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ON THE MERGING OF ANALYTICITY AND STATISTICS IN EXTRAPOLATIONS
AND REPRESENTATIONS OF DATA

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ABSTRACT

Basic ideas of the Cutkosky theory of the representation of data by analytic functions together with some additions are applied to some problems of analytic extrapolations.

The data and hypotheses about the behaviour of the extrapolated function are understood as determining the probability in a suitable space of analytic functions. Quantities of interest (values of a function in a point or average values over an arc) appear as expectation values obtained by the continuum (Wiener) integration over the space of analytic functions. The corresponding errors are defined in an unambiguous and "statistical" way as a generalization of usual standard deviations. The problems dealt with as applications of the formalism are of relevance for the determination of coupling constants from dispersion relations and for extrapolations of the Chew-Low-Goebel type.

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1. INTRODUCTION

Analyticity properties of scattering amplitudes and form factors are frequently the most reliable piece of a priori information. In correlating the data or in making inferences thereof, one has to perform some kind of an extrapolation. The extrapolation techniques proposed so far rely [with a notable exception ^{1),2)}] mostly on mathematics [represented, e.g., by Walsh's excellent book ³⁾] originally devised for interpolation and/or approximation by analytic functions. In this mathematical approach, one assumes that exact values of the function to be approximated are exactly given in some points. In a realistic situation, the data are always subject to experimental errors and, using this kind of mathematics, one has to interpret the experimental errors as rigorous upper bounds, which does not respect their statistical nature. The estimated errors of the extrapolated quantities appear then also in a form of rigorous upper bounds which are thus expected to be unrealistically large.

The way out of this unsatisfactory situation has been recently indicated in an important paper by Cutkosky ¹⁾, where a theory of representing the data by analytic functions was proposed.

Differences between the new approach and the traditional fitting procedures are best seen on a simple example. Suppose the data y_i with errors ϵ_i are given in points x_i on the real axis. To fit the data, one usually constructs a function $P(\alpha_1, \dots, \alpha_n; x)$ containing free parameters $\alpha_1, \dots, \alpha_n$ and minimizes

$$\chi^2 = \sum_i \epsilon_i^{-2} [P(\alpha_1, \dots, \alpha_n; x_i) - y_i]^2 \quad (1)$$

with respect to α_i . This actually amounts to looking for "the most probable function" in the functional space $P(\alpha_1, \dots, \alpha_n; x)$. The procedure may be quite satisfactory in interpolations or in cases when the form $P(\alpha_i; x)$ is given a priori from physical reasons. In extrapolations the model dependence [hidden in the particular form of $P(\alpha_i; x)$] may lead to serious biases.

To depress the model dependence, one has either to choose "the correct form" for $P(\alpha_i, x)$ (which, in most cases, is impossible due to lack of information) or to enlarge the space of functions.

For extrapolation purposes, a suitable space is a space ^{*)} of all functions analytic in a given domain and bounded by a common constant. This space was actually used in most of the recent works ^{**) 5)-16)} on analytic extrapolations.

A rigorous upper bound on all functions considered is not a suitable starting point for implementing statistical concepts. An alternative, and for statistical purposes much more suitable option, was proposed by Cutkosky ¹⁾. He considers a space of "all" functions analytic in a given domain, each function $f(z)$ being weighted by a factor proportional to the expression

$$P(f) \sim \exp \left[- \int_C |f(x) M^{-1}(x)|^2 |dx| \right] \quad (2)$$

where C is the boundary of the domain, and $M(x) > 0$ is a given function independent of $f(x)$.

Such a formulation takes into account "all" functions, suppressing those which in the previous formulation (based on the upper bound) were not accepted and assigning different weight to those which were previously accepted tacitly as equally probable.

The purpose to the present paper is to discuss in some detail the introduction of a probability into the space of analytic functions (Section 2). We shall interpret the experimental data as a source for introducing the probability to the functional space, deviating in this

*) The statement is due to Tikhonov. References to original papers and main ideas can be found in Ref. 4).

**) Some elements of the Cutkosky theory ¹⁾ have been used in the work on phase shift analyses and on the determination of coupling constants 2), 17)-19) based on the Cutkosky-Deo method 2), 19), 20).

respect from Cutkosky's approach ^{*)}. Having put all the information available into the probability in a given functional space, we can (Section 3) calculate average (expectation) values, variances, standard deviations, etc. In Section 4, we shall discuss along these lines the extrapolation which is involved in determinations of the coupling constants from dispersion relations and in Section 5 we shall deal with the one related to the Chew-Low-Goebel extrapolation. Section 6 contains comments and conclusions.

2. PROBABILITY IN A SPACE OF ANALYTIC FUNCTIONS

Suppose that the following two kinds of information about a function analytic in a region D with a boundary B are available :

- i) experimental data on a set $E: E \subset \bar{D} = D \cup B$. The set E can consist of a few points in which the measurement was done, or of arcs on which the data are given, e.g., in the form of a histogram;
- ii) some a priori information or hypotheses about the behaviour of the function on that part (denoted as H) of the boundary which is not included in E . The set $C = E \cup H$ thus contains the whole boundary B .

Point ii) is a mathematical necessity corresponding, in a sense, to the requirement of upper bounds ⁴⁾ which make the extrapolation stable. On the other hand, the assumption ii) may well correspond to reality. If one, for instance, writes a fixed t dispersion relation for πN scattering, the data at low energies fall under i) and hypotheses about a particular high energy behaviour as Regge poles, Regge cuts, etc., which are, of course, necessary if the dispersion relation is to make sense, enter ii).

^{*)} Another difference, more on a technical level, concerns the way how the integration in the functional space is performed.

Introduction of a probability into a functional space is a step which should rely on consistency of the definition and otherwise on intuition. It may therefore contain some arbitrariness.

Suppose first that a measurement was performed in only one point x_i , $x_i \in D$ giving the value y_i and the error ε_i . Assuming a Gaussian distribution of errors and vaguely invoking the Bayes theorem, it is natural to define

$$P(F) \sim \exp \left\{ - (1/2) \varepsilon_i^{-2} |F(x_i) - y_i|^2 \right\} \quad (3)$$

as a probability for a function $F(z)$ analytic in the region D to be consistent with the data. Note that $P(F)$ is a probability in a functional space and the proportionality constant in Eq. (2) can be given only after the integration in the functional space was specified.

For n independent measurements in points $x_i \in E$, with results y_i and errors ε_i , $i=1,2,\dots,n$ we shall write

$$P(F) \sim \exp \left\{ - \frac{1}{2} \sum_1^n \varepsilon_i^{-2} |F(x_i) - y_i|^2 \right\} \quad (3')$$

The internal consistency of the definitions in Eqs. (3) and (3') can be checked by considering two (or more) measurements performed in the same point. It is seen that the probability introduced by Eq. (3') is the same as the one introduced by Eq. (3) if the two measurements in the same point leading to values and errors (y_1, ε_1) and (y_2, ε_2) are interpreted as one measurement with value y and error ε given by the standard formulae

$$\varepsilon^{-2} = \varepsilon_1^{-2} + \varepsilon_2^{-2}, \quad y\varepsilon^{-2} = y_1\varepsilon_1^{-2} + y_2\varepsilon_2^{-2}$$

This requirement of consistency would allow to write Eq. (3') in a more general form

$$P(F) \sim \exp \left\{ - \frac{1}{2} \sum_1^n \varepsilon_i^{-2} \Delta_i |F(x_i) - y_i|^2 \right\}$$

with Δ_i depending on positions x_i and subject to the constraint $\sum_{i=1}^n \Delta_i = n$. It might be interesting to study the properties of allowed Δ_i , e.g., from the point of view of the conformal invariance of definitions in Eqs. (3) and (3'). We shall, however, forget about this possibility in the sequel and stick to Eq. (3') as it stands.

It happens frequently ^{*)} that the data are dense on an arc inside the analyticity region where the function is known to be smooth. In such a case it is more convenient to have the term in the exponent of Eq. (3') expressed as an integral rather than as a sum of many terms. We thus wish to find functions $\mathcal{E}(x)$ and $y(x)$ such that it holds [see Eq. (3')]

$$\sum_{i=1}^n \mathcal{E}_i^{-2} |F(x_i) - y_i|^2 = \int_{\ell} \mathcal{E}^{-2}(x) |F(x) - y(x)|^2 |dx| \quad (4)$$

where ℓ is the arc containing the points x_i , $i=1, \dots, n$. The function $y(x)$ is clearly a smooth interpolation to y_i ; so that $y(x_i) = y_i$, and $\mathcal{E}(x)$ should be a smooth function which satisfies the condition

$$\mathcal{E}(x_i) = \mathcal{E}_i \rho^{-1/2}(x_i) \quad (5)$$

where $\rho(x)$ is a density of the experimental points near the point x_i . The fact that the density $\rho(x)$ enters into the definition of the "effective smoothed error" is quite natural, since it is intuitively clear that the smoothed errors should be inversely proportional to the number of events along a unit length ^{**)}.

With this definition of a smoothed error $\mathcal{E}(x)$ we can rewrite the probability induced by "dense" experimental data [Eq. (3')] as

*) Examples for this are, e.g., data on nucleon form factors in the space-like region, or data on cross-section at a fixed energy, taken as function of $z = \cos \vartheta$.

**) To the best of our knowledge, this rather simple fact was so far neither used in calculations based on dispersion relations, nor discussed in extrapolation methods proposed. This is apparently due to, at most, intuitive connection between the experimental errors and mathematical errors taken as upper bounds.

$$P(F) \sim \exp \left\{ -\frac{1}{2} \int_{\ell} \varepsilon^{-2}(x) |F(x) - y(x)|^2 |dx| \right\} \quad (6)$$

If the experimental data are given also on a part of the boundary, we have to express the corresponding contribution to the probability in the form of Eq. (6), since the form used in Eq. (3') would leave some arcs on the boundary (those between the experimental points) where no restrictions on the function are given. This would, in a classical formulation, correspond to functions which are not bound in the analyticity region and any stable extrapolation would be impossible.

Note, however, that the arbitrariness in constructing the interpolation functions $y(x)$ and $\varepsilon(x)$ for data which are not dense is already reflected in the factor $\rho^{-\frac{1}{2}}$ entering Eq. (3'). Due to this factor, the smoothed error for data which are not dense will be anyway large.

As we have repeatedly mentioned above, the behaviour of the function has to be restricted along the whole boundary. For the part H of the boundary B which is not included in the experimental region E we shall thus write in analogy to Eq. (6)

$$P(F) \sim \exp \left\{ -\frac{1}{2} \int_H \varepsilon_H^{-2}(x) |F(x) - F_H(x)|^2 |dx| \right\} \quad (7)$$

Here $F_H(x)$ is the behaviour on H predicted by the hypothesis and $\varepsilon_H(x)$ is the estimated error of that hypothesis. Detailed estimates of values of $\varepsilon_H(x)$ and of $F_H(x)$ have to be done according to a specified situation, taking into account the analogy between Eq. (7) and Eqs. (6) and (5).

On the basis of Eqs. (3'), (6) and (7) we are finally left with the following expression for the probability in the functional space induced by the information contained in both experimental data and hypotheses made

$$P(F) \sim \exp \left\{ -\frac{1}{2} \chi_c^2(F) \right\} \quad (8)$$

where

$$\chi_c^2(F) = \sum_i \varepsilon_i^{-2} |F(x_i) - y_i|^2 + \frac{1}{L} \int_B |F(x) - \phi(x)|^2 M^{-2}(x) |dx| \quad (8')$$

Here L is the length of the boundary B ,

$$\begin{aligned} \phi(x) &= F_H(x), \quad L^{-1} M^{-2}(x) = \varepsilon_H^{-2}(x) \quad \text{for } x \in H \\ \phi(x) &= y(x), \quad L^{-1} M^{-2}(x) = \varepsilon^{-2}(x) \quad \text{for } x \in B-H \end{aligned} \quad (9)$$

and the meaning of ε_i and y_i is the same as in Eq. (3'). In some cases, the first term in the right-hand side of Eq. (8') can be written in the form of Eq. (6) and in cases when experimental data are given in the form of a histogram (data summarized over an interval) the first term in the right-hand side of Eq. (8') can be changed to

$$\bar{\varepsilon}_i^{-2} \left| \bar{y}_i - \int_{\ell_i} F(x) m_i(x) |dx| \right|^2 \quad (8'')$$

where $\bar{y}_i$ is the measured value of a histogram over a bin ℓ_i , $\bar{\varepsilon}_i$ is the error and $m_i(x)$ is a suitable weight over ℓ_i reflecting either the experimental situation or the way in which the "histogram" value is built from the function $F(x)$.

In what follows, we shall use $P(F)$ as given by Eqs. (8) and (8'), referring to χ_c^2 as to Cutkosky chi-squared. The function $\phi(x)$ is supposed to be square integrable on B with the weight $M^{-2}(x)$:

$$\int_B M^{-2}(x) |\phi(x)|^2 |dx| < \infty$$

In the space of all functions analytic in the region D and obeying the condition

$$\|F\|^2 \equiv \frac{1}{L} \int_B M^{-2}(x) |F(x)|^2 |dx| < \infty$$

we define the inner product as

$$(F, G) = \frac{1}{L} \int_B M^{-2}(x) F(x) G^*(x) |dx| + \sum_i \varepsilon_i^{-2} F(x_i) G^*(x_i) \quad (10)$$

and denote the corresponding Hilbert space as $D(2)$.

The inner product in Eq. (10) corresponds to the $\chi_c^2(F)$ given by Eq. (8'). Analogues of Eq. (10) corresponding to other expressions for $\chi_c^2(F)$ are easily written down, and in the next Section we shall mostly use the form of the scalar product for the case when there are no data inside the analyticity region :

$$(F, G) = \frac{1}{L} \int_B M^{-2}(x) F(x) G^*(x) |dx|$$

$$\chi_c^2(F) = (F, F) \quad (11)$$

Note finally that the term which enters Eqs. (10) and (11) through Eq. (6) is due to the definition of the smoothed error by Eq. (5) explicitly conformally invariant ^{*}). This indicates that the definition of smoothed errors in making the hypotheses [Eq. (7)] should be done along similar lines and the factor $\varrho^{-\frac{1}{2}}(x)$ should be included therein.

^{*}) In making the mapping, say, $x \rightarrow z = z(x)$ the term $\varrho(x) |dx| \rightarrow \varrho(x) |dx/dz| |dz|$ and $\varrho(x) |dx/dz|$ is just the density of data points in the new variable.

The term in Eq. (7) which corresponds to data given in a form of a histogram is not explicitly conformally invariant, but this is not surprising since the weights $m_i(x)$ are naturally changed when the transformation of variables is performed.

3. AVERAGE VALUES AND STANDARD DEVIATIONS OF FUNCTIONALS IN $D(2)$

We shall start by specifying the functional integration in the space $D(2)$. Let the set $\{P_n(x)\}_0^\infty$ be an orthonormal basis in $D(2)$ with respect to the inner product given by Eq. (11). Putting

$$F(x) = \sum_{n=0}^{\infty} b_n P_n(x), \quad F_N(x) = \sum_{n=0}^N b_n P_n(x) \quad (12)$$

one easily obtains

$$\chi_c^2(F_N) = \sum_0^N |b_n|^2 - 2 \operatorname{Re} \sum_0^N b_n B_n^* + |B|^2 \quad (13)$$

where

$$B_n = \frac{1}{L} \int_B M^{-2}(x) \phi(x) P_n^*(x) |dx|$$

$$|B|^2 = \frac{1}{L} \int_B M^{-2}(x) |\phi(x)|^2 |dx|$$

In the notation $c_n = b_n - B_n$ we obtain

$$\chi_c^2(F_N) = \sum_0^N |c_n|^2 + |B|^2 - \sum_0^N |B_n|^2 \quad (14)$$

We are now in a position to introduce in a convenient way the continuum integration. Let $R(F)$ be a functional in $D(2)$. Its average value $\langle R \rangle$ is defined as ^{*}

$$\langle R \rangle = \lim_{N \rightarrow \infty} \frac{\int d^2 F_N P(F_N) R(F_N)}{\int d^2 F_N P(F_N)} \quad (15)$$

with

$$d^2 F_N \equiv \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} d(Re c_0) d(Im c_0) \dots d(Re c_N) d(Im c_N) \quad (16)$$

The probabilities $P(F_N)$ discussed above were not normalized. This is the reason for the appearance of the denominator in Eq. (15). If the functions in $D(2)$ are further restricted, e.g., the region D is symmetric with respect to the real axis and $\phi(x)$ and $F(x)$ are real analytic; the coefficients C_n are real, then the terms $d(Im C_i)$ are omitted in Eq. (16).

Terms independent of C_n cancel in Eq. (15) and we can then omit them in Eq. (14). Therefore we take

$$P(F_N) \sim \exp \left\{ -\frac{1}{2} \sum_0^N |C_m|^2 \right\} \quad (16')$$

The standard deviation ΔR of a functional $R(F)$ is defined as

$$(\Delta R)^2 \equiv \langle |R|^2 \rangle - |\langle R \rangle|^2 \quad (17)$$

Let R be a linear functional in $D(2)$. Then there exists a function $R(x) \in D(2)$ such that

^{*}) Note the technical difference between the form used in Eq. (15) and that used by Cutkosky ¹⁾. Useful references about Wiener integrals, such as Eq. (15), are found in Ref. 21).

$$R(F) = (F, R) = \frac{1}{L} \int_B M^{-2}(x) F(x) R^*(x) |dx| \quad (18)$$

Expanding $R(x)$ into the orthogonal set $\{P_n(x)\}$

$$R(x) = \sum_0^\infty r_m P_m(x) \quad (19)$$

we obtain

$$R(F_N) = \sum_0^N r_m^* b_m = \sum_0^N r_m^* (c_m + B_m)$$

Inserting this into Eq. (15) we get

$$\langle R \rangle = \sum_0^\infty r_m^* B_m = (\bar{F}, R) = R(\bar{F}) \quad (20)$$

where

$$\bar{F}(x) = \sum_0^\infty B_m P_m(x) \quad (21)$$

It follows from Eq. (16') that $\chi_C^2(F_N)$ is minimal for $c_n = b_n - B_n = 0$. The function $\bar{F}(x)$ given by Eq. (21) thus minimizes $\chi_C^2(F)$. In order to determine $R(\bar{F})$ it is not necessary to perform explicitly the continuum integration in Eq. (15). It is more convenient to find first the function $\bar{F}(x)$ which minimizes $\chi_C^2(F)$ and then calculate $\langle R \rangle$ from Eq. (20).

We proceed in a similar way also in calculating $(\Delta R)^2$. It is useful to express first $|R(F_N)|^2$ in the form

$$|R(F_N)|^2 = \sum_0^N \sum_0^N r_k^* r_l (c_k + B_k)(c_l^* + B_l^*)$$

*) The function $\bar{F}(x)$ can be called "the most probable function".

The following relations are easily checked

$$\begin{aligned} \langle r_k^* r_\ell (c_k + B_k)(c_\ell^* + B_\ell^*) \rangle &= r_k^* r_\ell \{ \langle c_k c_\ell^* \rangle + \langle B_k B_\ell^* \rangle \} = \\ &= \sqrt{2\pi} (r_k^* r_\ell \delta_{k\ell} + r_k^* r_\ell B_k B_\ell^*) \end{aligned} \quad (22)$$

Hence it follows that

$$\langle |R|^2 \rangle = (R, R) + |\langle R \rangle|^2$$

and consequently

$$\langle (\Delta R)^2 \rangle \equiv \langle |R|^2 \rangle - |\langle R \rangle|^2 = \sum_0^\infty |r_k|^2 = \|R\|^2 \quad (23)$$

Equations (20) and (23) permit us to calculate the average values and standard deviations of linear functionals in $D(2)$ without actually performing the continuum integration. The simplicity is naturally due to the convenient choice of the basis in $D(2)$ and to the definition of continuum integration in terms of this particular basis. Equations (20) and (23) are written in a form which is explicitly basis independent. For completeness sake, we shall demonstrate here this independence in a more explicit way, starting from the definition of $\langle R \rangle$ in Eqs. (15) and (16).

Note first that we have above only assumed that $\{P_n(x)\}$ be an orthogonal basis with respect to the inner product in $D(2)$. Therefore the results derived are valid for any orthogonal basis.

Suppose now that $\{\hat{P}_n(x)\}$ is a basis in $D(2)$ which is not orthogonal. Proceeding as in Eq. (12), we define

$$\begin{aligned} F(x) &= \sum_0^\infty \hat{c}_m \hat{P}_m(x) \\ \hat{F}_N(x) &= \sum_0^N \hat{c}_m \hat{P}_m(x) \end{aligned} \quad (24)$$

The average value of a functional $R(\hat{F})$ is defined as

$$\langle R \rangle = \lim_{N \rightarrow \infty} \frac{\int d^2 \hat{F}_N P(\hat{F}_N) R(\hat{F}_N)}{\int d^2 \hat{F}_N P(\hat{F}_N)} \quad (25)$$

with

$$d^2 \hat{F}_N = \int_{-\infty}^{+\infty} \dots \int d(\text{Re} \hat{c}_0) \dots d(\text{Im} \hat{c}_N) \quad (26)$$

The non-orthogonal basis $\{\hat{P}_n\}$ can be orthogonalized by the Schmidt procedure

$$\pi_m(x) = \sum_0^m a_{mk} \hat{P}_k(x) \quad (27)$$

and the function $\hat{F}_N(x)$ can be written as

$$\hat{F}_N(x) = \sum_0^N \beta_k \pi_k(x) \quad (28)$$

where

$$\beta_k = \sum_{m=0}^N a_{mk} \hat{c}_m \quad (28')$$

It is now easy to see that

$$\frac{\int d^2 \hat{F}_N P(\hat{F}_N) R(\hat{F}_N)}{\int d^2 \hat{F}_N P(\hat{F}_N)} = \frac{\int d^2 \phi_N P(\hat{F}_N) R(\hat{F}_N)}{\int d^2 \phi_N P(\hat{F}_N)} \quad (29)$$

where

$$\int d^2 \phi_N = \int_{-\infty}^{+\infty} \dots \int d(\text{Re} \beta_0) \dots d(\text{Im} \beta_N) \quad (29')$$

since $d^2\phi_N$ and $d^2\hat{F}_N$ differ only by a multiplicative factor [a Jacobian of the transformation in Eq. (2.8')] and this factor cancels in Eq. (29). The right-hand side of Eq. (29) is however given in the orthogonal basis $\{\pi_n(x)\}$ and therefore all the preceding results are valid.

In the following two sections we shall apply the formalism described above to some, relatively simple, problems of analytic extrapolation. We shall mostly rely on results contained in Eqs. (20) and (23).

4. EXTRAPOLATION FROM A PART OF THE BOUNDARY TO A POINT INSIDE A (SIMPLY CONNECTED) ANALYTICITY REGION

This problem was treated in Refs. 4), 12), 16) and pertains, apart from other possible applications, to the determination of coupling constants from dispersion relations at fixed momentum transfer.

Any simply connected analyticity region can be conformally mapped onto the unit circle and we shall consider only this case explicitly. The open unit disc is denoted as D , its boundary as B . The symbol H is used for that part of B over which a hypothesis is being made and experimental data are available on $B-H$. In the notation of Eqs. (8') and (9) we write the Cutkosky $\chi_c^2(F)$ in the form

$$\chi_c^2(F) = \frac{1}{2\pi} \int_B |F(x) - \phi(x)|^2 M^{-2}(x) |dx| , \quad (30)$$

We shall first find within $D(2)$ the "most probable function" $\bar{F}(x)$, e.g., the one which minimizes $\chi_c^2(F)$ on $D(2)$.

For this purpose let us construct a function $g(x)$ which is

- i) analytic in D ;
- ii) without zeros in D ;
- iii) and satisfies the condition

$$|g(x)| = M(x) > 0 \quad \text{for } x \in B$$

up to an unimportant phase factor $g(z)$ is given [see, e.g., Ref. 22] as

$$g(x) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\vartheta} + x}{e^{-i\vartheta} - x} \ln M(e^{i\vartheta}) d\vartheta \right\} \quad (31)$$

Let $f(x)$ and $\varphi(x)$ be defined as

$$f(x) = F(x) g^{-1}(x), \quad x \in D \quad (32)$$

$$\varphi(x) = \phi(x) g^{-1}(x), \quad x \in B \quad (32')$$

The function $f(x)$ is analytic in D and can be written as a Taylor series

$$f(x) = \sum_0^{\infty} a_m x^m, \quad x \in D$$

and for $\varphi(x)$ we have a Laurent series

$$\varphi(x) = \sum_{-\infty}^{+\infty} c_m x^m, \quad x \in B \quad (33)$$

Then

$$\chi_c^2(f) \equiv \frac{1}{2\pi} \int_B |f(x) - \varphi(x)|^2 |dx| = \sum_1^{\infty} |c_{-m}|^2 + \sum_0^{\infty} |a_m - c_m|^2$$

The minimum of $\chi_c^2(F)$ within the space $D(2)$ occurs thus for

$$\bar{F}(x) = \bar{f}(x) g(x) \quad (34)$$

where

$$\bar{f}(x) = \sum_0^{\infty} c_n x^n = \frac{1}{2\pi i} \int_B \frac{\varphi(y)}{y-x} dy \quad (34')$$

The last equality follows from Eq. (33).

Having found the most probable function, we proceed to the calculation of average values of some functionals. Consider first

$$R(F) = F(x_0), \quad x_0 \in D. \quad (35)$$

According to Eq. (20), it holds

$$\langle R(F) \rangle \equiv \langle F(x_0) \rangle = \bar{f}(x_0) g(x_0) \quad (36)$$

The connection of this result with those obtained by methods of optimal weight functions ^{4),12),16)} is immediately seen by rewriting Eq. (36) as

$$\langle F(x_0) \rangle = \bar{F}(x_0) = \frac{1}{2\pi i} \int_B \frac{\bar{f}(x) g(x)}{x-x_0} dx$$

In fact, the $g(x)$ used above was the optimal weight function in the formulation by Ciulli and Fischer ¹⁶⁾, although it was used here in a different context.

To calculate the standard deviation

$$\langle (\Delta R)^2 \rangle \equiv \langle (\Delta F(x_0))^2 \rangle$$

we note first that functions $\{g(x)x^n\}$ form an orthonormal basis with respect to the scalar product

$$(F, G) = (2\pi)^{-1} \int M^{-2}(x) F(x) G^*(x) |dx|. \quad (37)$$

On the other hand

$$F(x) \equiv f(x)g(x) = \sum_0^{\infty} a_m g(x) x^m \quad (38)$$

so that

$$R(F) \equiv F(x_0) = \bar{f}(x_0)g(x_0) = \sum_0^{\infty} a_m g(x_0) x_0^m$$

writing $R(F)$ in the form of Eq. (20), e.g.,

$$R(F) = \sum r_m^* a_m$$

with $r_n^* = g(x_0)x_0^n$ and using Eq. (23) we find

$$\langle (\Delta R)^2 \rangle \equiv \langle (\Delta F(x_0))^2 \rangle = \sum_0^{\infty} |r_m^*|^2 = |g(x_0)|^2 \sum_0^{\infty} |x_0|^{2m}$$

and hence

$$\langle (\Delta F(x_0))^2 \rangle = \frac{|g(x_0)|^2}{1 - |x_0|^2} \quad (39)$$

It is worth noting that Eq. (39) is consistent with bounds obtained, Refs. 4), 12), 16), by the Nevanlinna principle.

In fact, assuming that

$$|g(x)| < \varepsilon \text{ for } x \in B-H, \quad |g(x)| < M \text{ for } x \in H$$

we get from the Nevanlinna principle 5), 8)

$$|g(x_0)| < \varepsilon^{\omega(x_0)} M^{1-\omega(x_0)}$$

where $\omega(x)$ is a corresponding harmonic measure. By Eq. (39) the bound on $g(x_0)$ is transferred onto $\langle \Delta^2 F(x_0) \rangle$.

As another example, let us consider the functional

$$\mathcal{R}(F) = F^{(m)}(0) \equiv \frac{d^m}{dx^m} F(x) \Big|_{x=0} \quad (40)$$

Using the same notation and proceeding as above, we obtain

$$\mathcal{R}(F) \equiv F^{(m)}(0) = m! \sum_0^m a_k g_{m-k} \quad (40')$$

where a_n and g_n are Taylor coefficients of $f(x)$ and $g(x)$ respectively.

Hence the coefficients r_k corresponding to $\mathcal{R}(F)$ are given as

$$r_k^* = m! g_{m-k} \text{ for } k = 0, 1, 2, \dots, m; \quad r_k^* = 0 \text{ for } k > m. \quad (41)$$

Thus

$$\langle \mathcal{R}(F) \rangle = \langle F^{(m)}(0) \rangle = m! \sum_0^m c_k g_{m-k}$$

and

$$\langle (\Delta \mathcal{R})^2 \rangle = \sum_0^\infty |r_k|^2 = m! \sum_0^m |g_k|^2$$

The results are formally similar to those obtained in weight function methods ^{4),13),16)}. The notable difference lies in the fact that the standard deviations of functionals calculated above are of a truly statistical nature, and that our approach accommodates well the statistical character of the information used at the input. Because of that, our output smoothed errors [see Eq. (5)] are also different from those which would be used in a method based on "mathematical" upper bounds. In our opinion the approach of this section is also simpler than the traditional one.

5. EXTRAPOLATION FROM A PART OF THE BOUNDARY TO A POINT INSIDE A DOUBLY CONNECTED REGION

The physical example of this problem is as follows. Suppose that the function we are interested in is analytic in the complex t plane with cuts, say, $(-\infty, -k_1)$ and (k_2, ∞) where $k_1, k_2 > 1$; the data are given in a smoothed form (see Section 2) on the physical region $<-1, 1>$ and some hypotheses about the behaviour of the function on the boundary (cuts) are available. Following the idea of Ciulli⁸⁾ we can now make an additional cut along the physical region and map the t plane onto the annular region. Proceeding in the same way as in Sections 2-4, we are led to the following problem.

Let D be the annular region $r < |z| < 1$, $\mathfrak{A}$ be a space of functions analytic in D and square integrable [with a weight $M(x)$] on the boundary. The probability in this functional space is given by the expression

$$\mathcal{P}(F) \sim \exp \left\{ -\frac{1}{2} \chi_c^2(F) \right\} \quad (42)$$

$$\chi_c^2(F) = \frac{1}{2\pi(1+r)} \int_C |F(x) - \phi(x)|^2 M^{-2}(x) |dx|, \quad (43)$$

$C = C_r \cup C_1$, where C_r and C_1 are circles with radii $r < 1$ and 1 respectively. Let x_0 be a point (for simplicity on the real axis) inside D and

$$\mathcal{R}(F) = F(x_0)$$

be a functional assigning to any $F \in \mathfrak{A}$ its value in x_0 . We wish to find the mean value of $\mathcal{R}(F)$ and its variance, which are defined by the general formalism of Sections 2 and 3. The solution of the problem runs along lines almost identical to those used in Section 4. We first find a suitable weight function which simplifies the expression for $\chi_c^2(F)$, then a suitable orthogonal basis is constructed and "the most probable function" found. The mean value

$\langle F(x_0) \rangle$ is then calculated by Eq. (20). The functional $R(F)$ is finally expressed as a scalar product

$$F(x_0) = R(F) = (F, R)$$

and the variance of $F(x_0)$ is calculated in terms of the norm of the function $R(x)$ by Eq. (23).

The steps mentioned will be now discussed in some detail. We shall mainly concentrate on differences between the present problem and the one dealt with in the previous section.

The weight function $g(x)$ is required to be analytic and free of zeros in D and should satisfy the requirements :

$$\begin{aligned} |g(x)| &= \alpha M(x) \quad \text{for } x \in C_n \\ |g(x)| &= M(x) \quad \text{for } x \in C_1 \end{aligned} \tag{44}$$

where α is a constant which will be shortly shown to be determined by the conditions imposed on $g(x)$.

Since $g(x)$ is free of zeros, we can write

$$g(x) = \exp(h(x)) \tag{45}$$

where $h(x)$ is analytic in D and obeys the conditions

$$\begin{aligned} \operatorname{Re} h(x) &= \ln \alpha + \ln M(x), \quad x \in C_n \\ \operatorname{Re} h(x) &= \ln M(x), \quad x \in C_1. \end{aligned} \tag{46}$$

Applying the Cauchy theorem to the function $h(x)(1/ix)$ analytic in D we obtain

$$\int_{C_n} \operatorname{Re} h(x) x^{-1} |dx| = \int_{C_1} \operatorname{Re} h(x) |dx|$$

and the constant $\mathcal{X}$ turns out to be given by the relation

$$\ln \mathcal{X} = \int_{-\pi}^{\pi} \ln \left[\frac{M(e^{i\varphi})}{M(re^{i\varphi})} \right] d\varphi \quad (47)$$

The explicit form of $g(x)$ can then be found by making a Laurent decomposition of $h(x)$ and of $M(x)$ on C_r and on C_1 . The procedure parallels the discussion in Appendix C of the second of Ref. 13) and we shall proceed further under the assumption that $g(x)$ is known ^{*}).

Coming back to Eq. (43) we write $F(x)$ and $\phi(x)$ in the form

$$F(x) = g(x)f(x) = g(x) \sum_{-\infty}^{+\infty} a_m x^m$$

$$\phi(x) = g(x)\varphi(x), \quad x \in C \quad (48)$$

where $f(x)$ is by definition analytic in D and $\varphi(x)$ is defined for $x \in C$ and can be written as

$$\varphi(x) = \sum_{-\infty}^{+\infty} c_m x^m, \quad x \in C_1$$

$$\varphi(x) = \sum_{-\infty}^{+\infty} d_m r^{-m} x^m, \quad x \in C_r. \quad (49)$$

Using Eqs. (43) and (47)-(49), we obtain

$$\chi_c^2 = \sum_{-\infty}^{+\infty} |a_m - c_m|^2 + \mathcal{X}^2 \sum_{-\infty}^{+\infty} |a_m r^m - d_m|^2$$

Requiring the minimum of χ_c^2 with respect to a_n, a_n^* we have

$$\frac{\partial \chi_c^2}{\partial a_m} = 0 \quad \frac{\partial \chi_c^2}{\partial a_m^*} = 0$$

^{*}) A problem similar to the one of constructing the function $g(x)$ is also discussed by Nenciu ¹⁴⁾.

and hence

$$a_m = \frac{c_m + x^2 r^m d_m}{1 + x^2 r^{2m}}$$

The most probable function $\bar{F}(x)$ is then found to be

$$\bar{F}(x) = \bar{f}(x) g(x) \quad (50)$$

with

$$\bar{f}(x) = \sum_{-\infty}^{+\infty} x^m \frac{c_m + x^2 r^m d_m}{1 + x^2 r^{2m}}$$

According to Eq. (49)

$$c_m = \frac{1}{2\pi i} \int_{C_1} \varphi(y) y^{-m-1} dy, \quad d_m = \frac{1}{2\pi i} \int_{C_r} \varphi(ry) y^{-m-1} dy \quad (51)$$

and inserting it into Eq. (50), we obtain

$$\bar{f}(x) = \frac{1}{2\pi i} \int_{C_1} dy y^{-1} [\varphi(y) \tau_x(x/y) + x^2 \varphi(ry) \tau(rx/y)] \quad (52)$$

where

$$\tau_x(z) = \sum_{-\infty}^{+\infty} \frac{z^m}{1 + x^2 r^{2m}} \quad (53)$$

The average value of a functional $R(F) = F(x_0)$ is then given as

$$\langle R(F) \rangle = \bar{F}(x_0) = g(x_0) \bar{f}(x_0). \quad (54)$$

In determining the variance, one makes use of the fact that the system

$$P_m(x) = \left[\frac{1+r}{1+x^2 r^{2m+1}} \right]^{1/2} x^m g(x) \quad (55)$$

is an orthogonal system in D . Since

$$R(F) = \sum_{-\infty}^{+\infty} A_m P_m(x_0) \quad \text{for} \quad F(x) = \sum_{-\infty}^{+\infty} A_m P_m(x) \quad (56)$$

we find $r_n^* = P_n(x_0)$ (the coefficients r_n are introduced in Section 3) and then, according to Eq. (23), we obtain

$$\langle (\Delta F(x_0))^2 \rangle = \sum_{-\infty}^{+\infty} |r_m|^2 = \sum_{-\infty}^{+\infty} |P_m(x_0)|^2. \quad (57)$$

Making use of Eqs. (53) and (55) we can rewrite Eq. (57) into a more convenient form

$$\langle (\Delta F(x_0))^2 \rangle = (1+r) |g(x_0)|^2 \mathcal{T}_{\text{ser}}(|x_0|^2). \quad (58)$$

Equations (51)-(54) and Eq. (58) are the solution of our problem.

6. COMMENTS AND CONCLUSIONS

Solutions to problems described in Sections 4 and 5 were obtained in a closed analytic form. This was mainly due to the use of the recently developed technique of weight functions ^{4),6),7),10)-16)}. On the other hand one can easily find problems which are relevant for applications but cannot be simplified by the technique of weight

functions. Although one can perhaps solve them in an approximate way ^{*)} within the present approach of continuum integration it would be better to have solutions in a closed form. As an example, we shall describe here a generalization of the problem discussed in Section 4. The probability in the space of analytic functions was there given as $P(F) \sim \exp(-\frac{1}{2}\chi_c^2)$ with [see Eq. (30)]

$$\chi_c^2(F) = \frac{1}{2\pi} \int_B \left\{ \left| \frac{\operatorname{Re} F(x) - \operatorname{Re} \phi(x)}{M(x)} \right|^2 + \left| \frac{\operatorname{Im} F(x) - \operatorname{Im} \phi(x)}{M(x)} \right|^2 \right\} |dx| \quad (59)$$

It is clear that, writing $\chi_c^2(F)$ in this form, one assumes that smooth errors of $\operatorname{Re} F(x)$ and of $\operatorname{Im} F(x)$ are the same. This may be a good approximation in some cases but is definitely bad in many others. For these one would like to be able to follow the procedure of Section 4 starting with $\chi_c^2(F)$ of the form

$$\chi_c^2(F) = \frac{1}{2\pi} \int_B \left\{ \left| \frac{\operatorname{Re} F(x) - \operatorname{Re} \phi(x)}{M_1(x)} \right|^2 + \left| \frac{\operatorname{Im} F(x) - \operatorname{Im} \phi(x)}{M_2(x)} \right|^2 \right\} |dx| \quad (60)$$

with $M_1(x) \neq M_2(x)$. Although the problem seems to be well posed, one cannot reduce it by a conformal mapping or by using weight functions to the simple case treated in Section 4. Apparently, a new technique is required for solving in a closed form the problems characterized by $\chi_c^2(F)$ in Eq. (60).

There are still some extrapolation problems which [under the assumptions essentially equivalent to $M_1(x) = M_2(x)$ above] can be treated along the lines of Sections 1-3 and whose solutions can be obtained in a closed form. Perhaps the most important of them is the extrapolation off a part of the boundary to the average values over the rest of the boundary. Within the approach of weight functions, the problem was recently discussed in Refs. 6), 7), 10)-13) and we shall describe the solution in the formulation of continuum integration elsewhere.

*) A recent reference on numerical minimization of functionals is Daniel ²³⁾ from where some of the earlier work can be traced down.

Let us finally add a few general remarks. A recourse to integration in functional spaces seems to be a natural way to avoid biases due to arbitrary or inadequate parametrizations. It is true that some elements of arbitrariness enter also the method based on continuum integration, either by the choice of the space to be used or by specifying the probability in that space. This kind of arbitrariness is, however, less dangerous and easier to get under control than the arbitrariness introduced by parametrizations ^{*}). The major advantage in using functional spaces, indicated already in the pioneering work by Cutkosky ¹⁾ lies in an unambiguous definition of the errors and in a clear description of the hypotheses underlying the problem. The mean values of functionals and the corresponding errors are defined here independently of the method used actually in the calculation. Such a definition is a necessary prerequisite for any reliable approximative method.

As a by-product of this definition, we have obtained above [see Eq. (5)] a clear description of what is meant under the "smoothed errors". This relation can be used also in calculations based on weight function techniques and may be of some help in, for instance, calculations of spectral functions of nucleon form factors by extrapolating the data from space-like regions [the method of Bowcock et al. ⁶⁾ or other of a similar spirit ^{11), 13), 14)}] or in calculating coupling constants from fixed t dispersion relations.

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^{*}) Criteria for choosing a functional space and for introducing a probability therein are discussed by A. Sard ²⁴⁾ who advocated the use of functional spaces in the theory of approximation of real functions.

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