ANALYTIC EQUIVALENCE OF WAVE FUNCTIONS AND INVARIANT AMPLITUDES FOR MASSIVE PARTICLES

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ABSTRACT

It is shown that (1) any pair of wave functions for a massive particle and (2) any pair of sets of invariant amplitudes without kinematical singularities or zeros, for massive particles, are mutually related by matrices whose components are polynomials in the momenta of the particles and therefore analytically equivalent.

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Let \( F_{\sigma_1 \ldots \sigma_n} (p_1, \ldots, p_n) \) be a function of the four-momenta \( p_1, \ldots, p_n \) and the three-component of the spins \( \sigma_1 \ldots \sigma_n \) of a system of \( n \) particles; for example, \( F \) could be a scattering amplitude or a vertex function (eventually with extra indices in the last case). It is known that one may write

\[
F_{\sigma_1 \ldots \sigma_n} (p_1, \ldots, p_n) = \psi_{\sigma_1} (p_1, \sigma_1) \psi_{\sigma_n} (p_n, \sigma_n) M_{\sigma_1 \ldots \sigma_n} (p_1, \ldots, p_n), \tag{1}
\]

where the \( \psi \) are wave functions, and \( M \) is covariant, i.e.,

\[
M_{\sigma_1 \ldots \sigma_n} (\Lambda p_1, \ldots, \Lambda p_n) = \sum_s S^{(s)}_{\sigma_1 \sigma_1'} (\Lambda) \cdots S^{(s')}_{\sigma_n \sigma_n'} (\Lambda) M_{\sigma_1' \ldots \sigma_n'} (p_1, \ldots, p_n). \tag{2}
\]

This follows from the transformation properties of the wave functions,

\[
\Lambda : \psi_{\sigma} (p, \sigma) \mapsto \sum_{\sigma'} S_{\sigma \sigma'} (\Lambda) \psi_{\sigma'} (\Lambda p, \sigma). \tag{3}
\]

From Eq. (2), and assuming certain analyticity properties of the \( M \), it follows \(^1\) that one can decompose \( M \) into products of covariant polynomials in the \( p_1^\mu \ldots p_n^\mu \), times invariant functions with the same analyticity properties as the \( M \).

It is obvious that such decomposition depends first of all on the kind of wave functions (spinorial, Rarita-Schwinger or Bargmann-Wigner e.g.) used to define the \( M \), and, on the other hand,
even starting with the same $M$, there are several different decompositions.

In the present note, it is shown that this arbitrariness does not present any problem, as different wave functions as well as covariant amplitudes are analytically equivalent among themselves. To be precise, we are going to show that (1) there exist the matrices $A_{ab}(p)$, $B_{ba}(p)$ whose elements are, on the mass shell, polynomials in the components $p^\mu$, and such that, if $\psi_a(p, \sigma)$, $\varphi_b(p, \sigma)$ are two different types of wave functions representing the same massive particle, then

$$\psi_a(p, \sigma) = \sum A_{ab}(p) \varphi_b(p, \sigma); \quad \varphi_b(p, \sigma) = \sum B_{ba}(p) \psi_a(p, \sigma);$$ \hspace{1cm} (4)

and (2) if

$$M_{a_1 \cdots a_m}(p_1, \cdots, p_m) = \sum X_{a_1 \cdots a_m}^{i}(p_1, \cdots, p_m) f_i(p_1 \cdots p_m);$$

$$M_{b_1 \cdots b_m}(p_1, \cdots, p_m) = \sum Y_{b_1 \cdots b_m}^{j}(p_1, \cdots, p_m) g_j(p_1 \cdots p_m);$$ \hspace{1cm} (5)

are two decompositions of $M$ into invariant amplitudes without kinematical singularities and zeros, then there exist the matrices $K_{ij}(p_1 \cdots p_n); L_{ji}(p_1 \cdots p_n)$ such that their elements are (on the mass shell) invariant polynomials in the $p_1^{\mu} \cdots p_n^{\mu}$ and one has
\[ f_i(p_1, \ldots, p_n) = \sum K_{ij}(p_1, \ldots, p_n) g_j(p_1, \ldots, p_n), \]
\[ g_j(p_1, \ldots, p_n) = \sum L_{ij}(p_1, \ldots, p_n) f_i(p_1, \ldots, p_n). \]

Before proving the theorem, a few remarks are due on conditions (2), (3).

It is clear that (2), (3) represent a restriction on admissible wave functions, and M functions. This restriction is essential to prove our results However, it is not necessary as analytically equivalent wave functions exist which do not satisfy (3). The motivation for requiring (2) or (3) is that, in a last analysis, the rationale for assuming analyticity properties of the M is that we can write

\[ M_{a_1 \ldots a_m}(p_1, \ldots, p_n) = \mathcal{F}\langle 0 | A^{(a_1)}(x_1) \ldots A^{(a_m)}(x_n) | 0 \rangle, \]

where \( \mathcal{F} \) means Fourier transform, and the \( A = A(x) \) are local fields. This implies local transformation properties for the \( A \)'s

\[ \Lambda: A_a(x) \rightarrow \sum S_{a a'}(\Lambda) A_{a'}(\Lambda x), \]

which in turn gives (2) and (3).

We now pass to the proofs.

*) And not, for example, \( \psi_{\alpha}(p, \sigma) \rightarrow \sum S_{\alpha \alpha'}(p \Lambda) \psi_{\alpha'}(\Lambda \rho, \sigma). \)
1. Equivalence of Wave Functions

From (3), it follows that the S form a representation of the Lorentz group. Thus, we may write

\[ \psi_{a}(p, \sigma) = \sum N_{a_{1} \cdots a_{q} \beta_{1} \cdots \beta_{r}} \psi_{a_{1} \cdots a_{q} \beta_{1} \cdots \beta_{r}}(p, \sigma), \]

where the \( \alpha, \beta \) are the standard spinorial indices \(^{1}\) and \( N \) is an invertible numerical matrix. Using now Stapp's symbols, we get, with \( m^2 = p^2 \)

\[ \psi_{\alpha_{1} \cdots \alpha_{q} \beta_{1} \cdots \beta_{r}}(p, \sigma) = \sum D^{\alpha_{1}}(i \sigma_{2}(\sigma \cdot p)/m)_{\beta_{1}} \cdots D^{\alpha_{q}}(i \sigma_{2}(\sigma \cdot p)/m)_{\beta_{r}} \psi_{\alpha_{1} \cdots \alpha_{q} \beta_{1} \cdots \beta_{r}}(p, \sigma). \]

It is to be noted that the \( D \)'s have components which are polynomials in the \( p^\mu \), and the same is true for its inverses,

\[ [i \sigma_{2}(\sigma \cdot p)/m]^{-1} = -i (\sigma \cdot p) \sigma_{2}/m; \]

here \( \sigma \cdot p = p^{0} - \vec{p} \cdot \vec{p}, \sigma \cdot p = p^{0} + \vec{p} \cdot \vec{p} \) and \( \sigma \) are the Pauli matrices.

Composing now the spins we see that

\[ \psi_{\alpha_{1} \cdots \alpha_{q} \beta_{1} \cdots \beta_{r}}(p, \sigma) = \sum C_{\alpha_{1} \cdots \alpha_{q} \beta_{1} \cdots \beta_{r}}^{(\sigma') \beta_{r}} \psi_{\alpha_{1} \cdots \alpha_{q} \beta_{1} \cdots \beta_{r}}(p, \sigma). \]
The \( \mathcal{J} \) are degeneracy parameters, and the C's are Clebsch-Gordan coefficients, hence numerical and invertible matrices. Since the wave functions are supposed to correspond to particles of well-defined spin (say \( s \)), it follows that enough subsidiary conditions must exist so that all \( \psi^{(s')}_{\mathcal{J}} = 0 \) except if \( s' = s \). Thus, we have shown that the original wave function is analytically related to the set \( \psi^{(s)}_{\mathcal{J}} \), \( \mathcal{J} = 1 \ldots \). All the \( \psi^{(s)}_{\mathcal{J}} \) transform similarly, so they have to be proportional and thus we can write

\[
\psi^{(s)}_{\alpha} (p, \sigma) = \gamma_{\mathcal{J}} \psi^{(s')}_{\alpha} (p, \sigma).
\]

The \( \psi^{(s)}_{\alpha} \) is what is known as a spinorial wave function, which we have thus shown to be related to \( \psi_{a} \) by

\[
\psi_{a} = \sum A_{a \alpha}^{i} \psi^{(s)}_{\alpha},
\]

\[
\psi^{(s)}_{\alpha} = \sum B_{\alpha a}^{i} \psi_{a}.
\]

The \( A_{a \alpha}^{i} \), \( B_{\alpha a}^{i} \) being polynomials in \( p^{\mu} \). The result, Eq. (3), then follows by the obvious transitivity of the relations above.

2. Equivalence of Invariant Amplitudes

From what has been said above, it is clear that we can just consider \( M^{\alpha_{1} \cdots \alpha_{n}} \), where the \( \alpha \) are undotted spinorial indices, or, composing the spins, \( M^{(s)}(p_{1} \cdots p_{n}) \).

We then assume that
\[ M_{\alpha}(p_1, \ldots, p_n) = \sum X_{\alpha}^i(p_1, \ldots, p_n) f_i(p_1, \ldots, p_n) \]  
\[ M_{\alpha}^{(a)}(p_1, \ldots, p_n) = \sum Y_{\alpha}^{ij}(p_1, \ldots, p_n) g_{ij}(p_1, \ldots, p_n). \]  
(8)

It is obvious that, knowing the \( g_{ij} \), one can recover the \( f_i \) and conversely; hence, \( f_i = f_i(g_{ij}) \). But Eqs. (8) are linear so that actually

\[ f_i(p_1, \ldots, p_n) = \sum T_{ij}(p_1, \ldots, p_n) g_{ij}(p_1, \ldots, p_n). \]  
(9a)

Here the \( T_{ij} \) must be analytic over the mass shell. Substituting into (8), it follows that, because of the linear independence of the \( Y \),

\[ Y_{\alpha}^{ij}(p_1, \ldots, p_n) = \sum X_{\alpha}^i(p_1, \ldots, p_n) T_{ij}(p_1, \ldots, p_n). \]  
(9b)

The \( X \) or \( Y \) transform covariantly; hence if \( Z \) is \( X \) or \( Y \), the quantities

\[ Z', Z'' = \sum D^{(a)}_{\alpha \alpha'}(i\sigma_{ij}) Z_{\alpha}^{i'} Z_{\alpha'}^{i''}, \]

are invariant. We may then form the set of equations
\[ x^{ij} y^{j} = \sum (x^i x^{i'}) T_{ij} , \]

which can be solved to give

\[ T_{ij}(p_1, \ldots, p_n) = \frac{P_{ij}(p_1, \ldots, p_n)}{Q_{ij}(p_1, \ldots, p_n)} . \]

where the \( P, Q \) are invariant polynomials. Because of the Hall-Wightman theorem, it follows that they are actually polynomials in the invariants, \( p^i p^j \) and \( \epsilon^{\mu \nu \rho \sigma} p^\mu p^\nu p^\rho p^\sigma \). However, the \( T_{ij} \) are analytic everywhere. This means that, if \( Q_{ij} \) has a zero of the order \( N \) at a point, \( P_{ij} \) must have a zero of at least the same order at the same point. But if two polynomials vanish of the same order at the same point, it is known that there must exist two other polynomials \( P_{ij}', Q_{ij}' \) with \( P_{ij}/Q_{ij} = P_{ij}'/Q_{ij}' \), and where \( Q_{ij}' \) does no more vanish. Therefore, we have arrived at the expression \( T_{ij} = F_{ij}/Q_{ij} \), where \( Q_{ij} \) never vanishes. But then it has to be constant: we have got that \( T_{ij} \) is itself an invariant polynomial. Thus, the first of the equalities in (6) is proved; by inverting the role of \( f, g \), the second would follow likewise: QED.

3. Comments

The results described here seem to be known to experts; however, they seem never to have been written in detail, and nobody appears to know the details of a proof. The author would like to thank Professor F. Lurçat and J.-L. Gervais for encouragement to

\(*) \) Otherwise, from Eq. (9a), the \( f \) would have kinematical poles.
publish these results, and to Professor J.S. Bell who reproduced part of our results in the Appendix to his paper in the Nuovo Cimento 61A, 541 (1969) for the same reason and for interesting discussions.

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