AN INTRODUCTION TO DUAL RESONANCE MODELS
IN MULTIPARTICLE PHYSICS *)

H. Satz
CERN -- Geneva

*) Lectures given at the Heidelberg-Karlsruhe Summer
Institute in Theoretical Physics, July 1970.

To be published in Springer Tracts in Modern
Physics.

TABLE OF CONTENTS

A.	INTRODUCTION	1
B.	THE MULTIPARTICLE VENEZIANO AMPLITUDE	3
1.	A review of B_4 essentials	3
2.	The construction of B_N	5
3.	Some properties of B_N	10
4.	Koba-Nielsen form and duality transformations	15
5.	Unitarity and other questions	19
C.	MULTIPARTICLE VENEZIANO PHENOMENOLOGY	23
1.	General aspects	23
2.	Classification of reactions	27
3.	Analyses of production experiments	29
4.	Present problems and outlook	35

A. INTRODUCTION

The collision of two hadrons at laboratory energies from a few GeV on up leads about four out of five times to the production of additional hadrons, and the average number of produced particles continues to grow with increasing energy. These empirical facts ¹⁾ make the theoretical description of multiparticle reactions one of the most important tasks of present-day hadron physics.

Unitarity inter-relates elastic and inelastic processes; the strong presence of production reactions thus prevents us from taking the classical road of first "solving" the dynamics in the elastic case, then going on to the more complex many-body problem. In fact, high energy elastic scattering at small momentum transfer appears to be due largely to the shadow of all inelastic reactions, and thus an understanding of multiparticle processes becomes necessary for that of an elastic two-body reaction.

Nevertheless, in many-body problems one still tries to avoid at the beginning the complicated relations provided by unitarity, and thus - hoping for a later "unitarization" - starts generally with a fixed N -body problem uncoupled from other N . Instead of the historical two- and many-body separation, it here turns out to be more fruitful to distinguish between diffractive and non-diffractive processes (Pomeron and non-Pomeron exchange); non-diffractive two-body reactions appear more similar to non-diffractive production reactions than to elastic scattering, and inelastic diffraction dissociation is difficult to accommodate in the usual models for non-diffractive production.

Starting from finite energy sum rules, the study of non-diffractive processes has in the past two years led to the fruitful concept of duality ²⁾ and to the development of the dual resonance model ³⁾⁻⁸⁾ - a crossing symmetric model which contains in a dual fashion both the low energy resonance and the high energy Regge behaviour. In this framework the inherent similarity of non-diffractive processes finds its expression in the so-called "bootstrap consistency" (factorization), which, for any N , allows the construction of the $(N-1)$ particle amplitude from that for N particles. The multiparticle dual resonance

model thus contains, in a way, "more" of the properties one expects of a correct description than any previous model - and the non-trivial nature of such a formulation is perhaps best illustrated by the fact that it is up to now not clear how to extend it from fictitious scalar bosons to physical particles in general.

The subsequent development took two main directions : on the one hand, one tries to use the fact that the narrow width approximation already yields so many of the wanted properties to embed, in a perturbative sense, the Veneziano formula as a Born term in a theory of strong interactions ⁹⁾; on the other hand, with complex trajectories as crude unitarization, the dual resonance model provides (at least in principle) a phenomenological model ¹⁰⁾⁻¹²⁾ incorporating many of the presently known basic empirical features. On a phenomenological level, we also find some attempts to extend duality ^{13),14)} and dual resonance descriptions ¹⁵⁾ to diffractive processes.

Both roads are at present rather blocked; exponential level degeneracy leading to divergent higher order perturbative terms, ghosts, no formulation for fermions, Pomeron questions - there remains more than enough to do. If nothing else, one now knows what the next model has to achieve to be considered as progress.

These lectures are intended as an introduction to the application of dual resonance techniques in multiparticle problems. As mentioned, these applications occur both in the more formal attempts to embed the Veneziano formula in a theory of strong interactions and to obtain a basis for phenomenological considerations. I have therefore attempted to keep the general arguments (Section B) as much as possible separate from the phenomenology (Section C), so that those mostly interested in the formal approach may, at the end of Section B, go on directly to Professor Fubini's lectures. Nevertheless, the twofold purpose will unavoidably have a (hopefully not entirely negative) effect upon the presentation of the subject.

B. THE MULTIPARTICLE VENEZIANO AMPLITUDE

1. A Review of B_4 Essentials

To illustrate the properties we would like our multiparticle amplitude to have, as well as to fix notation and terminology, we begin by briefly reviewing the essentials of B_4 .

Consider a system of neutral bosons, all states of which lie on a single straight Regge trajectory $\alpha(x)$ or on daughters thereof, with lowest state $J^P = 0^+$ of mass μ

$$\alpha(x) = \alpha(0) + \alpha' x \quad ; \quad \alpha(0) = -\mu^2 \alpha' \quad (\text{B-1.1})$$

The Veneziano amplitude ³⁾ for the non-diffractive reaction $0^+ + 0^+ \rightarrow 0^+ + 0^+$ then takes the form (cf. Fig. B-1.1)

$$T_4 = \beta [V(1234) + V(2134) + V(2314)] \quad (\text{B-1.2})$$

with constant β and with

$$V(1234) = B_4(-\alpha(s_{12}), -\alpha(s_{23})), \text{ etc.} \quad (\text{B-1.3})$$

$$B_4(x, y) = \int_0^1 du \, u^{x-1} (1-u)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

where $s_{ij} = (p_i + p_j)^2$.

Fig. B-1.1

We note in particular :

- a) $V(1234)$ is invariant under cyclic and anticyclic permutations of the external lines, since $B_4(x,y)$ remains invariant under the interchange of x and y . The $4! = 24$ permutations of $p_1, \dots, p_4$ are thus divided into three $(24/(4+4) = 3)$ equivalence classes, each of which leaves invariant one of the terms in (B-1.2) and hence can be represented by it or by the corresponding graph of Fig. B-1.1. Because of this cyclic invariance, the fully crossing symmetric amplitude, instead of being a sum of all 24 permutations, is given by the sum (B-1.2) of the three non-equivalent terms.
- b) $B_4(x,y)$ is analytic in x except for poles at $-x=0,1,2,\dots$; because of a), the same is true for the dual variable y . The poles occur since for $-x=0,1,\dots$, the integral (B-1.3) diverges at $u=0$, and for $-y=0,1,\dots$ at $v=(1-u)=0$; because $u=1$ at $v=0$ and vice versa, there can, by construction, be no double (coincident) poles in x and y . We can thus expand $B_4(x,y)$ as a sum of poles

$$B_4(x,y) = \sum_{k=0}^{\infty} R_k(y) \frac{1}{k+x} \quad (\text{B-1.4})$$

$$R_k(y) = (-1)^k \binom{y^{-1}}{k} \quad (\text{B-1.5})$$

with the residue at the k^{th} pole in x being a polynomial of degree k in y . The k^{th} term thus corresponds to the "exchange" of a family of particles (parent and daughters) of spin $j \leq k$. Since by symmetry a) the same holds for a pole expansion in y , we thus have a crossing symmetric resonance saturation picture.

- c) In the limit $\alpha(s) = \alpha(s_{12})$ going to infinity [in both $\text{Re } \alpha(s)$ and $\text{Im } \alpha(s)$] with $t = s_{23}$ fixed, we find from (B-1.2) and (B-1.3)

$$T \rightarrow \frac{\pi}{\Gamma(\alpha(t)-1)} \left[\frac{1 + e^{i\pi\alpha(t)}}{\sin \pi\alpha(t)} \right] [\alpha(s)]^{\alpha(t)} \quad (\text{B-1.6})$$

with analogous results for the limits in the other channels. [The specific approach to infinity with $\text{Im}\alpha(x)$ also diverging is necessary to obtain (B-1.6) and is considered as a "phenomenological unitarization"]. Since the amplitude T thus gives, in a crossing symmetric fashion, both low energy resonance and high energy Regge behaviour, it is in accord with what is generally considered as duality.

Summarizing : the essential points in the construction of a dual resonance amplitude for $N=4$ were invariance under cyclic and anti-cyclic permutations and a (trajectory determined) pole structure without coincident poles in dual variables.

In concluding this part, we make a remark on the uniqueness of the construction. We can replace $B_4(x,y)$ by

$$\begin{aligned}\tilde{B}_4(x,y) &= \int_0^1 du f(u) u^{x-1} (1-u)^{y-1} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta_{mn} B_4(x+m, y+n)\end{aligned}\tag{B-1.7}$$

and not alter the properties a) to c), provided $f(u)=f(1-u)$ and $f(u)$ is well behaved at $u=0$. The series in (B-1.7) contains $B_4(x,y)$ as first term; the absence of the higher ("satellite") terms in (B-1.3) thus is only a hopeful assumption.

2. The Construction of B_N

We now want to extend this picture to arbitrary N , still considering the system of neutral bosons introduced above. As of B_4 , we require of B_N invariance under cyclic and anticyclic permutations of the external lines; the fully crossing symmetric amplitude T_N can then be written as a sum over the $(N-1)!/2$ ($=N!/(N+N)$) non-equivalent permutations P

$$T_N = \beta_N \sum_{\{P\}} B_N(P)\tag{B-2.1}$$

Before going further, we now have to generalize the concept of dual variable. For $N=4$, we had called energy and momentum transfer dual to each other; but equally well we could have defined as dual those Mandelstam variables s_{ij} which in no Feynman tree diagram can have simultaneous poles: this in fact makes s_{12} , s_{23} and s_{13} (s , t and u) mutually dual. With

$$s_{ij} = (p_i + p_{i+1} + \dots + p_j)^2 \quad i < j \quad (\text{B-2.2})$$

the latter definition can be extended directly to general N : s_{ij} and s_{kl} are dual if no Feynman tree diagram with poles in both can be constructed. In practice, the labelling problem is easily solved using "dual diagrams"; associate with each ordered N point function $B_N(P)$ an N sided polygon $Q_N(P)$, with external lines of $B_N(P)$ corresponding to sides of $Q_N(P)$ (cf., Fig. B-2.1). All possible Mandelstam variables are then in a one-to-one correspondence with the diagonals linking non-adjacent vertices of Q_N , the diagonal $(1,3)$ being associated with $s_{13} = (p_1 + p_2 + p_3)^2$, etc. All intersecting diagonals of Q_N are now dual to each other, just as the two diagonals of the square for $N=4$. For each ordering P there remain $(N-3)$ non-intersecting, and hence mutually non-dual, variables which can have

Fig. B-2.1

coincident poles. For these, any $(N-3)$ non-intersecting variables can be chosen (cf., Figs. B-2.2 and B-2.3); the "multiperipheral" form shown in Fig. B-2.3 is, however, particularly easy to visualize and the set s_{ij} , $j=2,3,\dots,(N-2)$ is therefore often used. We return to this freedom of choice in subsection 4.

Fig. B-2.2

Fig. B-2.3

Introducing for each s_{ij} a trajectory

$$x_{ij} = \alpha(s_{ij}) = \alpha(0) + \alpha' s_{ij} \quad i < j \quad (\text{B-2.3})$$

and its conjugate variable u_{ij} , the essential problem now becomes quite clear : to write down an integral over the $(N-3)$ mutually non-dual u_{ij} such that we have : i) poles as fixed by $\alpha(x)$ for all corresponding x_{ij} ; ii) symmetry under cyclic and anticyclic permutations of the external lines, and iii) no coincident poles in dual variables.

Let us try to develop a pattern of solutions. We had

$$B_4(x_{12}, x_{23}) = \int_0^1 du_{12} \int_0^1 du_{23} u_{12}^{x_{12}-1} u_{23}^{x_{23}-1} \delta(u_{12} + u_{23} - 1) \quad (\text{B-2.4})$$

where the auxiliary variable u_{23} dual to u_{12} is fixed by forbidding coincident poles. Similarly we can write

$$B_5(x_{12}, x_{23}, x_{34}, x_{45}, x_{51}) = \int_0^1 du_{12} \int_0^1 du_{23} \int_0^1 du_{34} \int_0^1 du_{45} \int_0^1 du_{51} * \\ * u_{12}^{x_{12}-1} u_{23}^{x_{23}-1} u_{34}^{x_{34}-1} u_{45}^{x_{45}-1} u_{51}^{x_{51}-1} G(u_{12}, u_{23}, u_{34}, u_{45}, u_{51}) \quad (\text{B-2.5})$$

where $G(u_{12}, \dots, u_{51})$ now has to express the absence of poles in dual variables in such a way as to render B_5 invariant under cyclic and anticyclic permutations of external lines. We thus choose a particular set of the $N-3=2$ mutually non-dual u_{ij} , say u_{12} and $u_{13}=u_{45}$ (from the corresponding polygon it is easy to see that all $u_{i,i+1}$ with a common index are dual to each other). The three auxiliary variables, here u_{23} , u_{34} and u_{51} , are then in fact fixed by the requirement of no double poles in dual variables: when $u_{12}=0$ (pole in x_{12}), we must have u_{51} and $u_{23} \neq 0$, etc. The resulting conditions are fulfilled if

$$u_{12} = 1 - u_{51} u_{23} \quad (B-2.6a)$$

$$u_{23} = 1 - u_{12} u_{34} \quad (B-2.6b)$$

$$u_{34} = 1 - u_{23} u_{45} \quad (B-2.6c)$$

$$u_{45} = 1 - u_{34} u_{51} \quad (B-2.6d)$$

$$u_{51} = 1 - u_{45} u_{12} \quad (B-2.6e)$$

Of these relations, only three are independent; for our choice of u_{12} and u_{45} as independent variables, we find that (B-2.6b), (B-2.6c) and (B-2.6e) alone give

$$u_{23} = (1 - u_{12}) / (1 - u_{12} u_{45}) \quad (B-2.7a)$$

$$u_{34} = (1 - u_{45}) / (1 - u_{12} u_{45}) \quad (B-2.7b)$$

$$u_{51} = (1 - u_{45} u_{12}) \quad (B-2.7c)$$

and therewith solve, as is easily verified, all five relations (B-2.6). Hence we find as solution :

$$G(u_{12}, u_{23}, \dots, u_{51}) = \delta(u_{23} + u_{12} u_{34} - 1) \delta(u_{34} + u_{23} u_{45} - 1) \delta(u_{51} + u_{45} u_{12} - 1) \quad (B-2.8)$$

and with it the Bardakçi-Ruegg-Virasoro form ^{4),5)} of B_5

$$B_5(12345) = \int_0^1 du_{12} du_{45} u_{12}^{x_{12}-1} u_{45}^{x_{45}-1} (1-u_{12})^{x_{23}-1} (1-u_{45})^{x_{34}-1} (1-u_{12} u_{45})^{x_{51}-x_{23}-x_{34}+1} \quad (B-2.9)$$

which was the first generalization of the Veneziano amplitude to $N > 4$.

Let us now denote $x_{12}, \dots, x_{51}$ and $u_{12}, \dots, u_{51}$ by x_k and u_k , with $k=1, \dots, 5$; we can then write

$$B_5(x_1, \dots, x_5) = \int \prod_{k=1}^5 du_k u_k^{x_k-1} \prod_{\bar{k}=1}^5 \delta(u_{\bar{k}} + u_{\bar{k}-1} u_{\bar{k}+1} - 1) \quad (B-2.10)$$

where we take $u_0 = u_5$, $u_6 = u_1$ for compact notation. The second (primed) product in the integral is defined to run over all u_k except the two mutually non-dual (and hence non-adjacent) chosen as independent variables. The argument of the delta-function is of the form "variable plus product of all dual variables minus one". This formulation, the Chan form ⁶⁾ of B_5 , clearly exhibits both the invariance under cyclic and anticyclic permutations as well as the absence of double poles in dual variables.

From (B-2.10) we can now obtain in a straightforward fashion the corresponding Chan form for arbitrary N ^{6), 7)}. As for $N=4$ and 5 , we write an integral over all $R_N = N(N-3)/2$ conjugate Mandelstam variables of the problem, denoted by u_k , $k=1, 2, \dots, R_N$:

$$B_N(x_1, \dots, x_{R_N}) = \int \prod_{k=1}^{R_N} du_k u_k^{x_k-1} \left\{ \prod_{\bar{k}=1}^{R_N} \delta(u_{\bar{k}} + \prod_{\bar{\bar{k}}=1}^{R_N} u_{\bar{\bar{k}}} - 1) \right\} \quad (B-2.11)$$

where the primed product runs over all $\bar{k}$ except the $N-3$ mutually non-dual independent variables (whichever one we have chosen), and the doubly primed over all variables dual to $\bar{k}$. As before, the $u_k^{x_k-1}$ bring in the pole structure, while the product of delta functions enforces the absence of coincident poles in dual variables, thereby determining the $(N-2)(N-3)/2$ auxiliary variables. The invariance under cyclic and anti-cyclic permutations is assured by the invariance of the duality relations under these operations.

To obtain a specific representation, one can now carry out the integrations over all delta-functions. Choosing as independent variables the $(N-3)$ multiperipheral u_{1j} , $j=2, 3, \dots, N-2$ and defining

$$\omega_{qr} = u_{1,q} u_{1,q+1} \cdots u_{1,r} \quad q < r \quad (\text{B-2.12}),$$

we arrive at the generalized Bardakçi-Ruegg form ⁴⁾

$$\begin{aligned} B_N(1 \cdots N) = & \int_0^1 du_{12} du_{13} \cdots du_{1,N-2} u_{12}^{x_{12}-1} \cdots u_{1,N-2}^{x_{1,N-2}-1} * (1-u_{12})^{x_{23}-1} * \\ & (1-u_{13})^{x_{34}-1} \cdots (1-u_{1,N-2})^{x_{N-2,N-1}-1} * (1-\omega_{23})^{-2(p_2 p_4) \alpha' + \alpha_0 + \alpha'} * \\ & \cdots (1-\omega_{N-3,N-2})^{-2(p_{N-3} p_{N-1}) \alpha' + \alpha_0 + \alpha'} * (1-\omega_{24})^{-2(p_2 p_5) \alpha'} \cdots \\ & \cdots (1-\omega_{N-4,N-2})^{-2(p_{N-4} p_{N-1}) \alpha'} * \cdots * (1-\omega_{2,N-2})^{-2(p_2 p_{N-1}) \alpha'} \end{aligned} \quad (\text{B-2.13})$$

which has been used extensively in the first investigations of level structure ^{16),17)}; we shall come to this in the next subsection, and return to the question of other equivalent representations of B_N in subsection 4.

3. Some Properties of B_N

Let us consider the simpler case $N=5$. Expanding the Bardakçi-Ruegg form (B-2.9) about $u_{12} u_{45} = 0$, we obtain after integration ¹⁸⁾

$$B_5(x_{12}, \dots, x_{51}) = \sum_{k=0}^{\infty} (-1)^k \binom{x_{51} - x_{23} - x_{34}}{k} B_4(x_{12} + k, x_{23}) B_4(x_{45} + k, x_{34}) \quad (\text{B-3.1})$$

which, incidentally, is the form usually used for computer calculations of B_5 . For $x_{12} \rightarrow 0$ it yields

$$\begin{aligned} B_5(x_{12}, \dots, x_{51}) = & \frac{1}{x_{12}} \left[\sum_{k=0}^{\infty} (-1)^k \binom{x_{51} - x_{23} - x_{34}}{k} B_4(x_{45} + k, x_{34}) \frac{x_{12}}{x_{12} + k} * \right. \\ & \left. * \frac{\Gamma(x_{12} + k + 1) \Gamma(x_{23})}{\Gamma(x_{12} + x_{23} + k)} \right]_{x_{12}=0} = \frac{1}{x_{12}} B_4(x_{45}, x_{34}) \end{aligned} \quad (\text{B-3.2})$$

i.e., by restricting ourselves to the first pole of any variable, we regain the Veneziano four-point formula (cf., Fig. B-3.1). Thus we

Fig. B-3.1

first encounter here the bootstrap consistency mentioned in the Introduction : if we know the amplitude for $N=5$, it is uniquely fixed for $N=4$ as residue of the first pole.

In general we have with $x_{12} \rightarrow -k$

$$B_5(x_{12}, \dots, x_{51}) = \frac{1}{x_{12} + k} \left[(-1)^k \binom{x_{51} - x_{23} - x_{34}}{k} B_4(x_{45} + k, x_{34}) \right] \quad (B-3.3)$$

and can thus write analogously to (B-1.4) :

$$B_5(x_{12}, \dots, x_{51}) = \sum_{k=0}^{\infty} C_k(x_{23}, \dots, x_{51}) \frac{1}{x_{12} + k} \quad (B-3.4)$$

We see from (B-3.3) that the residues $C_k(x_{23}, \dots, x_{51})$ are polynomials of degree k in the "angle" variables x_{51} and x_{23} (cf., Fig. B-3.1), and thus the k^{th} pole corresponds, as in (B-1.5), to the exchange of a family of particles with spins from k down to zero (mother plus daughters).

By requiring bootstrap consistency in general (not only for the first poles or, equivalently, scalar external particles), we can obtain through factorization from (B-3.1) the Veneziano amplitude for three external particles of spin zero and one of spin j . For a vector and three scalar particles, we write (B-3.1)

$$[p_{(1)}^\mu - p_{(2)}^\mu] B_{\mu}^{(4)}(p_1, p_4, p_5) \quad (B-3.5)$$

to find for the ordering of Fig. B-3.1 through antisymmetrization in 1 and 2

$$B_{\mu}^{(4)}(p_3, p_4, p_5) = [p_{\mu}^{(3)} + p_{\mu}^{(5)}] B_4(x_{45}+1, x_{34}) \quad (B-3.6)$$

and, similarly for other orderings. This then yields

$$\begin{aligned} T_{\mu} &\equiv p_{\mu}^{(3)} A_1 + p_{\mu}^{(4)} A_2 + p_{\mu}^{(5)} A_3 \\ &= p_{\mu}^{(3)} [B_4(x_{34}, x_{45}) + B_4(x_{35}, x_{45})] + p_{\mu}^{(4)} [B_4(x_{34}, x_{35}) + \\ &\quad B_4(x_{35}, x_{45})] + p_{\mu}^{(5)} [B_4(x_{34}, x_{35}) + B_4(x_{34}, x_{45})] \end{aligned} \quad (B-3.7)$$

as the full Veneziano amplitude to be contracted with the polarization vector ϵ_{μ} of the external vector particle.

Finally, let us look at the various "high energy" limits of B_5 . Using standard formulae for beta and gamma functions, one can rewrite (B-3.1) in the form¹⁹⁾

$$B_5(12345) = B_4(12, 51) B_4(23, 34) {}_3F_2(-45+12+23, 51, 34; 12+51, 23+34; 1) \quad (B-3.8)$$

where ij is used to denote x_{ij} and

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)}{\Gamma(\alpha_1)} \frac{\Gamma(\alpha_2+n)}{\Gamma(\alpha_2)} \frac{\Gamma(\alpha_3+n)}{\Gamma(\alpha_3)} \frac{\Gamma(\beta_1)}{\Gamma(\beta_1+n)} \frac{\Gamma(\beta_2)}{\Gamma(\beta_2+n)} \frac{z^n}{n!} \quad (B-3.9)$$

is a generalized hypergeometric function. Using its limiting properties, one obtains for x_{45} and $x_{12} \rightarrow \infty$, x_{23} and $\eta = x_{12}/x_{45}$ fixed ("single Regge limit", cf., Fig. B-3.2)

$$B_5(1 \cdots 5) = B_4(45, 51) R(23, 34) \quad (B-3.10)$$

Fig. B-3.2

Here $B_4(45,51)$ is the Veneziano amplitude for the reaction $4+5 \rightarrow (23)+1$ [which, as we know, gives the usual $(x_{45})^{x_{51}} * f(x_{51})$ for large x_{45}] while the vertex function

$$R(23,34) = \sum_{m=0}^{\infty} \frac{C_m(34, \eta)}{x_{23} + m} \quad (\text{B-3.11})$$

is expressed as a sum over resonances in the (23) system. We thus find in fact the correct behaviour in the single Regge limit. For x_{45} , x_{12} and $x_{23} \rightarrow \infty$ with $\kappa = x_{12}x_{23}/x_{45}$ fixed ("double Regge limit") one finds in a similar fashion

$$\begin{aligned} B_5(1 \dots 5) &= B_4(45,51) B_4(23,34-51) f(34,51, \kappa) \\ &\simeq x_{12}^{x_{51}} x_{23}^{x_{34}} \tilde{f}(34,51, \kappa) \end{aligned} \quad (\text{B-3.12})$$

and thus the usual double Regge form. It should be noted that both Regge limits of B_5 introduce a well-defined dependence on the Toller angle ($\sim \eta, \kappa$); for details, see Ref. 19).

Thus our B_5 does have the essential properties we had wanted to generalize from B_4 : besides the dual pole structure, we find residues polynomial in angle variables and thus correct spin structure, factorization and thus bootstrap consistency, and finally high energy Regge behaviour. This all remains true for arbitrary N ; rather than repeating it here for the general case, we refer the reader to Ref. R1) and the original literature quoted there. Here we shall restrict ourselves to a rather qualitative introduction to some aspects of factorization.

We had seen for B_5 that at the first pole in any $x_{i,i+1}$ the residue factorizes into a part depending only on particles i and $i+1$ (here simply a constant) and a part depending on the other three particles (namely B_4). How does this look in general at higher poles? Consider the residue of B_N for arbitrary N at the pole $\alpha(s_{1i}) = -n$, $s_{1i} = (p_1 + \dots + p_i)^2$ (cf., Fig. B-3.3). The question is: can this residue be written as the sum of no less than $z(n)$ terms, with $z(n)$ independent of N and i , such that each term is the product of one factor depending only on the momenta of the external particles $1, \dots, i$ and one depending on $i+1, \dots, N$. If this is indeed the case, we can speak of factorization with $z(n)$ -fold degenerate level structure.

Fig. B-3.3

It has been shown ^{16), 17)}: a) B_N does in fact provide such a factorization; b) the number of degenerate levels grows as

$$z(n) \sim \exp a\sqrt{n} \quad (B-3.13)$$

i.e., the number of states at fixed large n grows exponentially with $\sqrt{n}$; since the number of spin states contributing to each n grows only linearly in n , the same is true generally for a spin j part of pole n ; c) there are in general ghosts, that is, in the sum corresponding to the factorized residue there are negative terms (negative coupling). It should be noted that both b) and c) are not true for the parent trajectory - its contributions are ghost-free and non-degenerate.

The further discussion of these questions forms an important part of the perturbative approach ⁹⁾, R2-R4) to a Veneziano "theory" and will presumably be considered in Professor Fubini's lectures. Let me just emphasize here that both factorization and the result (B-3.13) are first instances of some type of unitarity-like relation between amplitudes of different particle numbers, e.g., between B_4 and B_N : factorization requires B_N to be of such a form as to yield B_4 as residue of the proper pole, while the level degeneracy of a four-point amplitude can be determined only by studying factorization of the amplitude for arbitrary N .

In closing this Section, we remark that the satellite ambiguity found for $N=4$ also persists at higher N , although factorization here imposes some further restrictions ²⁰⁾ on the weight corresponding to $f(u)$ in (B-1.7).

4. Koba-Nielsen Form and Duality Transformations

Let us for the moment return to the four-point function

$$B_4(x_{12}, x_{23}) = \int_0^1 du_{12} u_{12}^{x_{12}-1} (1-u_{12})^{x_{23}-1} \quad (B-4.1)$$

We had suggested that the essential properties of this amplitude were :
a) invariance under cyclic and anticyclic permutations and, b) pole structure without coincident poles in dual variables.

Consider now the function

$$\tilde{B}_4(x_{12}, x_{23}) = \int_0^1 du_{12} e^{u_{12}(x_{12}+x_{23})} \quad (B-4.2)$$

It is invariant under cyclic and anticyclic permutations of the external lines, but not dual : it allows in neither x_{12} nor x_{23} a pole expansion with residues polynomial in the dual variables, does not give Regge behaviour, etc. On the other hand

$$\tilde{\tilde{B}}_4(x_{12}, x_{23}) = \int_0^1 du_{12} u_{12}^{x_{12}-1} (1-u_{12})^{x_{23}-1} f(u_{12}) \quad (B-4.3)$$

with $f(u) \neq f(1-u)$ is dual in the above sense, yet not invariant under cyclic and anticyclic permutations of the external lines. Hence these properties are independent.

The invariance of (B-4.1) under the permutation corresponding to $x_{12} \rightleftharpoons x_{23}$ is assured by the invariance of the integrand under the transformation $u \rightleftharpoons (1-u)$. Similarly, all other permutations can be associated with transformations of the integration variable ^{R4)}. This connects cyclic invariance and thus finally crossing symmetry with the invariance of the integrand under transformations of a permutation group.

The second property ("duality") is satisfied since u_{23} , the conjugate variable of x_{23} , is with

$$u_{23} = 1 - u_{12} \quad (\text{"duality condition"}) \quad (\text{B-4.4})$$

chosen such that it is unity when u_{12} vanishes. We can now ask whether it is possible to associate this property also with an invariance under some group.

This question finds at least a partial answer when one rewrites B_4 in terms of anharmonic ratios [Koba-Nielsen form ⁸⁾]. We associate with the cut x_{23} of Fig. B-4.1 the anharmonic ratio

$$Z(23,41) = \frac{\rho_1 - \rho_4}{\rho_1 - \rho_3} * \frac{\rho_2 - \rho_3}{\rho_2 - \rho_4} \quad (\text{B-4.5})$$

Fig. B-4.1

where each p_i is an arbitrary real number associated with external particle i ; in the numerator of (B-4.5) appears the difference of pairs on each side of the cut, in the denominator the "crossed" differences as seen from top or bottom.

The anharmonic ratio corresponding to the cut x_{12} then is

$$z(12,34) = \frac{p_2 - p_1}{p_2 - p_4} * \frac{p_3 - p_4}{p_3 - p_1} = 1 - z(23,41) \quad (\text{B-4.6})$$

and thus $z(12,34)$ and $z(23,41)$ fulfil the duality relation for the conjugate variables of x_{12} and x_{23} . We can therefore write

$$B_4(x_{12}, x_{23}) = \int_0^1 dz(12,34) [z(12,34)]^{x_{12}-1} [z(23,41)]^{x_{23}-1} \quad (\text{B-4.7})$$

Now, it can easily be shown that the anharmonic ratios are invariant under projective transformations, i.e.,

$$z(12,34) = z(p_1, p_2; p_3, p_4) = z(p'_1, p'_2; p'_3, p'_4) \quad (\text{B-4.8})$$

with

$$p'_i = \frac{ap_i + b}{cp_i + d} \quad ; \quad ad - bc \neq 0 \quad (\text{B-4.9})$$

Since, on the other hand, anharmonic ratios form in the Veneziano model the solution of the "duality condition", the group $L(2, R)$ of projective transformations appears to be closely related to what we consider as duality. The study of these "duality transformations" thus forms a rather central part of much present work on the formal aspects of the Veneziano model and may perhaps lead to a better understanding of what is really meant by duality.

The requirement $0 \leq z(12,34) \leq 1$ for the region of integration implies an ordering for the p_i ; we must have either $p_1 < p_2 < p_3 < p_4$ or $p_4 < p_3 < p_2 < p_1$. Now one can always find a projective transformation which takes any given $\{p_i\}$ of this order into $p_1 = 0$, $p_3 = 1$, $p_4 = \infty$; hence we can write

$$B_4(x_{12}, x_{23}) = \int_0^1 d\rho_2 [z(12, 34)]^{x_{12}-1} [z(23, 41)]^{x_{23}-1} \quad (B-4.10)$$

Another, formally more general form, would be

$$B_4(x_{12}, x_{23}) = \frac{1}{V} \int_{-\infty}^{\infty} \prod_{k=1}^4 d\rho_k \theta(\rho_k - \rho_{k-1}) [z(12, 34)]^{x_{12}-1} [z(23, 41)]^{x_{23}-1} \quad (B-4.11)$$

Because of the invariance of anharmonic ratios under $L(2, R)$ this implies, however, an infinite normalization volume V .

In closing this subsection, we briefly sketch how the general N particle duality condition

$$(u_p + \prod_{\bar{p}} u_{\bar{p}} - 1) = 0 \quad (B-4.12)$$

[cf., (B-2.11)] is also solved by anharmonic ratios. For this, we associate with an x_{ij} cut (cf., Fig. B-4.2) the anharmonic ratio

$$u_{ij} \equiv z(i+1, j-1, j, i) = \frac{\rho_i - \rho_j}{\rho_i - \rho_{j-1}} * \frac{\rho_{i+1} - \rho_{j-1}}{\rho_{i+1} - \rho_j} \quad (B-4.13)$$

and obtain from it a factor $(u_{i,j})^{x_{i,j}-1}$ in the integrand of B_N . As before, a pole in $x_{i,j}$ will result from $u_{i,j} = 0$. From (B-4.13) this means either $\rho_i = \rho_j$ or $\rho_{i+1} = \rho_{j-1}$. A generalization of the $\rho_i > \rho_{i-1}$ ordering found for $N=4$ then requires $\rho_i = \rho_j = \rho_r$, $i \leq r \leq j$. This means for a dual variable, e.g., $x_{i+1, j+1}$ (cf., Fig. B-4.2) that

$$u_{i+1, j+1} = \frac{\rho_{i+1} - \rho_{j+1}}{\rho_{i+1} - \rho_j} * \frac{\rho_{i+2} - \rho_j}{\rho_{i+2} - \rho_{j+1}} = 1 \quad (B-4.14)$$

Fig. B-4.2

and hence prevents any simultaneous poles in variables dual to $x_{i,j}$. The $u_{i,j}$ defined by (B-4.13) thus fulfil by construction the N particle duality condition.

For the further development of the N particle Koba-Nielsen form (construction of the integration measure), we refer the reader to Refs. 8), 21).

5. Unitarity and Other Questions

The most striking shortcoming of the Veneziano amplitude is that as narrow resonance approximation it violates unitarity already in giving rise to unbounded partial waves. Essentially three different approaches towards an improvement of the unitarity situation for N point functions have so far been suggested.

- a) "Complex trajectories" : this is what is used in Veneziano phenomenology - one chooses the trajectory $\alpha(x)$ complex when x is an energy, with $\text{Im } \alpha(x)$ increasing with x and determined such as to reproduce correctly all observed resonance widths. This trick of course takes the poles off the real axis in the physical region and at the same time assures correct high energy Regge behaviour.

- b) "Smoothing Veneziano" : one can consider B_4 as a distribution to be smeared out by convolution with a suitable weight function ϕ to obtain, e.g., ²²⁾

$$W_4(x_{12}, x_{23}) = \int_{x_0}^1 dx \phi(x) B_4(xy_{12}, xy_{23}) \quad (B-5.1)$$

$$y_{12} = \alpha(s_{12} - 4/3) = x_{12} - 4\alpha'/3, \text{ etc.}$$

as new four-point amplitude. While this procedure also moves the poles out of the physical region, it modifies the high energy Regge behaviour.

Both a) and b) are essentially phenomenological approaches in the sense that although they make the partial waves finite, they do not indicate how the actual unitarity condition expressing the inter-relation of amplitudes for all particle numbers is to be solved.

- c) "Perturbative unitarization" : this much more ambitious approach considers, as pointed out in the Introduction, the Veneziano amplitude as a skeleton theory in the Born term sense ⁹⁾, for which one should now calculate self-energy-like corrections - which should then in principle provide both finite width resonances and the right unitarity coupling of all channels. Whether or not this can be done, appears at present completely open.

In summarizing, it can be noted that unitarity remains perhaps the most fundamental open problem in the multiparticle Veneziano model - just as it was in previous attempts (multiperipheral model, multi-Regge model).

The other problems we want to consider here are concerned with the extension of the dual resonance picture to physical particles : the fermion question and the problem arising from spin one lowest particle states.

So far, no (fully satisfactory) formulation of a Veneziano model for a reaction involving fermions has been given (e.g., for pion-nucleon scattering). One can, of course, take a Veneziano form for an invariant amplitude free of kinematical singularities, similar to what is done in

a Regge picture. While this, in fact, leads to the desired pole structure, the straight line trajectories in s , t and u give rise to an amplitude invariant under change of sign of the amplitude $W = \sqrt{s}$ and thus by MacDowell symmetry ²³⁾ to parity doubling. On the other hand, trajectories of the form $\alpha(x) = \alpha_0 + \alpha' x^\nu$ with $\nu < 1$ no longer give rise to residues polynomial in the dual variable and hence destroy the spin structure ("ancestors"); they also modify the high energy Regge behaviour.

Finally we want to illustrate, by considering a specific example, the type of difficulties that arise when one tries to extend the B_N constructed above even to physical bosons. Let us look at the amplitude for the $\sigma 4\pi$ system ²⁴⁾⁻²⁶⁾, where σ denotes a $J^P = 0^+$ boson; the relevant trajectories are shown in Fig. B-5.1. If we take $B_5(\alpha_\pi^{12}, \alpha_\rho^{23}, \alpha_\rho^{34}, \alpha_\rho^{45}, \alpha_\pi^{51})$ as Veneziano form for this amplitude, with $\alpha_\rho(x) = \alpha_0 + \alpha' x$, $\alpha_0 = 1 - \alpha' m_\rho^2$ and with physically reasonable values of

Fig. B-5.1

α' and m_ρ , then we introduce a spurious state of negative mass at $\alpha_\rho(x) = 0$. If we try to circumvent this by starting the rho trajectory at one, $B_5(\alpha_\pi^{12}, \alpha_\rho^{23-1}, \alpha_\rho^{34-1}, \alpha_\rho^{45-1}, \alpha_\pi^{51})$, then we have to multiply, e.g., by α_π^{12} to obtain indeed spin one at the $\alpha_\rho^{23} = 1$ pole; but then the $\alpha_\pi^{51} = 0$ pion pole is not of spin zero, as it should be.

A possible solution is given by ²⁵⁾⁻²⁶⁾ : $\alpha_\rho^{34} B_5(\alpha_\pi^{12}, \alpha_\rho^{23-1}, \alpha_\rho^{34-1}, \alpha_\rho^{45-1}, \alpha_\pi^{51})$, since the multiplicative factor kills the $\alpha_\rho^{34} = 0$ ghost and provides in fact the correct spin for all the rho poles without altering that of the pion poles.

We can see from this, and similar work on N pion amplitudes²⁷⁾, that even the construction of the Veneziano N point function for general physical bosons is quite far from being a trivial extension of (B-2.11).

C. MULTIPARTICLE VENEZIANO PHENOMENOLOGY

1. General Aspects

One of the main reasons for the great interest in the Veneziano model is, as already mentioned, that by introducing a crude "unitarization" in form of complex trajectories, one arrives at a model in principle directly applicable to the description of experiments - a model which contains low energy resonance and high energy Regge behaviour in a dual (and crossing symmetric) picture and thus exhibits already many features that one would want to find in a theoretical description of production processes.

The question then becomes : applicable to what reactions, and with what further approximations ? That some further ones are necessary is evident : all measurable hadron processes involve fermions, for which, as we saw above, no Veneziano model free from parity doubling has so far been constructed. So we know there will be spin approximations.

Furthermore, we have the problem of how to deal with diffractive processes in such a picture, i.e., how to combine dual resonance models and Pomeron exchange. The reason why this question is of such considerable importance in the actual analysis of reactions is that there are very few processes in which Pomeron exchange contributions can be excluded; and these are not the most common or the best measured ones.

That the Pomeron plays a different role in any dual picture is clear from the fact that it is also dominantly present in reactions which, to our present knowledge, do not contain direct channel resonances (e.g., elastic PP or K^+P scattering) - so it cannot be built up from resonances as are the normal Regge trajectories. There are attempts^{13),14)} to incorporate the Pomeron in a dual scheme by relating it to non-resonant background, e.g., in FESR form

$$\int_{\nu_0}^N d\nu \, \Im A(\nu, t) = \int_{\nu_0}^N d\nu \, \left[\Im A_{\text{Res}}(\nu, t) + \Im A_{\text{Bg}}(\nu, t) \right] \quad (C-1.1)$$

$$\int_{\nu_0}^N d\nu \, \gamma_m A_{Res}(\nu, t) \sim \beta(t) N^{\alpha_x(t)+1}$$

$$\int_{\nu_0}^N d\nu \, \gamma_m A_{Bg}(\nu, t) \sim \gamma(t) N^{\alpha_P(t)+1}$$

with α_x and α_P denoting normal and Pomeron trajectory, respectively. This conjecture is in accord with the approximate constancy of total PP and K^+P cross-sections, with reality properties of some amplitudes at high energies, etc.; nevertheless, it is more a reformulation of where one should expect resonance saturation to work, more a restriction on the "usual" duality picture, than an actual tool in the analysis of Pomeron dominated reactions, the non-resonant background being a rather elusive thing. Moreover, it has recently been shown that in a crossing symmetric picture, the duality between Pomeron and non-resonant background is not compatible with resonance saturation²⁸⁾; if non-resonant background is present in Pomeron channels, crossing puts it also in channels where no Pomeron is possible, i.e., channels which should be resonance saturated.

So we find that up to now the role of the Pomeron in the duality picture remains rather mysterious. In the remainder of this more general part we want to consider the consequences of the absence of exotic states and then the extension of B_N to particles with isospin.

By exotic states we mean mesons which cannot be accommodated in the usual $SU(3)$ classification scheme [mesons as $SU(3)$ singlets or octets and baryons as singlets, octets, or decuplets] - or, in other words, mesons or baryons that cannot be represented as quark-antiquark ($q\bar{q}$) or three quark (qqq) system, respectively. The fact that such states have so far not been observed leads us to postulate their absence in all channels and derive from this condition on non-diffractive scattering amplitudes^{29),30)}. Since in a resonance saturation picture the imaginary part of the amplitude is built up from direct channel resonances, the amplitude must be purely real if the direct channel is exotic. Since the real part of the amplitude is built up from the resonances in crossed channels, the amplitude must vanish if these are also exotic.

For actual checks, the so-called duality diagrams of Harari²⁹⁾ and Rosner³⁰⁾ prove to be quite helpful. One represents each particle in the process by its quark picture (oppositely directed arrows for q and $\bar{q}$). If the reaction can be drawn such that : i) no quark lines cross, ii) no lines start and end at the same particle, and iii) any "cut" results in a permissible quark state ($q\bar{q}$, qqq , $\bar{q}\bar{q}\bar{q}$), then the corresponding permutation contributes to the amplitude, otherwise not (cf., Fig. C-1.1). For the above reaction $K^+N \rightarrow K^0P$, we find thus as only possible diagram the one for $K^-P \rightarrow \bar{K}^0N$ (cf., Fig. C-1.2), i.e., the one obtained through s-u crossing. Similarly one can in higher

Fig. C-1.1

Duality diagrams - (a) and (b) allowed,
(c) forbidden

Fig. C-1.2

particle number amplitudes exclude such permutations as shown in Fig. C-1.3; as is evident from the duality diagram, all orderings are excluded in which K and $\bar{K}$ are not adjacent. We shall see further down how this is used to greatly reduce the number of permutations in the analysis of three-body final states.

Let us consider now the question of how to bring isospin into the Veneziano picture, that is, we want to determine an isospin projection

Fig. C-1.3

coefficient $C_N(P)$ multiplying the $B_N(P)$ corresponding to permutation P of external particles

$$T_N = \sum_{\{P\}} C_N(P) B_N(P) \quad (C-1.2)$$

such that : i) $C_N(P)B_N(P)$ remains invariant under cyclic and anti-cyclic permutations; ii) factorization is retained, and iii) no exotics occur in any channel.

We shall treat here only the case of N external isovector particles; condition iii) then requires the absence of states with $I > 1$. A solution fulfilling all requirements is given by ^{31),32)}

$$C_N(12 \cdots N) = \frac{1}{2} \text{Tr} (\tau_{a_1} \tau_{a_2} \cdots \tau_{a_N}) \quad (C-1.3)$$

where τ_{a_i} denotes the 2×2 Pauli matrix representing the isospin state a_i of external particle i . From

$$\text{Tr } ABC = \text{Tr } CAB \quad (\text{C-1.4})$$

we immediately have i), while iii) follows from the closure under multiplication of the 2×2 Pauli matrices. Finally one can prove

$$\begin{aligned} \frac{1}{2} \text{Tr}(\tau_{a_1} \dots \tau_{a_N}) &= \frac{1}{4} \left[\text{Tr}(\tau_{a_1} \dots \tau_{a_M}) \text{Tr}(\tau_{a_{M+1}} \dots \tau_{a_N}) \right] \\ &+ \frac{1}{4} \sum_{x=1}^3 \left[\text{Tr}(\tau_{a_1} \dots \tau_{a_M} \tau_x) \text{Tr}(\tau_x \tau_{a_{M+1}} \dots \tau_{a_N}) \right] \end{aligned} \quad (\text{C-1.5})$$

and thus factorization. The first term here corresponds to a singlet (isospin zero) and the sum of the three terms to a triplet (isospin one) intermediate state. We thus find in general each pole contributing both to $I=0$ and $I=1$, i.e., we have an isospin degeneracy. Upon summing over all permutations, it turns out, however, that the two states have different signature, so that we have in effect found the ρ - f_0 degeneracy from the isospin factor.

Such considerations can be extended to isospin zero and one-half external particles, as well. Let us perhaps note again finally, however, that all this is only a solution of the pure isospin problem, not an answer to the question of how to construct Veneziano amplitudes for physical particles, e.g., pseudoscalar mesons !

2. Classification of Reactions

We want here to briefly look at production processes under the aspect of where Pomeron exchange can be excluded, and how.

a) No Pomeron exchange

Hyperon production by the interaction of nucleons and antikaons

$$\bar{K}N \rightarrow Y S \pi, \quad S = 1, 2, 3, \dots \quad (\text{C-2.1})$$

and baryon-antibaryon annihilation into mesons

$$N \bar{N} \rightarrow S \pi + r (K + \bar{K}) \quad (C-2.2)$$

are the only observable cases with arbitrary particle number where no Pomeron exchange is possible, because of strangeness and baryon number exchange, respectively. In the three-particle final state process

$$\pi N \rightarrow K \bar{K} N \quad \Delta Q_N \neq 0 \quad (C-2.3)$$

the Pomeron is forbidden if the nucleon changes its charge; charge exchange plays a similar role in

$$\pi N \rightarrow \gamma K \pi \quad \Delta Q_\pi \neq 0 \quad (C-2.4)$$

$$N N \rightarrow \gamma K N \quad \Delta Q_N \neq 0 \quad (C-2.5)$$

This essentially exhausts the list of measured reactions where Pomeron exchange is strictly forbidden.

b) Probably no Pomeron exchange

Current concepts of diffraction dissociation generally include the so-called Gribov ³³⁾ - Morrison ³⁴⁾ rule, which permits only natural parity changes for the diffractively produced system

$$\Delta P = (-1)^{\Delta J} \quad (C-2.6)$$

with ΔP and ΔJ denoting the parity and spin change between incident particle and diffractively excited final system. This rule excludes Pomeron exchange in several other reactions; in particular

$$K N \rightarrow K N \pi \quad \Delta Q_K \neq 0 \quad (C-2.7)$$

$$\bar{K} N \rightarrow \bar{K} N \pi \quad \Delta Q_{\bar{K}} \neq 0 \quad (C-2.8)$$

$$\pi N \rightarrow K \bar{K} N \quad (C-2.9)$$

are then non-diffractive because of the transition from one to two pseudoscalar mesons.

c) Pomeron exchange

The bulk of the measured processes, in particular such reactions as

$$\pi N \rightarrow N S \pi \quad S = 2, 3, \dots \quad (C-2.10)$$

$$N N \rightarrow N N (S-1) \pi \quad (C-2.11)$$

$$K N \rightarrow N K (S-1) \pi \quad (C-2.12)$$

$$\bar{K} N \rightarrow N \bar{K} (S-1) \pi \quad (C-2.13)$$

can thus contain Pomeron exchange, and various investigations³⁵⁾⁻³⁷⁾ have indicated that even at intermediate energies they do in fact contain it to a considerable extent. On the other hand, one can generally not exclude ordinary Regge exchanges in channels where Pomeron exchange is possible, except at very high energies (e.g., $P_{lab} \sim 20-30$ GeV/c for three-body final states, higher for more). Thus one really needs both the N point Veneziano amplitude and a dual model for diffraction dissociation in order to obtain a dual resonance description for production processes in general. In the following subsection we shall survey the present phenomenological attempts to solve this. We shall not consider here baryon-anti-baryon annihilation investigations^{10),38)}; for reviews of work on these processes, we refer, e.g., to the contribution of Dr. Lusignoli at this meeting.

3) Analyses of Production Experiments

A detailed discussion of all applications of Veneziano amplitudes to production reactions is certainly outside the scope of these lectures; we shall therefore instead concentrate on some interesting aspects in one or two applications of a pure B_5 model, of a dual resonance model for diffraction dissociation, and of a combination of these two. For further details and other applications, we shall attempt to list the original references.

a) $\underline{\underline{A}}_{\underline{\underline{B}}_5} \underline{\underline{\text{model}}}$ (no Pomeron exchange)

There are, in our opinion, essentially two features in which a Veneziano analysis is expected to be an improvement over previous multi-Regge descriptions : the Veneziano amplitude is applicable to the entire reaction, not just resonance production or background or high energy behaviour; because of crossing symmetry, the same amplitude (with fixed parameters) should describe the whole set of reactions obtainable from each other by crossing.

The B_5 amplitude was first applied in the analysis of a production experiment by Petersson and Törnqvist ¹¹⁾, who studied the reaction

$$K^- P \rightarrow \Lambda \pi^+ \pi^- \quad (C-3.1)$$

which appears to be quite suitable for such an analysis because of the absence of both Pomeron and pion exchange, the position of the latter in a duality scheme also being somewhat unclear. The resulting fits are remarkably good and encouraged further applications ³⁹⁾. Since reaction (C-3.1) could at least in principle have Pomeron contributions in a crossed channel and since the actual analysis encounters some difficulties with the hypothesis of no exotics, we prefer to consider in more detail a subsequent analysis of a different set of reactions. Chan, Raitio, Thomas and Törnqvist ¹²⁾ have investigated the $(K^{\pm 0} \pi^{\pm} PP)$ system, which via Gribov-Morrison rule appears to be the only case with no Pomeron in any possible channel. We thus want to look at reactions of the kind

$$K^+ P \rightarrow K^0 \pi^+ P \quad (C-3.2)$$

$$K^- P \rightarrow \bar{K}^0 \pi^- P \quad (C-3.3)$$

$$\pi^- P \rightarrow K^0 K^- P \quad (C-3.4)$$

$$\pi^+ P \rightarrow \bar{K}^0 K^+ P \quad (C-3.5)$$

$$P \bar{P} \rightarrow K^- K^0 \pi^+ \quad (C-3.6)$$

which are different channels of the same five-point amplitude.

Let us see first how many diagrams in fact contribute. We know that there are $(N-1)!/2 = 12$ inequivalent permutations of external lines; of these, all leading to exotic channels should, however, vanish. This leaves only the three diagrams shown in Fig. C-3.1, as is easily seen by looking at the corresponding Harari-Rosner duality diagrams; only the two topologies of Fig. C-3.2 are permitted. It becomes clear here how restrictive the absence of exotics turns out to be.

Fig. C-3.1

Fig. C-3.2

The choice of trajectories to be taken into account in each diagram is, analogously to that question in Reggeology, dictated to a large extent by experiment. We shall not enter here into this discussion, but simply indicate in Fig. C-3.1 the choice used in Ref. 12). With this choice, one then takes, e.g., for the first diagram of Fig. C-3.1 the form

$$(P) = \varepsilon_{\mu\nu\rho\sigma} p_{\pi}^{\mu} p_K^{\nu} p_{\bar{K}}^{\rho} p_P^{\sigma} *$$

$$B_5(1-\alpha_{K^*}, 1-\alpha_{A_2}, \frac{1}{2}-\alpha_{\Lambda}, 1-\alpha_{\omega}, \frac{1}{2}-\alpha_{\rho})$$

(C-3.7)

This form gives us poles at the right places, and because of the ε factor with the right angular behaviour (meson trajectories start with p wave resonances) without destroying the cyclic symmetry. The spin of the external baryons, however, is totally neglected.

Finally, one fixes the relative weights of the three terms by requiring the wrong signature contributions of N and Δ to vanish, obtaining

$$T = \beta [(P) + (R) + (S)] \quad (C-3.8)$$

Since the trajectory parameters are known, we are left with only one open parameter, the over-all normalization β .

In Figs. C-3.3 - C-3.5 we show now representative results obtained in Ref. 12). While there are discrepancies in details (in particular somewhat too flat momentum transfer distributions in channels where the possible pion exchange has been neglected), the over-all agreement with the data is rather promising, keeping in mind the roughness of the approximation. The reason for the over-all discrepancy of roughly a factor two in the reaction C-3.4 is not clear; since the over-all values for reaction (C-3.6) appear, with very sparse data, also rather much larger in theory than in experiment, it may be that the present forms of the model encounter difficulties with very "central" interactions ($K\bar{K}$ formation, $P\bar{P}$ annihilation).

By taking the amplitude (C-3.8) at the nucleon or Λ pole, one can now get predictions for two-body reactions of the kind ⁴⁰⁾

$$K^- P \rightarrow \bar{K}^0 N \quad (C-3.9)$$

$$\bar{\pi}^- P \rightarrow K^0 \Lambda \quad (C-3.10)$$

The resulting agreement is fairly good.

In conclusion, let us note other analyses and extensions. In Ref. 41) a similar analysis to the one in Ref. 12) was carried out. The problem of baryon spins was taken up in considerable detail in Refs. 42) and 43); while not solving the problem of parity doubling of baryon resonances, one may hope that this will eventually lead to a fuller understanding of spin effects and of the relations between B_5 and B_4 subreactions ⁴⁴⁾.

b) A dual resonance approach to diffraction dissociation

The next classes of processes we want to consider are those where the Pomeron can be taken as dominant whenever it can occur, i.e., high energy diffraction dissociation (cf., Fig. C-3.6) such as

$$\mathcal{N}\mathcal{N} \rightarrow \mathcal{N}(\pi\mathcal{N}) \quad (C-3.11)$$

$$\pi\mathcal{N} \rightarrow \pi(\pi\mathcal{N}) \quad (C-3.12)$$

$$K\mathcal{N} \rightarrow K(\pi\mathcal{N}) \quad (C-3.12)$$

$$\pi\mathcal{N} \rightarrow (3\pi)\mathcal{N}; \quad K\mathcal{N} \rightarrow (K2\pi)\mathcal{N} \quad (C-3.14)$$

and also, assuming the photon to behave essentially as a hadron

$$\gamma\mathcal{N} \rightarrow (2\pi)\mathcal{N} \quad (C-3.15)$$

A conclusive test for strong Pomeron dominance is, of course, the constancy of the cross-section; given a measurement at one energy only, one can, however, also compare different charge configurations ⁴⁵⁾ - thus Pomeron dominance implies for reaction (C-3.11) the ratio

$$\sigma(PP \rightarrow PN\pi^+)/\sigma(PP \rightarrow PP\pi^0) = 2/1.$$

In the diffraction dissociation vertex V (cf., Fig. C-3.6), one observes in general both direct and crossed channel exchange effects: while in, e.g., (C-3.15) the direct channel ρ is strongly dominant, there are clearly observable shifts in the two-pion mass distribution

indicating the presence of pion exchange contributions; on the other hand, while, e.g., (C-3.11) can be rather well described by reggeized pion exchange in V , there are N^* resonance effects which in this fashion are neglected ($PP \rightarrow PN_{1470}^*$, PN_{1518}^*).

Pokorski and the present author ¹⁵⁾ have therefore attempted to describe diffraction dissociation reactions by an amplitude of the form (cf., Fig. C-3.6)

$$T \sim f(t_{AA}) \bar{s} V_{PB \rightarrow CD} \quad (C-3.16)$$

where $f(t_{AA})$ denotes a form factor for the hadron-hadron-Pomeron vertex which can, by factorization, be determined from elastic scattering; the factor $\bar{s} = (p_A + p_B)^2$ is to account for the Pomeron propagator ($\bar{s}^{\alpha_P} \approx \bar{s}$); finally, $V_{PB \rightarrow CD}$ denotes the amplitude for the "reaction" $P+B \rightarrow C+D$. For the latter, we want to take a dual resonance form and thus obtain a Veneziano description of diffraction dissociation (DRDD).

Assuming the Pomeron to behave roughly as a $J^P = 0^+$ particle, we thus take for (C-3.15) for example the form ⁴⁶⁾

$$V_{\pi\pi \rightarrow \pi^+\pi^-} \sim [B_4(1-\alpha_\rho^s, -\alpha_\pi^t) + B_4(1-\alpha_\rho^s, -\alpha_\pi^u)] \quad (C-3.17)$$

with ρ and π trajectories dominant in direct and crossed channels of $\pi\pi \rightarrow \pi^+\pi^-$, respectively.

In Figs. C-3.7 - C-3.10 we again show some representative results obtained in such an approach; again the results are on the whole quite satisfactory.

Besides the application to $NN \rightarrow N(\pi N)$ ¹⁵⁾, $\pi N \rightarrow \pi(\pi N)$ ⁴⁷⁾, and two-pion ⁴⁶⁾ as well as two-kaon ⁴⁸⁾ photoproduction, work is in progress also on the diffractive dissociation into three particles (taking $V \sim B_5$) and appears to yield rather good agreement for $KN \rightarrow (K2\pi)N$ ⁴⁹⁾.

Fig. C-3.6

c) Models B_5 plus DRDD

As already mentioned, one has to combine models such as those described under C-3.a) and C-3.b) to obtain a description applicable to production reactions in the intermediate energy range. A number of such analyses is presently in progress; work on $K^\pm P \rightarrow K^\pm \pi^0 P$ ⁵⁰⁾ and on $K^\pm P \rightarrow K^\pm \pi^\pm N$ ⁵¹⁾ has just appeared and seems to give fairly good agreement in most instances. In the second of these analyses pion exchange has been included and is found to give in fact strong contributions.

While there are still many open ends, it thus appears that one now has on a crude phenomenological level an over-all outline for the dual resonance analysis of production processes; only further investigations can tell us how critical the approximations are that we have made to obtain this scheme, and to what extent particularly spin effects can eventually be taken into account.

4) Present Problems and Outlook

In closing this introduction to dual resonance models in multi-particle phenomenology, we want to emphasize once more what the main unsolved problems are.

As in previous descriptions, the unitarity question remains completely open, and primitive unitarizations such as the complex trajectories used above may soon encounter at least two difficulties: it appears that more detailed analyses will require different widths for

parents and daughters ^{52),53)}, and it is not clear what this will do in general to the properties of B_N ; unitarizations neglecting the inter-relation between various particle number channels probably will not give rise to the smooth energy behaviour of the over-all inelastic cross-section observed experimentally. Moreover, it is not clear to what extent the separation into diffractive and non-diffractive processes is possible in the long run; here one encounters such questions as whether the Pomeron is built up, through unitarity, mainly from non-diffractive many-body reactions, or whether diffraction dissociation will provide significant parts in this build-up ; (looking at low energy K^+P scattering, which is already strongly Pomeron dominated in spite of essentially no inelastic Pomeron reactions, one might hope for a construction via bootstrap from non-diffractive reactions only).

Secondly, the fermion question will certainly become more and more important as the analyses become more detailed. It remains to be seen if the present attempts to remove parity doublets in Regge theory can, in a workable fashion, be extended to dual models - i.e., if parity doublets can be removed in a crossing symmetric fashion.

Thirdly, as we saw in reactions involving $K\bar{K}$ and $P\bar{P}$ systems, it is not clear if the present form of the Veneziano amplitude is applicable both to very peripheral and to very central reactions.

Finally, we should emphasize the fact that the complexity of the expression and consequently the amount of technical effort increase greatly when one just increases the particle number by one or two. It is difficult to conceive calculations of six or seven-body final states with the present formulae. Thus we would consider an equally important question that of how to simplify the Veneziano form for large N without losing its essential aspects.

ACKNOWLEDGEMENTS

It is a great pleasure to thank many colleagues at CERN, in particular Chan Hong-Mo, K. Kajantie, B. Petersson, H. Ruegg and K. Schilling, for stimulating conversations on various aspects of "Venezianology".

LITERATURE

- REVIEWS : R1) Chan Hong-Mo, CERN preprint TH.1057, July 1969
(Royal Society Meeting Lecture).
- R2) C. Lovelace, CERN preprint TH.1123, January 1970
(Irvine Regge Pole Meeting Lecture).
- R3) V. Alessandrini, D. Amati, M. Le Bellac and D. Olive,
CERN preprint TH.1160, May 1970.
- R4) G. Veneziano, July 1970
(Erice Lectures).

REFERENCES

- 1) Cf., e.g., O. Czyzewski, Rapporteur's talk, Proceedings of Vienna conference (1968).
- 2) R. Dolen, D. Horn and C. Schmid, Phys.Rev. 166, 1768 (1968).
- 3) G. Veneziano, Nuovo Cimento 57A, 190 (1968).
- 4) K. Bardakçi and H. Ruegg, Phys.Letters 28B, 342 (1968);
K. Bardakçi and H. Ruegg, Phys.Rev. 182, 1884 (1969).
- 5) M. Virasoro, Phys.Rev.Letters 22, 37 (1969).
- 6) Chan Hong-Mo, Phys.Letters 28B, 425 (1969);
Chan Hong-Mo, Tsou Sheung Tsun, Phys.Letters 28B, 485 (1969).
- 7) C.L. Goebel and B. Sakita, Phys.Rev.Letters 22, 259 (1969).
- 8) Z. Koba and H.B. Nielsen, Nuclear Phys. B10, 633 (1969).
- 9) K. Kikkawa, B. Sakita and M.A. Virasoro, Phys.Rev. 184, 1701 (1969).
- 10) C. Lovelace, Phys.Letters 28B, 264 (1968).
- 11) B. Petersson and N.A. Törnqvist, Nuclear Phys. B13, 629 (1969).

- 12) Chan Hong-Mo, R.O. Raitio, G.H. Thomas and N.A. Törnqvist, Nuclear Phys. B19, 173 (1970).
- 13) H. Harari, Phys.Rev.Letters 20, 1395 (1968).
- 14) P.G.O. Freund, Phys.Rev.Letters 20, 235 (1968).
- 15) S. Pokorski and H. Satz, Nuclear Phys. B19, 113 (1970).
- 16) S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969).
- 17) K. Bardakçi and S. Mandelstam, Phys.Rev. 184, 1640 (1969).
- 18) J.L. Hopkinson and E. Plahte, Phys.Letters 28B, 489 (1969).
- 19) A. Białas and S. Pokorski, Nuclear Phys. B11, 479 (1969).
- 20) D. Gross, CERN preprint TH.1048 (1969).
- 21) Z. Koba and H.B. Nielsen, Z.Phys. 229, 243 (1969).
- 22) A. Martin, Phys.Letters 29B, 431 (1969).
- 23) S.W. MacDowell, Phys.Rev. 116, 774 (1959).
- 24) C. Savoy, Nuovo Cimento Letters 2, 870 (1969).
- 25) R.E. Waltz, Nuclear Phys. B18, 61 (1970).
- 26) J. Dethlefsen, "A₁ parameters from a modified five-point function" Copenhagen preprint (1969).
- 27) D.J. Dorren, V. Rittenberg and H.R. Rubinstein, CERN preprint TH.1192 (1970), and further references therein.
- 28) E. Del Giudice and G. Veneziano, Nuovo Cimento Letters 3, 363 (1970).
- 29) H. Harari, Phys.Rev.Letters 22, 563 (1969).
- 30) J.L. Rosner, Phys.Rev.Letters 22, 689 (1969).
- 31) J.E. Paton and Chan Hong-Mo, Nuclear Phys. B10, 516 (1969).
- 32) D.E. Neville, Phys.Rev.Letters 22, 494 (1969).

- 33) V.N. Gribov, Yad.Fiz.USSR 5, 197 (1967).
- 34) D.R.O. Morrison, Phys.Rev. 165, 1699 (1968).
- 35) H. Satz, Nuclear Phys. B14, 366 (1969).
- 36) Scandinavian Collaboration, Phys.Letters 30B, 369 (1969).
- 37) L. Van Hove, CERN preprint TH.1178 (1970);
cf., also : A. Białas et al., Nuclear Phys. B11, 479 (1969).
- 38) G. Altarelli and H.R. Rubinstein, Phys.Rev. 183, 1469 (1969);
H.R. Rubinstein, E.J. Squires and M. Chaichian, Phys.Letters 30B,
191 (1969).
- 39) P. Hoyer, B. Petersson and N.H. Törnqvist, CERN preprint TH.1159
(1970).
- 40) B. Petersson and G.H. Thomas, Nuclear Phys. B20, 451 (1970).
- 41) J. Bartsch et al., Nuclear Phys. B20, 63 (1970).
- 42) I. Bender, H.G. Dosch, V.F. Müller and H.J. Rothe, Z.Phys. 237, 192
(1970); and "Production of $\pi\pi$ resonances in a dual model...",
Heidelberg preprint (July, 1970).
- 43) A.C. Hirshfeld and M.G. Schmidt, "The dual $\bar{K}N\tilde{\Lambda}\pi\pi$ amplitude with
spin", Heidelberg preprint (July, 1970).
- 44) A.C. Hirshfeld, "Relations between scattering processes derivable
from a dual five-point amplitude with spin", Heidelberg preprint
(July, 1970).
- 45) H. Satz, Phys.Letters 29B, 38 (1969).
- 46) H. Satz and K. Schilling, Nuovo Cimento 67A, 511 (1970).
- 47) P.K. Schilling, DESY Int.Rep.R1/70/3 (February, 1970).
- 48) H. Satz and K. Schilling, Nuovo Cimento Letters 3, 723 (1970).

- 49) G. Otter and S.J. Goldsack, "Veneziano model calculations for the decay $Q \rightarrow K \pi \pi$ ", Imperial College London preprint (1970); and G. Otter, private communication.
- 50) K. Kajantie and S. Papageorgiou, CERN preprint TH.1170 (1970).
- 51) Aachen-Berlin-CERN-London-Vienna collaboration, CERN/D.PH.II/Phys.70-13, (June, 1970).
- 52) P. Hoyer, A. Lee, J.E. Paton, B. Petersson and G.H. Thomas, to appear soon.
- 53) S. Pokorski, M. Szeptycka and A. Znieminski, to appear soon.

Fig. C-3.3 : Cross-sections for $K^\pm p \rightarrow K^0(\bar{K}^0) \pi^\pm p$ and $\pi^- p \rightarrow K^0 K^- p$, from Ref. 12).

Fig. C-3.4 : Proton-proton and kaon-kaon momentum transfer distributions in $K^+P \rightarrow K^0 \bar{\nu}^+P$, from Ref. 12).

Fig. C-3.5 : Invariant mass distributions in $K^+ p \rightarrow K^0 \pi^+ p$,
from Ref. 12).

Fig. C-3.7 : $N \pi^+$ mass distribution in $PP \rightarrow PN \pi^+$, from Ref. 15).

Fig. C-3.8 : Proton-proton momentum transfer distribution in
 $PP \rightarrow PN \pi^+$, from Ref. 15).

Fig. C-3.9 : Two-pion mass distribution in $\gamma P \rightarrow P \pi^+ \pi^-$,
from Ref. 46).

Fig. C-3.10 : Proton-proton momentum transfer distribution for various two-pion masses, in $p p \rightarrow p \pi^+ \pi^-$ from Ref. 46).