag{5.15}$$

Pais and Treiman suggest to write the decay distribution as ²⁵⁸⁾

$$\begin{aligned}
 I(M_{\pi\pi}, M_{\ell\nu}, \theta_{\pi}, \theta_{\ell}, \varphi) &= I_1 + I_2 \cos \theta_{\ell} + I_3 \sin^2 \theta_{\ell} \cos 2\varphi \\
 &+ I_4 \sin 2\theta_{\ell} \cos \varphi + I_5 \sin \theta_{\ell} \cos \varphi + I_6 \cos \theta_{\ell} \\
 &+ I_7 \sin \theta_{\ell} \sin \varphi + I_8 \sin 2\theta_{\ell} \sin \varphi + I_9 \sin^2 \theta_{\ell} \sin 2\varphi
 \end{aligned}
 \tag{5.16}$$

where the I_i 's may be given as complicated expressions involving kinematical factors and F, G and H , but are depending only on $M_{\pi\pi}$, $M_{\ell\nu}$ and θ_{π} and may in principle all be obtained from the data. Integrating over $M_{\ell\nu}^2$ and $\cos \theta_{\pi}$ to obtain $\langle I_i \rangle$'s, Eqs. (5.14) imply

$$\begin{aligned}
 \tan(\delta_0^0(M_{\pi\pi}) - \delta_1^1(M_{\pi\pi})) &= \langle I_7 \rangle / 2 \langle I_4 \rangle \\
 \tan(\delta_0^0(M_{\pi\pi}) - \delta_1^1(M_{\pi\pi})) &= 2 \langle I_8 \rangle / \langle I_5 \rangle
 \end{aligned}
 \tag{5.17}$$

This is an extremely model-independent method for obtaining $(\delta_0^0 - \delta_1^1)$ as a function of $M_{\pi\pi}$ but it depends on correlations in the (θ_{ℓ}, φ) distribution that are not clearly visible in the meagre data. Therefore mainly one-dimensional distributions have been

studied ^{260),264),266)}. The form factors $f_s, \dots, h_p$ are assumed constant and $\delta_0^0 - \delta_1^1$ parametrized by a Chew-Mandelstam effective range formula neglecting δ_1^1 . The most valuable information comes from the ϕ distribution. Writing

$$\frac{dT}{d\phi dM_{\pi\pi}^2} = a + b \sin \phi + c \cos \phi \quad (5.18)$$

a and c contribute symmetrically around $\phi = 0$ and are almost independent of $\delta_0^0 - \delta_1^1$. For b on the other hand ²⁶⁴⁾ we have

$$b = \frac{\pi}{4} K f_s g_p \sin(\delta_0^0 - \delta_1^1) \quad (5.19)$$

where K is a positive definite kinematical function. Figure 31 shows the experimental distribution and from the asymmetry, $\langle b \rangle$ is seen to be positive. Thus two solutions are found :

$$\begin{aligned} \text{(I)} \quad g_p / f_s > 0, \quad a_0^0 > 0 \\ \text{(II)} \quad g_p / f_s < 0, \quad a_0^0 < 0 \end{aligned} \quad (5.20)$$

Solution (I) agrees with Weinberg and (II) does not. Form factors are determined from the other distribution and the ambiguity may be resolved by the θ_ℓ distribution, but statistics do not allow to do that. Table III shows the results. The fits are satisfactory and no disagreement is found with the constraint $a_0^0 r_0^0 \approx 1$ implied by the Chew-Mandelstam form.

Figure 31 shows (dotted line) the Veneziano model fit ²⁶³⁾ that does have large negative effective range. Considerations on quantum number and angular momentum contents of poles in the various possible channels suggest to write

$$F \propto G \propto H \propto B(1 - \alpha_p(s), 1 - \alpha_{K^*}(t)) \quad (5.21)$$

where $s = M_{\pi\pi}^2$ and t is the momentum transfer between the leptons and one pion. Soft pion theorems may be imposed and the form (5.14) K matrix unitarized. Equation (5.15) is approximately satisfied and the only free parameter is the relative normalization between H and (F,G).

The fit has no very strong bearing on Veneziano theory as such, but it does show that existing data can be described by a Weinberg type $\pi\pi$ amplitude and perfectly decent form factors.

5.4. - Miscellaneous additional reactions.

$K^0 \rightarrow 2\pi$ decays depend to some extent on the phase difference $\delta_0^0(\pi\pi) - \delta_0^2(\pi\pi)$. The status is largely unchanged since the review of Kalmus²⁶⁵⁾.

For $K_L^0 \rightarrow 2\pi$ the usual CP violating amplitudes η_{+-} and η_{00} satisfy²⁶⁷⁾

$$\eta_{+-} - \eta_{00} \propto i e^{i(\delta_0^2(m_K) - \delta_0^0(m_K))} \quad (5.22)$$

However, at this time, experiments are in agreement with the "superweak" prediction²⁶⁸⁾

$$\eta_{+-} = \eta_{00}$$

in which case Eq. (5.22) is empty²⁶⁹⁾.

For $K_S^0 \rightarrow 2\pi$ the ratio

$$R(\Gamma(K_S^0 \rightarrow \pi^+\pi^-)/\Gamma(K_S^0 \rightarrow \pi^0\pi^0))$$

satisfies

$$R - 2 = 6\sqrt{2} \operatorname{Re}(A_2/A_0) \cos(\delta_0^2 - \delta_0^0) + 2f \quad (5.23)$$

where A_0 (real) and A_2 are the amplitudes for di-pion $I = 0, 2$.
 $R-2 \equiv 0$ if the $|\Delta I| = \frac{1}{2}$ rule was exactly valid. $\text{Im } A_2 \approx 0$ is found experimentally and assumed. A_2 is measured from $K^+ \rightarrow \pi^+ \pi^0$, however, lack of knowledge on $\Delta I = 5/2$, contributes a theoretical uncertainty of ± 0.02 and the same is true with model-dependent electromagnetic corrections [these effects are lumped together under f in Eq. (5.23)]. Altogether one obtains ^{265), 270)}

$$\delta_0^0(m_K) - \delta_0^2(m_K) = \pm (40^\circ + 15^\circ_{-20^\circ}) \text{ mod } (180^\circ) \quad (5.24)$$

Due to the theoretical uncertainties the errors cannot be drastically decreased and thus this piece of information is unfortunately unable to resolve the up-down ambiguity below the ρ .

$K_{\ell 3}$ decays have been used by many people in attempts to extract $K\pi$ phase shifts. At this time the theoretical understanding of the experimental results is as confused as ever and statements on $K\pi\pi$ interaction can hardly be made ^{62), 262)}.

Assuming pure vector transition, $|\Delta I| = \frac{1}{2}$ rule and CP conservation, the amplitude is determined by

$$(2\pi)^3 \sqrt{4E_K E_\pi} \langle \pi^+(q) | V_\mu^+(0) | \bar{K}^0(k) \rangle = \\ (k+q)_\mu f_+(t) + (k-q)_\mu f_-(t) \quad (5.25)$$

with $t \equiv (k-q)^2$.

Parameters $\lambda_\pm, \lambda_0$ and ξ are defined by

$$f_\pm(t) \simeq f_\pm(0) (1 + \lambda_\pm t/\mu^2) \\ f(t) \equiv f_+(t) + \frac{t}{m_K^2 - \mu^2} f_-(t) = f_+(0) (1 + \lambda_0 t/\mu^2) \quad (5.26) \\ \xi = f_-(0) / f_+(0)$$

The Callan-Treiman relation ²⁷¹⁾ suggests

$$f_+(m_K^2) + f_-(m_K^2) = F_K / F_\pi \approx f(m_K^2) \quad (5.27)$$

Experimentally, assuming Cabibbo theory with one angle ²⁷²⁾

$$f(0) = f_+(0) = F_K / F_\pi [1.28 \pm 0.06]^{-1} \quad (5.28)$$

whereupon

$$\lambda_0 \approx 0.28 \mu^2 / m_K^2 \quad (5.29)$$

results. Dispersion relations for $f_+(t)$ and $f(t)$ are determined respectively by spin 1 and 0 intermediate states. Assuming for $f(t)$ an unsubtracted form dominated by a χ meson :

$$f(t) = f_+(0) \frac{m_\chi^2}{m_\chi^2 - t} \quad \text{or } m_\chi = 1.06 \text{ GeV} \quad (5.30)$$

obtains. This appears to be in fairly nice agreement both with experiment (subsection 4.8) and Veneziano theory (here $m_\chi = m_{K^*}(890)$, however, being a wide object the χ may easily be fitted below threshold by poles at either 900 MeV or 1100 MeV). The trouble is that by now direct experimental measurements of λ_0 appear to prefer a negative value (errors are large) ²⁶²⁾. Should that be taken as evidence for a negative $\delta_0^{\frac{1}{2}}(K\pi)$? We make the following comments suggesting the answer is no :

- (i) For $f_+(t)$ there is not much question about $\delta_1^{\frac{1}{2}}(K\pi)$ but $K^*(890)$ dominance ($\lambda_+ \simeq 0.02$, $\xi = 0$) is in equally bad agreement with experiment ($\lambda_+ \approx 0.05 \pm 0.02$, $\xi = -0.9 \pm 0.3$).

- (ii) Everything may be saved by allowing more complicated behaviours for f , f_+ and models exist that suggest this to be the case ; however, the theoretical understanding is so poor that $\delta_{0,1}^{\frac{1}{2}}(K\pi)$'s cannot be obtained.
- (iii) A model-independent extrapolation of the data in the decay region onto the $K\pi\pi$ cut would give the low energy phases. In principle this method is very similar to the application on backward πN scattering to obtain information on $\delta_{\ell}^I(\pi\pi)$. Since the data are so enormously much poorer, any attempt would be meaningless, however.

$K_3\pi$ decay was treated among others by Lovelace ⁴²⁾ in terms of Fig. 32 with $X=K$ and the square representing weak interactions and for the blob using a Veneziano form with "one heavy pion". In such a framework the decay distribution would measure $\pi\pi$ scattering. The model has subsequently been shown to lead to severe difficulties ^{273),274)} partly having to do with inclusion of the diagram, Fig. 33.b and partly with off-shell variations that are experimentally required to be unreasonably large. Similar treatments have been tried with $\eta \rightarrow 3\pi$ but troubles appear to be even greater. Possible cures have been conceived ^{275),276)}.

We consider these reactions at present to yield no useful information on meson meson scattering. Anyway their use is distinctly different from that of all other so far described processes in that even in principle, experimental information alone, however, accurate, will be insufficient to solve the difficulties : a more or less complete theory for relating the amplitude at the one experimental point, $m_x = m_K$ (or m_η) to the $\pi\pi$ scattering point $m_x = \mu$ is needed, whereas previously the amplitude could be measured in a non-vanishing region and the principle of analytic continuation be invoked.

Similar remarks apply to the now famous Veneziano studies of $\bar{N}N$ annihilations ⁴²⁾ to three mesons ($X = \bar{N}N$ with $J^P = 0^-$ in Fig. 33.a). Here, using the model, one has the exciting possibility of measuring meson meson scattering in the "double spectral function region".

Of course, the mass extrapolation penalties are uncontrollable, but this kind of study is no doubt of the greatest interest in a more general hadrodynamical (dual) content. New studies ²⁷⁷⁾ of $\bar{p}p \rightarrow \eta \pi^+ \pi^-$ ²⁷⁸⁾ according to the Veneziano model for $\eta\pi$ scattering ¹²²⁾ indicate qualitative agreement with theory (provided δ decouples ¹²²⁾).

Similar but much more complicated Veneziano fits ²⁷⁹⁾ to the process (again at rest)

$$\bar{p}p \rightarrow \pi^+ \pi^- \omega \quad (5.31)$$

may be of greater interest. The isolation of the mode (5.31) from the data on

$$\bar{p}p \rightarrow 2\pi^+ 2\pi^- \pi^0$$

is ambiguous [several $(3\pi)^0$'s with mass $\approx m_\omega$ can co-exist] and the ω peak is seen only above a very large background. For these reasons a genuine quantitative analysis has formidable difficulties. The fits indicate dominance of $\rho^0(765)$, $f(1260)$ and $B(1230)$. However, more interesting for our purpose is an observed discrepancy in the $\pi\pi$ mass spectrum around the 1 GeV region: enhancements at 850 and 1115 MeV are seen, separated by a non-fitted dip at 940 MeV and this structure is blamed on the $\delta^0(\pi\pi)$. Similar conclusions were obtained in final state resonance model fits ²⁸⁰⁾ and in a similar analysis of

$$\bar{p}p \rightarrow \pi^+ \pi^- \pi^+ \pi^-$$

assuming the initial 3S_1 state to go predominantly to ρ plus an $I = 0$ di-pion ²⁸¹⁾. This is just what is expected from the combined $\mathcal{E} + S^* + K\bar{K}$ threshold anomaly (cf. subsections 4.6 and 4.7) and the effect was qualitatively fitted by Morgan ²³²⁾ in these terms. When taken together with these other pieces of information (subsection 4.7) this seems more satisfactory than a fit using two simple elastic resonances ^{279), 281)}.

The annihilation of colliding e^+e^- beams gives of course valuable information on the ρ parameters^{282),283)} but has so far mainly been used to test VMD predictions. Several suggestions for using the process^{284),285)}

$$e^+e^- \rightarrow \pi^+\pi^-\gamma \quad (5.32)$$

for studying $\delta_0^0(\pi\pi)$ have been made and cross-sections estimated using the model (in the ϕ region)

$$e^+e^- \rightarrow \phi \rightarrow \pi^+\pi^-\gamma \quad (5.33)$$

However, in conformal invariant (scale invariant) theories the coupling is forbidden by gauge invariance and current conservation when one ϕ is on-shell and the other has zero mass²⁸⁶⁾. This is true for $\epsilon\rho\rho$ and $\epsilon\omega\omega$ couplings as well. In dual theories $\epsilon\phi\phi$ would be forbidden in tree-diagrams automatically by the Harari-Rosner rules⁵⁶⁾ but not $\epsilon\rho\rho$ or $\epsilon\omega\omega$ ²⁸⁶⁾.

Di-pion photoproduction has been studied for some time in the ρ region and recent experiments on nuclei produced some weak but interesting evidence for an enhancement between 1300 and 1800 MeV²⁸⁷⁾ with a production cross-section, however, several hundred times smaller than for ρ . This, however, is perfectly compatible with the ρ' required in the two-component dual picture²⁸⁸⁾.

Finally, experiments of the kind

$$p+d \rightarrow He^3 + 2\pi \quad (5.34)$$

$$N+N \rightarrow d + 2\pi \quad (5.35)$$

looking for missing mass to the final nucleons, give much higher values than Weinberg for a_0^0 ,

$$a_0^0 \approx 1 \text{ to } 2 \mu^{-1}$$

In the spirit of this paper, we shall tentatively consider the observed effect to be of nuclear physics origin, for which there is some excuse in the fact that huge complex anomalous threshold cuts are present in the $M_{\pi\pi}$ variable.

5.5. - Dispersion relation phenomenology.

Dispersion relations may be used to sharpen results coming out of the experimental analysis just as is well known in low energy meson nucleon scattering. For meson meson scattering this has so far only been done to any appreciable extent for $\pi\pi$ scattering. Soon, however, two-channel treatments of $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow K\bar{K}$ and $K\pi \rightarrow K\pi$ should be useful.

In $\pi\pi$ scattering ⁵⁾ forward dispersion relations have in recent time been preferred to partial wave ones due to the more powerful way crossing symmetry enters and due to the fact that unitarity is a manageable problem at low energies since so few partial waves are important.

In addition to the usual (s,u,t) variables the following are useful

$$\begin{aligned} z &= (s-u)/4\mu^2 \\ \nu &= q^2/\mu^2 = (s-4\mu^2)/4\mu^2 \\ \omega &= \mu \cdot z = \text{pion lab. energy } (t=0) \\ k^2 &= \omega^2 - \mu^2 \end{aligned} \quad (5.36)$$

Then for a forward $(t=0)$ s channel amplitude $F^I(z)$ of definite isospin I, an unsubtracted dispersion relation reads

$$F(z) = \frac{1}{\pi} \int_1^\infty \frac{\text{Im} F^I(z'+i\epsilon)}{z'-z} dz' - \frac{1}{\pi} \int_1^\infty \frac{\text{Im} F^I(-z'+i\epsilon)}{z'+z} dz' \quad (5.37)$$

with

$$\begin{aligned} \text{Im } F^I(-z' + i\epsilon) &= -\text{Im } F^I(-z' - i\epsilon) \\ &= -\sum_{I'} \mathcal{G}_{su}(I, I') \text{Im } F^{I'}(z' + i\epsilon) \end{aligned}$$

[cf. Eq. (2.33)]. With normalization $N = 2$ [Eq. (2.3)], subtracted forms are [see also Refs. 4), 291]

$$A^I(\nu) = \begin{pmatrix} a_0^0 \\ 0 \\ a_0^2 \end{pmatrix} + J^I(\nu) + \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} J(\nu) \quad \text{for } I = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad (5.38)$$

where

$$J^I(\nu) = \frac{\nu(\nu+1)}{4\pi^2} \int_0^\infty d\nu' \frac{2\nu'+1}{\sqrt{\nu'(\nu'+1)}} \frac{\sigma^I(\nu')}{(\nu'+\nu+1)(\nu'-\nu)}$$

$$J(\nu) = \frac{\nu}{\pi} \int_0^\infty \frac{d\nu'}{(\nu'+1)^2} \text{Im} \left[\frac{2}{3} A^0(\nu') + A^1(\nu') - \frac{5}{3} A^2(\nu') \right] \quad (5.39)$$

The following sum rules are important :

- (1) evaluating A_t^1 (the amplitude with t channel isospin $I_t = 1$) at $t = 0$, $s = 4\mu^2$ gives

$$\begin{aligned} 2a_0^0 - 5a_0^2 &= \frac{12}{\pi} \int_1^\infty \frac{d\omega}{\omega^2-1} \text{Im} \left\{ \frac{1}{3} A_s^0 + \frac{1}{2} A_s^1 - \frac{5}{6} A_s^2 \right\} \\ &= \frac{12}{\pi} \int_0^\infty \frac{dk}{\omega k} \text{Im} A_\ell^1 \\ &= \frac{3}{4\pi^2} \int_0^\infty d\nu \frac{\sigma_{tot}^{\pi^+\pi^-}(\nu) - \sigma_{tot}^{\pi^+\pi^+}(\nu)}{\sqrt{\nu(\nu+1)}} \end{aligned} \quad (5.40)$$

- (2) evaluating $\text{Re } A_t^1(\nu)/\nu$ at $\nu = 0$ gives

$$\begin{aligned} 3a_1^1 &= \frac{2}{\pi} \int_1^\infty d\omega \left\{ \frac{\text{Im} A_s^1}{(\omega-1)^2} + \frac{\text{Im} \left(\frac{1}{3} A_s^0 - \frac{1}{2} A_s^1 - \frac{5}{6} A_s^2 \right)}{(\omega+1)^2} \right\} \\ &= \frac{8}{\pi} \int_0^\infty \frac{dk}{k^3} \text{Im} A_s^1 + \frac{2}{\pi} \int_1^\infty \frac{d\omega}{(\omega+1)^2} \text{Im} A_\ell^1 \\ &= \frac{1}{4\pi^2} \int_0^\infty d\nu \frac{(2\nu+1) \sigma^1(\nu)}{[\nu(\nu+1)]^{3/2}} + \frac{1}{8\pi^2} \int_0^\infty d\nu \frac{\sigma_{tot}^{\pi^+\pi^-} - \sigma_{tot}^{\pi^+\pi^+}}{\sqrt{\nu(\nu+1)}} \frac{\nu}{\nu+1} \end{aligned} \quad (5.41)$$

The twice subtracted form (5.38) is known to rigorously exist (cf. subsection 3.4). From a Regge pole picture, the following asymptotic relations are expected for forward amplitudes for $s \rightarrow \infty$

$$\begin{aligned} A_{\ell}^0 &\rightarrow i \gamma_p \frac{s}{s_0} + \gamma_p [i - \cot \frac{\pi}{2} \alpha_{p,0}] (s/s_0)^{\alpha_{p,0}} \\ A_{\ell}^1 &\rightarrow \gamma_p (i + \tan \frac{\pi}{2} \alpha_{p,0}) (s/s_0)^{\alpha_{p,0}} \end{aligned} \quad (5.42)$$

$$A_t^2 = \gamma_2 (i - \cot \frac{\pi}{2} \alpha_{\text{eff}}) (s/s_0)^{\alpha_{\text{eff}}}$$

where $\alpha_p(0) = 1$ has been used and where $\alpha_{p,0} \approx 0.5$ and α_{eff} is expected to be negative.

Similarly, for example, relations for derivatives may be written down.

Goebel and Shaw²⁹²⁾ have made important use of dispersion relations for inverse amplitudes. For a forward amplitude describing scattering of physical particles (F^+) the left-hand cut describes the corresponding anti-particle reaction (F^-) so that the imaginary part on both is positive from the optical theorem. Then the sign of $\text{Im}(F^{-1})$ is fixed (negative). The observation is made that for reasonable asymptotic behaviour the number of zeros of $F^+(z)$ is determined from the values

$$a^{\pm} \equiv F(\pm i\mu).$$

For $a^{\pm}$ both negative (no zeros) the following relation is suggested

$$-(a^{\pm})^{-1} = \frac{1}{\pi} \int_1^{\infty} dz \left[\frac{R^{\pm}(z)}{z-1} + \frac{R^{\mp}(z)}{z+1} \right] \quad (5.43)$$

with

$$R \equiv \text{Im}(F^{-1}) = \frac{q}{\omega} \frac{\sigma_{\text{tot}}}{\pi \left(\frac{d\sigma_{\text{el}}}{d\Omega} \right) \Big|_{t=0}}$$

i.e., positive definite. Thus any knowledge on the cross-sections puts a negative lower bound on $a^\pm$. The same is trivially satisfied if $a^\pm$ are not both negative. With information available at the time, the result

$$a_0^0 \geq -0.5 \mu^{-1} \quad (5.44)$$

was obtained thus ruling out several suggestions for large negative values of δ_0^0 [Refs. cited in Ref. 292]. Equation (5.44) is much stronger than the rigorous bounds Eq. (3.74).

Olsson²⁹¹⁾ argued that as a means of determining a_1^1 , Eq. (5.41) is rather insensitive to assumed s waves. He obtained the value

$$a_1^1 = (0.040 \pm 0.005) \mu^{-3} \quad (5.45)$$

is compared to Weinberg's predictions [Eq. (3.14)] for which

$$L \equiv \frac{1}{6} (2a_0^0 - 5a_0^2) = (0.10 \pm 0.01) \mu^{-1} \quad (5.46)$$

and

$$a_1^1 = (0.033 \pm 0.003) \mu^{-3}$$

are equivalent.

The most recent work of Morgan and Shaw appears to prove that at least for L , a value can be obtained which is indeed very independent of assumed s waves and high energy behaviours²⁹³⁾.

The general objective is to take available information on δ_1^1, δ_0^2 and δ_0^0 in the energy region

$$500 \text{ MeV} < \sqrt{s} < 850 \text{ MeV} \quad (5.47)$$

and deduce low energy extrapolations and d waves. In the forward dispersion relations, the integration range was divided into three pieces : low, intermediate (5.47) and high. In the first two regions s , p and d waves were parametrized and some parameters fixed by the input. The contribution in the low energy region from the high energy region was treated in a near model-independent way and parametrized by polynomials in z (5.36).

We first consider results for the s wave scattering lengths a_0^0 and a_0^2 (Fig. 33). For each assumed δ_0^0 in the intermediate region (5.47) (classified according to up and down forms defined by Fig. 13 - "between" - forms were allowed as well in the below ρ region), the parameter m_ρ is the energy at which δ_0^0 passes through 90° and for the broad resonant solutions need have no strong relation to the resonance pole parameters. The striking point about Fig. 33 is that despite the very large range of allowed input forms of δ_0^0 and the high energy contributions (fixed by requiring unitarity of the dispersion relation output), the quantity L is accurately determined,

$$L \equiv \frac{1}{6}(2a_0^0 - 5a_0^2) = (0.095 \pm 0.015) \mu^{-1} \quad (5.48)$$

This result appears to depend on the fact that only δ_0^0 forms passing through the up and down bands were allowed ²⁹⁴⁾.

Another interesting observation is that all δ_0^0 solutions produce $\pi\pi$ amplitudes having small d waves and satisfying several crossing and positivity tests in the Mandelstam triangle. However, the solutions D UI, D UII, DD with a_0^0/a_0^2 very different from the isoscalar σ term prediction, $-7/2$, need high energy contributions that could not be explained in terms of known high mass resonances (f, g) or Regge exchanges. This is in agreement with the more theoretical quasi-theorem of subsection 3.4, that the ρ meson together with crossing and unitarity appear "reasonably" to constrain low energy scattering to the Weinberg form.

Excluding the unsatisfactory solutions or imposing the "experimental" determinations of a_0^0/a_0^2 [Eqs. (4.35), (4.56)] δ_0^0 is essentially uniquely determined below the ρ . Above, the down

branch was preferred from comparison with the $\pi^0\pi^0$ mass spectrum of Deinet et al.²⁰⁸⁾ about which, however, there is no longer universal agreement (subsection 4.6), but as we have seen a number of other arguments exist for believing in that (subsections 4.7, 5.2). Then the final forms for δ_0^0 , δ_0^2 , δ_1^1 and δ_2^0 results (Figs. 34, 35) (δ_2^2 was consistent with zero). The resulting low energy parameters were

$$a_0^0 = (0.16 \pm 0.04)\mu^{-1}, a_0^2 = (-0.05 \pm 0.01)\mu^{-1}, a_1^1 = (0.035 \pm 0.002)\mu^{-3} \quad (5.49)$$

in remarkable agreement with Weinberg, Eqs. (5.46), (3.15). Further

$$\delta_0^0(m_K) = 33^\circ \pm 5^\circ \text{ and } \delta_0^0(m_K) - \delta_0^2(m_K) = 41^\circ \pm 6^\circ \quad (5.50)$$

[cf. Eq. (5.24)].

Castoldi's²⁹⁵⁾ suggestion for a positive a_0^2 despite the negative values found for δ_0^2 above 500 MeV appears then to result from incomplete imposition of crossing (only the $I = 2$ channel was considered).

Pisut's²⁹⁶⁾ suggestion for a "turn-over" δ_0^0 ($a_0^0 < 0$) partly suffers the same criticism and in Ref. 293), it is possible only for DD, DUI, DUII solutions (where it is moreover required) which were considered undesirable for other reasons.

6. - CONCLUSION

In this Section we attempt a qualitative and semi-quantitative description in words of those meson meson partial waves for which this makes sense. That will partly represent the conclusion of some of the material collected in the paper but to a greater if not extreme extent will it represent the author's prejudice, and warnings against that could not be over-emphasized. For a more reasonable evaluation of uncertainties and alternatives, reference must be made to the appropriate portions of the text.

$\pi\pi \rightarrow \pi\pi$

At low energies, no strong evidence exists against Weinberg's amplitude [subsection 3.1, Eqs. (3.14), (3.15)]. Theoretically it is not only consistent with unitarity but is "almost" required by unitarity and the existence of the ρ (subsection 3.5, Fig. 5). Experimentally, several pieces of evidence in its favour exist [Eqs. (4.35), (4.55), (5.45), (5.48), (5.49), Figs. 29, 33].

δ_0^2 has a value of -15° to -20° at the ρ position (Figs. 2, 11, 12, 35) and may level off in the 1 GeV region (Fig. 11) above which it is unknown. $\eta_0^2 \approx 1$ throughout.

δ_1^1 is of course dominated by the ρ (Figs. 11, 34). Uncertainties at the 20 MeV level continue to exist. Above 1 GeV, it is surprisingly poorly known. Suggestions for moderate (Figs. 20, 18) and large (Fig. 17) inelasticities exist. A ρ' signal may have been seen in photoproduction (subsection 5.4) but almost not in pion production (subsection 4.6) and the pwa is quite small at the $f^0(1260)$ (Fig. 19).

δ_0^0 . Below the ρ there is increasing agreement about between forms both from theory (Fig. 2), experiment [Figs. 11, 14, Eq. (5.24)] and dispersion relation phenomenology [Fig. 35, Eq. (5.50)]. K_{e4} experiments are so far unable to confirm that (Table III) but indicate no disagreement (Fig. 31). Above the ρ the down branch is barely

preferred up to 850 MeV (subsections 4.6, 4.7, 5.2, 5.6). Mass and width values for a possible $\mathcal{E}$ resonance can hardly be inferred from available data alone and Breit Wigner phenomenology would be grossly misleading. For semi-theoretical suggestions, cf. Table II and in particular Eqs. (3.78), (3.79). Above 850 MeV, a large acceleration due to the S^* inelasticity above the $K\bar{K}$ threshold (995 MeV) is likely to shoot up δ_0^0 to around 180° at threshold. Above 1 GeV, Fig. 23 (rotated 180°) gives qualitative ideas about the behaviour depending on details of S^* . η_0^0 seems remarkably close to 1 below 1 GeV (subsection 4.6) but drops quickly above [Eq. (4.56)] mainly due to $K\bar{K}$ inelasticity but 4π 's account for an increasingly large fraction, possibly as much as 30% at 1100 MeV. At the $f^0(1260)$ the pwa is somewhere in the upper half of the Argand plot (Fig. 19). The zero (CDD) in the pwa near 1 GeV may account for a dip seen in NN annihilation at rest (subsection 5.4, cf. also Fig. 2, curve L).

δ_2^0 is dominated by the near elastic $f^0(1260)$ and is non-negligible at the ρ position ($\approx 2^\circ$, Figs. 11, 34).

δ_2^2 is negligible below 1 GeV and may be negative above (Fig. 11).

$\pi\pi \rightarrow K\bar{K}$ (subsection (4.7))

Only for the $I = 0$ s wave does the phase shift language have meaning (Fig. 22) but ambiguities are still too large for numbers to be selected. Combined analysis with $\pi\pi \rightarrow \pi\pi$ is compulsory but 4π inelasticities may not be negligible [Eq. (4.56)]. There appears good evidence that the S^* corresponds to a pair of complex second sheet poles, but whether to the right (resonance) or the left (scattering length) of the $K\bar{K}$ threshold is not known - it may even be considered of secondary importance. However, they are probably on the second $K\bar{K}$ sheet rather than on the second $\pi\pi$ sheet.

$K\pi \rightarrow K\pi$

Again no evidence exists against the soft meson low energy form [subsection 3.1, Eqs. (3.18), (3.20)-(3.22)], but unlike in $\pi\pi$ no good quantitative results strongly support it. At medium energies, however, the pwa's are sufficiently analogous to $\pi\pi$ ones that one may feel confident about the low energy amplitudes as well.

$\delta_{11}^{\frac{3}{2}}$ is consistent with zero (subsection 4.8) as expected [Eq. (3.22), Fig. 3].

$\delta_{00}^{\frac{3}{2}}$ is negative, $-20^\circ \pm 15^\circ$, up to 1050 MeV (subsection 4.8) in agreement with expectation [Eq. (3.22), Fig. 3].

$\delta_{11}^{\frac{1}{2}}$ is dominated below 1050 MeV by the elastic $K^*(890)$. The pwa seems to be small in the $K^{**}(1420)$ region (Figs. 26, 28).

$\delta_{00}^{\frac{1}{2}}$ (Fig. 27) is different from Fig. 3 whether the up or down solution is preferred. However, it is positive and of appreciable magnitude as expected [Eq. (3.22)]. In analogy with ϵ one might expect the down branch to be the easier to get on with, but no dispersion relation phenomenology exists so far. In that case a broad κ state (in qualitative agreement with Table II) is indicated but far from established. K_{l3} data (subsection 5.4) are unfortunately inconclusive on the κ . Above 1100 MeV inelasticities presumably become important, but no good values exist (subsection 4.8). The pwa is of considerable size at the $K^{**}(1420)$ (Figs. 26, 28) and may even resonate at 1370 MeV (fig. 28).

$\delta_{22}^{\frac{1}{2}}$ is dominated by the 50% inelastic $K^{**}(1420)$.

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	1 st	2 nd	Particle ($Y \geq 0$)	Antiparticle ($Y < 0$)
x	$(-)^{I_a + I_x - I_1}$	1	$(-)^{I_x}$	1
y	1	$(-)^{I_b + I_y - I_1}$	$(-)^{I_y}$	1
$\bar{x}$	1	$(-)^{I_b + I_x - I_2}$	1	$(-)^{I_x}$
$\bar{y}$	$(-)^{I_a + I_y - I_2}$	1	1	$(-)^{I_y}$

TABLE I : Rules for constructing the phase factor ϕ of Eq. (2.28). For each of the four particles, x , y , $\bar{x}$, $\bar{y}$, the two appropriate factors from the corresponding line in the Table are selected. All of these eight factors are then multiplied together to give ϕ . Note that ϕ depends on whether the particles participating in the crossing process are "1st" (beam in initial state, forward in final state) or "2nd" (target in initial state, backward in final state).

Name	J^P	I^b	Mass		Decay mode	Partial width	
			Theory	Expt		Theory	Expt
ρ	1^-	1^+	749	765 ± 10	$\pi\pi$	128	125 ± 20
ω	1^-	0^-	807	784	—	—	—
f	2^+	0^+	1289	1264 ± 10	$\pi\pi$	98	123 ± 20
					$K\bar{K}$	10	7 ± 1
A_2	2^+	1^-	1292	$<1310>$	$K\bar{K}$	7.2	≈ 4
ξ	3^-	1^+	1671	1660 ± 20	$\pi\pi$	38	$\lesssim 200$
					$K\bar{K}$	11	$\lesssim 14$
K^*	1^-	$\frac{1}{2}$	895	893	$K\pi$	38	50
K^{**}	2^+	$\frac{1}{2}$	1381	1408	$K\pi$	36	64 ± 10
K^{***}	3^-	$\frac{1}{2}$	1737	—	$K\pi$	24	—
ϕ	1^-	0^-	1009	1019	$K\bar{K}$	1.42	3.2 ± 0.3
f'	2^+	0^+	1455	1514 ± 5	$\pi\pi$	0	<14
					$K\bar{K}$	27	50 ± 16
ϵ	0^+	0^+	698	≈ 750	$\pi\pi$	407	$\gg 100$
δ	0^+	1^-	752	962 ± 5	—	—	—
ρ'	1^-	1^+	1288	—	$\pi\pi$	105	—
					$K\bar{K}$	36	—
ϕ'	1^-	0^-	1280	—	$K\bar{K}$	22	—
f''	2^+	0^+	1661	—	$\pi\pi$	63	—
					$K\bar{K}$	20	—
ϕ^*	3^-	0^-	1673	—	$K\bar{K}$	11	—
κ	0^+	$\frac{1}{2}$	899	—	$K\pi$	214	—
K^{*1}	1^-	$\frac{1}{2}$	1380	—	$K\pi$	94	—
K^{**1}	2^+	$\frac{1}{2}$	1737	—	$K\pi$	49	—
S^*	0^+	0^+	911	1070 ± 30	$\pi\pi$	15	<150

TABLE II : Updated version of Table I, Ref. 47), using particle data group tables (April, 1971). For further information (in particular on κ , ϵ and S^*), see sections 4, 5 and 6.

	Berends (I)	Ely (I)	Berends (II)	Ely (II)	Weinberg
$a_0^0 (\mu^{-1})$	0.71 ± 0.37	$1.26 \begin{smallmatrix} + 0.68 \\ - 0.52 \end{smallmatrix}$	-0.57 ± 0.22	$- 0.89 \pm 0.28$	0.2
g_p/f_s	2.00 ± 0.90	1.33 ± 0.17	-2.09 ± 0.42	$- 1.61 \pm 0.15$	1
f_p/f_s	-0.34 ± 0.37	-3.1 ± 1.1	1.95 ± 0.30	12.5 ± 1.2	0
h_p/f_s	-4.70 ± 1.96	-1.09 ± 0.81	-4.80 ± 1.66	$- 2.41 \pm 0.86$	-
f_s	1.06 ± 0.23	1.18 ± 0.08	1.05 ± 0.16	1.72 ± 0.10	0.8

TABLE III : Results of Berends et al. ²⁶⁴) and Ely et al. ²⁶⁰) for the two solutions Eq. (5.20). Predictions of Weinberg ²⁶¹), ¹⁶) are shown for comparison.

REFERENCES

- 1) Meson Spectroscopy, Ed. C. Baltay and A.H. Rosenfeld, New York, Benjamin, 1968.
- 2) Proceedings of the Argonne Conference on $\pi\pi$ and $K\pi$ Interactions, May 1969, Ed. F. Loeffler, E. Malamud.
- 3) Experimental Meson Spectroscopy, Ed. C. Baltay and A.H. Rosenfeld, Columbia University Press, 1970.
- 4) J.L. Basdevant and J. Reigner, Herceg-Novti Lecture Notes (1970).
- 5) D. Morgan and J. Pišút, Springer Tracts in Modern Physics, Vol. 55 (1970).
- 6) M. Jacob, Herceg-Novti Lecture Notes (1970).
- 7) S. Mandelstam, Phys.Rev.Letters 4, 84 (1960).
- 8) D.E. Neville, Phys.Rev. 160, 1375 (1967).
- 9) J.J. De Swart, Rev.Mod.Phys. 35, 916 (1963).
- 10) P.A. Carruthers and J.P. Krisch, Ann.Phys. 33, 1 (1965).
- 11) R.J. Eden et al., The Analytic S-Matrix (Cambridge University Press, 1966).
- 12) M. Rotenberg et al., The 3 - j and 6 - j Symbols (The Technology Press, Cambridge, Massachusetts 1959).
- 13) G.F. Chew and S. Mandelstam, Phys.Rev. 119, 467 (1960).
- 14) G.P. Canning, Nuclear Phys. B14, 437 (1969).
- 15) H. Osborn, Nuclear Phys. B15, 501 (1970).
- 16) S. Weinberg, Phys.Rev.Letters 17, 616 (1966).
- 17) R.W. Griffith, Phys.Rev. 176, 1705 (1968).
- 18) G. Ebel et al., Compilation of Coupling Constants and Low Energy Parameters, Ed. G. Höhler, Springer Tracts in Modern Physics, Vol.55 (1970).

- 19) M. Gell-Mann et al., Phys.Rev. 175, 2195 (1968) ;
J. Ellis, Nuclear Phys. B13, 153 (1969).
- 20) R.A. Brandt and G. Preparata, Ann.Phys. 61, 119 (1970).
- 21) T.P. Cheng and R. Dashen, Phys.Rev.Letters 26, 594 (1971).
- 22) G.F. Chew et al., Phys.Rev. 119, 478 (1960) ;
G.F. Chew and S. Mandelstam, Nuovo Cimento 19, 752 (1961).
- 23) R.G. Levers, Nuovo Cimento 60A, 575 (1969).
- 24) I. Bars, Phys.Rev. D2, 1630 (1970).
- 25) S. Krinsky, Phys.Rev. D2, 1168 (1970).
- 26) M.G. Olsson and L. Turner, Phys.Rev. 181, 2141 (1969) ;
Phys.Rev.Letters 20, 1127 (1968).
- 27) J.H. Scharenguivel et al., Nuclear Phys. B22, 16 (1970).
- 28) B. Diu et al., Nuovo Cimento 35, 460 (1965).
- 29) S. Okubo, Progr.Theoret.Phys. 27, 949 (1962).
- 30) R. Dalitz, ref. 1), p. 497.
- 31) S. Okubo, Phys.Letters 5, 165 (1963).
- 32) K. Kawarabayashi and M. Suzuki, Phys.Rev.Letters 16, 255 (1966) ;
E : 16, 384 (1966).
- 33) Fayyazuddin and Riazuddin, Phys.Rev. 147, 1071 (1966).
- 34) L.S. Brown and R.L. Goble, Phys.Rev.Letters 20, 346 (1968).
- 35) A.N. Kamal, Phys.Rev. 180, 1454 (1969).
- 36) R.C. Johnson, Nuclear Phys. B23, 247 (1970).
- 37) G.J. Gounaris and J.J. Sakurai, Phys.Rev.Letters 21, 244 (1968).
- 38) F.J. Gilman and H. Harari, Phys.Rev. 165, 1803 (1968) ;
H. Harari, Proceedings of the 14th International Conference
on High Energy Physics, Vienna 1968, edited by
J. Prentki and J. Steinberger.
- 39) S. Weinberg, Phys.Rev. 177, 2604 (1969) and Proceedings of
the Vienna Conference, op. cit.

- 40) S.L. Adler, Phys.Rev. 140B, 736 (1965) ; Phys.Rev.Letters 14, 1051 (1965).
- 41) J.E. Bowcock and G. John, Nuclear Phys. B11, 659 (1969) ;
- 42) C. Lovelace, Phys.Letters 28B, 264 (1968).
- 43) D.Y. Wong, Phys.Rev. 183, 1412 (1969).
- 44) K. Kawarabayashi et al., Phys.Letters 28B, 432 (1969).
- 45) H. Osborn, Nuovo Cimento Letters 1, 513 (1969).
- 46) H. Osborn, Nuclear Phys. B17, 141 (1970).
- 47) C. Lovelace, Ref. 2), p. 562, CERN preprint TH.1041 (1969).
- 48) C. Lovelace, Irvine Conference on Regge poles (1969), CERN preprint TH.1123 (1970).
- 49) M. Jacob, Proceedings of the Lund International Conference on Elementary Particles, Lund (1969), Ed. G. von Dardel.
- 50) O.W. Greenberg, Proceedings of the Lund Conference op. cit.
- 51) S.L. Glashow and R.H. Socolow, Phys.Rev.Letters 15, 329 (1965).
- 52) C.B. Chiu and J. Finkelstein, Phys.Letters 27B, 510 (1968).
- 53) M. Ademollo et al., Phys.Rev.Letters 22, 83 (1969).
- 54) N. Tokuda, Progr.Theor.Phys. 41, 853 (1969).
- 55) R.L. Sugar and J.D. Sullivan, Phys.Rev. 166, 1515 (1968).
- 56) J.L. Rosner, Phys.Rev.Letters 22, 689 (1969) ;
H. Harari, Phys.Rev.Letters 22, 562 (1969) .
- 57) P. Brooker and J.C. Taylor, Nuclear Phys. B17, 467 (1970).
- 58) M. Kugler, Phys.Letters 31B, 379 (1970).
- 59) J. Yellin, Phys.Rev. 182, 1482 (1969) ; E : Phys.Rev. D1, 375 (1970).
- 60) J.A. Cronin and K. Kang, Phys.Rev.Letters 23, 1004 (1969).
- 61) F. Csikor, Phys.Letters 31B, 141 (1970).
- 62) Fayyazuddin, Rawalpindi preprint (1971);
J. Ellis and H. Osborn, Phys.Letters 31B, 580 (1970);
E : Phys.Letters 33B, 540 (1970).

- 63) L.M. Chounet and M.K. Gaillard, CERN report 70-14 (1970).
- 64) M. Ademollo and R. Gatto, Phys.Rev.Letters 13, 264 (1964).
- 65) V. Alessandrini et al., CERN preprint TH.1160 (1970).
- 66) R. Arnovitt et al., Phys.Rev. D1, 1813 (1970).
- 67) T. Yoneya et al., Nuovo Cimento Letters 4, 459 (1970).
- 68) D.W. McKay et al., Phys.Rev. D2, 742 (1970).
- 69) A. Kanazawa and M. Haruyama, Purdue University preprint (1971).
- 70) V. Alessandrini and D. Amati, CERN preprint TH.1278 (1971).
- 71) P.N. Dobson Jr., Nuovo Cimento Letters 2, 761 (1969).
- 72) R. Baier et al., Nuovo Cimento Letters 3, 594 (1970).
- 73) E.P. Tryon, Columbia preprint NYO - 1932 (2)-177, (1970).
- 74) K. Kang, Nuovo Cimento Letters 3, 576 (1970).
- 75) P. Gensini and G. Soliani, Nuovo Cimento 68A, 293 (1970).
- 76) A. Martin, Scattering Theory : Unitarity Analyticity and Crossing (Springer, 1969).
- 77) A. Martin and F. Cheung, Analytic Properties and Bounds on the Scattering Amplitude, New York, Gordon and Breach (1970).
- 78) G. Wanders, Springer Tracks of Modern Physics, Vol.57 (1970).
- 79) A. Martin, Invited talk of the Kiev Conference, 1970, TH.1232-CERN.
- 80) D. Atkinson, Nuclear Phys. B23, 397 (1970).
- 81) J. Kupsch, Nuovo Cimento 66A, 202 (1970).
- 82) R. Roskies, Phys.Rev. D2, 247 (1970) ; Phys.Rev. D2, 1649 (1970).
- 83) G. Wanders, Nuovo Cimento 63A, 108 (1969).
- 84) A.P. Balachandran and J. Nuyts, Phys.Rev. 172, 1821 (1968).

- 85) R. Roskies, Phys.Letters 30B, 42 (1969) ; Nuovo Cimento 65A, 467 (1970).
- 86) J.L. Basdevant et al., Nuovo Cimento 64A, 585 (1969).
- 87) A. Martin, Nuovo Cimento 63A, 167 (1969).
- 88) F.J. Yndurain, Nuovo Cimento 64A, 225 (1969).
- 89) A.K. Common and F.J. Yndurain, Commun.Math.Phys. 18, 171 (1970).
- 90) O. Piguet and G. Wanders, Phys.Letters 30B, 418 (1969).
- 91) A.P. Balachandran and M.L. Blackmon, Phys.Letters 31B, 655 (1970).
- 92) G. Auberson, Nuovo Cimento 68A, 281 (1970).
- 93) G. Auberson et al., Nuovo Cimento 65A, 743 (1970).
- 94) M.R. Pennington, Nuclear Phys. B25, 621 (1971) ; Nuclear Phys. B24, 317 (1970).
- 95) L. Zukaszuk and A. Martin, Nuovo Cimento 52A, 122 (1967).
- 96) B. Bonnier and R. Vinh Mau, Phys.Rev.165, 1923 (1968).
- 97) A.K. Common, Nuovo Cimento 63A, 451 (1969).
- 98) S.M. Roy, Phys.Rev. 183, 1403 (1969).
- 99) R. Wit, Nuclear Phys. B23, 168 (1970).
- 100) A.K. Common and R. Wit, CERN preprint TH.1217 (1970).
- 101) J.R. Fulco and D.Y. Wong, Phys.Rev.Letters 19, 1399 (1967).
- 102) R.H. Graham and R.C. Johnson, Phys.Rev. 188, 2362 (1969) ;
E : D2, 2114 (1970).
- 103) R.G. Levers, Phys.Rev. 179, 1528 (1969).
- 104) E.P. Tryon, Phys.Rev.Letters 20, 769 (1968) ;
- 105) S. Kinsky, Phys.Rev. D2, 1168 (1970).
- 106) J.L. Basdevant and B.W. Lee, Saclay preprint
D.Ph.T/70/11/JLB/BWL, (1970).
- 107) J.L. Basdevant and J. Zinn-Justin, Saclay preprint
D.Ph.T/70-56, (1970).

- 108) J.C. Le Guillou et al., CERN preprint TH.1260, (1970).
- 109) J. Zinn-Justin, Springer Tracts in Modern Physics, Vol. 57, (1970).
- 110) J. Zinn-Justin, Physics Reports 1C, 55 (1971).
- 111) B. Bonnier and P. Gauron, Nuclear Phys. B21, 465 (1970).
- 112) M.G. Olsson, Wisconsin preprint COO-282, (1970).
- 113) G. Auberson et al., Phys.Letters 28B, 41 (1968).
- 114) F. Arbab and J.T. Donohue, Phys.Rev. D1, 217 (1970).
- 115) J.B. Carotte and R.C. Johnson, Phys.Rev. D2, 1945 (1970).
- 116) G. Maiella, Nuovo Cimento, 70A, 507 (1970).
- 117) B. Bonnier, Thesis, Paris (1971).
- 118) H. Navelet, private communication.
- 119) J. Dilley, Nuclear Phys. B25, 227 (1971).
- 120) J. Iliopoulos, Nuovo Cimento 52A, 192 (1967) ; Nuovo Cimento 53A, 552 (1968).
- 121) P.D.B. Collins and R.C. Johnson, Phys.Rev. 177, 2472 (1969) ; Phys.Rev.182, 1755 (1969) ; Phys.Rev. 185, 2021 (1969).
- 122) J. Baacke et al., Nuovo Cimento 62A, 332 (1969).
- 123) G. Costa, Nuovo Cimento Letters 3, 329 (1970).
- 124) P. Brooker and J.C. Taylor, Nuclear Phys. B17, 461 (1970).
- 125) L. Turner, Phys.Rev. D1, 3466 (1970).
- 126) A.M. Harun-Ar Rashid, Nuovo Cimento 64A, 985 (1969).
- 127) R. Brout et al., Phys.Letters 29B, 590 (1969) ; 29B, 686 (1969).
- 128) G.V. Dass et al., Nuovo Cimento 67A, 429 (1970).
- 129) G.P. Canning, Phys.Rev. D2, 1146 (1970).
- 130) J. Mandula et al., Phys.Rev.Letters 23, 266 (1969).
- 131) J. Finkelstein, Nuclear Phys. B28, 73 (1971).

- 132) H.J. Schnitzer and S. Weinberg, Phys.Rev. 164, 1828 (1967).
- 133) R. Arnowitt, Ref. 2), p. 619.
- 134) S. Gasiorowicz and D.A. Geffen, Rev.Mod.Phys. 41, 531 (1969).
- 135) A.M.S. Amatya et al., Phys.Rev. 172, 1755 (1968).
- 136) J.J. Brehm et al., Phys.Rev. Letters 23, 666 (1969).
- 137) D.A. Nutbrown, Phys.Rev. D2, 1140 (1970).
- 138) P.V. Collins, Nuclear Phys. B21, 577 (1970).
- 139) B. Zumino, Rapporteur's talk at the 15th International
Conference on High Energy Physics, Kiev ; CERN preprint
TH.1246 (1970).
- 140) J. Ellis, paper presented at the Coral Gables Conference on
Fundamental Interactions at High Energies ; CERN preprint
TH.1289 (1970).
- 141) P.E. Schlein, Ref. 1), p.161.
- 142) P.E. Schlein, Ref. 2), p.1.
- 143) P.E. Schlein, Lecture Notes, Proc.Int.School of Subnuclear
Phys. Erice, (1970).
- 144) G.L. Kane, Ref. 3), p.1.
- 145) L.J. Gutay et al., Nuclear Phys. B12, 31 (1969).
- 146) K. Gottfried and J.D. Jackson, Nuovo Cimento 33, 309 (1964).
- 147) C. Goebel, Phys.Rev.Letters, 1, 337 (1958).
- 148) G.F. Chew and F.E. Low, Phys.Rev. 113, 1640 (1959).
- 149) E. Ferrari and F. Selleri, Nuovo Cimento Suppl. 24, 453 (1962).
- 150) H.G. Dosch and V.F. Müller, Z.Physik 236, 192 (1970).
- 151) M. Jacob and G.C. Wick, Ann.Phys. 7, 404 (1959).
- 152) G. Cohen-Tannoudji et al., Ann.Phys. 46, 239 (1968).
- 153) M.E. Rose, Elementary Theory of Angular Momentum, John Wiley
and Sons, 1957.
- 154) J.L. Petersen, Nuclear Phys. B13, 73 (1969).

- 155) H. Navelet and E. Pittet, CERN preprint TH.1295 (1971).
- 156) B.E.Y. Svensson, Nuclear Phys. B15, 93 (1970).
- 157) J.D. Jackson and G.E. Hite, Phys.Rev. 169, 1248 (1968).
- 158) T.L. Trueman, Phys.Rev. 173, 1684 (1968) ; E : 181, 2154 (1969).
- 159) D.M. Levine, Nuclear Phys. B11, 641 (1969).
- 160) T.W.B. Kibble, Phys.Rev. 117, 1159 (1960).
- 161) G. Cohen-Tannoudji et al., Nuovo Cimento 55A, 412 (1968).
- 162) H.P. Dürr and H. Pilkuhn, Nuovo Cimento 40A, 899 (1965).
- 163) J. Benecke and H.P. Dürr, Nuovo Cimento 56A, 269 (1968).
- 164) G. Wolf, Phys.Rev. 182, 1538 (1969) ; Ref. 2), p.689.
- 165) Z. Ming Ma et al., Phys.Rev.Letters 23, 342 (1969).
- 166) F. Wagner, Nuovo Cimento 64A, 189 (1969).
- 167) L.J. Gutay et al., Phys.Rev.Letters 23, 431 (1969).
- 168) L.L. Loos et al., Phys.Letters 31B, 470 (1970).
- 169) N.N. Biswas et al., Phys.Letters 27B, 513 (1968), Ref. 2), p.55.
- 170) S. Marateck et al., Phys.Rev.Letters 21, 1613 (1968).
- 171) L. Chan et al., Phys.Rev. D2, 583 (1970).
- 172) A.M. Boyarski et al., Phys.Rev.Letters 20, 300 (1968).
- 173) J.S. Ball, Phys.Rev. 124, 2014 (1961).
- 174) J.S. Ball and M. Jacob, Nuovo Cimento 54A, 620 (1968).
- 175) D. Horn and M. Jacob, Nuovo Cimento 56A, 83 (1968).
- 176) M. Le Bellac and G. Plaut, Nuovo Cimento 64A, 95 (1969).
- 177) C.F. Cho and J.J. Sakurai, Phys.Letters 30B, 119 (1969) ;
Phys.Rev D2, 517 (1970).
- 178) C.D. Froggatt and D. Morgan, R.H.L.-preprint, 1971, to be
published in Phys.Letters.
- 179) H. Harari and B. Horovitz, Phys.Letters 29B, 314 (1969).
- 180) J.D. Jackson and C. Quigg, Phys.Letters 29B, 236 (1969).

- 181) B.D. Hyams et al., Nuclear Phys. B7, 1 (1968).
- 182) P.K. Williams, Phys.Rev. D1, 1312 (1970).
- 183) K. Gottfried and J.D. Jackson, Nuovo Cimento 34, 735 (1964).
- 184) G.V. Dass and C.D. Froggatt, Nuclear Phys. B8, 661 (1968).
- 185) J.S. Ball et al., Phys.Rev.Letters 20, 518 (1968).
- 186) M. Le Bellac, Phys.Letters 25B, 524 (1967) ;
M. Aderholz et al., Phys.Letters 27B, 174 (1968).
- 187) G.L. Kane and M. Ross, Phys.Rev. 177, 2353 (1969).
- 188) J.D. Kimel, Phys.Rev. D2, 862 (1970).
- 189) R.E. Cutkosky and B.B. Deo, Phys.Rev. 174, 1859 (1968).
- 190) S. Ciulli, Nuovo Cimento 61A, 787 (1969).
- 191) J. Pisut, Springer Tracts in Modern Physics, Vol.55, (1970).
- 192) R.E. Cutkosky, Ann.Phys. 54, 350 (1969).
- 193) Chia Chang Shih, Carnegie-Mellon University preprint, (1970).
- 194) P.E. Schlein, Phys.Rev.Letters 19, 1052 (1967).
- 195) C.D. Froggatt and D. Morgan, Phys.Rev. 187, 2044 (1969).
- 196) B. Diu and M. Le Bellac, Nuovo Cimento 53A, 158 (1968).
- 197) J.H. Scharenguivel et al., Phys.Rev.Letters 24, 332 (1970).
- 198) B.Y. Oh et al., Phys.Rev. D1, 2494 (1970).
- 199) S.J. Barish et al., Phys.Rev. 184, 1375 (1969) ;
B.G. Reynolds et al., Phys.Rev. 184, 1424 (1969).
- 200) J.P. Baton et al., Phys.Letters 33B, 525 and 528 (1970).
- 201) E. Colton et al., University of California preprint, UCAL -
20053, (1970).
- 202) J.H. Scharenguivel et al., Phys.Rev. 186, 1387 (1969).
- 203) B. Maglic, Ref. 49), p.269.
- 204) E. Malamud and P.E. Schlein, Ref. 2), p.93.
- 205) D. Cline et al., Nuclear Phys. B18, 77 (1970).

- 206) M.M. Makarov et al., Phys.Letters 31B, 666 (1970).
- 207) E. Colton and E. Malamud, University of California preprint, UCRL-20111, (1970).
- 208) W. Deinet et al., Phys.Letters 30B, 359 (1970).
- 209) E.I. Shibata et al., Phys.Rev.Letters 25, 1227 (1970).
- 210) B.D. Hyams et al., Ref. 3), p.41.
- 211) M.J. Hones et al., Phys.Rev. D2, 827 (1970).
- 212) J.S.M. Flatté, paper presented at the Bologna Conference on Meson Resonances, April 1971.
- 213) J.A. Gaidos et al., Purdue preprint, (1971).
- 214) J.V. Beaupre et al., ABC collab. Nuclear Phys. B28, 77 (1971).
- 215) B.G. Reynolds et al., Nuclear Phys. B21, 77 (1970).
- 216) D.H. Miller et al., Phys.Rev.Letters 21, 1489 (1968).
- 217) P.H. Stuntebeck et al., Phys.Letters 32B, 391 (1970).
- 218) J.S.M. Flatté et al., University of California preprint, UCRL-20273, (1971).
- 219) J. Bartsch et al., ABC collab., Nuclear Phys. B22, 109 (1970).
- 220) B.D. Hyams, Communication to the Bologna Conference on Meson Resonances, April 1971.
- 221) C. Caso et al., Genova preprint, IS FG 70/5, (1970).
- 222) E. Schrauner, Phys.Rev.Letters 20, 1258 (1968).
- 223) W.J. Abbe, Phys.Rev. 160, 1519 (1967).
- 224) K.W. Lai, Ref. 2), p.477.
- 225) A. Astier, Proceedings of the XVth International Conference on High Energy Physics, Kiev, 1970, to be published.
- 226) A. Barbaro-Galtieri, Ref. 3), p.331.

- 227) M. Alston-Garnjost et al., Phys.Letters 33B, 607 (1970) ;
G. Grayer et al., CERN preprint,
K.J. Foley et al., Phys.Rev.Letters 26, 413 (1971) ;
G. Lutz, paper presented at the Bologna Conference on Meson
Resonances, April 1971.
- 228) W. Beusch, Ref. 3), p.185.
- 229) T.F. Hoang et al., Phys.Rev. 184, 1363 (1969) ;
T.F. Hoang, Nuovo Cimento 64A, 55 (1969).
- 230) S. Margulies et al., Nuovo Cimento 63A, 1124 (1969).
- 231) D. Morgan, Paper submitted to the Bologna Conference on
Meson Resonances, April 1971.
- 232) H. Dietl, paper presented at the Bologna Conference on
Meson Resonances, April 1971.
- 233) P.E. Schlein, Ref. 2), p. 446.
- 234) T.G. Trippe et al., Phys.Letters 28B, 203 (1968).
- 235) W. De Baere et al., Nuovo Cimento 61A, 397 (1969).
- 236) B. Werner et al., Nuclear Phys. B23, 37 (1970).
- 237) G.L. Kane, Ref. 3), p. 1.
- 238) C. Fu et al., Nuclear Phys. B18, 93 (1970).
- 239) A.M. Bakker et al., Nuclear Phys. B24, 211 (1970).
- 240) H.H. Bingham, E. Colton, Y. Goldschmidt-Clermont, P. Herquet,
D. Linglin, F. Muller, P. Schlein, W. Slater, T. Trippe,
Private Communication.
- 241) P. Herquet and T.G. Trippe, paper presented at the XVth
International Conference on High Energy Physics, Kiev
1970, CERN/D.Ph.II/Phys 70-29, 1970.
- 242) Particle Data Group, Phys.Letters 33B, 1 (1970).
- 243) M. Aguilar-Benitez et al., Phys.Rev.Letters 25, 1362 (1970).
- 244) A. Firestone et al., University of California preprint,
UCRL-20261, 1971.

- 245) P.J. Davis et al., Phys.Rev.Letters 23, 1071 (1969).
- 246) J. Kirz et al., Phys.Rev. 130, 2481 (1963).
- 247) R.G. Roberts and F. Wagner, Nuovo Cimento 64A, 206 (1969).
- 248) Particle Data Group, University of California preprint, UCRL-20030 π N, (1970).
- 249) T. Maung et al., Phys.Letters 33B, 521 (1970).
- 250) J.C. Botke, Nuclear Phys. B23, 253 (1970).
- 251) A.A. Ansel'm and V.N. Gribov, Soviet Phys., JETP 10, 354 (1960).
- 252) I.M. Blair et al., Phys. Letters 32B, 528 (1970).
- 253) J. Hamilton et al., Phys.Rev. 128, 1881 (1962).
- 254) A. Donnachie et al., Phys.Letters 22, 332 (1966).
- 255) H. Nielsen et al., Nuclear Phys. B22, 525 (1970);
J. Hamilton, Springer Tracts in Modern Physics, Vol.55, (1970);
J.L. Petersen, Talk at the 1970 Prague-meeting, CERN preprint TH.1244, (1970).
- 256) J.L. Petersen and J. Pisut, to be published.
- 257) J. Engels, Nuclear Phys. B25, 141 (1970).
- 258) A. Pais and S.B. Treiman, Phys.Rev. 168, 1858 (1968).
- 259) R.W. Birge et al., Phys.Rev. 139B, 1600 (1965).
- 260) R.P. Ely et al., Phys.Rev. 180, 1319 (1969).
- 261) S. Weinberg, Phys.Rev.Letters 17, 336 (1966) ; E : Phys.Rev. 18, 1178 (1967).
- 262) M.K. Gaillard, CERN preprint, TH.1292 (1971).
- 263) R.G. Roberts and F. Wagner, Proc.Topical Conference on Weak Interactions, CERN Report 69-7 (1969), p.281.
- 264) F.A. Berends et al., Phys.Rev. 171, 1457 (1968).
- 265) T.D. Lee and C.S. Wu, Ann.Rev.Nuclear Sci. 16, 471 (1966).
- 266) N. Cabibbo and A. Maksymowicz, Phys.Rev. 137, B438 (1965).
- 267) T.T. Wu and C.N. Yang, Phys.Rev.Letters 13, 380 (1964).
- 268) L. Wolfenstein, Phys.Rev.Letters 13, 562 (1964).

- 269) J.M. Gaillard, CERN preprint, (1971).
- 270) B. Gobbi et al., Phys.Rev.Letters 22, 682 (1969) ; E : 865.
- 271) C.G. Callan and S.B. Treiman, Phys.Rev.Letters 16, 153 (1966).
- 272) N. Brene et al., Nuclear Phys. B6, 255 (1968).
- 273) D.G. Sutherland, Nuclear Phys. B13, 45 (1969).
- 274) M. Jacob et al., Nuovo Cimento 63A, 574 (1969).
- 275) B.H. Kellett, Nuclear Phys. B21, 413 (1970).
- 276) S. Makajan, Purdue preprint, (1971).
- 277) S.U. Chung et al., CERN/D.Ph.II/Phys. 71-1, (1971).
- 278) P. Espigat et al., CERN/D.Ph.II/Phys. 71-7, (1971).
- 279) S.U. Chung et al., CERN/D.Ph.II/Phys. 71-5, (1971).
- 280) R. Bizzarri et al., Nuclear Phys. B14, 169 (1969).
- 281) J. Diaz et al., Nuclear Phys. B16, 239 (1970).
- 282) J.E. Augustin et al., Phys.Letters 28B, 508 (1969) ;
Nuovo Cimento Letters 2, 214 (1969).
- 283) M. Roos and J. Pisut, Nuclear Phys. B10, 563 (1969).
- 284) M.J. Creutz and M.B. Einhorn, Phys.Rev. D1, 2537 (1970).
- 285) A.Q. Sarker, Phys.Rev.Letters 25, 1527 (1970).
- 286) J. Ellis, Nuclear Phys. B22, 478 (1970).
- 287) F. Bulos et al., Phys.Rev.Letters 26, 149 (1971) ;
H. Alvensleben et al., Phys.Rev.Letters 26, 273 (1971).
- 288) P. Frampton et al., CERN preprint, in preparation.
- 289) N.E. Booth and A. Abashian, Phys.Rev. 132, 2314 (1963).
- 290) J.H. Hall et al., Nuclear Phys. B12, 573 (1969).
- 291) M.G. Olsson, Phys.Rev. 162, 1338 (1967).
- 292) C.J. Goebel and G. Shaw, Phys.Letters 27B, 291 (1968).
- 293) D. Morgan and G. Shaw, Nuclear Phys. B10, 261 (1969).
- 294) M.G. Olsson, Wisconsin Preprint C00.270 (1970).
- 295) P. Castoldi, Nuclear Phys. B12, 567 (1969).
- 296) J. Pisut, Nuclear Phys. B8, 159 (1968).

FIGURE CAPTIONS

- Figure 1 : α_ρ , α_{K^*} and α_ϕ as determined from Eq. (3.47). Experimental states are indicated as $M^2 \pm M\Gamma$ in units $(\alpha')^{-1}$. Dashed lines are first daughters. Circles indicate predicted resonances lacking conclusive experimental evidence (black) or any at all (open). Names from Ref. 47).
- Figure 2 : $\pi\pi$ phase shifts δ_0^0 and δ_0^2 predicted by various models: LW - single channel unitarized Veneziano theory ²⁴⁷⁾;
L - double channel unitarized Veneziano theory ⁴⁷⁾;
 σ - Ref. 106), subsection 3.4;
 ρ - Ref. 107), subsection 3.4;
GMN - Ref. 108), subsection 3.4.
- Figure 3 : $K\pi$ phase shifts predicted by unitarized Veneziano theory ⁴⁷⁾.
- Figure 4 : Mandelstam diagram for $\pi\pi \rightarrow \pi\pi$.
- Figure 5 : Regions in the (a_0^0, a_0^2) plane allowed by low energy crossing and unitarity as obtained by Dilley ¹¹⁹⁾. Branch I (indicating author's artistic view) was found also by Chew et al. ²²⁾. CA is Weinberg's current algebra prediction, Eq. (3.15).
- Figure 6 : Kinematical notation pertaining to di-meson production.
- Figure 7 : Triangle graphs giving rise to anomalous Δ^2 thresholds, Eqs. (4.18).
- Figure 8 : Δ_0^2 vs. s Ref. 168). Straight line fits and Veneziano model predictions are shown. Dashed and dotted lines include d and f waves.

- Figure 9 : Predictions of cross-sections (a) and density matrix elements (b), for the process $\pi^-p \rightarrow \pi^+\pi^-n$ at 11.2 GeV/c using photoproduction data and VMD¹⁷⁷⁾. The symbols s, t, (J), correspond to W^2 , $-A^2$ and (GJ) in the text.
- Figure 10 : Model predictions (full curves) of $\rho_{00}^H - \rho_{11}^H$ (a) and the forward backward asymmetry (b) in $\pi^-p \rightarrow \pi^+\pi^-n$ using a simplified absorption calculation¹⁸²⁾. Dashed lines indicate "smooth" extrapolations.
- Figure 11 : Phase shifts $\delta_1^1, \delta_2^0, \delta_2^2, \delta_0^0, \delta_0^2$ (P_1, D_0, D_2, S_0, S_2) from Baton et al.²⁰⁰⁾. Dotted curves correspond to zero inelasticity and the bands result when inelasticity is taken into account.
- Figure 12 : Results for δ_0^2 from Ref. 201) and several earlier determinations.
- Figure 13 : Results for δ_0^0 deduced from the forward backward asymmetry, Ref. 202).
- Figure 14 : Results for δ_0^0 from forward backward asymmetry, Ref. 27). Data points are from other earlier determinations.
- Figure 15 : Extrapolated values for the $\pi^-\pi^+ \rightarrow \pi^0\pi^0$ cross-section compared with predictions from up and down forms of δ_0^0 , Ref. 208).
- Figure 16 : Extrapolated values for $\frac{1}{4}|e^{2i\delta_0^0} - e^{2i\delta_0^0}|^2$ cross-section for $\pi^-\pi^+ \rightarrow \pi^0\pi^0$, Ref. 209).
- Figure 17 : Inelasticity parameters pertaining to Fig. 11, Ref. 200).
- Figure 18 : Phase shifts and inelasticities from Ref. 197).
- Figure 19 : $f^0 \rightarrow \pi^+\pi^-$ decay angular distribution, Ref. 214).
- Figure 20 : Phase shifts and inelasticities from Ref. 214).
- Figure 21 : Total and elastic $\pi^+\pi^-$ cross-sections from Ref. 221), ($\sigma_{\text{tot}} > \sigma_{\text{el}}$).

- Figure 22 : (a) Phases deduced from s-d interference and d wave phase shift expected from the f^0 ;
(b) s wave $\pi\pi \rightarrow K\bar{K}$ phases and $\pi\pi \rightarrow \pi\pi$ phases [Ref. 228)].
- Figure 23 : Model Argand diagram for I=0 s wave $\pi\pi$ amplitude. Full curve: S^* resonance; dotted curve: scattering length with effective range [Ref. 228)].
- Figure 24 : Density matrix elements for $K^{*0}(890)$ production in reaction (4.61a) [Ref. 235)].
- Figure 25 : The $|t'|$ distribution (A^2 in text) for various kinematical cuts. Numbers associated with exponential fits (straight lines) are exponent coefficients [Ref. 238)].
- Figure 26 : $\cos\theta$ distributions corresponding to various kinematical cuts in $K^{*0}(890)$ decays [Ref. 238)].
- Figure 27 : Values for $\delta_{00}^{\frac{1}{2}}(K^*)$ (δ_0^1 in the Figure) from the international K^* collaboration ²⁴⁰⁾ [see also Ref. 241)].
- Figure 28 : $K\pi$ decay angular distributions and $d\sigma/ds d\Delta^2$ in the $K\pi$ mass bands 1.3-1.4 GeV (a, c) and 1.4-1.5 GeV (b, d) respectively, [Ref. 244)].
- Figure 29 : Summary of results obtained from low-energy inelastic πN scattering: Maung et al., ²⁴⁹⁾, Botke ²⁵⁰⁾ [using Eq. (5.4)], Olson and Turner ²⁶⁾, Weinberg ³⁹⁾ (OT) and Roberts and Wagner ²⁴⁷⁾ (RW) (not a fit to the data). Weinberg's prediction is shown for comparison.
- Figure 30 : Results for f^0 obtained from backward πN dispersion relations ²⁵³⁾. In (b) curve B is the result of the direct truncation of Eq. (5.7) whereas curve B_1 is the result of the more correct prescription (see text). Curves D and B_2 represent equally likely possibilities [see Ref. 253)].

- Figure 31 : ϕ distribution in K_{e4} decay. Dotted lines are from the Roberts-Wagner fit ²⁶³⁾.
- Figure 32 : Model diagrams for $X \rightarrow 3\pi$ decay.
- Figure 33 : Plot of solutions obtained in Ref. 293) for various input forms for δ_o^0 (U: up, D: down, B: between). Curves are for $m_\sigma = 900$ MeV (dashed line), $m_\sigma = 765$ MeV (solid line) and $m_\sigma = 600$ MeV (dotted line). LV corresponds to curve LW of Fig. 2.
- Figure 34 : Final results for δ_1^1 and δ_2^0 , Ref. 293).
- Figure 35 : Final results for δ_o^0 and δ_o^2 [Ref. 293)] compared with the data of Fig. 13 [Ref. 202)].

FIG.1

FIG 2

$m_{K\pi}$ (GeV)

FIG. 3

FIG.4

FIG.5

FIG. 6

FIG. 7

FIG. 8

FIG. 9a

FIG. 9b

FIG. 10a

FIG. 10 b

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