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ANALYTICITY OF A SCATTERING AMPLITUDE
IN AN EXTERNAL MASS VARIABLE *)

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A B S T R A C T

We study the analyticity in an external mass variable of a four-point function by looking at simple perturbation theory diagrams. This analyticity information is required in order to apply models like vector meson dominance and PCAC to two-body scattering amplitudes. A complex singularity is found on the physical sheet but at high energies it moves far from the physical region. The role of such singularities in nuclear shadowing is briefly discussed.

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1. INTRODUCTION

In this paper we discuss the properties of a two-body scattering amplitude (four-point function) considered as an analytic function of an external mass variable. Our discussion is restricted to singularities generated by triangle and box diagrams of perturbation theory.

Interest in analytic properties in an external mass variable arises naturally when one considers the application of pole dominance models like vector meson dominance (VMD) and PCAC to scattering amplitudes (e.g., $\gamma + p \rightarrow \pi^+ + n$).

The predictions of the VMD model for this latter process has generated a great deal of discussion about spin and gauge invariance^{1),2)}. As fascinating as these discussions may be they are not the complete story. The implementation of any pole dominance model calls for the writing (either implicitly or explicitly) of an unsubtracted dispersion relation in the variable in question and the assumption that in some region of that variable the amplitude is well approximated by pole terms corresponding to known particles. The often used word "smoothness" is not a substitute for a dispersion relation; it means a dispersion relation and nothing less.

Problems of spin and gauge invariance are best handled by decomposing the full amplitude in terms of invariant (Mandelstam) amplitudes¹⁾. Before writing the dispersion relations for these invariant amplitudes it is necessary to know what, if any, singularities they have in addition to the normal threshold singularities known from simple arguments. The identification and role of these other singularities is the program of this paper.

As will become clear below, the amplitude has complex singularities on the physical sheet in addition to the usual normal threshold singularities. In light of this rather complicated analytic structure in even the simple diagrams we consider, it seems impossible at least with tools so far developed to investigate analyticity in an

external mass variable using the machinery of axiomatic field theory in a way analogous to the well-known proofs of analyticity in an energy variable ³⁾. Hence the recourse to perturbation theory.

But even though our investigation is thus limited and far from complete, we feel our results are not without interest. Indeed the analytic structure of the full amplitude is expected to have all the singularities of the perturbation series, perhaps more ⁴⁾. (That is, the singularities of the perturbation series are the simplest set consistent with crossing, unitarity, etc.) Moreover, perturbation theory diagrams of higher order than those we explicitly consider have among their singularity structure the triangle and box singularities discussed here. Namely, our diagrams may be considered to have arbitrary "blobs" at their vertices. In this sense one learns from our work about some of the singularities of a rather large class of diagrams. It remains unclear, however, in what sense the singularities we discuss are the most important ones.

We feel the most serious limitations to our work (and to singularity hunting in general) is one's inability to say anything meaningful about the size of discontinuities across branch cuts. Experience in the complex energy plane (here we have in mind the often-successful resonance saturation approximation) has taught us that the most relevant information physically is the knowledge of where the discontinuity across a branch cut is large and not where its end points, the singularities, happen to be ⁵⁾. For certain special cases of the problem we are considering there is, perhaps, some reason for hope. The (complex) singularities we encounter are closely related to the (real) anomalous singularities which play a key role in the elastic electromagnetic form factors of the deuteron ⁶⁾. The presence of these anomalous singularities corresponds physically in a well-known way to the fact the deuteron is a weakly bound system and the discontinuity across the anomalous cut is related to the deuteron bound state wave function. In our case, therefore, it is reasonable to expect that if we consider scattering from a loosely bound system we can learn something about discontinuities in terms of wave functions and non-relativistic quantum mechanics. We leave for

another time a detailed investigation of this question, but we do use our results to discuss nuclear shadowing ⁷⁾ from a dispersion theoretic point of view. In our examples we will mostly talk about VMD, but the reader can easily translate the results to PCAC, etc.

2. METHOD AND KINEMATICS

The singularity structure of the triangle and box diagrams can be examined in many ways. The most straightforward is to examine the explicit closed-form expressions for the diagrams after all integrations have been performed. The triangle diagram has been given by Källén and Wightman ⁸⁾ as the sum of 16 Spence ⁹⁾ (di-logarithm) functions, and the box diagram was shown by Wu ¹⁰⁾ to be expressible as the sum of 192 Spence functions. Unfortunately, these forms (especially for the box) are not particularly transparent since, as discussed by these authors, individual terms have singularities which can cancel among sets of terms. The presence or non-presence of these singularities in the full amplitude depends on the particular Riemann sheet one is examining. Painstaking attention to phases is required. Thus, the closed form expressions offer no simple answer to the perpetual problem of deciding which singularities are on the physical sheet.

Instead we chose to use methods already developed for determining the location of singularities of perturbation diagrams without explicit evaluation of Feynman parameter and/or loop variables ⁴⁾. The algebraic equations which determine the location of the singularities in all variables of the triangle and box diagrams are already known. We decide which of these are on the physical sheet by beginning with the variables, other than the external mass of interest, adjusted to values such that the analytic structure is simple (only normal threshold singularities on the physical sheet). We then analytically continue these other variables to their final physical values and watch which singularities move onto the physical sheet of the external mass variable.

The analytic structure of perturbation diagrams has been subject to intensive investigation in the past ¹¹⁾. However, most of this work was devoted to proving the existence of certain representations, and interest was usually lost in cases having complex singularities. The studies of production amplitudes, especially the Herculean work of Cook and Tarski ¹²⁾, are the most closely related to our problem. Let us turn to our task.

Consider the scattering $p_1 + p_2 \rightarrow p_3 + p_4$ of spinless, unequal mass particles, $p_i^2 = M_i^2$, proceeding by the Feynman diagram of Fig. 1. There is no loss of generality in neglecting spin since we are interested in the analytic structure of invariant amplitudes. We adopt, as closely as possible, the notation of Collas and Norton ¹³⁾ who also studied analyticity in an external mass but in a situation which has no connection to ours. We seek the singularities of the amplitude described by Fig. 1 as a function of complex M_3^2 at fixed M_1^2, M_2^2, M_4^2 , $s = (p_1 + p_2)^2$, $t = (p_2 - p_4)^2$, and fixed internal masses m_1, m_2, μ_1 and μ_2 . We are ultimately interested in values of s and t appropriate for the s channel physical region: $s \geq (M_1 + M_2)^2 + i\epsilon$, $t \leq 0$ and especially the high energy limit $s \rightarrow \infty + i\epsilon$.

Let us first discuss the choice of variables which are to be held fixed. The reason for keeping the other external masses M_1^2, M_2^2 and M_4^2 constant is obvious. In VMD, for example, we are interested in relating a process initiated by a real or virtual photon to the same process initiated by a vector meson. Experiment will measure the two processes with particles 1, 2 and 4 on mass shell. (An obvious exception is the amplitude for virtual forward Compton scattering where one is interested in analyticity in $M_3^2 = M_2^2$ with $M_1^2 = M_4^2$, and $t = 0$. A complete investigation of the triangle and box singularities for complex M_3^2 and s has been given recently by Suri ¹⁴⁾. The triangle singularities at fixed s have also been given by one of us ¹⁵⁾.) We elect to keep s and t fixed so that singularities in these variables do not become additional singularities in M_3^2 . Perhaps by relaxing this latter condition one might in some sense ¹⁶⁾, achieve "better" analyticity in M_3^2 . We have not finished investigating

this point, and will not discuss it further here. The study of the singularities of perturbation theory is also simplest if s and t are held fixed, another reason for our doing so.

To express the singularity equations in the simplest form it is standard practice to introduce the Karplus, Sommerfeld, Wichmann variables ¹⁷⁾

$$\left. \begin{aligned} x &= \frac{s - m_1^2 - m_2^2}{2m_1 m_2} & y &= \frac{t - \mu_1^2 - \mu_2^2}{2\mu_1 \mu_2} \\ x_1 &= \frac{M_1^2 - \mu_1^2 - m_1^2}{2\mu_1 m_1} & x_2 &= \frac{M_2^2 - \mu_1^2 - m_2^2}{2\mu_1 m_2} \\ x_3 &= \frac{M_3^2 - \mu_2^2 - m_2^2}{2\mu_2 m_2} & x_4 &= \frac{M_4^2 - \mu_2^2 - m_1^2}{2\mu_2 m_1} \end{aligned} \right\} \quad (2.1)$$

The conditions described above become in terms of the x_i 's.

$$\left. \begin{aligned} x_3 & \text{ arbitrary and complex} \\ x_1, x_2, x_4, x, y & \text{ fixed with} \\ x & > 1 + i\epsilon, \quad y < -1. \end{aligned} \right\} \quad (2.2)$$

Finally we consider only situations for which the "stability" conditions

$$|x_i| < 1, \quad i = 1, 2, 4$$

hold. One may relax the stability conditions at the end if desired. In our work we have found it simplest to consider the general mass

case (i.e., all internal and external masses distinct) since this avoids artificial and troublesome confluences of singularities. Obviously one can specialize to equal mass situations at the end. Along the way it is sometimes useful to consider near equal mass examples $x_{1,2,4} \approx -\frac{1}{2}$ in order to get a feel for the situation. Typically our figures will be drawn for such values.

By contracting one of the internal lines of the box diagram one obtains four distinct triangle diagrams. Only the two of Fig. 2 have singularities in x_3 . Diagram 2a is independent of s and may be described by the variables x_2, x_3, y . Similarly, diagram 2b is independent of t and is a function only of x, x_4 and x_3 .

Contracting once again one finds only one bubble diagram, Fig. 2c, which has singularities in x_3 . Diagram 2c depends only on x_3 ; its singularities are at $x_3 = \pm 1$. The singularity at $x_3 = +1$ ($M_3^2 = (\mu_2 + m_2)^2$) is the expected normal threshold branch point and always lies on the physical sheet. The pseudothreshold $x_3 = -1$ never lies on the physical sheet.

As a realistic example let μ_2 and m_2 be pion masses and draw the square root branch cut from $x_3 = +1$ to $+\infty$. The rho meson is a pole found on the second Riemann sheet reached by dropping through this cut from positive imaginary ($+i\epsilon$) values. Of course, the rho pole is not present in our simple diagram. But the (infinite set) diagrams which generate the rho in any calculation will all have the $x_3 = +1$ branch point; only the discontinuity changes. Naturally these diagrams will have other x_3 as well as x and y singularities.

3. TRIANGLE SINGULARITIES

We first examine the singularities of diagrams 2a and 2b; such triangle diagrams have been extensively studied in the past ⁴⁾ and we need give only a discussion which emphasizes the features of special interest in our case.

Consider first diagram 2a. Its leading singularities as determined by the Landau equations ^{4),13)} are at $x_3 = L_{y2}^{\pm}$ and $x_3 = L_{y2}^{-}$ where

$$L_{y2}^{\pm} = -yx_2 \pm \sqrt{(1-y^2)} \sqrt{(1-x_2^2)}. \quad (3.1)$$

Begin with y real and adjusted to some value satisfying the stability condition $|y| < 1$. (Recall $|x_2| < 1$ by hypothesis.) Phases are fixed by choosing both square roots in Eq. (3.1) positive for $|x_2| < 1$, $|y| < 1$; for all other x_2 , y , phases are determined by continuity. Equation (3.1) describes an ellipse inscribed inside the unit square in the real x_3 , y plane whose orientation depends on the sign of x_2 . For $x_2 < 0$ we have the situation shown in Fig. 3a and for $x_2 > 0$ the situation of Fig. 3b.

With no loss of generality we may also suppose y chosen to satisfy $x_2 + y < 0$. Under such conditions it is well known ⁴⁾ that neither triangle singularity $L_{y2}^{\pm}$ is on the physical sheet of x_3 . In fact $x_3 = L_{y2}^{+}$ is located on sheet II, the Riemann sheet reached by passing through the normal threshold cut $1 \leq x_3 < \infty$ from the upper half of sheet I, the physical sheet. (In all our work the name sheet II will be reserved exclusively for this unique sheet. Sheet I will always mean the physical sheet.) Since the normal threshold cut is a square root cut one can also reach sheet II by passing through the cut from the lower half of sheet I.

The particular subsheet on which L_{y2}^{-} is located depends on the sign of $y - x_2$. For example, in the case illustrated in Fig. 3a where $y - x_2 > 0$ L_{y2}^{-} is found on sheet III, the sheet reached by going through the (square root) pseudothreshold cut $-\infty \leq x_3 \leq 1$ on sheet II ¹⁸⁾. Fig. 3b illustrates the case $y - x_2 < 0$ for which L_{y2}^{-} is found on sheet II.

With the location of all singularities thus known we proceed to continue in y to the region of interest $y < -1$ (i.e., $t < 0$). Before $y = -1$ is reached, necessarily the condition $y - x_2 < 0$ will be met; if indeed it was not satisfied at the outset. As $y - x_2$ decreases

through zero L_{y2}^- loops -1 and moves up onto sheet II. Things go smoothly until $y = -1$ at which point $L_{y2}^\pm$ collide at $L_{y2}^\pm = x_2$. As y is further decreased $L_{y2}^\pm$ move off into the complex plane as a pair of complex conjugate singularities dragging their cuts behind them. See Fig. 4.

By following the phase conventions we prescribed above, one can easily show that if y is decreased below -1 with a small positive imaginary part ($y < -1 + i\epsilon$ for short) L_{y2}^+ moves into the upper half plane and conversely for $y < -1 - i\epsilon$. Now, since there is no singularity at $y = -1$ on sheet I of the y variable we know it cannot matter whether y passes over or under the point -1 . Alas, this corresponds precisely to a real collision of the singularities $L_{y2}^\pm$ on the same sheet (II) when $y = -1$. The two singularities lose their identities in the collision and it is merely a matter of convention which is called L_{y2}^+ and which L_{y2}^- afterwards. Physics, properly, is indifferent. It is also amusing to note that it is necessary to have $L_{y2}^\pm$ complex conjugates on the same Riemann sheet to protect the real analyticity of the amplitude generated by diagram 2a.

By inspection we have seen that all the action takes place on sheet II; no singularity ever moves onto sheet I. Hence, we may conclude that diagram 2a, in the region of interest $y < -1$, has no singularities on the physical sheet of x_3 except the normal threshold singularity $x_3 = +1$. This is the expected result; internal instability is never harmful.

Next, consider diagram 2b where the results will be very different. Its leading singularities are at $x_3 = L_{x4}^\pm$ where

$$L_{x4}^\pm = -xx_4 \pm \sqrt{(1-x^2)} \sqrt{(1-x_4^2)} \quad . \quad (3.2)$$

Again we begin at a known configuration by choosing $|x| < 1$ and the signs of the square roots in Eq. (3.2) positive for $|x|, |x_4| < 1$, and $x + x_4 < 0$. Because of the last condition, L_{x4}^+ is on sheet II.

Moreover, $L_{x_4}^-$ is on sheet II (III) for $x-x_4 < 0$ ($x-x_4 > 0$). Continue in x along the positive real axis giving x a small positive imaginary part. Eventually one reaches $x+x_4=0$. At this value of x , $L_{x_4}^+$ has just reached $x_3=1$ and is about to move up onto the physical sheet. Because we have given x a small positive imaginary part $L_{x_4}^+$ actually loops $x_3=+1$ as illustrated in Fig. 5. Had we instead continued x under $-x_4$, $L_{x_4}^+$ would have looped up onto sheet I by the complex conjugate path; the resulting amplitude would be the same, however, reflecting the fact it is real analytic in x_3 for $|x|, |x_4| < 1$. (Note that the direction through which singularities loop lower order singularities can be read directly from the curves in the real plane. For example, consider for L_{x_4} the analogue of Fig. 3a obtained by the replacements $y \rightarrow x$, $x_2 \rightarrow x_4$ and $L_{y2} \rightarrow L_{x4}$. By Taylor's theorem if we let $x \rightarrow x + i\epsilon$ then

$$L_{x_4}(x+i\epsilon) = L_{x_4}(x) + i\epsilon \frac{\partial L_{x_4}(x)}{\partial x}.$$

Hence

$$\text{Im } L_{x_4}^{\pm} = \left(\frac{\partial L_{x_4}^{\pm}(x)}{\partial x} \right) \text{Im } x$$

except at the points $x=\pm 1$ where $L_{x_4}^{\pm}$ is not an analytic function of x . Now, for $x+x_4 < 0$ we have by inspection of the analogue of Fig. 3a $\partial L_{x_4}^+ / \partial x > 0$ so $\text{Im } L_{x_4}^+ / \text{Im } x > 0$. Conversely, for $x+x_4 > 0$, $\text{Im } L_{x_4}^+ / \text{Im } x < 0$.)

As we further increase x , $L_{x_4}^+$ the well-known anomalous singularity of elastic form factors, Refs. 4),6), moves down the real axis below the normal threshold with its cut following. And, if it has not already happened, $L_{x_4}^-$ loops -1 and goes onto sheet III when x passes through $x-x_4=0$.

Finally, we reach the critical point $x=1$. Continuing above $x=1$ with a small positive imaginary part we read from Eq. (3.2) that $L_{x_4}^+$ moves off into the lower half complex sheet I and $L_{x_4}^-$ moves along a complex conjugate path in the upper half sheet III. At high energies $x \rightarrow 0$ the singularity is on the physical sheet at

$$L_{x_4}^+ \xrightarrow{x \rightarrow \infty + i\epsilon} x(-x_4 - i\sqrt{(1-x_4^2)}) \quad (3.3)$$

and moves far from the real axis, to the left for $x_4 > 0$, to the right for $x_4 < 0$. Fig. 5 illustrates the case $x_4 < 0$.

This time, because there is a singularity in x at $x=1$, the answer depends on how we pass this point. We chose to pass over $x=1$ because this corresponds to the $s+i\epsilon$ prescription which selects the physical amplitude in the s channel. Had we passed under $x=1$, $L_{x_4}^+$ would have moved into the upper half of the physical sheet and we would be describing the complex conjugate of the amplitude. Clearly, the absorptive part in the s channel, i.e., the difference of the amplitude taken $s+i\epsilon$ and $s-i\epsilon$, has a pair of complex conjugate singularities on sheet I of the variable x_3 (as well as their partners on sheet III). Note also that for $x > 1+i\epsilon$ the amplitude is no longer a real analytic function of x_3 . This is as it should be. For example, the electroproduction amplitude for virtual $\gamma + p \rightarrow \pi^+ + n$ is not real for spacelike photon masses when we are above threshold in s . Real analyticity is recovered if we complex conjugate the x variable as well as x_3

$$B(x_3, x, x_4) = B^*(x_3^*, x^*, x_4) \quad (3.4)$$

for $-1 < x_4 < 1$. (Actually $-\infty < x_4 < 1$.)

Finally note that the branch cut which runs between $L_{x_4}^+$ and $L_{x_4}^-$ is curiously trapped in the Riemann sheet structure. In the final configuration it begins at $L_{x_4}^+$ on the physical sheet, goes onto sheet II by looping the normal threshold, loops the pseudo-threshold $x_3 = -1$ and goes onto sheet III where it ends at $L_{x_4}^-$. As $x \rightarrow \infty$ this cut gets stretched more and more but it is for ever wound around the $x_3 = \pm 1$ singularities.

4. BOX SINGULARITIES

Having examined carefully all normal threshold and triangle singularities, we are in a position to study the leading singularities of the box diagram, Fig. 1. To make maximum use of what is already known we first exploit the fact that the amplitude corresponding to that box diagram is formally symmetric under interchange of any pair of the variables $(x,y) \leftrightarrow (x_1,x_3) \leftrightarrow (x_2,x_4)$ ¹³⁾. This may be seen, for example, in the unintegrated Feynman parameter form. Thus the amplitude of diagram 1 is identical to that of Fig. 6, where now the variable x_3 (or equivalently M_3^2) appears in the more familiar role of an energy variable. Of course we are interested in values of x and y which are outside the familiar stability regions so we still have some work to do. We will see that in the final configuration the box singularities in x_3 remain real, but only one is on the physical sheet (boundary). The variable x_1 , which we call the partner to x_3 , plays a key role in the determining the locations of the singularities in x_3 . For a while we will leave its value completely free.

The box singularities ^{4),13)} as determined by the Landau equations, or any other method, are $x_3 = M_{\pm}$ where

$$M_{\pm} = (x_1^2 - 1)^{-1} \left\{ x_1 (xy + x_2 x_4) + y x_2 + x x_4 \right. \\ \left. \pm \sqrt{(x_1 - L_{y4}^{\pm})(x_1 - L_{y4}^{\mp})(x_1 - L_{x2}^{\pm})(x_1 - L_{x2}^{\mp})} \right\}. \quad (4.1)$$

The quantities $L_{y4}^{\pm}$ and $L_{x2}^{\pm}$ in Eq. (4.1) are the positions of the triangle singularities in the variable x_1

$$L_{y4}^{\pm} = -y x_4 \pm \sqrt{(1-y^2)(1-x_4^2)} \\ L_{x2}^{\pm} = -x x_2 \pm \sqrt{(1-x^2)(1-x_2^2)}. \quad (4.2)$$

As in the last section we begin with initial values of x and y in the stability region $|x| < 1$, $|y| < 1$ and specify phases by choosing all square roots in Eqs. (4.1) and (4.2) to be positive in this region. Later when we continue $y < 1$ and $x > 1 + i\epsilon$, $L_{y4}^{\pm}$ and $L_{x2}^{\pm}$ will undergo the same behaviour in the x_1 plane that we saw for $L_{y2}^{\pm}$ and $L_{x4}^{\pm}$ in the x_3 plane.

The curve Γ in the real x_1, x_3 plane as specified by Eq. (4.1) consists of five separate arcs for $|x|, |y|, |x_2|, |x_4| < 1$ and was first described by Tarski¹⁹⁾. The various arcs have complex connections which we need not be concerned with in our work. The precise form of the curve Γ depends in detail upon the parameters x, y, x_2 and x_4 . Since we need to be precise about the location of all singularities before continuing in x and y let us consider some of these details. Useful algebra is given in our Appendix.

As a consequence of the sign conventions adopted above, $L_{ij}^- < L_{ij}^+$ for any of the quantities in Eqs. (3.1), (3.2) and (4.2). If we pick a given ordering of the triangle singularities in the variable x_1 only two orderings of the triangle singularities in x_3 are possible¹⁹⁾. Since we are ultimately going to continue to $x > 1$, $y < -1$ it makes sense to pick the initial ordering in x_1 as $L_{y4}^- < L_{y4}^+ < L_{x2}^- < L_{x2}^+$ and in x_3 to pick out of the two possibilities $L_{y2}^- < L_{y2}^+ < L_{x4}^- < L_{x4}^+$. (See Appendix.)

Further choose the initial situation such that none of the triangle singularities L_{ij}^+ are on the physical sheets. This requires $x + x_4 < 0$, $y + x_2 < 0$, $x + x_2 < 0$, and $y + x_4 < 0$. And for definiteness we rather arbitrarily choose $x - x_4 > 0$, $x - x_2 > 0$, $y - x_2 < 0$, $y - x_4 < 0$. The totality of our choices implies $y, x_2, x_4 < 0$. This entails no loss of generality since we are going to continue in y below -1 anyway, and furthermore after the x and y continuations have been performed one can rather easily continue x_2 and x_4 to positive values if desired.

Having made these choices, one has fixed all the significant features of the curve Γ and we draw it in Fig. 7a. Asymptotically Γ is tangent to the normal threshold and pseudothreshold lines $x_1, x_3 = \pm 1$ and each triangle singularity $L_{ij}^\pm$ is tangent to Γ once and only once ¹⁹⁾.

Under the conditions just prescribed it is well known ⁴⁾ that only the arc γ_1 of Γ is singular when approached from the physical sheets of x_1 and x_3 . Each point actually corresponds to two distinct singularities; one reached via $x_1(I) + i\epsilon, x_3(I) + i\epsilon$ and another distinct singularity reached via $x_1(I) - i\epsilon, x_3(I) - i\epsilon$. The crossed limits $x_1(I) \pm i\epsilon, x_3 \mp i\epsilon$ are not singular ²⁰⁾. Furthermore, there are no complex singularities for x_1, x_3 on their respective physical sheets. Indeed one has a Mandelstam representation where the integration extends over the shaded region of Fig. 7.

Perform first the continuation in y to values $y < -1$. The points $L_{y2}^\pm$ and $L_{y4}^\pm$ approach each other and simultaneously collide at $y = -1$, Fig. 7b, on their respective sheet II's and afterwards become complex conjugate pairs ²¹⁾. During this process the arcs γ_3 and γ_5 become fused into a single arc γ_{35} as illustrated in Fig. 7c. The arcs $\gamma_1, \gamma_2, \gamma_4$ suffer only minor deformations and no changes in their singular and non-singular characters.

It remains to perform the x continuation. As x is increased the picture of Fig. 7c deforms smoothly until one of the triangle singularities reaches $+1$. For definiteness suppose L_{x2}^+ does this first (i.e., $x_2 > x_4$). As $x + x_2$ passes through 0, and L_{x2}^+ loops up onto the physical sheet, the point of tangency of L_{x2}^+ jumps from arc γ_4 to arc γ_1 by way of infinity. Actually since we give x a small positive imaginary part to avoid having L_{x2}^+ pass right through 1 the point of tangency traces out a large semi-circle in the lower half complex x_3 plane.

Further increase in x brings about the analogous situation in the x_3 plane and for sufficiently large x (but $|x| < 1$) we end up with the configuration of Fig. 8a. A doubly anomalous Mandelstam representation in x_1, x_3 exists with the spectral function non-vanishing in the shaded area. It is useful to pause here to establish the location of the box singularities corresponding to all parts of γ_1 before making the final continuation to $x > 1$.

It is known ⁴⁾ that γ_1 between its contacts with L_{x2}^+ and L_{x4}^+ (see Fig. 8a) is singular when approached in the limits $\text{Im } x_1 / \text{Im } x_3 > 0$ from the physical sheets of x_1 and x_3 . It is non-singular when approached from the crossed limits $\text{Im } x_1 / \text{Im } x_3 < 0$. Conversely, the other two pieces of γ_1 are singular in the limit $\text{Im } x_1 / \text{Im } x_3 < 0$ but are non-singular when approached $\text{Im } x_1 / \text{Im } x_3 > 0$. This change of behaviour comes about because the box singularities loop the triangle singularities at their point of tangency since there $(\partial x_3 / \partial x_1) \gamma_1$ changes sign. The pattern of singular and non-singular limits described is of course necessary and sufficient for the existence of the (doubly anomalous) Mandelstam representation mentioned already.

At this point it is good to pick a convenient value of x_1 before finishing the x continuation. Pick x_1 anywhere in the interval

$$L_{x2}^+ < x_1(I) + i\epsilon < N^+(L_{x4}^+)$$

where $N^\pm$ are the locations of the box singularities in x_1 as a function of x_3

$$N^\pm(x_3) = (x_3^2 - 1)^{-1} \left\{ x_3(x_4 + x_2 x_4) + y x_4 + x x_2 \right. \\ \left. \pm \sqrt{(x_3 - L_{y2}^-)(x_3 - L_{y2}^+)(x_3 - L_{x4}^-)(x_3 - L_{x4}^+)} \right\}$$

(4.3)

The corresponding box singularities are $M_a = M^-(x_1 + i\epsilon)$ and $M_b = M^+(x_1 + i\epsilon)$ both reached by approaching the real x_3 axis from the lower half of sheet I. For $\epsilon \neq 0$ $\text{Im} M^-(x_1 + i\epsilon) > 0$ so M_a is actually found just above the real axis on sheet IIa. (Sheet IIa is the Riemann sheet reached by passing up through the anomalous cut $L_{x4}^+ < x_3 < 1$ from below.) Similarly, M_b is just above the real axis on sheet II.

As x is increased to 1 we reach the transition configuration of Fig. 8b. Continuing to $x_1 > 1 + i\epsilon$ the final pattern, Fig. 8c develops. Recalling the results of Section 3, we know that L_{x4}^+ has gone complex in the lower half physical sheet of x_3 and L_{x4}^- is at a complex conjugate position on some subsheet III²¹). The same phenomena occurring in the x_1 plane causes no problem since we are only interested in real x_1 approached via $x_1(I) + i\epsilon$.

For the initial value of x_1 that we have chosen the box singularities move smoothly as we pass above $x=1$, and, in particular, they do not change sheets since they nowhere come into contact with any lower order branch points. Hence M_b and M_a remain real and as $x \rightarrow \infty + i\epsilon$ one sees that they move off to $+\infty$ and $-\infty$ respectively

$$M_{(b)} = M_{\mp} \xrightarrow{x \rightarrow \infty} \frac{x}{(x_1^2 - 1)} \left\{ \pm |(x_1 - Ly\bar{4})(x_1 - Ly\dot{4})|^{\frac{1}{2}} + (x_1 y + x_4) \right\} \rightarrow \mp \infty \quad (4.4)$$

since

$$|(x_1 - Ly\bar{4})(x_1 - Ly\dot{4})|^{\frac{1}{2}} > |x_1 y + x_4| \quad (4.5)$$

for $|y| > 1$, $|x_1|, |x_4| < 1$.

As noted above for $\text{Im} x_1 = \epsilon \neq 0$ since M_a is on sheet II_{-a} we see that as $x \rightarrow \infty$ it rather quickly disappears from the boundary of the physical sheet. M_b , however, remains on the lower boundary for all $x > 1 + i\epsilon$. Having now completed our discussion of the singularities in x_3 generated by the box diagram we consider the x_3 dispersion relation.

5. DISPERSION RELATION

With the location of all singularities on the physical sheet known a simple application of Cauchy's theorem gives the dispersion relation in x_3 . Assuming no contribution from the circle at infinity we have

$$B(x_3) = \frac{1}{2\pi i} \int_C \frac{du}{u-x_3} B(u) \quad (5.1)$$

where the contour C is shown in Fig. 9a.

The expression (5.1) naturally divides into two pieces

$$B(x_3) = B_N + B_A \quad (5.2)$$

where

$$B_N(x_3) = \frac{1}{\pi} \int_1^{\infty} \frac{du}{u-x_3} \text{disc}_1 B(u) \quad (5.3)$$

and

$$B_A(x_3) = \frac{1}{\pi} \int_{L_{x_4}^+}^1 \frac{du}{u-x_3} \text{disc}_{L_{x_4}^+} B(u). \quad (5.4)$$

In Eq. (5.3) the discontinuity across the normal threshold cut is

$$\text{disc}_1 B(u) = \frac{1}{2i} \{ B(u+i\epsilon) - B(u-i\epsilon) \} \quad (5.5)$$

where it is understood that $B(u-i\epsilon)$ is to be reached by going around the singularity $L_{x_4}^+$ on the physical sheet. Similarly the discontinuity across the anomalous cut is

$$\text{disc}_{L_{x_4}^+} B = \frac{1}{2i} \{ B(u_1) - B(u_2) \}$$

where u_1 and u_2 are corresponding points on the opposite sides of the anomalous cut as illustrated in Fig. 9a. In writing our dispersion relation we have drawn the portion of the anomalous cut which lies on the physical sheet along the route swept out by $L_{x_4}^+$ during the x continuation. This is highly non-unique; we return to the point shortly. First a few remarks.

Since B is no longer real analytic in x_3 alone $\text{disc}_1 B(u) \neq \text{Im} B(u+i\epsilon)$ in contrast to the case for three point functions (form factors). However, $\text{disc}_1 B$ will still be large at resonance positions $u=u_R$ because the resonance pole on sheet II at $u_R - i\gamma_R$ ($\gamma_R > 0$) is near the point $u_R + i\epsilon$ for narrow ($\gamma_R \ll u_R$) resonances. In the form factor case $B(u_R - i\epsilon)$ is also large due to another pole on sheet II at $u_R + i\gamma_R$. Presumably this happens for scattering amplitudes too although the situation is more complex because at large x the box singularity M_b passes by with its attached cut. The presence of this additional cut means that one does not go onto sheet II but onto some other sheet when one moves upward through the cut(s) $1 \leq x_3 < \infty$. In any case a pole dominance model (like VMD) consists of the approximation

$$\text{disc}_1 B(u) = b \delta(u - u_R) \quad (5.6)$$

where b depends on all remaining variables x, y, x_1, x_2 and x_4 .

It is difficult to say much of anything about the strength of the anomalous contribution B_A given by Eq. (5.4). Unless additional singularities are encountered, the integration contour for B_A can be deformed to alternate configurations; only its end points are fixed²²⁾. For example see Fig. 9b. Note that in such deformations areas previously assigned to the physical sheet slide onto a subsheet and are replaced by areas formerly part of a subsheet (the shaded region in Fig. 9b). Clearly without additional physical input the notion of "physical sheet" is somewhat arbitrary.

For the point-vertex triangle and box diagrams we have seen that there are no additional singularities which would block the deformation of the anomalous contour. However, in a more realistic case with strong interaction "blobs" at the vertices the discontinuity, computed from Cutkosky's rules, will have additional x_3 singularities and some of these will be encountered in deformations of the anomalous contour.

Obviously it is of paramount importance to know the strength of the anomalous cut contribution. Unfortunately we have little useful experience to draw upon. Consider the resonance saturation hypothesis which in many cases seems to give a good approximation to discontinuities across normal threshold cuts in energy variables. This idea is no more than the instruction to deform the dispersion contour as we discussed above and to keep only the poles (and other singularities) crossed in the process. But as we have already said no one knows what singularities will be encountered under the anomalous cuts in a realistic example. This situation is typical of S matrix theory and is obviously unsatisfactory.

An alternate way of thinking about cut contributions is to use a two-dimensional electrostatic analogue. The cut represents a sheet of charge with charge density proportional to the cut discontinuity. If the total charge on the anomalous cut is roughly constant then at high energies $x \rightarrow \infty$ it will be spread more and more thinly as the cut stretches and the charge is located farther and farther from the real axis. (Recall that physically accessible regions are real x_3 with $|x_3| > 1$.) If such a picture is correct then at high energies the anomalous contribution B_A would fade away and leave the normal threshold contribution B_N (pole dominated?) in control. Of course, nature need not work this way, but it might.

One thing is clear, however. Pole dominance models applied to four-point functions involve not only the usual approximation, Eq. (5.6) which is familiar from form factor applications, but also the additional assumption $B_A(x_3) \approx 0$. The latter, if it is true anywhere, is more likely to be true at high energies when the branch point has moved far from the physical region.

It is perhaps interesting to consider the explicit form of the amplitude from the box diagram, Fig. 1. In the stability region $|x| < 1$, $|y| < 1$ the x_3 dispersion relation is well known^{23),13)}

$$B(x_3) = \frac{1}{\pi} \int_1^{\infty} \frac{du}{u-x_3} \text{disc } B(u)$$

$$\text{disc } B(u) = \frac{1}{\pi} \left((u^2-1)(N^+-x_1)(N^--x_1) \right)^{-\frac{1}{2}} \ln \mathcal{E}$$

$$\mathcal{E} = \frac{N^+ + N^- - 2x_1 + 2\sqrt{(N^+u)(N^-u)}}{N^+ + N^- - 2x_1 - 2\sqrt{(N^+-u)(N^--u)}}$$

(5.7)

where $N^{\pm}$ are the quantities of Eq. (4.3) evaluated at $x_3 = u$. All square roots in Eq. (5.7) are positive and the logarithm is on its principal branch.

The continuation to $y < -1$ is trivial. When we continue to $x > 1 + i\epsilon$ we must deform the integration contour in Eq. (5.7) into the lower half plane to avoid the moving singularity L_{x4}^+ . The result is

$$B = B_A + \frac{1}{\pi} \int_1^{\infty} \frac{du}{u-x_3} \text{disc}_1 B(u)$$

(5.8)

where

$$\text{disc, } B(u) = -\frac{1}{2\pi} (u^2 - 1)(x_1 - N^-)(x_1 - N^+)^{-\frac{1}{2}} \ln \left\{ \frac{N^+ + N^- - 2x_1 - 2\sqrt{(x_1 - N^+)(x_1 - N^-)}}{N^+ + N^- - 2x_1 + 2\sqrt{(x_1 - N^+)(x_1 - N^-)}} e^{-2\pi i} \right\} \quad (5.9)$$

for $u < M^-(x_1) \equiv M_b$

$$\text{disc, } B(u) = \frac{i}{2\pi} (u^2 - 1)(N^+ - x_1)(x_1 - N^-)^{-\frac{1}{2}} \ln \left\{ \frac{N^+ + N^- - 2x_1 - 2i\sqrt{(N^+ - x_1)(x_1 - N^-)}}{N^+ + N^- - 2x_1 + 2i\sqrt{(N^+ - x_1)(x_1 - N^-)}} \right\} \quad (5.10)$$

for $u > M_b$, and ²⁴⁾

$$B_A = \frac{i}{\pi} \int_{L_{x_4}^+}^1 \frac{du}{u - x_3} \frac{1}{\sqrt{(1 - u^2)(x_1 - N^+)(N^- - x_1)}} \quad (5.11)$$

In the $x \rightarrow \infty$ limit one sees that the anomalous contribution is $O(x^{-1})$ and hence negligible compared to the leading $O(x^{-1} \ln x)$ contribution of the normal threshold term. Moreover, the parts of the dispersion integral (5.11) having $u \approx L_{x_4}^+$ give a very small $O(x^{-3})$ contribution at large x . What relevance this has for realistic cases is unclear.

At low energies it seems more certain that the anomalous singularities play an important role; they can be quite close to the experimental region. (Most applications of PCAC in current algebra calculations are in the low energy region ²⁵⁾). Consider pion photoproduction. The diagram of Fig. 10a generates a singularity in the photon mass variable at

$$k^2 \equiv M_3^2 = 2m_\pi^2 \left\{ 1 + x \frac{m_\pi}{2m_N} - i \sqrt{(x^2-1) \left(1 - \frac{m_\pi^2}{4m_N^2} \right)} \right\} \quad (5.12)$$

which near threshold $x \gtrsim 1$ is quite close the spacelike region ($k^2 \leq 0$) probed in electro and photoproduction. It ought to have a observable effect.

Nuclear targets offer more striking examples. For example, consider the diagram of Fig. 10b where the incoming line labelled A is a nucleus of A nucleons and the line A-1 is some state, perhaps excited, of a nucleus with A-1 nucleons. The particular final state $f = \pi + A, \pi + N + (A-1)$, etc., will not matter here. It is easy to work out that

$$x_4 \cong 1 - \frac{\delta}{m_N} \left(\frac{A}{A-1} \right) \quad (5.13)$$

where δ is the binding energy defined by $\delta = M_{A-1} + m_N - M_A$. For energies s near threshold ($x \gtrsim 1$) the singularity in the photon mass is at

$$k^2 = 2m_N \delta \left(\frac{A}{A-1} \right) \quad (5.14)$$

Taking a typical value $\delta = 6$ MeV and $A \gg 1$ we see $k^2 \cong 0.6 m_\pi^2$ which is well below the normal threshold at $4m_\pi^2$ and way below the rho meson enhancement at $k^2 = m_\rho^2 \cong 31 m_\pi^2$. Clearly, at low energies the anomalous singularities are of utmost importance. Since there are A such contributions²⁶⁾ coming from the A constituents of the target nucleus, the net effect is an amplitude $\sim A$ as expected.

At high energies these triangle singularities move far away and if their contributions fade away the physics will be controlled by normal threshold contributions such as the rho meson. These latter generate an amplitude $\sim A^{2/3}$ in the idealized case at large A.

We see here then the dispersion theoretic description of the "switching on" of the nuclear shadowing with energy which has been discussed already from other points of view ²⁷⁾. Crudely one would estimate the transition to occur at energies for which the vector meson pole(s) and triangle singularity are equidistant from the physical region. If there are other important normal threshold contributions more massive than the known vector mesons (the behaviour of the elastic nucleon form factors almost guarantees this) this transition will be pushed to higher energies. A gross difference in the strength of the numerators (discontinuities) of the normal and anomalous contributions would affect the position of this transition also.

If we approximate the normal cut by the rho pole alone and assume the strengths of the anomalous and normal cut discontinuities are the same except in their A dependences, our value for the transition energy is the usual one

$$E_{\gamma}(\text{lab}) \approx A^{\frac{1}{3}} \frac{m_{\rho}^2}{2m_N}. \quad (5.15)$$

6. SUMMARY

We have examined the analytic structure in an external mass variable of the simplest non-trivial perturbation theory diagrams for a two-body scattering amplitude. In addition to the expected normal threshold branch points we have shown that a complex triangle singularity occurs on the physical sheet in the lower half plane. The box singularities seem relatively uninteresting. Only one is accessible to the physical sheet; it remains real and is located by approaching the normal threshold cut from the $-i\epsilon$ direction.

The presence of other singularities means that dispersion relations in an external mass variable have additional contributions which do not occur in the more familiar case of vertex functions. (However, similar additional contributions are quite familiar in the study of form factors of weakly bound systems ⁶⁾.) It is a necessary (but not sufficient!) condition that these additional contributions be weak in order that models like VMD and PCAC work for four-point functions. This is more likely to occur at high energies because then the additional singularities are far from the physical region.

As an example we have briefly discussed the role played by the triangle singularities in amplitudes for photon induced processes on nuclei. As the triangle singularity contribution supposedly fades away with increasing energy the amplitude changes from an A to an $A^{2/3}$ behaviour (nuclear shadowing).

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A P P E N D I X

We give in this Appendix some useful algebraic results related to the curve Γ . Some of these expressions are given in Ref. 19) in terms of angles defined by $\cos \theta_i = -x_i$. We have found it easier and more convenient to avoid introducing these angles.

First consider the ordering of the triangle singularities. In Section 4 we choose the order in x_1 as $L_{y4}^- < L_{y4}^+ < L_{x2}^- < L_{x2}^+$. Introduce the positive quantities $z_i = + \sqrt{(1-x_i^2)}$, $z_x = \sqrt{(1-x^2)}$, etc. Then $L_{y4}^+ < L_{x2}^-$ becomes

$$z_y z_4 + z_x z_2 < yx_4 - xx_2. \quad (\text{A.1})$$

Squaring both sides of Eq. (A.1) one finds

$$z_y z_2 + z_x z_4 < |yx_2 - xx_4|. \quad (\text{A.2})$$

There are two cases: (i) $yx_2 - xx_4 > 0$ in which case Eq. (A.2) says $L_{y2}^+ < L_{x4}^-$, and (ii) $yx_2 - xx_4 < 0$ for which Eq. (A.2) says $L_{x4}^+ < L_{y2}^-$. Now, since $L_{ij}^- < L_{ij}^+$ always, the complete order of the triangle singularities in x_3 is fixed. Our choice in Section 4 was case (i).

Next consider the points at which the real arcs of Γ are tangent to the lower order triangle singularities. We take first the tangency to $x_1 = L_{x2}^+$. This occurs at [see Eq. (4.1)]

$$M(L_{x2}^+) \equiv M^\pm(L_{x2}^+) = (L_{x2}^{+2} - 1)^{-1} \left\{ L_{x2}^+ (yx + x_2 x_4) + yx_2 + x x_4 \right\}. \quad (\text{A.3})$$

The expression (A.3) is not very transparent. However, it becomes so if we go to the situation in which L_{x2}^+ is about to loop the normal threshold. To this end let $x+x_2 = -\epsilon$ and use

$$L_{x2}^+ = 1 - \frac{\epsilon^2}{2(1-x^2)} + O(\epsilon^3) \quad (A.4)$$

to simplify Eq. (A.3) to

$$M(L_{x2}^+) = \frac{(1-x^2)(y+x_4)}{\epsilon} + O(\epsilon^0). \quad (A.5)$$

Thus, for $y+x_4 < 0$ and ϵ sufficiently small but positive $M < -1$; this says γ_4 is tangent to L_{x2}^+ . For $\epsilon < 0$ we see that arc γ_1 is tangent to L_{x2}^+ . The situation for non-infinitesimal ϵ follows by continuity.

In an identical manner one works out that the point of tangency to L_{y4}^- occurs at

$$M(L_{y4}^-) = - \frac{(1-y^2)(x-x_2)}{\epsilon} + O(\epsilon^0) \quad (A.6)$$

where $y-x_4 = -\epsilon$ and hence for $\epsilon > 0$, $M < -1$, which implies that arc γ_3 is tangent to L_{y4}^- . The tangent points to the lines L_{y2}^- and L_{x4}^+ are readily obtained also. In this manner one establishes the pattern we have drawn in Fig. 7a.

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- 19) J. Tarski - J.Math.Phys. 1, 154 (1960).

- 20) When we wish to indicate that a variable is on Riemann sheet n we will use the notation : variable (n) .

- 21) The Riemann sheeting is more complicated for the box diagram than the simple point-vertex triangle diagram. The box diagram has two pairs of triangle singularities. Members of one set can pass through the cut generated by the other set. It is unproductive to go into details of this here. We will continue to use the names sheet II, III, etc., in a symbolic sense.

- 22) Our division, Eq. (5.2), is not unique. One clearly need not draw the anomalous cut through the branch point at $x_3 = 1$. One could choose any point on the real axis $1 < x_3 - i\epsilon < \infty$. This amounts to a different partition of the amplitude between B_N and B_A .

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FIGURE CAPTIONS

Figure 1 : The box diagram which we study as a function of complex M_3^2 at fixed M_1^2, M_2^2, M_4^2, s and t .

Figure 2 :

- 2a A triangle diagram which has no anomalous singularities in x_3 on the physical sheet.
- 2b A triangle diagram which has a complex singularity in x_3 on the physical sheet.
- 2c The bubble diagram which has only the normal threshold branch point $x_3 = 1$.

Figure 3 : The triangle singularity ellipse for: (a) $x_2 < 0$ and (b) $x_2 > 0$. At the value of y indicated by the dotted line the singularity L_{y2}^- is on sheet III in case (a) and on sheet II in case (b). In both cases L_{y2}^+ is on sheet II at this value of y .

Figure 4 : Motion of the triangle singularities $L_{y2}^\pm$ on sheet II as the continuation to $y < -1$ is carried out. The dotted path is on sheet III and corresponds to the region $y - x_2 > 0$.

Figure 5 : Motion of the triangle singularities $L_{x4}^\pm$ as the continuation to $x > 1$ is carried out. The case $x_4 < 0$ is pictured. The solid (dotted) path of L_{x4}^+ is on sheet I (II). L_{x4}^- is on sheet III.

Figure 6 : The box diagram which is equivalent to Fig. 1.

Figure 7 : Intersection of the leading singularity surface of the box diagram with the real x_1, x_3 plane for $x+x_2 < 0$, $x+x_4 < 0$ and: (a) $|x| < 1$, $|y| < 1$, (b) $|x| < 1$, $y = -1$, (c) $|x| < 1$, $y < -1$.

Figure 8 : Intersection of the leading singularity surface of the box diagram with the real x_1, x_3 plane for (a) $y < -1$, $|x| < 1$, $x+x_2 > 0$, $x+x_4 > 0$, (b) $y < -1$, $x = 1$, (c) $y < -1$, $x > 1$.

Figure 9 :

- 9a A possible contour for the dispersion relation in x_3 .
- 9b The result of deforming the contour. The shaded area formerly was assigned to a different Riemann sheet.

Figure 10 : Triangle diagrams giving complex singularities in the photon mass variable for: (a) pion production $\gamma + N \rightarrow \pi + N$ (b) some photonuclear reaction $\gamma + A \rightarrow f$.

FIG.1

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FIG.10