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A GENERAL FORMULATION OF INCLUSIVE SUM RULES

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A B S T R A C T

Sum rules for inclusive reactions, which follow from conservation of linear momentum and/or of any other additive quantum number, are obtained in a general and compact form.

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Some attention has been recently focused on sum rules which relate different inclusive cross-sections in a way independent of the multiplicity ^{1),2)}. Although such equations are a mere consequence of kinematics (i.e., conservation of four-momentum) their content looks quite non trivial if one expresses inclusive cross-sections as discontinuities of elastic amplitudes ³⁾. Conditions on Regge coupling constants can thus be obtained ²⁾.

Moreover, one can prove ⁴⁾ that the full set of sum rules (including those for discontinuities of inelastic amplitudes) imply the basic discontinuity formula :

$$\text{Disc}_{(P_a, P_b)^2} T(\alpha + \beta \rightarrow \gamma + \delta) = i(2\pi)^4 \sum_n T(\alpha \rightarrow \gamma + n) T^*(\beta + n \rightarrow \delta) \delta^{(4)}(P_a + P_b - P_n) \quad (1)$$

which, in turn, is a consequence of unitarity ⁵⁾.

More phenomenological implications of these sum rules ⁶⁾ concern the question ⁷⁾ of which inclusive cross-sections exhibit a limiting distribution at subasymptotic energies. Finally, as it will be shown here, the sum rules have strong implications on the correlation functions in n particle distributions.

It looks then worthwhile to write down the sum rules in a general, compact and possibly simple form.

In order to do so, we start with an algebraic identity. For any set of complex numbers Y_i , the following algebraic identity holds :

$$(Y_1 Y_2 \dots Y_n - 1) = \sum_{k=1}^n \sum_{i_1 < i_2 < \dots < i_k} (Y_{i_1} - 1)(Y_{i_2} - 1) \dots (Y_{i_k} - 1) \quad (2)$$

Equation (2) is true for $n = 1, 2$ by inspection and is proven in general by induction.

We now recall the definition, in terms of the T matrix, of the total cross-section σ_{ab} for $a + b \rightarrow \text{anything}$ and of the general inclusive cross-section $(d\sigma_{ab})/(dP_1 \dots dP_K)$ for $a + b \rightarrow P_1 + P_2 + P_K + \text{anything}$ ⁸⁾

$$\sigma_{ab} = \sum_{n=2}^{\infty} \frac{1}{n!} \int dP_1 dP_2 \dots dP_n |T_{ab \rightarrow P_1 + P_2 + \dots + P_n}|^2 \delta^{(4)}(P_a + P_b - \sum_1^n P_i) \quad (3)$$

$$\frac{d\sigma_{ab}}{dP_1 dP_2 \dots dP_K} = \sum_{n=0}^{\infty} \frac{1}{n!} \int dP_{K+1} dP_{K+2} \dots dP_{K+n} |T_{ab \rightarrow P_1 + \dots + P_K + P_{K+1} + \dots + P_{K+n}}|^2 \delta^{(4)}(P_a + P_b - \sum_1^{n+K} P_i) \quad (4)$$

where $dP_i \equiv \frac{d^3 P_i}{2E_i}$.

Define $P = P_a + P_b$ and $Y_i = e^{P_i z_\mu}$ with z_μ arbitrary. Then we multiply both sides of Eq. (3) by $(e^{Pz} - 1)$ and get

$$(e^{Pz} - 1) \bar{\sigma}_{ab} = \sum_{n=2}^{\infty} \frac{1}{n!} \int dP_1 dP_2 \dots dP_n |T|^2 \delta^{(4)}(P - \sum_{i=1}^n P_i) (Y_1 Y_2 \dots Y_n - 1) \quad (5)$$

where we have used momentum conservation in the right-hand side. We now use Eq. (2) and note that, because of symmetry, each term in the $\sum_{i_1 < i_2 < \dots < i_k}$ contributes the same to Eq. (5). There are $n!/(n-k)! k!$ such terms. We then get, using Eq. (4)

$$\begin{aligned} (e^{Pz} - 1) \bar{\sigma}_{ab} &= \sum_{n=2}^{\infty} \frac{1}{n!} \int dP_1 \dots dP_n |T|^2 \delta^{(4)}(P - \sum_{i=1}^n P_i) \sum_{k=1}^n \frac{n!}{(n-k)! k!} (Y_1 - 1)(Y_2 - 1) \dots (Y_k - 1) = \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} \int dP_1 \dots dP_k (Y_1 - 1) \dots (Y_k - 1) \cdot \frac{1}{(n-k)!} \int dP_{k+1} \dots dP_n |T|^2 \delta^{(4)}(P - \sum_{i=1}^n P_i) = \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \int dP_1 \dots dP_k (Y_1 - 1) \dots (Y_k - 1) \frac{d\bar{\sigma}_{ab}}{dP_1 \dots dP_k} \end{aligned} \quad (6)$$

Therefore, we get, for any z :

$$(e^{Pz} - 1) \bar{\sigma}_{ab} = \sum_{k=1}^{\infty} \frac{1}{k!} \int dP_1 \dots dP_k \frac{d\bar{\sigma}_{ab}}{dP_1 \dots dP_k} (e^{P_1 z} - 1) \dots (e^{P_k z} - 1) \quad (7)$$

This is our final result. It can be further generalized by using the same technique on Eq. (4) instead of Eq. (3). Then we get :

$$(e^{(P - \sum_{i=1}^k P_i)z} - 1) \frac{d\bar{\sigma}_{ab}}{dP_1 \dots dP_k} = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \int dP_{k+1} \dots dP_{k+\ell} \frac{d\bar{\sigma}_{ab}}{dP_1 \dots dP_k dP_{k+1} \dots dP_{k+\ell}} (e^{P_{k+1} z} - 1) \dots (e^{P_{k+\ell} z} - 1) \quad (8)$$

As a check of Eq. (7), let $P = (E, 0, 0, 0)$ and $z \rightarrow -\infty$. Then Eq. (7) reads :

$$-\bar{\sigma}_{ab} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int dP_1 \dots dP_k \frac{d\sigma}{dP_1 \dots dP_k} \quad (9)$$

The integral on the right-hand side counts each cross-section $\sigma_n = \sigma(ab \rightarrow n \text{ particles})$ $n!/(n-k)!$ times ; thus we get the identity

$$-\bar{\sigma}_{ab} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \sigma_n = \sum_{n=2}^{\infty} \sigma_n \sum_{k=1}^n (-1)^k \binom{n}{k} = - \sum_n \sigma_n \quad (10)$$

Differentiating Eq. (7) with respect to z_μ once or twice we get the two lowest moment sum rules of Ref. 2) :

$$P_\mu \bar{\sigma}_{ab} = \int dP_1 P_{1\mu} \frac{d\bar{\sigma}_{ab}}{dP_1} \quad (11)$$

$$P_\mu P_\nu \sigma_{ab} = \int dP_1 P_{1\mu} P_{1\nu} \frac{d\sigma}{dP_1} + \int dP_1 dP_2 P_{1\mu} P_{2\nu} \frac{d\sigma}{dP_1 dP_2} \quad (12)$$

The two sum rules can be used to prove that the two-particle correlation function, defined as :

$$\frac{1}{\sigma_{ab}} \frac{d\sigma_{ab}}{dP_1 dP_2} = \left(\frac{1}{\sigma_{ab}} \frac{d\sigma_{ab}}{dP_1} \right) \cdot \left(\frac{1}{\sigma_{ab}} \frac{d\sigma_{ab}}{dP_2} \right) + \rho^{(2)}(P_1, P_2) \quad (13)$$

cannot be zero. Indeed if $\rho^{(2)} \equiv 0$ one gets immediately

$$0 = \int dP_1 P_{1\mu} P_{1\nu} \frac{d\sigma}{dP_1} \quad (14)$$

which is false for $\mu = \nu = 0$. Actually, by considering Eq. (12) for $\mu = \nu = 0$ and $\mu = \nu = 3$ (3 being the direction of motion of particles a, b) one gets :

$$\int_{P_1^L \cdot P_2^L < 0} dP_1 dP_2 E_1 E_2 \rho^{(2)}(P_1, P_2) = 0 \quad (15)$$

$$\int_{P_1^L \cdot P_2^L > 0} dP_1 dP_2 E_1 E_2 \rho^{(2)}(P_1, P_2) = - \int dP_1 E_1^2 \frac{d\sigma_{ab}}{dP_1} < 0 \quad (16)$$

In other words, the sum rules are compatible with no correlation in the double fragmentation region but demand negative correlation if both detected particles are fragments of the same initial particle. Indeed this is obvious : the two detected particles cannot have both $(P_L/2\sqrt{s}) \lesssim +1$ (energy conservation) and therefore correlations must exist. On the other hand, absence of correlation in the double fragmentation region has already been found in recent calculations⁹⁾. Taking $\mu = \nu = 1, 2$ in Eq. (12) and summing over μ one gets

$$0 = \int dP_1 |P_1^L|^2 \frac{d\sigma}{dP_1} + \int dP_1 dP_2 |P_1^L| |P_2^L| \cos \varphi \frac{d\sigma}{dP_1 dP_2} \quad (17)$$

It was pointed out recently¹⁰⁾ that $d\sigma/dP_1 dP_2$ must have azimuthal dependence. Indeed in the opposite case, the second term in Eq. (17) vanishes and the equation itself is violated.

Since correlations must be there, we now look for sum rules directly in terms of correlations. This is obtained simply from Eq. (7) if one defines¹¹⁾

$$\begin{aligned} \frac{1}{\sigma} \frac{d\sigma}{dP_1} &= \rho^{(1)}(P_1) ; \quad \frac{1}{\sigma} \frac{d\sigma}{dP_1 dP_2} = \rho^{(1)}(P_1) \rho^{(1)}(P_2) + \rho^{(2)}(P_1, P_2) ; \\ \frac{1}{\sigma} \frac{d\sigma}{dP_1 dP_2 dP_3} &= \rho^{(1)}(P_1) \rho^{(1)}(P_2) \rho^{(1)}(P_3) + \rho^{(1)}(P_1) \rho^{(2)}(P_2, P_3) + \rho^{(1)}(P_2) \rho^{(1)}(P_1, P_3) + \rho^{(1)}(P_3) \rho^{(2)}(P_1, P_2) + \rho^{(3)}(P_1, P_2, P_3) \end{aligned} \quad (18)$$

and, in general,

$$\frac{1}{\sigma} \frac{d\sigma}{dP_1 \dots dP_K} = \sum_{\sum i \lambda_i = K} \sum_{\lambda_i} \sum_{\text{Perm } j_i} P^{(1)}(P_{j_1}) \dots P^{(1)}(P_{j_{\lambda_1}}) P^{(2)}(P_{j_{\lambda_1+1}}, P_{j_{\lambda_1+2}}) \dots P^{(2)}(P_{j_{\lambda_1+2\lambda_2-1}}, P_{j_{\lambda_1+2\lambda_2}}) \dots P^{(k)}(P_{j_{\lambda_1+2\lambda_2+\dots+i\lambda_i-1}}, \dots, P_{j_{\lambda_1+2\lambda_2+\dots+i\lambda_i}}) \dots P^{(K)}(P_{j_1}, \dots, P_{j_K}). \quad (19)$$

Using the symmetry of the integral in Eq. (7) the $\sum_{\text{Perm}}$ gives $\frac{K!}{i!} \lambda_i! \lambda_i!$ times a representative term which we choose to be the natural ordering term. Thus we get :

$$(e^{Pz} - 1) = \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\lambda_i} \prod_i \frac{K!}{\lambda_i! (i!)^{\lambda_i}} \int dP_1 \dots dP_K P^{(1)}(P_1) \dots P^{(K)}(P_K). \quad (20)$$

$$(e^{Pz} - 1) \dots (e^{P_K z} - 1) = \sum_{K=1}^{\infty} \sum_{\lambda_i} \prod_i \frac{1}{\lambda_i! (i!)^{\lambda_i}} [\tilde{\rho}^{(i)}(z)]^{\lambda_i}$$

where we have defined :

$$\tilde{\rho}^{(i)}(z) = \int dP_1 \dots dP_i P^{(i)}(P_1 \dots P_i) (e^{P_1 z} - 1) \dots (e^{P_i z} - 1) \quad (21)$$

Equation (20) can be summed up (since explicit dependences on K have dropped out) and gives ¹²⁾ :

$$(e^{Pz} - 1) = \prod_i \left(\sum_{\lambda_i=0}^{\infty} \frac{[\tilde{\rho}^{(i)}(z)]^{\lambda_i}}{\lambda_i! (i!)^{\lambda_i}} \right) - 1 = \prod_i \exp \left(\frac{\tilde{\rho}^{(i)}(z)}{i!} \right) - 1 \quad (22)$$

The final result is thus simply :

$$P.z = \sum_{i=1}^{\infty} \frac{1}{i!} \tilde{\rho}^{(i)}(z) \quad (23)$$

One can check easily that Eq. (23) gives (when expanded to lowest powers of z) Eqs. (11) and (12).

It is also immediate to verify that, if $\tilde{\rho}^{(i)}(z) = 0$ for $i \geq 2$, one gets an inconsistency, i.e. :

$$P.z = \tilde{\rho}^{(1)}(z) = \int dP \frac{d\sigma}{\sigma \cdot dP} (e^{Pz} - 1) \quad (24)$$

which cannot hold for all values of z ¹³⁾.

We have said that z is arbitrary. It is amusing to specify it in the form

$$z_\mu = \beta P_\mu / P^2 \quad (25)$$

Thus $P \cdot z = \beta$ and Eq. (7) becomes

$$(e^{\beta} - 1) \sigma_{ab} = \sum_{k=1}^{\infty} \frac{1}{k!} \int dP_1 \dots dP_k \frac{d\sigma_{ab}}{dP_1 \dots dP_k} (e^{\beta \bar{x}_1/2} - 1) \dots (e^{\beta \bar{x}_k/2} - 1) \quad (26)$$

where we have defined

$$\bar{x}_i = \frac{2 P_i \cdot P}{P^2} = \frac{2 E_i}{E_{\text{Tot}}} \Big|_{\text{c.m.}}$$

Explicit dependence on E_{tot} has disappeared on both sides showing that strong interactions scaling, defined as

$$\frac{1}{\sigma} \frac{d\sigma}{dP_1 \dots dP_k} \xrightarrow{E \rightarrow \infty} f(\bar{x}_i, P_i^+) \quad (27)$$

is consistent with (and indeed suggested by) the sum rules.

Finally, we want to note possible further generalizations of our results. Generalizations to non-forward matrix elements of TT^+ are obvious.

If we want to include other additive conserved quantum numbers, like charge, hypercharge, baryonic number, it is simply enough to replace everywhere $P \cdot z$ (and $P_i \cdot z$) with $\hat{P} \cdot \hat{z}$ ($\hat{P}_i \cdot \hat{z}$) where $\hat{P} \cdot \hat{z}$ are now vectors with $4 + \sigma$ components ¹⁴⁾ (σ being the number of extra-conserved quantities besides four-momentum). The sum rules obtained in this way are likely to bring information on the question of exotic channels and early scaling ⁷⁾.

Finally, our considerations are certainly not limited to strong processes. We can consider the case of a massive photon (either spacelike or timelike) scattering on a hadron, i.e., deep inelastic scattering. The sum rules will then suggest again in which variable scaling may be expected.

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- 8) For simplicity, we consider the case of identical particles. Generalizations to the case of different final particles are straightforward. We use the normalizations of Ref. 5) and we set equal to one the flux factor which is always the same in both sides of the sum rules.
- 9) D.Z. Freedman, C.E. Jones, F.E. Low and J.E. Young - Phys.Rev.Letters 26, 1197 (1971).
In the dual framework this factorization has also been noticed by several authors (e.g., B. Hasslacher, C.S. Hsue and D.K. Sinclair - SUNY Preprint, 1971, and Chian-li Jen, K. Kang, P. Shen and C.I. Tan - Brown University Preprint, 1971).
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One of us (G.V.) is grateful to them for an interesting discussion.
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- 12) Indeed, at this point, our calculation reduces to the one of Ref. 11), if we replace all $(e^{P_{12}} - 1)$ in Eq. (7) with a constant h . In this case the left-hand side of Eq. (7) is the function ¹¹⁾ $\psi(h) \equiv \sum_n (1+h)^n \sigma_n / \sigma_{ab}$, which, expanded in powers of h , generates the moments of the multiplicity distributions. Our result, Eq. (22) gives, as in Ref. 11)

$$\psi(h) = \exp \left(\sum_i h^i \int P_i^{(u,v)} dP_i \right)$$
- 13) Mathematically, the only solution is

$$\frac{1}{\sigma} \frac{d\sigma}{dP_1} \propto \delta(P_1^+) \delta(P_1^-/E)$$

and $P_1^2 = 0$. This is the only situation in which all σ_n can be different from zero at all energies.
- 14) This has an amusing similarity with the definition of the Grand Potential in Statistical Mechanics.