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REMARKS ON THE RADIATIVE CORRECTIONS
OF ORDER α^2 TO MUON DECAY
AND THE DETERMINATION OF G_μ

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ABSTRACT

We discuss several theoretical aspects of the study of the radiative corrections to the total decay probability P in μ -decay. On the basis of the theorem on cancellation of mass singularities, we obtain by a simple argument that part of the radiative corrections of order α^2 to P which contain mass singularities (i.e. terms which diverge in the mathematical limit $m_e \rightarrow 0$) in an expansion in powers of the renormalized charge. We also state a theorem on the effect of the finite electron mass on the radiative corrections of order α . We then discuss an exponentiation method recently applied to study the corrections to P . We show by direct calculation that the method, as presently implemented, is ambiguous and that it leads, at order α^2 , to a violation of the conclusions obtained on the basis of the theorem on mass singularities.

Finally we give what we consider to be the best value presently available for G_μ in the framework of the local V-A theory, and briefly discuss the expected order of magnitude of the theoretical error.

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1. INTRODUCTION

As is well known, in the framework of the local V-A theory, the weak interaction coupling constant G_μ is determined from the knowledge of the muon lifetime τ_μ by means of the formula

$$\frac{1}{\tau_\mu} = \frac{G_\mu^2}{192 \pi^3} m_\mu^5 \left(1 - 8 \frac{m_e^2}{m_\mu^2} \right) (1 + \delta), \quad (1)$$

where m_μ and m_e stand for the muon and electron masses and δ is the radiative correction to the total decay probability. In Eq. (1) we have not included negligible contributions of order $O[(m_e/m_\mu)^4 \ln(m_\mu/m_e)]$ and higher. In units of $\hbar = c = 1$, the most recent values of $\tau_\mu^{(1)}$, $m_\mu^{(2)}$ and the fundamental constants²⁾ determine

$$\begin{aligned} G_\mu &= (1.4324 \pm 0.0003) (1 + \delta)^{-1/2} \cdot 10^{-49} \text{ erg cm}^3 \\ &= (1.1635 \pm 0.0002) (1 + \delta)^{-1/2} \cdot 10^{-5} \text{ GeV}^{-2}. \end{aligned} \quad (2)$$

To first order in α , and neglecting small contributions which vanish in the mathematical limit $m_e/m_\mu \rightarrow 0$, δ is given by^{3,4)}:

$$\delta = \delta_1 = - \frac{\alpha}{2\pi} \left(\pi^2 - \frac{25}{4} \right) \approx -4.2 \cdot 10^{-3}, \quad (3)$$

where the subscript 1 indicates that we have only included the corrections of first order in α . Using this value the corrected G_μ becomes

$$G_\mu = 1.4354 \cdot 10^{-49} \text{ erg cm}^3 = 1.1660 \cdot 10^{-5} \text{ GeV}^{-2}. \quad (4)$$

A question of considerable theoretical and practical interest is the following: what is the order of magnitude of the corrections of order α^2 to the total decay probability P in μ -decay? We note that the only large parameter likely to appear in the calculation is $\omega \equiv \ln(m_\mu/m_e) \approx 5.332$. Barring exceptionally large numerical coefficients one expects the corrections of order α^2 to P to be quite small, say, of relative order $\lesssim 10^{-4}$ unless there exist contributions of order $\alpha^2 \omega^n$ with a relatively

large exponent n . These terms exhibit an interesting feature: they diverge in the mathematical limit $m_e \rightarrow 0$. Throughout this paper we will refer to these terms as *mass singularities*, although they are certainly finite in the real world! It is well known that terms of order $\alpha\omega$ exist in the radiative corrections to the electron spectrum in μ -decay. Moreover, one knows that such mass singularities of order α cancel exactly after integration over all electron energies^{*}). This is clearly exhibited in Eq. (3), where the analytical expression for δ_1 does not contain mass singularities.

Observe that the existence of terms of order $\alpha^2 \omega^n$ with $n = 4$, say, would make the corrections of order α^2 to P potentially as large as the corrections of order α given in Eq. (3)! Fortunately, nature does not appear to be so malicious. On the basis of the theorem on cancellation of mass singularities derived to all orders in (unrenormalized) perturbation theory by T. Kinoshita⁵⁾ and by T.D. Lee and M. Nauenberg⁶⁾, we obtain in Section 2, by means of a simple argument, that part of the radiative corrections of order α^2 to P which contain mass singularities in an expansion in powers of the renormalized (i.e. physical) charge. The result is:

$$\delta_2^{\text{m.s.}} = \frac{2\alpha}{3\pi} \omega \delta_1 = -\frac{\omega}{3} \left(\frac{\alpha}{\pi}\right)^2 \left(\pi^2 - \frac{25}{4}\right) \approx -3.5 \cdot 10^{-5}, \quad (5)$$

where the superscript m.s. and the subscript 2 indicate that we have only included the contributions of order α^2 containing mass singularities. The fact that $\delta_2^{\text{m.s.}}$ involves only ω raised to the first power and the observation that Eq. (5) is indeed very small for the actual values of m_μ and m_e are particularly comforting. It should be clearly understood that this result applies only to the total decay probability, i.e. the corrections to the inverse lifetime τ_μ^{-1} . This includes a summation over all final states accessible to order α^2 , i.e. $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$, $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu + \gamma$, $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu + \gamma + \gamma$ and $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu + e^+ + e^-$. For partial decay rates, for the corrections to the electron spectrum in μ decay, etc., the theorem on cancellation of mass singularities is not applicable and we expect terms of order $\alpha^2 \omega^n$

^{*}) Mass singularities and their cancellations in the radiative corrections of order α are discussed in Section 3 of Ref. 4.

with $n > 1$. We emphasize that the subtle cancellations implied by this theorem involve hard as well as soft photons and e^+e^- pairs^{*)}.

Another question one may like to ask is: what is the error arising from the neglect of the finite electron mass in Eq. (3)? This question is clearly not on the same footing as the previous one, because the radiative corrections of order α have been computed exactly and, therefore, can be completely controlled at present^{**)}. We state our result in the form of the following theorem: in the V-A theory, and to first order in α , the contributions to δ_1 arising from the finite electron mass are of order of magnitude $(\alpha/\pi) (m_e/m_\mu)^2 \omega$, $(\alpha/\pi) (m_e/m_\mu)^2$ and smaller^{†)}. Because of the smallness of $(m_e/m_\mu)^2$ we see that the effect of the finite electron mass in the calculation of δ_1 can be safely neglected.

*) For example an approximate calculation for the total rate of $\mu \rightarrow e + \nu_e + \nu_\mu + e^+ + e^-$ using the method of S.G. Eckstein and R.H. Pratt⁷⁾ suggests that this partial decay rate contains terms of order $\alpha^2 \omega^3$. According to the theorem on mass singularities these terms must cancel similar contributions of order α^2 to the total decay rate arising, for example, from the virtual radiative corrections. This is not surprising on the basis of experience gathered in the study of the corrections of order α : for example, to order α the virtual radiative corrections plus soft photons contain terms of order $\alpha \omega^2$ which are cancelled by the contributions from hard real photons.

***) See Behrends et al.⁸⁾ and Grotch⁹⁾. The function D in Eq. (19) of Ref. 8 must be replaced by $D + C$ where C is given in Eq. (C.4) of Ref. 4.

†) The main point of theoretical interest in this theorem is the following: in the charge retention order, the radiative corrections to the axial vector amplitude in μ decay are obtained from the corrections to the vector amplitude by changing the sign of the electron mass. The general argument that leads to this conclusion was given in Section III of Berman and Sirlin¹⁰⁾. Since in the charge retention order there is no interference between the V and A interaction in the calculation of the decay spectrum of unpolarized muons, it follows that in the V-A theory the terms odd in m_e cancel in the corrections to the spectrum. It is still necessary to check that the further integration over the electron momentum to obtain the corrections to the integrated probability does not introduce contributions of order $\alpha (m_e/m_\mu)^n$ with $n < 2$. This we have done by studying the explicit expressions for the corrections to the spectrum as given in Ref. 8.

These discussions lead us to the conclusion that Eq. (3) is very likely a good approximation to the radiative corrections to the total decay probability and that, therefore, Eq. (4) is the best value at present for G_μ , in the framework of the local V-A theory. In the light of the previous discussion, we expect the theoretical error in Eq. (4) to be roughly of relative magnitude $\lesssim 10^{-4}$, which seems to be a reasonable estimate of the neglected contributions of order α^2 .

Recently, δ has been re-evaluated¹¹⁾ by means of an exponentiation method which includes some contributions of order α^2 and higher and the claim has been made that $\delta \approx -2 \times 10^{-3}$ rather than the value given in Eq. (3). In Section 3 we show by direct calculation that the exponentiation method as implemented in Ref. 11 is ambiguous (i.e. the same general arguments may be used to derive very different answers). Furthermore, we point out that this method leads, in its present form, to contributions to the total decay rate of order $\alpha^2 \omega^n$, $0 \leq n \leq 4$, in sharp disagreement with the conclusions obtained in Section 2 from the theorem on cancellation of mass singularities.

2. DETERMINATION OF $\delta_2^{m.s.}$

Suppose that the total decay probability for μ decay is expanded in the form

$$P = \overset{\circ}{P} \left[1 + \frac{e_0^2}{4\pi} M_1 + \frac{e_0^4}{(4\pi)^2} M_2(\Lambda) + \dots \right], \quad (6)$$

where $\overset{\circ}{P}$ is the uncorrected decay probability, e_0 is the bare charge and Λ is an ultraviolet cut-off which will be removed at a later stage of the argument. The reason why a cut-off is necessary in M_2 when the expansion is done in terms of the bare charge is that in the contributions of fourth order there are many graphs involving vacuum polarization. In the pure electrodynamics of electrons and photons, as is well known, the divergences associated with the electron loops are, in effect, "absorbed" in the charge renormalization of lower order diagrams. The same is true in the case of μ decay. (See Fig. 1.) As we have not yet performed the charge renormalization in Eq. (6) and M_1 is finite, a cut-off is clearly needed in M_2 .

The theorem on cancellation of mass singularities as applied to μ decay, asserts that each term in the "unrenormalized" perturbation expansion of Eq. (6) for the total decay probability is free from electron mass singularities^{5,6}). In particular, this is the case for $M_2(\Lambda)$. Let us perform now the conventional charge renormalization:

$$\frac{e_o^2}{4\pi} = \frac{e^2}{4\pi Z_3} = \alpha \left[1 + \frac{2\alpha}{3\pi} \ln\left(\frac{\Lambda}{m_e}\right) + \frac{2\alpha}{3\pi} \ln\left(\frac{\Lambda}{m_\mu}\right) + \dots \right] \quad (7)$$

In Eq. (7) we have only retained those terms of order α^2 which exhibit the ultraviolet divergence and the mass singularity. Inserting Eq. (7) into Eq. (6):

$$P = P^0 \left\{ 1 + \alpha M_1 + \frac{2\alpha^2}{3\pi} \left[2 \ln\left(\frac{\Lambda}{m_\mu}\right) + \omega \right] M_1 + \alpha^2 M_2(\Lambda) + \dots \right\} \quad (8)$$

It is well known that the perturbation expansion for μ -decay in powers of the renormalized charge is finite to all orders^{10,12}). Therefore the term $(4\alpha^2/3\pi) \ln(\Lambda/m_\mu)$ cancels the cut-off dependence of $M_2(\Lambda)$. Equation (8) now becomes

$$P = P^0 \left[1 + \alpha M_1 + \frac{2\alpha^2}{3\pi} \omega M_1 + \alpha^2 M_2^f + \dots \right], \quad (9)$$

where

$$M_2^f = M_2(\Lambda) + \frac{4}{3\pi} \ln\left(\frac{\Lambda}{m_\mu}\right). \quad (10)$$

Note that M_2^f is free from ultraviolet divergence and mass singularities! Thus the only contribution of order α^2 with mass singularity in Eq. (9) is:

$$\delta_2^{m.s.} = \frac{2\alpha^2}{3\pi} \omega M_1 = \frac{2\alpha}{3\pi} \omega \delta_1. \quad (11)$$

Combining Eqs. (3) and (11) we obtain the result of Eq. (5).

The following observations are in order:

- i) Note that the mass singularity is introduced because of our use of conventional charge renormalization [See Eq. (7)]. As pointed out in Ref. 6 it is possible, for theoretical purposes, to obtain a renormalized perturbation expansion free from mass singularity if one renormalizes the photon propagator by choosing a subtraction point $q^2 = M^2 \neq 0$. In our case, however, it is most convenient to perform the conventional charge renormalization because we wish to identify α with the usual fine structure constant.
- ii) In evaluating Eq. (11) we have assumed that the weak vertex is not renormalized in the calculation. Such a procedure is the appropriate one when G_μ in Eq. (1) stands for the "bare" coupling of the weak interactions, "unrenormalized" with respect to electromagnetism. As is well known, a renormalization of the weak vertex is not necessary to obtain finite answers, because the corresponding divergences are automatically cancelled by those arising from the wave function renormalization of the electron and muon fields^{10,12}).
- iii) The method used in this Section to determine $\delta_2^{m.s.}$ is, to a considerable extent, analogous to T. Kinoshita's discussion of the contributions of order α^2 and higher in the calculation of the μ magnetic moment¹³).

3. REMARKS ON THE EXPONENTIATION METHOD

To first order in α , the electron spectrum in the decay of an unpolarized muon can be written as

$$dP = dP^0 \left[1 + \frac{\alpha}{\pi} C_2(E) + \frac{\alpha}{\pi} C_1(E) \ln \left(\frac{q^2}{m_\mu m_e} \right) \right], \quad (12)$$

where dP^0 is the uncorrected spectrum, E is the electron energy and $q^2 = (P_\mu - P_e)^2 = m_\mu^2 + m_e^2 - 2m_\mu E$. The function $C_1(E)$ is given by

$$C_1(E) = \frac{1}{\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) - 2$$

where $\beta = v/c$ is the electron velocity in units of c , and $C_2(E)$ is a very complicated function^{*)} which simply consists of all the corrections of order α (arising from both virtual and real photons) which are not contained in the last term of Eq. (12). It has been known for a long time that the last term in Eq. (12) shows an anomalous behaviour in the immediate vicinity of the point $q^2 = 0$, i.e. at the very tip of the electron spectrum. As q^2 approaches zero, this term becomes very large and negative and it clearly leads to a breakdown of perturbation theory^{**)}. It has been pointed out⁴⁾ that, in practice, this does not cause problems because the interval in which the unphysical behaviour occurs is enormously smaller than the present experimental resolution^{**)}.

If one is interested in the study of the electron spectrum one must consider the convolution of Eq. (12) with the resolution distribution and no problem arises, at least at present. If one measures the lifetime without determining the electron spectrum one can simply integrate Eq. (12) over all electron energies, as the singularity of the last term is of a logarithmic and, therefore, mild and integrable character. Thus, to order α , the corrections δ_1 to the total decay probability, obtained by integrating Eq. (12) over the electron energy, are perfectly finite and well defined.

It is still interesting to inquire what is the mechanism that removes the anomalous behaviour as $q^2 \rightarrow 0$ in the electron spectrum, in the ideal case of perfect electron resolution. As perturbation theory breaks down

*) The function $C_2(E)$ can be determined, for example, by comparing Eq. (12) with the expressions of Ref. 4.

**) The region of the spectrum in which this happens is, however, extremely small. For example the corrections of order α become of magnitude $\gtrsim 1$ roughly in the interval $(E_{\max} - E)/E_{\max} \lesssim e^{-50}$, where $E_{\max} \approx m_\mu/2$ is the maximum possible energy for the electron. The reason for the occurrence of the $\ln q^2$ term is that at $q^2 = 0$ the emission of real photons becomes impossible and, therefore, the infrared divergence associated with the virtual quanta is not cancelled.

in this limit and as the phenomenon in question is clearly related to infrared behaviour, it is extremely plausible that the solution to this problem involves consideration of multiple soft photon emission in those configurations in which the electron energy is very close to the tip of the electron spectrum.

In order to implement this idea the following procedure was proposed in Ref. 11: one rewrites Eq. (12) as

$$dP = dP^0 \left[1 + \frac{\alpha}{\pi} C_1(E) \ln \left(\frac{q^2}{m_\mu m_e} \right) \right] \left[1 + \frac{\alpha}{\pi} C_2(E) \right], \quad (13)$$

and exponentiates the first factor, so that Eq. (13) is replaced by

$$dP_{(1)} = dP^0 \exp \left[\frac{\alpha}{\pi} C_1(E) \ln \left(\frac{q^2}{m_\mu m_e} \right) \right] \left[1 + \frac{\alpha}{\pi} C_2(E) \right]. \quad (14)$$

The unphysical behaviour as $q^2 \rightarrow 0$ has been removed in Eq. (14) and, in fact, the spectrum tends now to zero as $q^2 \rightarrow 0$, a theoretically desirable result. However, numerical evaluations using Eq. (14) lead to results for δ very different from Eq. (3). As these two expressions coincide to order α , one must conclude that Eq. (14) leads to contributions of order α^2 , say, of the same magnitude as those of order α . This apparently mysterious and remarkable phenomenon can readily be understood if one studies the contributions of order α^2 arising from Eq. (14) as a expansion in powers of ω . After performing the integration over the electron energies using the explicit expression for $C_2(E)$ we find that Eq. (14) leads to contributions to P of order $(\alpha/\pi)^2 \omega^n$ with $0 \leq n \leq 4$. (Note that $\omega^4 \approx 8.4 \times 10^2$, which is close to $2\pi/\alpha$.) These observations explain qualitatively why Eq. (14) gives rise to such large corrections of order $(\alpha/2\pi)^2$. Clearly this behaviour is in sharp disagreement with the conclusions obtained from the theorem on mass singularities (see Section 2).

Independently of this theorem it is not difficult to show that the exponentiation method as implemented in Ref. 11 is ambiguous. For example, we can rewrite Eq. (12) as

$$\hat{aP} = \hat{aP} \left[1 + \frac{\alpha}{\pi} \hat{C}_2(E) + \frac{\alpha}{\pi} C_1(E) \ln \left(\frac{q^2}{m_\mu^2 + m_e^2} \right) \right] \quad (15)$$

with

$$\hat{C}_2(E) = C_2(E) + C_1(E) \left[\omega + \ln \left(1 + \frac{m_e^2}{m_\mu^2} \right) \right],$$

and exponentiate as follows:

$$\hat{aP}_{(2)} = \hat{aP} \exp \left[\frac{\alpha}{\pi} C_1(E) \ln \left(\frac{q^2}{m_\mu^2 + m_e^2} \right) \right] \left[1 + \frac{\alpha}{\pi} \hat{C}_2(E) \right]. \quad (16)$$

Equation (16) also removes the anomalous behaviour of the spectrum as $q^2 \rightarrow 0$ and is identical to Eq. (14) to order α . However, detailed examination shows that Eqs. (14) and (16) give rise to very different contributions of order α^2 *). We find by explicit analytical calculation **):

$$\begin{aligned} \frac{P_{(1)} - P_{(2)}}{\hat{P}} &= -2 \left(\frac{\alpha}{\pi} \right)^2 (\omega^4 - 6\omega^3 + 13.8\omega^2 - 12.1\omega) + O(\alpha^3) \\ &= -2.4 \cdot 10^{-3} + \dots \end{aligned} \quad (17)$$

*) In Ref. 11 the claim is made that the ambiguity of the exponentiation method is small, roughly of the order of 1 part in 10^4 . Our detailed result of Eq. (17) obviously disagrees with that statement. We note that Eq. (6.11) of Ref. 11, which was used as a rough estimate of the ambiguity, was greatly underestimated numerically in Eqs. (6.12) and (6.14) of that paper. It is easy to see why Eqs. (14) and (16) give rise to such different corrections of order α^2 : both the second and third terms in the square bracket of Eq. (12) contain terms of order $\alpha\omega^2$, while their sum contains at most contributions of order $\alpha\omega$; instead the second and third terms of Eq. (15) contain only terms of order $\alpha\omega$. When the exponentiation is carried through, this leads to contributions of order $\alpha^2\omega^n$ with $n \leq 4$ in $P_{(1)}$ and $n \leq 2$ in $P_{(2)}$

**) In computing $P_{(1)} - P_{(2)}$ we have included both the contributions of order α^2 in Eqs. (14) and (16) which obtain in the Taylor expansion of the exponentials and the cross terms between the terms of order α in the exponentials and the terms $(\alpha/\pi) C_2(E)$ and $(\alpha/\pi) \hat{C}_2(E)$. These cross terms are negative and large and dominate the positive contributions of the first type. This explains the negative sign in Eq. (17). Should we arbitrarily dismiss the cross terms, the contributions of order α^2 to either $P_{(1)}$ or $P_{(2)}$ are necessarily positive: we find that the ambiguity in this case is still very large but of opposite sign to that given in Eq. (17).

Here $P_{(1)}$ and $P_{(2)}$ refer to the total decay probability calculated on the basis of Eqs. (14) and (16), respectively, and we have neglected terms that vanish as $m_e/m_\mu \rightarrow 0$. We note that the ambiguity in the α^2 terms, as measured by Eq. (17), is of the same order of magnitude as the corrections of order α given in Eq. (3)! **) Although we have chosen to illustrate the ambiguity by means of a simple example, it is clear that the arbitrariness of the method, as applied to μ decay, is of a more general kind. Observe, for instance, that exponentiations such as given in Eqs. (14) and (16) affect to some extent the entire energy spectrum from $q^2 = 0$ to $q^2 = (m_\mu - m_e)^2$ and that in the region where q^2 is significantly different from zero, they involve hard as well as soft photons. We note in passing that contrary to Eq. (14), Eq. (16) leads to contributions of order α^2 to P significantly smaller than the terms of order α , a property which conforms more closely to intuitive expectations **). In our opinion, however, neither Eqs. (14) nor (16) should be used to compute the corrections to P because both contradict in order α^2 the conclusions obtained from the theorem on mass singularities. It is indeed very plausible that some kind of exponentiation involving $(\alpha/\pi) C_1(E) \ln q^2$ actually occurs, but if this is so there should exist additional contributions of order α^2 which cancel the anomalous mass singularities generated to this order by the exponential.

It should be clear from the above discussion that the inclusion of an arbitrarily selected and partial set of terms of order α^2 and higher, involving large powers of ω as in Eqs. (14) and (16) is highly misleading in the evaluation of the corrections to P in μ decay. If one wishes to include terms of higher order it seems necessary in this case either to use a general theorem such as the theorem on mass singularities, which applies to the sum total of all the contributions of a given order, or otherwise to do a fairly complete calculation of these corrections. These observations suggest also the importance of expansions of the form $(\alpha/\pi)^2 \sum_n C_n \omega^n$ in the evaluation of the corrections of order α^2 . Corrections of such order to the energy-angle distribution in μ decay may be indeed very useful for practical as well as theoretical reasons.

**) See previous footnote.

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Figure caption

Fig. 1 : Two related diagrams of second and fourth order. The divergences associated with the e and μ loops in diagram (b) are "absorbed" in the charge renormalization of diagram (a).