POMERANCHUK THEOREM VIOLATION AND NEAR-FORWARD ZEROS
OF THE DIFFERENTIAL CROSS SECTION AT HIGH ENERGIES

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ABSTRACT

It is shown that, if

$$\lim_{s \rightarrow \infty} [\sigma_{\text{tot}}^{\pi^- p}(s) - \sigma_{\text{tot}}^{\pi^+ p}(s)] = \Delta\sigma,$$

$$\lim_{s \rightarrow \infty} [\sigma_{\text{tot}}^{\pi^- p}(s) + \sigma_{\text{tot}}^{\pi^+ p}(s)] / \ln s = 0,$$

and

$$\lim_{s \rightarrow \infty} \text{Inf} [(2/\pi^3) m_\pi^2 (\Delta\sigma)^2 / \sigma_{\text{non-flip}}^{\pi^- p \rightarrow \pi^0 n}(s)] \equiv R,$$

and if isospin invariance holds, then the helicity non-flip differential cross-section ratios

$$[(d\sigma/dt)/(d\sigma/dt)_{t=0}]^{\pi^\pm p \rightarrow \pi^\pm p}$$

must have:

i) zeros at momentum transfers $t \underset{s \rightarrow \infty}{\sim} -\text{const.} (\ln s)^{-2}$ if

$$R \geq 0.865,$$

ii) forward peaks with widths proportional to $(\ln s)^{-2}$ if $R > 1/4$.

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The Serpukhov total cross-section data ¹⁾ suggest a possible violation of the Pomeranchuk theorem ²⁾. Some important consequences of such a violation have been established ^{3),4)}. Further, the Pomeranchuk theorem violation models of Finkelstein ⁵⁾, and Casella ⁶⁾ have raised the important possibility that the differential cross-sections have zeros at high c.m. energies $\sqrt{s}$ at low momentum transfers $t \xrightarrow{s \rightarrow \infty} \text{const}/(\ln s)^2$.

In the present work, we prove that such zeros are indeed implied by sufficiently large violations of the Pomeranchuk theorem. In the case of the particle and antiparticle total cross-sections approaching unequal constants asymptotically, Eden and Kaiser ⁷⁾ have proved that the scattering amplitudes $T(s,t)$ must have an infinite number of zeros t_j in the complex t plane with $|t_j|(\ln s)^2 \xrightarrow{s \rightarrow \infty} \text{const}$, and Auberson, Kinoshita and Martin ⁸⁾ have proved that $[T(s,t)/T(s,0)]$ must approach an entire function of $[t(\ln s)^2]$ with an infinite number of zeros in the left half $[t(\ln s)^2]$ plane. We prove under slightly more general assumptions on the individual total cross-sections that sufficiently large violations of the Pomeranchuk theorem would force some zeros to be in the physical region and hence to show up in the differential cross-section. We first derive the main result needed on $\text{Re}T(s,t)$ and then summarize the consequences. We will assume (i) the analyticity properties and asymptotic bounds given by axiomatic field theory ⁹⁾, (ii) crossing and (iii) unitarity.

Bounds on $\text{Re}T(s,t)$

Consider for simplicity the spinless scattering processes $\pi^+K^+ \rightarrow \pi^+K^+$, $\pi^-K^+ \rightarrow \pi^-K^+$ and $\pi^-K^+ \rightarrow \pi^0K^0$ with amplitudes denoted respectively by $T_1(s,t)$, $T_2(s,t)$ and $T_3(s,t)$ and normalized such that

$$\left(\frac{d\sigma}{d\Omega}\right)_i = \left| \frac{T_i(s,t)}{\sqrt{s}} \right|^2, \quad i = 1, 2, 3. \quad (1)$$

and if asymptotic isospin invariance holds

$$\lim_{s \rightarrow \infty} \sqrt{2} T_3(s, t) / [T_1(s, t) - T_2(s, t)] = 1. \quad (2)$$

Our discussion will be limited to the types of Pomeranchuk theorem violation defined by the following conditions on the asymptotic cross-sections ¹⁰⁾

$$\lim_{s \rightarrow \infty} [\sigma_{\text{tot}}^{\pi^- K^+}(s) - \sigma_{\text{tot}}^{\pi^+ K^+}(s)] = \Delta\sigma \neq 0, \quad (3)$$

$$\lim_{s \rightarrow \infty} [\sigma_{\text{tot}}^{\pi^- K^+}(s) + \sigma_{\text{tot}}^{\pi^+ K^+}(s)] / (\ln s) = 0, \quad (4)$$

and

$$\sigma_3(s) \equiv \int \left(\frac{d\sigma}{d\Omega}(s, \Omega) \right)_3 d\Omega < \text{const.} \quad s \rightarrow \infty \quad (5)$$

Dispersion relations in s then imply that ^{3), 4)}

$$\lim_{\substack{s \rightarrow \pm i\infty \\ t(\ln s)^2 \text{ fixed}}} [T_1(s, t) + T_2(s, t)] / (s \ln s) = 0, \quad (6)$$

$$\lim_{s \rightarrow \pm i\infty} [T_1(s,0) - T_2(s,0)] / (s \ln s) = \frac{\Delta\sigma}{4\pi^2}, \quad (7)$$

and that there must exist sequences of values of $s \rightarrow +\infty$, such that

$$\lim_{\substack{s \rightarrow +\infty \\ t(\ln s)^2 \text{ fixed}}} [T_1(s,t) + T_2(s,t)] / (s \ln s) = 0, \quad (8)$$

and

$$\lim_{s \rightarrow +\infty} [T_1(s,0) - T_2(s,0)] / (s \ln s) = \frac{\Delta\sigma}{4\pi^2}. \quad (9)$$

In this paper, we shall use " $s \rightarrow +\infty$ " to mean $s \rightarrow +\infty$ through sequences such that Eqs. (8) and (9) hold. We then prove the following theorem.

Theorem

If the conditions (2)-(5) hold, then, for real (positive and negative) values of t ,

$$\begin{aligned} B(x) &\equiv 2 \frac{J_1(x)}{x} - \sqrt{\left(\frac{1}{R}-1\right) \left[J_0^2(x) + \left(1 - \frac{4}{x^2}\right) J_1^2(x) \right]} \leq \frac{\operatorname{Re} T_i(s,t)}{\operatorname{Re} T_i(s,0)} \\ &\leq \frac{2 \frac{J_1(x)}{x} + \sqrt{\left(\frac{1}{R}-1\right) \left[J_0^2(x) + \left(1 - \frac{4}{x^2}\right) J_1^2(x) \right]}}{2 \frac{J_1(x)}{x} - \sqrt{\left(\frac{1}{R}-1\right) \left[J_0^2(x) + \left(1 - \frac{4}{x^2}\right) J_1^2(x) \right]}} \equiv A(x), \end{aligned} \quad (10)$$

$\begin{matrix} \leq \\ s \rightarrow +\infty \\ t(\ln s)^2 \text{ fixed} \end{matrix}$

where

$$x \equiv \frac{1}{2m_\pi} \sqrt{-t(\ln s)^2}, \quad (11)$$

$$R \equiv \lim_{s \rightarrow \infty} \text{Inf} \left[\frac{2m_\pi^2}{\pi^3} (\Delta\sigma)^2 / \sigma_3(s) \right], \quad (12)$$

and the $J_0(x)$ and $J_1(x)$ are Bessel functions.

We note that the argument of the square root appearing in Eq. (10) is necessarily positive because of the following result of Roy and Singh ⁴⁾

$$R \leq 1, \quad (13)$$

and necessarily finite because of the assumptions (3) and (5).

Proof of the theorem

We use the partial wave expansions

$$\text{Re } T_3(s, t) = \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l+1) \text{Re } a_l(s) P_l\left(1 + \frac{t}{2k^2}\right), \quad (14)$$

$$\sigma_3(s) = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) |a_l(s)|^2, \quad (15)$$

where the $a_l(s)$ are the charge exchange partial wave amplitudes, and k is the c.m. momentum. Further, the Jin-Martin theorem ⁹⁾ that the amplitudes $T_{1,2}$ satisfy twice subtracted dispersion relations in s for $|t| < 4m_\pi^2$ yields

$$s^2 \geq \lim_{s \rightarrow \infty} T_1(s, t_0) \geq \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l+1) |a_l(s)|^2 P_l\left(1 + \frac{t_0}{2k^2}\right) \equiv G(s, t_0),$$

$$0 < t_0 < 4m_\pi^2. \quad (16)$$

Treating the $a_l(s)$ as variational parameters, it follows from the method of Lagrange multipliers ¹¹⁾ or from the "direct subtraction method" ¹²⁾ that upper and lower bounds on $\text{Re} T_3(s, t)$, given $\text{Re} T_3(s, t=0)$, $\sigma_3(s)$ and $G(s, t_0)$ are obtained by the choice

$$\text{Re } a_l(s) = \frac{\alpha + \beta P_l\left(1 + \frac{t}{2k^2}\right)}{P_l\left(1 + \frac{t_0}{2k^2}\right) + \gamma}, \quad \gamma > -1,$$

$$\text{Im } a_l(s) = 0, \quad (17)$$

where α, β, γ are determined from the given values of $\text{Re} T_3(s, t=0)$, $\sigma_3(s)$ and $G(s, t_0)$ and the solutions with $\beta > 0$ and $\beta < 0$ correspond respectively to the upper and lower bounds. Straightforward evaluation then yields the above theorem for the amplitude T_3 . The corresponding results for T_1 and T_2 follow immediately on using Eqs. (2) and (8).

Generalization to higher spins

This is straightforward in the case of helicity non-flip scattering. In particular, for the cases of πN and KN scattering the results (10)-(12) hold, with (T_1, T_2, T_3) denoting respectively the helicity non-flip amplitudes

$$(T^{\pi^+p \rightarrow \pi^+p}, T^{\pi^-p \rightarrow \pi^-p}, T^{\pi^-p \rightarrow \pi^0 n})$$

and

$$(T^{K^0p \rightarrow K^0p}, T^{\bar{K}^0p \rightarrow \bar{K}^0p}, \sqrt{2} T^{K_L p \rightarrow K_S p}),$$

$\Delta\sigma$ denoting respectively

$$\lim_{s \rightarrow \infty} [\sigma_{tot}^{\pi^-p}(s) - \sigma_{tot}^{\pi^+p}(s)]$$

and

$$\lim_{s \rightarrow \infty} [\sigma_{tot}^{K^0p}(s) - \sigma_{tot}^{\bar{K}^0p}(s)]$$

and σ_3 denoting respectively the helicity non-flip integrated cross-sections, $\sigma^{\pi^-p \rightarrow \pi^0 n}$ and $2\sigma^{K_L p \rightarrow K_S p}$.

Conclusions

The above theorem has the following important consequences:

- i) If the Pomeranchuk theorem violation is maximal, i.e., if $R = 1$, then, for real values of t ,

$$\lim_{\substack{s \rightarrow \infty \\ t(\ln s)^2 \text{ fixed}}} \left[\frac{T_i(s, t)}{T_i(s, 0)} \right] = \lim_{\substack{s \rightarrow \infty \\ t(\ln s)^2 \text{ fixed}}} \left[\frac{\operatorname{Re} T_i(s, t)}{\operatorname{Re} T_i(s, 0)} \right] = 2 \frac{J_1(x)}{x} \quad (18)$$

Adapting the method of Ref. 8) to our case we can show that the left-hand side of (18) is an entire function of x , and

hence the result (18) must hold, by analytic continuation, also for complex values of t . Hence, the amplitude ratios in (18) and, consequently, the asymptotic differential cross-section ratios

$$\left[\frac{(d\sigma_i/dt)}{(d\sigma_i/dt)_{t=0}} \right]$$

have an infinite number of zeros t_j on the negative real t axis given by the zeros of $J_1(x)$; the first zero is at

$$t_1 \simeq - 58.53 \, m_\pi^2 / (\ln s)^2 .$$

Further, the amplitude ratios in (18) have no zeros for complex t and for positive values of t .

- ii) If the Pomeranchuk theorem violation is sufficiently large, such that $R \geq 0.865$, then the upper bound in (10) becomes negative for some negative values of $t(\ln s)^2$ and hence the asymptotic differential cross-section ratios $[(d\sigma_i/dt)/(d\sigma_i/dt)_{t=0}]$ must have at least one zero in the near forward physical region. Larger number of zeros in the physical region can be proved for larger values of R , the number becoming infinite for $R \rightarrow 1$. The first zero ¹³⁾ t_1 of the differential cross-section ratio must obey the following bound

$$(-t_1) \leq_{s \rightarrow \infty} 4 m_\pi^2 x_1^2 / (\ln s)^2 , \quad (19)$$

where x_1 is the smallest positive solution of the equation

$$A(x) = 0 . \quad (20)$$

We note that

$$x_1 \approx 3.832 + 4.756 \sqrt{1-R}, \text{ if } R \approx 1. \quad (21)$$

iii) The near-forward differential cross-sections must obey the bounds

$$\theta(B(x)) B^2(x) \underset{\substack{s \rightarrow \infty \\ t(\ln s)^2 \text{ fixed}}}{\leq} \left[\frac{(d\sigma_i/dt)}{(d\sigma_i/dt)_{t=0}} \right] \underset{\substack{s \rightarrow \infty \\ t(\ln s)^2 \text{ fixed}}}{\leq} \max[A^2(x), B^2(x)] \quad (22)$$

The differential cross-section ratios in (22) can, therefore, have no zero in the region $B(x) > 0$. Further, if it has any zeros, the first zero t_1 must obey the bound

$$(-t_1) \underset{s \rightarrow \infty}{\geq} 4m_\pi^2 y_1^2 / (\ln s)^2, \quad (23)$$

where y_1 is the smallest positive solution of the equation

$$B(y) = 0 \quad (24)$$

We note that

$$y_1 \approx 3.832 - 4.756 \sqrt{1-R}, \text{ if } R \approx 1, \quad (25)$$

and

$$y_1 \approx 2 [12 R]^{1/4}, \text{ if } R \approx 0. \quad (26)$$

The lower bound (23) improves the lower bound given by Eden and Kaiser ⁷⁾. For $R \approx 0$ the improvement is by a factor

$$e^2 \sqrt{24} \sqrt{\frac{\max(\sigma_{tot}^{\pi^+\kappa^+}, \sigma_{tot}^{\pi^-\kappa^+})}{\sigma_3}}$$

The result (23) correspondingly yields a larger domain of validity for Kinoshita's Pomeranchuk-like theorem ³⁾ than that given by Eden and Kaiser ⁷⁾. We now obtain

$$\lim_{\substack{s \rightarrow \infty \\ t(\ln s)^2 \text{ fixed}}} \left[\frac{(d\sigma_1/dt)}{(d\sigma_2/dt)} \right] = 1, \text{ if } (-t)(\ln s)^2 < 4 m_\pi^2 y_1^2. \quad (27)$$

iv) The "diffraction peak width ω_i " for the elastic and charge exchange cross-sections ¹⁴⁾ must obey the bound

$$\frac{(\ln s)^2}{16 m_\pi^2} \left[1 - \sqrt{\frac{1-R}{3R}} \right] \leq \omega_i^{-1} \equiv \left. \frac{d}{dt} \ln \left(\frac{d\sigma_i}{dt} \right) \right|_{t=0} \leq \frac{(\ln s)^2}{16 m_\pi^2} \left[1 + \sqrt{\frac{1-R}{3R}} \right] \quad (28)$$

The bounds (28) show that there must be a peak in the forward direction, i.e., ω_i must be positive if $R > 1/4$.

The width of the peak must be proportional to $1/(\ln s)^2$. Such a shrinkage of the diffraction peak width has been suggested before ^{3),7)}. Now the bounds (28) are the best possible within the present assumptions, because they can be reached by choosing the set of partial waves (12). We, therefore, also conclude that the existence of a peak at $t = 0$ in the differential cross-section or the $(\ln s)^{-2}$ shrinkage of its width as defined ¹⁴⁾ by (28) cannot be proved for sufficiently small Pomeranchuk theorem violation such that $R < 1/4$, without including more specific assumptions or information than in the present work ¹⁵⁾.

Comparison with the experiment

For this we need in addition to precise values for the asymptotic particle-antiparticle total cross-section differences, the high-energy values of the integrated $\pi^- p \rightarrow \pi^0 n$ and $K_L^- p \rightarrow K_S^0 p$ cross-sections. In the absence of such data, we can only remark that the present trend of the data suggests that R might be of the order of unity for πN ⁴⁾ and KN ¹⁶⁾ scattering.

Finally, we note that, in the theoretically interesting case of the total cross-sections attaining the Froissart bound, analogous results on the zeros of $\text{Im}T(s,t)$ can be derived.

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- 13) By "the first zero" we mean "the zero closest to $t = 0$ ".
- 14) This definition of ω_i is taken, for example, in S.W. MacDowell
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 $t = 0$, we are prepared to accept an alternative definition
of the "diffraction peak width", for example, as the momentum
transfer region within which $\left[\frac{d\sigma_i}{dt} \right] / \left(\frac{d\sigma_i}{dt} \right)_{t=0} > 1/e$,
then the works of Kinoshita³⁾ and Eden and Kaiser, Ref. 7),
prove that this width must be proportional to $(\ell \ln s)^{-2}$
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