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SUM RULES FOR INCLUSIVE REACTIONS AND DISCONTINUITY FORMULAE

G. Veneziano *)
CERN - Geneva

A B S T R A C T

The basic discontinuity formula, generally regarded as the building block of unitarity, is shown to be equivalent to a set of linear sum rules, plus the existence of isolated, factorizable poles in the physical region.

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*) Present address : Centre for Theoretical Physics, M.I.T., Cambridge, Mass.
On leave of absence from the Weizmann Institute of Science, Rehovoth.

Sum rules relating different inclusive cross-sections have been recently derived ¹⁾⁻³⁾ from energy momentum conservation and the definition of inclusive cross-sections in terms of exclusive ones.

The content of these sum rules becomes non-trivial once we compute inclusive cross-sections without explicit reference to exclusive processes. Such is, for instance, the approach of Mueller ⁴⁾, in which, by use of unitarity, the inclusive cross-section is computed as a certain discontinuity of an elastic amplitude. In such a case the sum rules become linear conditions (in the T matrix) relating discontinuities of different amplitudes to each other. In this note we shall investigate further the dynamical content of these latter sum rules.

In order to gain the generality we need, we consider the process (Fig. 1a) $\alpha + \beta \rightarrow \gamma + \delta$ where α, β (γ, δ) represent any collection of incoming (outgoing) particles ⁵⁾. We now define a quantity $t(\alpha, \beta \rightarrow \gamma, \delta)$ as

$$t(\alpha, \beta \rightarrow \gamma, \delta) = i(2\pi)^4 \sum_n \frac{1}{n!} \int d^3q_i \dots d^3q_n T(\alpha \rightarrow \gamma + n) T(\beta + n \rightarrow \delta) \delta(P_\alpha - P_\gamma - \sum_i q_i) \quad (1)$$

where $d^3q_i \equiv d^3q_i / (2\pi)^3 2E_i$ and $T(i \rightarrow f) = \langle f | T | i \rangle$ is an ordinary T matrix element in the notations of Eden et al. ⁶⁾. If $(P_\alpha - P_\gamma)^2 = (P_\beta - P_\delta)^2 < 0$ one has $t(\alpha, \beta \rightarrow \gamma, \delta) = 0$. If $(P_\alpha - P_\gamma)^2 > 0$ and $E_\alpha > E_\gamma$ t , as defined in (1) is generally non-zero. If $(P_\alpha - P_\gamma)^2 > 0$ and $E_\alpha < E_\gamma$, we also define $t(\alpha, \beta \rightarrow \gamma, \delta) = 0$. In such case $t(\beta, \alpha \rightarrow \delta, \gamma)$ will not be generally zero since $E_\beta > E_\delta$. Following Ref. 2), it is immediate to prove that, as a consequence of Eq. (1) and four-momentum conservation, one has

$$(P_\alpha - P_\gamma)_\mu t(\alpha, \beta \rightarrow \gamma, \delta) = \int d^3P_1 \dots d^3P_n t(\alpha, \beta + 1 \rightarrow \gamma + 1, \delta) \quad (2)$$

where the state $\beta + 1$ ($\gamma + 1$) is obtained from β (γ) by adding an extra particle 1 with incoming (outgoing) momentum P_1 (Fig. 1b).

What makes Eq. (2) non-trivial is unitarity, which identifies $t(\alpha, \beta \rightarrow \gamma, \delta)$ [or $t(\beta, \alpha \rightarrow \delta, \gamma)$] with a particular discontinuity of $\langle \gamma, \delta | T | \alpha, \beta \rangle$. Recent investigations ⁷⁾ appear to confirm that such discontinuity can be unambiguously defined in terms of the analytically continued amplitude $T(\alpha + \beta \rightarrow \gamma + \delta)$. The right definition is indicated in Fig. 2. We shall then write, as a consequence of unitarity ⁸⁾:

$$\text{Disc}_{(P_\alpha - P_\gamma)^2} T(\alpha + \beta \rightarrow \gamma + \delta) = \begin{cases} t(\alpha, \beta \rightarrow \gamma, \delta); & E_\alpha > E_\gamma \\ t(\beta, \alpha \rightarrow \delta, \gamma); & E_\alpha < E_\gamma \end{cases} \quad (3)$$

with the understanding that the symbol Disc is to be interpreted according to Fig. 2. One then rewrites Eq. (2) in terms of discontinuities as :

$$(P_\alpha - P_\gamma)_\mu \text{Disc}_{(P_\alpha - P_\gamma)^2} T(\alpha + \beta \rightarrow \gamma + \delta) = \int dP_1 P_{1,\mu} \text{Disc}_{(P_\alpha - P_1)^2} T(\alpha + \beta + 1 \rightarrow \gamma + \delta) \quad (4)$$

where, if $E_\alpha > E_\gamma$, we have to limit ourselves to $E_\alpha > E_\gamma + E_1$ in the right-hand side of Eq. (4). To be definite we shall assume $E_\alpha > E_\gamma$ in the following.

We have thus proven that, if Eq. (3) (i.e., unitarity) holds, then Eq. (4) must also hold. Our purpose now is to show that the reverse is true, namely that Eq. (4) implies the discontinuity formula Eq. (3).

In order to prove that, we need a further assumption, that is the existence of isolated factorized poles in the physical region due to single particle exchange. This is also a known consequence of unitarity⁶⁾. We shall assume therefore that, in the region $0 < (P_\alpha - P_\gamma)^2 < 4\mu^2$, the only singularity of $T(\alpha + \beta \rightarrow \gamma + \delta)$ in $(P_\alpha - P_\gamma)^2$ is a pole at $(P_\alpha - P_\gamma)^2 = \mu^2$, with

$$\begin{aligned} \text{Disc}_{(P_\alpha - P_\gamma)^2} T(\alpha + \beta \rightarrow \gamma + \delta) &= i \int dP_1 T(\alpha \rightarrow \gamma + 1) T^\dagger(1 + \beta \rightarrow \delta) \delta^{(4)}(P_\alpha - P_\gamma - P_1) = \\ &= 2i\pi \delta((P_\alpha - P_\gamma)^2 - \mu^2) T(\alpha \rightarrow \gamma + 1) T^\dagger(1 + \beta \rightarrow \delta). \end{aligned} \quad (5)$$

Consider now the region $4\mu^2 < (P_\alpha - P_\gamma)^2 < 9\mu^2$. Then in the process $\alpha + \beta + 1 \rightarrow 1 + \gamma + \delta$ one has $0 < (P_\alpha - P_\gamma - P_1)^2 < 4\mu^2$ and therefore according to Eq. (5), we can write :

$$\text{Disc}_{(P_\alpha - P_\gamma - P_1)^2} T(\alpha + \beta + 1 \rightarrow \gamma + \delta + 1) = i(2\pi)^4 \int dP_2 T(\alpha \rightarrow \gamma + 1 + 2) T^\dagger(1 + 2 + \beta \rightarrow \delta) \delta^{(4)}(P_\alpha - P_\gamma - P_1 - P_2) \quad (6)$$

Using Eq. (4) and Eq. (6) one has :

$$\begin{aligned} (P_\alpha - P_\gamma)_\mu \text{Disc}_{(P_\alpha - P_\gamma)^2} T(\alpha + \beta \rightarrow \gamma + \delta) &= \\ &= i(2\pi)^4 \int dP_1 dP_2 P_{1,\mu} T(\alpha \rightarrow \gamma + 1 + 2) T^\dagger(1 + 2 + \beta \rightarrow \delta) \delta^{(4)}(P_\alpha - P_\gamma - P_1 - P_2) = \\ &= i(2\pi)^4 \frac{1}{2} \int dP_1 dP_2 (P_1 + P_2)_\mu T(\alpha \rightarrow \gamma + 1 + 2) T^\dagger(1 + 2 + \beta \rightarrow \delta) \delta^{(4)}(P_\alpha - P_\gamma - P_1 - P_2) \end{aligned} \quad (7)$$

where we have used symmetry in $P_1 \leftrightarrow P_2$.

Using $P_1 + P_2 = P_\alpha - P_\gamma$ and the fact that $(P_\alpha - P_\gamma)_0 \neq 0$ one finds :

$$\text{Disc}_{(P_\alpha - P_\gamma)^2} T(\alpha + \beta \rightarrow \gamma + \delta) \Big|_{4\mu^2 < (P_\alpha - P_\gamma)^2 < 9\mu^2} = i(2\pi)^4 \frac{1}{2} \int dP_1 dP_2 T(\alpha \rightarrow \gamma + 1 + 2) T^\dagger(1 + 2 + \beta \rightarrow \delta) \delta^{(4)}(P_\alpha - P_\gamma - P_1 - P_2) \quad (8)$$

which coincides with Eqs. (3) and (1).

In general, assume that Eq. (3) is valid if $(P_\alpha - P_\gamma)^2 < (n+1)^2 \mu^2$. Then to extend its validity to the region $(n+1)^2 \mu^2 < (P_\alpha - P_\gamma)^2 < (n+2)^2 \mu^2$ we use Eq. (4). In the right-hand side, we can use $\text{Disc } T(\alpha + \beta + 1, 1 + \gamma + \delta) = t(\alpha, \beta + 1 \rightarrow \gamma + 1, \delta)$ and obtain with trivial manipulations :

$$(P_\alpha - P_\gamma)_\mu \text{Disc}_{(P_\alpha - P_\gamma)^2} T(\alpha + \beta \rightarrow \gamma + \delta) = i \int dP_1 P_{1,\mu} \sum_{m=1}^n \frac{1}{m!} \int dP_2 \dots dP_{m+1} \cdot$$

$$\cdot T(\alpha \rightarrow \gamma + 1 + 2 + \dots + (m+1)) T_{(1+2+\dots+m) \rightarrow \beta \rightarrow \delta}^+ (2\pi)^4 \delta^{(4)}(P_\alpha - P_\gamma - P_1 - P_2 - \dots - P_{m+1}) = \quad (9)$$

$$= (P_\alpha - P_\gamma)_\mu i \sum_{p=2}^{n+1} \frac{(2\pi)^4}{p!} \int dP_1 \dots dP_p T(\alpha \rightarrow \gamma + 1 + \dots + p) T_{(1+2+\dots+p) \rightarrow \beta \rightarrow \delta}^+ \delta^{(4)}(P_\alpha - P_\gamma - P_1 - \dots - P_p)$$

Recalling that, by assumption, we have to add the term with $p = 1$, we see that Eq. (9) coincides with Eq. (3) in the extended energy region. The whole Eq. (4) is thus proven by induction. We have therefore shown that Eqs. (4) and (5) imply Eq. (3) with $t(\alpha\beta \rightarrow \gamma\delta)$ defined in Eq. (1). One can therefore think of the sum rules as a complete set of equations having the full content of the discontinuity formula or, if we wish, of a unitarity constraint.

The most apparent advantage in using the sum rule (4) instead of Eq. (3) is the linear nature of the former. As a consequence, no summation over intermediate states appears in the sum rules. We want to stress, however, that, although linear in T , our sum rules are in general non linear in the coupling constants (e.g., Regge coupling constants) and are therefore able in principle to constraint them. The sum rules could then provide an easier way to carry out multiperipheral bootstrap programs ⁹⁾.

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FIGURE CAPTIONS

Figure_1   a) Diagramatic representation of the scattering amplitude $T(\alpha + \beta \rightarrow \gamma + \delta)$;
              b) Same for the amplitude $T(\alpha + \beta + 1 \rightarrow \gamma + \delta + 1)$.

Figure_2   Definition of the discontinuity used in this paper.

(a)

(b)

- Figure 2 -