Estimating non-gaussianity in the microwave background

A.F. Heavens
Institute for Astronomy, University of Edinburgh, Blackford Hill, Edinburgh EH9 3HJ, U.K.

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ABSTRACT

The bispectrum of the microwave background sky is a possible discriminator between inflationary and defect models of structure formation in the Universe. The bispectrum, which is the analogue of the temperature 3-point correlation function in harmonic space, is zero for most inflationary models, but non-zero for non-gaussian models. The expected departures from zero are small, and easily masked by noise, so it is important to be able to estimate the bispectrum coefficients as accurately as possible, and to know the errors and correlations between the estimates so they may be used in combination as a diagnostic to rule out non-gaussian models. This paper presents a method for estimating in an unbiased way the bispectrum from a microwave background map in the near-gaussian limit. The method is optimal, in the sense that no other method can have smaller error bars, and in addition, the covariances between the bispectrum estimates are calculated explicitly. The method deals automatically with partial sky coverage and arbitrary noise correlations without modification. A preliminary application to the Cosmic Background Explorer 4-year dataset shows no evidence for non-gaussian behaviour.

1 INTRODUCTION

The question of whether or not the microwave background sky is well-approximated by a gaussian random field is important for distinguishing inflationary and defect models of the early Universe. Inflationary models predict immeasurably small non-gaussian components, arising from gravity waves (?) and the Rees-Sciama effect (?). The difficulty which besets such tests is that the predicted departures from gaussian behaviour for most defect models are small (e.g. ?), and are correspondingly difficult to detect in the presence of noise, which may be instrumental or cosmic variance. This makes it extremely important to be able to calculate the statistical properties of the non-gaussian discriminants. In particular, it will probably be necessary to combine the results of a large number of estimates, to obtain a statistically significant departure from gaussianity (if it exists), or to make a convincing case that the sky is indeed gaussian.

Ironically, the best evidence for an inflationary model may well come not from a specific test for non-gaussianity, but rather from the power spectrum. For inflationary models, the power spectrum of temperature fluctuations is predicted to have a reasonable amount of structure in it, with multiple acoustic oscillation peaks which should be measurable by future satellites such as MAP and Planck (?). Should such structures be found, and found to agree with the inflationary model predictions within the errors, the case against any form of defect model would be strong, and even the present knowledge of the power spectrum appears already to rule out many defect models (?). In this case, the absence of non-gaussian signatures would be a useful confirmation. However, if the power spectrum turns out not to be well-fitted by inflationary models, the question of the gaussian nature or otherwise of the fluctuations becomes correspondingly more important. There are many ways to approach the problem of determining whether fluctuations are gaussian, in large-scale structure and in the CMB. These include the 3-point function (e.g. ?, ?, ?), the genus and Euler-Poincaré statistic (?), ?, ?), peak statistics (?, ?, ?) and studies of tensor modes in the CMB (?). The approach we take here is to investigate the bispectrum, for which some studies have been made in the large-scale structure literature (?), ?) and in the CMB (?). There may be sharper tools for detecting specific non-gaussian models, but the rationale for this approach is that the bispectrum offers a generic test for non-gaussian models, in the following sense: a general field will have non-zero connected n-point functions at all orders, and the bispectrum is the lowest statistic (with n > 1) for which a gaussian field has zero expectation value.

The principal advantages of the approach detailed in this paper are that it deals automatically with masked regions of the sky and correlated noise, and that the estimates of the bispectrum coefficients come with error bars and covariances between the errors. This last point is particularly important when one realises that a single bispectrum coefficient estimate is unlikely to rule out a model, because the cosmic variance is often larger than the expected signal, and so one is going to need many coefficients in practice. A final point is that, for gaussian fluctuations, the estimator below cannot be improved, in the sense of having a smaller error bar.
2 Method

The optimisation procedure in this paper is a generalisation of the optimal quadratic estimator for the power spectrum, presented by \(A.F.\) Heavens. For consistency, we follow his notation as far as possible. Let \(x_i\) be the temperature fluctuation \(\Delta T/T\) in some sky pixel \(i\). The temperature map is expanded in spherical harmonics \(Y^m_l(\Omega)\) in the usual way:

\[
a_{\ell m}(\Omega) = \int d\Omega \Delta T/T Y^{m*}_l(\Omega) \approx \sum x_i Y^{m*}_l(\hat{\Omega}_i, \phi_i) \Delta \Omega_i
\]

where \(d\Omega, \Delta \Omega\) represent elements of solid angle, and \(\hat{\Omega}_i\) and \(\phi_i\) are polar coordinates. The power spectrum is defined as

\[
C_{\ell} = \langle |a_{\ell m}|^2 \rangle
\]

where the angle brackets indicate an ensemble average. Expectation values of products of distinct spherical harmonic coefficients are zero by isotropy, independently of whether the temperature map is gaussian or not. The bispectrum is defined as

\[
B(\ell_1, \ell_2, \ell_3, m_1, m_2, m_3) = \langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3} \rangle
\]

It is zero, unless the indices comply with the following triangle closure constraints (e.g., \(7\)): \(m_1 + m_2 + m_3 = 0;\) \(\ell_1 + \ell_2 + \ell_3 = \text{even};\) \(|\ell_i - \ell_j| \leq \ell_k \leq \ell_i + \ell_j\) for \(i, j, k = 1, 2, 3\).

We seek an estimator of \(B\) which is lossless, if possible, in the sense that it contains as much information as the bispectrum estimates, but will not be the bispectrum estimates in the sense that it contains as much information as the triplet data covariance matrix: \(\langle i, j, k \rangle\) defined as

\[
E_{ijk}^\alpha = \sum_{\text{pixels}} E_{ijk}^\alpha x_i x_j x_k
\]

We will find that the \(y_{\alpha}\) are related to the bispectrum estimates, but will not be the bispectrum estimators themselves. We introduce the shorthand notation \(\alpha \equiv \{\ell_1, \ell_2, \ell_3, m_1, m_2, m_3\}\), and we also combine the list of data triplets into a data vector with elements labelled by \(A\):

\[
D_A \equiv \{x_i x_j x_k\}
\]

where \(A\) represents some triplet \(\{i, j, k\}\). The \(E_{ijk}^\alpha\) are coefficients to be determined. The mean of \(y_{\alpha}\) involves the 3-point function, which may be written in terms of the bispectra as follows:

\[
\mu_A \equiv \langle x_i x_j x_k \rangle = \sum_{\alpha} B_{\alpha} R_{ijk}^\alpha
\]

where we have assumed that the noise has a zero 3-point function. If it is known and non-zero, it may be added. The functions connecting the 3-point functions in real and harmonic space are (7):

\[
R_{ijk}^\alpha = \frac{\pi}{2} N(\gamma_{ij}, \gamma_{jk}, \gamma_{ki}) W_{\ell_1} W_{\ell_2} W_{\ell_3}
\]

where

\[
H_{\ell_1, \ell_2, \ell_3}^{m_1, m_2, m_3} = \int d\Omega Y_{\ell_1}^{m_1*} Y_{\ell_2}^{m_2*} Y_{\ell_3}^{m_3*}
\]

and can be related to Clebsch-Gordon coefficients. The effect of beam-smearing (here modelled by a gaussian) is through the window functions

\[
W_\ell = \exp\left[-\ell(\ell + 1)a^2/2\right].
\]

\(\gamma_{ij}\) is the angle between pixels \(i\) and \(j\), and

\[
N^2(\gamma_{ij}, \gamma_{jk}, \gamma_{ki}) = 1 - \cos^2 \gamma_{ij} - \cos^2 \gamma_{jk} - \cos^2 \gamma_{ik} + 2 \cos \gamma_{ij} \cos \gamma_{jk} \cos \gamma_{ik}.
\]

2.1 Optimal estimator \(y_\alpha\)

We wish to minimise the variance of \(y_\alpha\) (cf \? for power spectrum), which involves the 6-point function. The means are

\[
\langle y_\alpha \rangle = \sum_{\alpha \neq jk} B_{\alpha}^* R_{ijkl} E_{ijkl}^\alpha.
\]

The covariance between the \(y_\alpha\) is \(C_{\alpha \alpha'} \equiv \langle y_\alpha y_{\alpha'} \rangle - \langle y_\alpha \rangle \langle y_{\alpha'} \rangle\) which we obtain from the triplet data covariance matrix:

\[
\langle x_i x_j x_k x_x x_x x_x \rangle - \langle x_i x_j x_k \rangle \langle x_x x_x x_x \rangle
\]

We now make an assumption concerning the departures from gaussianity. Since these are expected to be small, we approximate the covariance matrix by the covariance matrix for a gaussian field with the same power spectrum. This assumes that the bispectrum is small compared with the cosmic variance, and also assumes that the connected 4-point function is small. Strictly, this method is optimal for testing the hypothesis that the field is gaussian, but it should be very close to optimal for practical cases, since the expectation is that the bispectrum will be small. If the assumption is not justified, and the bispectrum is not small compared with the cosmic variance, detection will not be difficult in any event. If this turns out to be the case, it will be important to check that the estimator remains unbiased in the case of large intrinsic bispectrum.

In the gaussian approximation, \(\langle x_i x_j x_k \rangle = 0\), and we use Wick’s theorem to write

\[
\langle x_i x_j x_k x_x x_x x_x \rangle = \xi_{ij} \xi_{kx} \xi_{j'k'} + \text{permutations (15 terms)}.
\]

where we have defined the 2-point function of the temperature field:

\[
P_\ell = \text{Legendre polynomials and } N_{ij} = \text{the noise covariance matrix}.
\]

We can then compute the covariance matrix for \(y_\alpha\):

\[
V_{\alpha \alpha'} = \langle y_\alpha y_{\alpha'} \rangle = \langle \xi_{ij} \xi_{kx} \xi_{j'k'} + \text{permutations} \rangle E_{ijkl} E_{i'j'k'}.\]

and we have from now adopted the summation convention for repeated pixel indices, and also, unless stated otherwise, \(\alpha\) indices. The products of \(\xi\) terms are of two types: there are 6 terms like \(\xi_{ij} \xi_{j'k'}\), with one of each pair of subscripts from each distinct \(E\), and 9 terms of the form \(\xi_{ij} \xi_{kx} \xi_{j'k'}\), where only one \(\xi\) mixes dashed and undashed indices. Since the \(E_{ijkl}^\alpha\) are symmetric to permutations in \(\{i, j, k, l\}\), we get

\[
V_{\alpha \alpha'} = (6 \xi_{ij} \xi_{kx} + 9 \xi_{jk} \xi_{i'k'}) E_{ijkl} E_{i'j'k'}.\]
We now minimise the variance \( V_{\alpha\alpha} \) (not summed) with respect to \( E_{\alpha j k} \), subject to a normalisation constraint on \( E \) to ensure it is not driven to zero. We choose

\[
R_{ij k}^\alpha E_{\alpha j k} = 1,
\]

giving

\[
\xi_{ij r} (6 \xi_{j r s} \xi_{i k s} + 9 \xi_{i r s} \xi_{j k s}) E_{ij r s k}^\alpha = \lambda R_{ij k}^\alpha
\]

where \( \lambda \) is a Lagrange multiplier. Multiplying by \( \xi_{ij r}^{-1} \xi_{k r}^{-1} \) and summing over \( i \), gives

\[
6 \xi_{k hr}^\alpha E_{ij r s k}^\alpha + 9 \xi_{i r s}^\alpha \delta_{ij k}^\alpha E_{ij r s k}^\alpha = \lambda \xi_{j r s}^{-1} \xi_{i r k}^{-1} R_{ij k}^\alpha.
\]

Taking the trace of this equation, and inserting it into the second term gives the coefficients we require for \( \alpha \) to have its minimum error bar:

\[
E_{ij k} = \frac{1}{6} \xi_{l i r}^{-1} \left[ \xi_{k r}^{-1} \xi_{l i r}^{-1} \right] R_{il k}^\alpha.
\]

In this expression, \( \lambda \) has been set to unity for convenience, and \( N \) is the number of pixels in the map. Provided that \( \lambda \) is finite, its value does not affect the bispectrum information content of \( y_{\alpha} \); any multiple of \( y_{\alpha} \) contains the same information as \( y_{\alpha} \) itself. This makes obvious sense, and can be shown formally via the Fisher information matrix (below).

### 3 LOSSLESS ESTIMATOR OF THE BISPECTRUM

In order to estimate how well the \( y_{\alpha} \) will perform in estimating the desired parameters \( B_{\alpha} \), we compute the Fisher information matrix (7)

\[
F_{\alpha\alpha} = -\frac{\partial^2 \ln p}{\partial B_{\alpha} \partial B_{\alpha'}}
\]

where \( p \) is the posterior probability distribution for the parameters (equal to the likelihood, if uniform priors for the parameters are assigned). For a data vector with components with means \( \mu \) and covariance matrix \( C \) the Fisher matrix is

\[
F_{\alpha\alpha} = \frac{1}{2} \text{Tr} \left[ C^{-1} \frac{\partial C}{\partial B_{\alpha}} C^{-1} \frac{\partial C}{\partial B_{\alpha'}} + 2 C^{-1} \frac{\partial \mu}{\partial B_{\alpha}} \frac{\partial \mu}{\partial B_{\alpha'}} \right].
\]

The error on the parameters \( B_{\alpha} \) is contained in this matrix: if all other parameters are known, the minimum error is the conditional one, \( \sigma_{B_{\alpha}} = 1 / \sqrt{F_{\alpha\alpha}} \). If all parameters are to be estimated from the data, then the appropriate error is the marginal error \( \sqrt{F_{\alpha\alpha}} \). This assumes the probability surface is adequately approximated by a second-order Taylor expansion at the peak.

As expected for the ‘near-gaussian’ approximation, the covariance matrix for either the triplets \( x_i x_j \) or the \( y_{\alpha} \) does not depend on the parameters to be estimated – the Fisher matrix is determined only by the derivatives of the mean values. For the triplets,

\[
F_{\alpha\alpha} = C_{ij k'}^{\alpha-1} R_{ij k'}^\alpha R_{ij k'}^\alpha
\]

\[
= \left[ \xi_{ij r} (6 \xi_{j r s} \xi_{i k s} + 9 \xi_{i r s} \xi_{j k s}) \right]^{-1} R_{ij k}^\alpha R_{ij k}^\alpha
\]

A similar procedure to the computation of \( E \) above gives the inverse covariance matrix

\[
C_{ij k' j' k'}^{-1} = \frac{1}{6} \xi_{ij r} \xi_{j r k'} \xi_{i r k'} - \frac{1}{2 (2 + 3 N)} \xi_{ij r} \xi_{j r k'} \xi_{i r k'}
\]

This is an important simplification for computational reasons: without it, the inversion of an \( N^3 \times N^3 \) matrix \( C \) would be impractically slow. Decomposing its inverse into \( N \times N \) matrices \( \xi^{-1} \) is much faster.

Since we can recreate the original temperature map \( \{ \xi_i \} \) from the triplets, (24) is also the Fisher information matrix for the original, entire map. We now make a comparison with the Fisher matrix for the \( y_{\alpha} \) – are their errors as small as is possible with the entire map? The covariance matrix for the \( y_{\alpha} \) is, for the optimal choice of \( E \) coefficients (21),

\[
V_{\alpha\alpha} = \xi_{ij r} (6 \xi_{j r s} \xi_{i k s} + 9 \xi_{i r s} \xi_{j k s})
\]

\[
C_{ij k' j' k'} R_{ij k'}^\alpha C_{ij k' j' k'}^\alpha R_{ij k'}^\alpha
\]

\[
= F_{\alpha\alpha}
\]

Since the ensemble average of \( y_{\alpha} \) is

\[
\langle y_{\alpha} \rangle = B_{\alpha} F_{\alpha\alpha}^{-1} y_{\alpha}
\]

\[
\langle y_{\alpha} \rangle = B_{\alpha} F_{\alpha\alpha}^{-1} y_{\alpha}
\]

i.e. we obtain the simple result

\[
\langle y_{\alpha} \rangle = B_{\alpha} F_{\alpha\alpha}^{-1} y_{\alpha}
\]

Consequently, we can use the vector

\[
\tilde{B}_{\alpha} = F_{\alpha\alpha}^{-1} y_{\alpha}
\]

as an estimator of the bispectrum. It is unbiased:

\[
\langle \tilde{B}_{\alpha} \rangle = B_{\alpha}
\]

\[
\langle \tilde{B}_{\alpha} \rangle = B_{\alpha}
\]

and the bispectrum estimates also have calculable covariance properties:

\[
C_{\alpha\alpha'} = \langle \tilde{B}_{\alpha} \tilde{B}_{\alpha'} \rangle - \langle \tilde{B}_{\alpha} \rangle \langle \tilde{B}_{\alpha'} \rangle
\]

\[
C_{\alpha\alpha'} = \langle \tilde{B}_{\alpha} \tilde{B}_{\alpha'} \rangle - \langle \tilde{B}_{\alpha} \rangle \langle \tilde{B}_{\alpha'} \rangle
\]

\[
C_{\alpha\alpha'} = \langle \tilde{B}_{\alpha} \tilde{B}_{\alpha'} \rangle - \langle \tilde{B}_{\alpha} \rangle \langle \tilde{B}_{\alpha'} \rangle
\]

\[
= \langle \tilde{B}_{\alpha} \tilde{B}_{\alpha'} \rangle - \langle \tilde{B}_{\alpha} \rangle \langle \tilde{B}_{\alpha'} \rangle
\]

This also proves that the estimators are optimal, by the Fisher-Cramer-Rao inequality.

### 4 APPLICATION TO COBE 4-YEAR DMR DATA:

We illustrate the method by applying the method to the COBE DMR 4-year data, focussing on measuring low-order coefficients. For this experiment, the width of the approximately gaussian beam is \( \sigma = 3.2^\circ \) (7). The method is computationally expensive, and is in the process of being optimised, but for the moment, the approach taken is to average the \( \sim 4000 \) unmasked pixels of the COBE dataset into larger pixels of roughly 12 degrees square. This introduces an additional effective gaussian smoothing for large scale coefficients of \( 12^\circ / \sqrt{12} \), which is added in quadrature to the COBE beam. We shall see the effect of this additional pixelisation below, in the form of an error bar larger than that.

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of cosmic variance, especially for the higher harmonics. The effective beam suppresses contributions to the bispectrum from harmonics with $\ell$ larger than $\ell(\ell + 1)\sigma_{\ell m}^2/2 \approx 1$, i.e. $\ell > 17$. We therefore truncate the summations in the estimator for $\Omega$, and the Fisher matrix (scalar in this case) at the conservative limit of $\ell_{\text{max}} = 40$. Pixel errors are taken from the COBE DMR datasets, and assumed to be independent. Averaging is done by inverse-variance weighting. The power spectrum is assumed to have a Harrison-Zeldovich spectrum, and the normalisation is $Q_{\text{rms}} = 18.4 \mu K$ (\text{"}). Coefficients are chosen for which non-Gaussian predictions are quoted in (\text{"}).

\[
\begin{align*}
B(222, 11 - 2) &= (2.5 \pm 4.1) \times 10^{-15} \\
B(444, 22 - 4) &= (5.4 \pm 6.5) \times 10^{-15} \\
B(666, 33 - 6) &= (12.4 \pm 8.9) \times 10^{-15} \\
B(888, 44 - 8) &= (1.5 \pm 11.6) \times 10^{-15} \\
B(101010, 5510) &= (-16.3 \pm 16.6) \times 10^{-15}
\end{align*}
\]

(32)

from which we see that there is no evidence of non-gaussianity, at least for these coefficients. Note that for the first bispectrum estimate, the cosmic variance corresponds to an error of $1.4 \times 10^{-15}$ (e.g. ?). These very large-scale modes are not ideal for this sort of study – a higher-resolution experiment generally has higher signal-to-noise (\text{"}). A recent preprint (\text{"}) claims a detection of non-gaussianity at $\ell = 16$, a mode which is not probed here.

5 SUMMARY

In this paper, we see that it is possible to construct an estimator for the bispectrum which is lossless, in the sense that it contains as much information on the bispectrum as the entire map. It is also unbiased, and the covariance properties of the estimators are calculable. The estimator involves one approximation – that the departures from gaussianity are small. In this limit, there is no other method which will lead to smaller error bars. The fact that the covariance properties are known is important in practical cases, because the bispectrum may well be small in comparison to cosmic variance, so a single estimate is unlikely to be sufficient to rule out many non-gaussian models. Many estimates (with different $\ell_1 \ell_2 \ell_3 m_1 m_2 m_3$) would be required. As a test for non-gaussianity, this method has the advantage that confidence levels can be computed analytically, without recourse to Monte Carlo simulation.

The notion of an optimal method is defined rather precisely in terms of information content and bias, but the issue of whether a method is good or not is wider than this. There is no doubt that the number of computations required to do this analysis is very large, dominated for reasonable pixel counts by computation of the $R$ coefficients. This can be aided by precomputing Clebsch-Gordan coefficients (\text{"}) and using a packed-storage algorithm, and by using parallel computers, for which this problem is ideal. However, it is clear that it will not be possible to deal directly with the entire dataset from Planck without some form of pre-compression, perhaps along the lines above for the low-$\ell$ modes. For high-$\ell$ modes, subdivision of the sky into essentially independent datasets may be required. This is an inherent problem for high-order statistics, but it may also be required even for the power spectrum.

A further point to note is that the method deals automatically with sky coverage which is not complete, and arbitrary noise correlations. These are the two crucial tests of any method, since they will be a feature of future high-quality experiments such as Planck and MAP. Three-point function estimators have traditionally not computed the covariances directly, but rely on simulation tests to decide significance (e.g. ?). This is the disadvantage of many methods (e.g. genus, extrema correlation functions) for which the evaluation of errors may be difficult analytically. Against this one has, of course, to balance speed advantages.

A preliminary application to the COBE 4-year data shows large-scale bispectrum coefficients consistent with zero. However, the errors are sufficiently large that these coefficients do not rule out non-gaussian models with confidence.