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THEORETICAL MODELS OF DIFFRACTION SCATTERING

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1. INTRODUCTION

The remarks presented here address themselves to only a very limited set of questions having to do with diffraction at very high energies. A large number of interesting topics will be touched on only lightly, or ignored altogether. Among these are the spin structure of diffraction; diffraction dissociation; the relations between diffractive processes involving different particles; diffraction from compound systems, such as nuclei; diffraction at less than asymptotic energies. We shall instead concentrate exclusively on the question of the fundamental origin of high-energy diffractive processes, on the connections between diffraction and production processes, and on the general characteristics of diffraction as evidenced in the asymptotic behaviour of cross-sections; or equivalently by the j-plane behaviour of cross-channel partial wave amplitudes.

In discussing questions of this nature, we shall take it for granted that differences between the high-energy diffractive behaviour of different hadrons having different spins and internal quantum numbers are unimportant; it will therefore be sufficient for our purposes to discuss diffraction as if there were only one kind of particle present in the world, and as if this particle were spinless¹⁾.

It must be admitted that these limitations are severe; however, the subject in its entirety is a large one, and for some of the facets of it omitted here excellent reviews exist to which anyone interested can address himself.

* * *

REFERENCES AND FOOTNOTES

- 1) After all, we have it on the best authority (G.F. Chew, "The Analytic S-matrix", W.A. Benjamin, Inc., N.Y., 1966) that "spin is an inessential complication in strong interaction dynamics".

2. EXPERIMENTAL REVIEW

From an experimental point of view, diffractive processes may be characterized as those for which the cross-section does not disappear at high energies, but approaches a constant (or nearly constant) value¹⁾. We may distinguish the following three sub-classes of diffractive processes.

2.1 Elastic processes

These, of course, are reactions of the form $A + B \rightarrow A + B$. All hadronic total cross-sections seem to approach constants (or at least to vary quite slowly) asymptotically. Hence, through the optical theorem, we deduce that the imaginary parts of forward elastic amplitudes rise approximately linearly with s , the square of the total centre-of-mass energy. Furthermore, elastic differential cross-sections all have forward peaks, which shrink, if at all, quite slowly with energy; hence elastic total cross-sections are also nearly constant at high energies.

These comments are illustrated by the representative experimental data shown in Figs. 1 to 13. Fig. 1(a) shows total cross-sections versus energy for pp , $\bar{p}p$, π^-p , π^+p , K^-p and K^+p scattering. In all cases the cross-sections are obviously not vanishing rapidly with energy, and in fact are quite consistent with constant asymptotic values. To illustrate that the same behaviour is also characteristic of other, more complicated, structures as well, Fig. 1(b) shows total cross-sections for π^- and K^- on deuterium.

In Figs. 2, 3, 4, 5, 6, and 7 are plotted differential cross-sections for pp , $\bar{p}p$, π^-p , π^+p , K^-p and K^+p elastic scattering. The large forward peak is evident. To summarize the available information on shrinkage of this peak with energy, we show Figs. 8, 9, and 10, which plot the "slope parameter" $b(s)$, defined by

$$b(s) = -\frac{1}{2t} \log \frac{(d\sigma/dt)_{t=0}}{(d\sigma/dt)}, \quad (2.1)$$

where t is the invariant momentum transfer, versus $\log s$, for the same six processes. This definition, obviously, is based on the fact that an

exponential form $e^{2b(s)t}$ gives a good fit to the forward peaks, out to $-t \lesssim 1 \text{ (GeV)}^2$. As can be seen from these pictures, for large s , $b(s)$ is no more than a linear function of $\log s$, and it may even be conceivable that in the truly asymptotic region $b(s)$ becomes independent of s altogether.

Finally, we show elastic total cross-sections for the same six reactions in Figs. 11, 12, and 13.

2.2 Diffraction dissociation in quasi-2-body processes

In these reactions the final state differs from the initial state, so that the process is not elastic; indeed, the final state will normally contain more than two particles. However, the final particles are selected so that they can be formed into one or two resonances, and the reaction may therefore be described as quasi-2-body, of the form

$$A + B \rightarrow A^* + B^* .$$

Now, for a given choice of A and B , certain of these reactions, for particular A^* and B^* , have cross-sections holding up at high energy, while for other A^* and B^* they do not. Examples are shown in Fig. 14.

Reactions for which the cross-section stays up are called 2-body diffraction dissociation reactions. The rule for deciding which reactions will be of this kind, and which will not, is not fully understood. One feature which all observed diffractive reactions (including, of course, elastic processes) have in common is that in the crossed reaction $A + \bar{A}^* \rightarrow \bar{B} + B^*$, the channel with no internal quantum numbers (that is, the vacuum channel) is accessible. Thus at the very least we may say that every observed diffractive process allows the "exchange of the vacuum channel". Conversely, if the crossed reaction for a process necessarily couples only to channels having at least one non-zero internal quantum number the process is never diffractive; the cross-section always vanishes at large energy like a power of the energy.

While the rule just described produces powerful constraints on what the particles A^* and B^* in a diffractive process can be, it does not allow us to uniquely predict A^* and B^* , because the spin structure of diffractive processes is not yet entirely clear. At least three rules

have been proposed. One is that the spin and parity difference between A and A^* must satisfy $(-)^{\Delta J} = \Delta P$ (and similarly for B and B^*)²⁾. Another conjectured rule is that A and A^* (and similarly B and B^*) must be members of the same $SU(6)_w$ multiplet -- as if the crossed-channel must contain only the vacuum quantum number (i.e. the representation 1) of $SU(6)_w$ as well as of purely internal symmetries³⁾. A third rule which has been proposed⁴⁾ is that the product of all four parities in a quasi-2-body diffraction dissociation process must be positive if the cross-section for the process is not to vanish in the forward direction.

Unfortunately, while these rules do differ, they do not differ in their predictions as to the best measured diffraction dissociation processes; furthermore, $SU(6)_w$ is of course broken, so the $SU(6)_w$ rule cannot really be exact. The situation must be left to further experiments to clean up.

We may note, incidentally, that non-elastic quasi-2-body diffraction dissociation processes show large forward peaks in the differential cross-section, as do the elastic ones. Some examples are shown in Fig. 15. While the data are not terribly accurate, it seems that the peaks here are also rather well fitted by exponentials in t , and that the slope parameters are noticeably smaller than those seen in elastic scattering, with one exception: the slope parameter for production of the $N^*(1405)$ resonance is much larger than in the elastic cases⁵⁾.

2.3 Diffraction production processes

These are processes of the form $A + B \rightarrow$ more than two particles, where the final particles cannot be grouped with two resonances. Here again certain cross-sections seem to stay up (as shown, for example, in Fig. 16) while others do not (see, for example, Fig. 17). The rule for deciding which states may be diffractively produced from a given initial pair of particles is, apparently, much the same as in two-body reactions. There are now, of course, several possible crossed reactions (for example, for $A + B \rightarrow C + D + E$ we can have $A + \bar{C} \rightarrow \bar{B} + D + E$, or $A + \bar{D} \rightarrow \bar{B} + C + E$, or $A + \bar{E} \rightarrow \bar{B} + C + D$ as crossed reactions) and the rule seems to be that if at least one of them contains the vacuum channel,

then the process is diffractive. (What happens if more than one crossed process contains the vacuum channel is somewhat unclear theoretically -- however, the process is still diffractive, and not qualitatively different experimentally.)

* * *

REFERENCES AND FOOTNOTES

- 1) We do not insist on absolute constancy of the cross-sections in this definition, in order to accommodate various theoretically inferred possibilities, such as a logarithmic variation with energy, or a decrease with a very small power of the energy. However, the deviation from constancy must be very small to be compatible with the experimental situation.
- 2) D.R.O. Morrison, Phys. Rev. 165, 1699 (1968).
- 3) R. Carlitz, S. Frautschi and G. Zweig, Phys. Rev. Letters 23, 1134 (1969).
- 4) T.T. Chou and C.N. Yang, Phys. Rev. 175, 1832 (1968).
- 5) No theoretical explanation of this fact, to my knowledge, exists. It is perhaps to be understood through the quark model, in which $N^*(1405)$ is usually distinguished as a radial excitation, while the other N^* 's are presumably orbital excitations.

3. CONSEQUENCES OF BASIC PRINCIPLES

We shall take it for granted at the outset that the Froissart bound¹⁾ is satisfied; thus underlying everything we have to say will be the belief that the total cross-section cannot grow faster than $(\log s)^2$,

3.1 Analyticity and crossing

Let us begin with the requirements of analyticity. We believe that the scattering amplitude $T(s,t)$ for an elastic scattering process $A + B \rightarrow A + B$ is an analytic function of s for fixed t , and that we can write

$$\begin{aligned} T(s,t) = & a(t) + b(t)s \\ & + \frac{s^2}{\pi} \int_{s_0}^{\infty} \frac{A(s',t)}{s'^2(s' - s)} ds' \\ & + \frac{u^2}{\pi} \int_{u_0}^{\infty} \frac{\bar{A}(u',t)}{u'^2(u' - u)} du' . \end{aligned} \quad (3.1)$$

We have made two subtractions to allow for the experimental fact that $T \sim s$ at large s ; according to the experiments, and to the Froissart bound, it seems that more than two subtractions are unnecessary.

A possibility which leaps immediately to mind is that the subtraction term in Eq. (3.1) is the one responsible for diffraction scattering. It would give us an amplitude linear in s ; $T(s,t) \rightarrow b(t)s$. However, analyticity in t requires $b(t)$ to be real below t -channel threshold, and in particular at and just below $t = 0$. Therefore the subtraction term by itself would produce, for $t \leq 0$, a real amplitude linear in s ; it could not, in particular, contribute to a constant (or nearly constant) total cross-section. It could not explain the fact that near $t = 0$ elastic amplitudes are primarily imaginary. (In fact, if we are to have $\text{Re } T/\text{Im } T \rightarrow 0$, and if total cross-sections are constant, b must be absent altogether.) We must, therefore, look elsewhere for diffraction.

Because of the optical theorem, we know that on the right-hand cut, for s above threshold, the absorptive part $A(s,t)$ of the amplitude satisfies

$$A(s,0) = s \sigma_{AB}(s) , \quad (3.2)$$

where $\sigma_{AB}(s)$ is the total cross-section for A on B .

Because of crossing, interchanging s and u is equivalent to interchanging the reactions $A + B \rightarrow A + B$ and $A + \bar{B} \rightarrow A + \bar{B}$; thus on the left-hand cut, for $u > u_0$, we have for the absorptive part

$$\bar{A}(u,0) = u \sigma_{A\bar{B}}(u) , \quad (3.3)$$

where $\sigma_{A\bar{B}}(u)$ is the total cross-section for A on $\bar{B}$.

More generally, for $t \neq 0$ and s above threshold, $A(s,t)$ is the imaginary span of the $A + B \rightarrow A + B$ amplitude while $\bar{A}(u,t)$ is the imaginary part of the $A + \bar{B} \rightarrow A + \bar{B}$ amplitude for u above threshold.

Now, experimentally, it is suggested that $\sigma_{AB}(s) \rightarrow \text{const.}$ as $s \rightarrow \infty$, as does $\sigma_{A\bar{B}}(u)$ as $u \rightarrow \infty$. Let us for the moment believe this to be true. The dispersion integrals, at $t = 0$, then behave like $s \log s$ and $u \log u$, for the right-hand cut and left-hand cut terms respectively, at large s and large u . Then as $s \rightarrow \infty$, at $t = 0$ we have

$$T(s,0) \rightarrow a + bs - \frac{\sigma_{AB}}{\pi} s \log(-s) - \frac{\sigma_{A\bar{B}}}{\pi} u \log(-u) . \quad (3.4)$$

But $u \rightarrow -s$ in this limit, so that in fact

$$T(s,0) \rightarrow a + bs + i s \sigma_{AB} - s \frac{(\sigma_{AB} - \sigma_{A\bar{B}})}{\pi} \log s . \quad (3.5)$$

Hence if $\text{Re } T(s,0)/\text{Im } T(s,0) < \log s$, we must conclude that $\sigma_{AB} = \sigma_{A\bar{B}}$ -- i.e. that particle and antiparticle total cross-sections approach each other. This is called the Pomeranchuk theorem²⁾.

The experimental situation may be reviewed by referring to Figs. 1, (i) through (vi). It can be seen that the data do not give unqualified support to the validity of the Pomeranchuk theorem; it is clearly necessary to have measurements of σ_{π^+p} and σ_{K^+p} at high energies to decide for certain whether or not it holds.

There is also experimental information on the question of the relative size of $\text{Re } T$ and $\text{Im } T$ at $t = 0$, as shown in Fig. 18. It can be seen that the data is quite consistent with $\text{Re } T/\text{Im } T \rightarrow 0$, thus supporting the Pomeranchuk result. However, it must also be admitted that models may be constructed in which the Pomeranchuk theorem is violated, and yet $\text{Re } T/\text{Im } T$ does not begin to grow until energies higher than are now available³⁾.

We may note that if $\sigma_{AB} \neq \sigma_{\bar{A}\bar{B}}$, so that $\text{Re } T(s,0) \sim s \log s$ as $s \rightarrow \infty$, then the forward peak in elastic scattering must shrink at least like $\log^2 s$ [that is, the slope parameter $b(s)$ must grow at least as fast as $\log^2 s$] to prevent σ_{el} from growing with s , and thus becoming bigger than the total cross-section itself.

We have mentioned earlier (and will again) that some theoretical models suggest that $\sigma_T \nrightarrow \text{const.}$, but rather behaves like $1/(\log s)^\alpha$, or like $s^{-\epsilon}$ [or even like $(\log s)^2$] as $s \rightarrow \infty$. Experiment does not unequivocally rule out these possibilities [naturally, theorists choose the numbers α and ϵ so that it does not; or if they cannot be so chosen, then theorists cast aspersions on the accuracy of the experiments], so it may be worth while to mention their consequences for the Pomeranchuk theorem.

Firstly, if σ behaves like a power of s (a falling power, naturally, to avoid violation of the Froissart bound), then $A(s,t)/s$ vanishes asymptotically. From Eq. (3.1) we then conclude that $\text{Re } T(s,t)/s$ vanishes in the same way as does $A(s,t)/s$, so that $\text{Re } T/\text{Im } T \rightarrow \text{const.}$, whether or not particle and antiparticle cross-sections become equal.

Secondly, if σ behaves like a power of $\log s$ (this power must be less than or equal to two to conform to Froissart), we find that $\text{Re } T/\text{Im } T \rightarrow \sim \log s$ unless the Pomeranchuk theorem holds, and if it does hold, then $\text{Re } T/\text{Im } T \rightarrow \text{const.}$

Therefore, in the case of the power behaviour, the Pomeranchuk theorem is not correlated with the ratio $\text{Re } T/\text{Im } T$, while in the case of powers of logarithms the conclusions are the same as those for constant asymptotic cross-sections.

We may also look at the behaviour of $T(s,t)$ for large s with $t \neq 0$, using Eq. (3.1). If for $t \neq 0$ it is true that $\text{Re } T/\text{Im } T \rightarrow 0$, then we see that $A(s,t) \rightarrow \bar{A}(s,t)$ as $s \rightarrow \infty$ for all t . Thus we must expect forward peaks for particle and antiparticle cross-sections to become equal. Whether or not this is plausible can be inferred from the experimental data on the slope parameters, shown in Figs. 8, 9, and 10. Asymptotic equality seems to be by no means out of the question.

3.2 t-channel unitarity

The dispersion relation (3.1) may be used to give us some information about the cross-channel partial wave amplitudes, as follows. We believe, from analyticity, that the amplitude $T(s,t)$ can be analytically continued from the $s > \text{threshold}$, $t < 0$ physical region where it describes the reaction $A + B \rightarrow A + B$, to the region $t > \text{threshold}$, $s < 0$ where it is the amplitude for the crossed reaction $A + \bar{A} \rightarrow B + \bar{B}$. In this region, furthermore,

$$s = -2q_t^2(1 - x_t), \quad (3.6)$$

where q_t is the t-channel centre-of-mass momentum and x_t is the cosine of the t-channel centre-of-mass scattering angle.

In order to keep track properly of the signature, we will as usual decompose $T(s,t)$ into even and odd functions of x_t :

$$\begin{aligned} T(s,t) = & \frac{1}{2} \left[T^{(+)}(t, x_t) + T^{(+)}(t, -x_t) \right] \\ & + \frac{1}{2} \left[T^{(-)}(t, x_t) - T^{(-)}(t, -x_t) \right]. \end{aligned} \quad (3.7)$$

Here, for convenience, we use as independent variables t and x_t rather than t and s .

We may now write dispersion relations in x_t for fixed t ; we have, in conformity with Eq. (3.1),

$$T^{(+)}(t, x_t) = \bar{a}(t) + \frac{x_t^2}{\pi} \int_{x_0}^{\infty} \frac{A^{(+)}(t, z)}{z^2(z - x_t)} dz \quad (3.8)$$

and

$$T^{(-)}(t, x_t) = \bar{b}(t)x_t + \frac{x_t^2}{\pi} \int_x^{\infty} \frac{A^{(-)}(t, z)}{z^2(z - x_t)} dz. \quad (3.9)$$

Thus

$$\begin{aligned} T(t, x_t) &= \bar{a} + \bar{b}x_t \\ &+ \frac{x_t^2}{\pi} \int_{x_0}^{\infty} \frac{dz}{z(z - x_t)} \frac{1}{2} \left[A^{(+)}(t, z) + A^{(-)}(t, z) \right] \\ &+ \frac{x_t^2}{\pi} \int_{x_0}^{\infty} \frac{dz}{z^2(z + x_t)} \frac{1}{2} \left[A^{(+)}(t, z) - A^{(-)}(t, z) \right]. \end{aligned} \quad (3.10)$$

Comparing this with Eq. (3.1), and noting that for large s , $x_t = s/2q_t^2 = -u/2q_t^2$, we see that $\bar{a}(t) = a(t)$, $\bar{b}(t) = 2q_t^2 b(t)$,

$$A(s', t) = \frac{1}{2} \left[A^{(+)}(t, z) + A^{(-)}(t, z) \right]$$

and

$$\bar{A}(u', t) = \frac{1}{2} \left[A^{(+)}(t, z) - A^{(-)}(t, z) \right]$$

with

$$s' = 2q_t^2 z \quad \text{and} \quad u' = -2q_t^2 z$$

Now let us suppose, as we did before, that $A^{(+)}(t, z)$ dominates over $A^{(-)}(t, z)$ at large z , in order that one may obtain the Pomeranchuk theorem. The properties of $T^{(+)}(t, x_t)$ are then what interest us⁴⁾.

Let us decompose the amplitudes into t -channel partial waves. We define

$$T^{(\pm)}(t, x_t) = \sum_j (2j + 1) P_j(x_t) T^{(\pm)}(t, j) . \quad (3.11)$$

The partial wave amplitudes themselves can be read off from the dispersion relations in x_t . From Eq. (3.8), we evidently find that

$$\begin{aligned} T^{(+)}(t, j) = & \left[\bar{a}(t) - \frac{1}{\pi} \int_{x_0}^{\infty} \frac{A^{(+)}(t, z)}{z} dz \right] \delta j_0 \\ & + \frac{1}{\pi} \int_{x_0}^{\infty} A^{(+)}(t, z) Q_j(z) dz , \end{aligned} \quad (3.12)$$

and from Eq. (3.9) that

$$\begin{aligned} T^{(-)}(t, j) = & \frac{1}{3} \left[\bar{b}(t) - \frac{1}{\pi} \int_{x_0}^{\infty} \frac{A^{(-)}(t, z)}{z^2} dz \right] \delta j_1 \\ & + \frac{1}{\pi} \int_{x_0}^{\infty} A^{(-)}(t, z) Q_j(z) dz . \end{aligned} \quad (3.13)$$

[Note that if $A^{(+)}(z, t) \sim z$ as $z \rightarrow \infty$, the separate terms on the right-hand side of these equations are infinite, though the entire combination is nevertheless finite.]

The last term in Eqs. (3.12) and (3.13) can be used to define partial wave amplitudes continued analytically in j , for $\text{Re } j > 1$, since if $A^{(\pm)} \sim z$ at large z , the integrals exist for $\text{Re } j > 1$. Whether or not the partial wave amplitudes can also be continued to $\text{Re } j \leq 1$ we cannot, in general, say. This depends on dynamics.

Two choices are open to us. We can assume that $T^{(\pm)}(t, j)$ can be continued analytically down to values of j below $j = 1$; in this event the subtraction constants $\bar{a}(t)$ and $\bar{b}(t)$ will have been carefully arranged so as to eliminate the Kronecker delta terms in $T^{(\pm)}(t, j)$.

Or, we can assume that $T^{(\pm)}(t, j)$ cannot be analytically continued, and that at least some vestige of the Kronecker deltas remains. However, these deltas cannot be used to describe diffraction, because a Kronecker delta term at $j = 1$ in $T^{(+)}$ does not survive to produce anything visible in $T(s, t)$ at large s , and though a Kronecker delta at $j = 1$ in $T^{(-)}(t, j)$ could produce a term linear in s in $T(s, t)$, it could produce only a real term, not an imaginary -- or even complex -- one, as we have seen before. It could, therefore, not easily be reconciled with experiment.

Consequently the assumption that $T^{(\pm)}(t, j)$ can be continued in j is definitely to be preferred.

Once the analytic extension of $T^{(\pm)}(t, j)$ to complex j is established, the large x_t behaviour of $T^{(\pm)}(t, j)$ is obtained by applying the Sommerfeld-Watson transformation to the partial wave expansion: thus as $s \rightarrow \infty$, for fixed t , we have from Eq. (3.11)

$$T^{(\pm)}(t, x_t) \rightarrow \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(2j+1) dj}{\sin \pi j} \left(-\frac{s}{2q_t^2} \right)^j \times \left\{ \frac{2^j}{\sqrt{\pi}} \frac{\Gamma(j + \frac{1}{2})}{\Gamma(j+1)} T^{(\pm)}(t, j) \right\} \quad (3.14)$$

and hence

$$T(s, t) \rightarrow \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj \left[\frac{(-s)^j + s^j}{2} \right] \tilde{T}^{(+)}(t, j) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj \left[\frac{(-s)^j - s^j}{2} \right] \tilde{T}^{(-)}(t, j),$$

where we define

$$\tilde{T}^{(\pm)}(t, j) = \frac{2j+1}{\sin \pi j} \left(\frac{1}{2q_t^2} \right)^{2j} \frac{\Gamma(j + \frac{1}{2})}{\sqrt{\pi} \Gamma(j+1)} T^{(\pm)}(t, j) . \quad (3.16)$$

In all the above equations, c is a number larger than the real part of any singularity in j of the partial wave amplitudes.

Equation (3.14) is in the form of an inverse Mellin transform in which $\tilde{T}(t, j)$ is the Mellin transform of $T(s, t)$. That is, if we write

$$T^{(\pm)}(s, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{T}^{(\pm)}(t, j) s^j dj , \quad (3.17)$$

then we may invert this Mellin transform to obtain an expression for the partial wave amplitude valid at large s : namely,

$$\tilde{T}^{(\pm)}(t, j) = \int_0^\infty T^{(\pm)}(s, t) s^{-j-1} ds . \quad (3.18)$$

We can now connect the large s behaviour of $T(s, t)$ with the furthest to the right (in the j plane) singularities $\tilde{T}^{(\pm)}(t, j)$ or of $T^{(\pm)}(t, j)$. Specifically:

a) If $T^{(\pm)}(t, j)$ has a pole at $j = \alpha(t)$, we have

$$T(s, t) \rightarrow \frac{\beta(t)}{\sin \pi \alpha(t)} \left[\frac{(-s)^{\alpha(t)} \pm s^{\alpha(t)}}{2} \right] , \quad (3.19)$$

where $\beta(t)/\sin \pi \alpha(t)$ is the residue of the pole in $\tilde{T}^{(\pm)}(t, j)$.

b) If $T^{(\pm)}(t, j)$ has a branch point at $j = \alpha_c(t)$, we have

$$T(s,t) \rightarrow \frac{1}{\pi} \int_{-\infty}^{\alpha_c(t)} dj f(t,j) \left[\frac{(-s)^j \pm s^j}{2} \right], \quad (3.20)$$

where $f(t,j)$ is the discontinuity across the cut in $\tilde{T}^{(\pm)}(t,j)$.

The precise asymptotic behaviour associated with the cut depends on how $f(t,j)$ behaves at $j = \alpha_c$. If the cut is a non-integer power, so that $f(t,j) \sim (j - \alpha_c)^\nu$, then $T(s,t) \sim s^{\alpha_c}/(\log s)^{\nu+1}$. If the cut is logarithmic, so that $f(t,j) \sim \log(j - \alpha_c)$, then $T(s,t) \sim s^{\alpha_c}/\log s$. More generally, if the cut is a power of a logarithm, so that $f(t,j) \sim [\log(j - \alpha_c)]^\nu$, then we find

$$T(s,t) \sim s^{\alpha_c} \frac{(\log \log s)^{\nu-1}}{\log s}. \quad (3.21)$$

In any event, for the case of diffraction we expect (on experimental grounds) the leading singularity to be in $T^{(+)}$, not in $T^{(-)}$, and we expect it to be in the vicinity of $j = 1$. It may be a pole at $j = \alpha_p(t)$ with $\alpha_p(0) = 1$ (this yields $\sigma_T \rightarrow \text{constant}$); or it may be a pole with $\alpha_p(0)$ slightly below one. [As we have noted before, because of the Froissart bound we cannot have $\alpha_p(0) > 1$.] The singularity may, on the other hand, be a cut -- either with $\alpha_c(0) = 1$ or with $\alpha_c(0) < 1$; or it may be some combination of pole and cut. It has to be something.

[A special case of some interest is that of a factorized asymptotic form: $T(s,t) \rightarrow F(s)f(t)$ as $s \rightarrow \infty$ for fixed t . From what we have just said, it is clear that this situation corresponds to the existence of a fixed singularity in j , not varying with t , in $T^{(+)}(t,j)$. Such fixed singularities are associated with non-shrinking diffraction peaks.]

Without dynamics, we cannot say what the singularity must be; however, we can say a few things that, according to t -channel unitarity, it cannot be.

When t is between elastic and inelastic threshold, the partial wave amplitudes, of both even and odd signature, satisfy the (analytically continued in j) elastic unitarity relations

$$T^{(\pm)}(t, j) - T^{(\pm)*}(t, j^*) = 2i\rho(t) T^{(\pm)}(t, j) T^{(\pm)*}(t, j^*) . \quad (3.22)$$

These equations come about because we know that $T^{(+)}(T^{(-)})$ satisfies unitarity at all even (odd) integer values of j , since it coincides with the physical partial wave amplitudes there. But then the same unitarity relations must also hold at all values of j , provided only that $T^{(\pm)}(t, j)$ behaves properly as $j \rightarrow \infty$ ⁵⁾.

Now if $T^{(+)}$ has a pole at $j = \alpha_p(t)$, then near the pole the unitarity relation states that

$$\frac{\beta(t)}{j - \alpha_p(t)} = 2i\rho(t) \frac{\beta(t)}{j - \alpha_p(t)} T^{(+)}(t, \alpha_p^*(t))^* , \quad (3.23)$$

so that we must have

$$T^{(+)}(t, \alpha_p^*(t)) = - \frac{1}{2i\rho(t)} . \quad (3.24)$$

Evidently, then, $\text{Im } \alpha_p(t)$ cannot be zero, otherwise the left-hand side of this equation is infinite and cannot match the right-hand side. To say in another way, if $\text{Im } \alpha_p(t) = 0$, then the left-hand side of the unitarity relation has a single pole at $j = \alpha_p(t)$, while the right-hand side has a double pole, and this is impossible.

Thus no poles are allowed with $\text{Im } \alpha(t) = 0$ for any t between elastic and inelastic threshold, unless $\beta(t)$ vanishes at that value of t as well.

Next, suppose $T^{(+)}(t, j)$ has a branch point at $j = \alpha_c(t)$. If the branch point is such that $T^{(+)}(t, j)$ becomes infinite at the tip of the cut, then the same argument applies as in the case of the pole, and we must have $\text{Im } \alpha_c(t) \neq 0$ for t in the elastic range. If, on the other hand, the cut is "soft", i.e. if $T^{(+)}(t, j)$ vanishes at the tip, then no such restriction applies.

To summarize, allowed by t -channel unitarity are ⁶⁾:

- i) complex poles
- ii) complex branch points, both "hard" and "soft"
- iii) real "soft" branch points.

Prohibited are:

- i) real poles
- ii) real "hard" branch points.

(It should be noted that these restrictions apply only to singularities on the physical sheet of the partial wave amplitude; on unphysical sheets the same constraints, obviously, do not exist.)

Generally, we expect that moving poles, for which $\alpha_p(t)$ is not constant, have non-zero $\text{Im } \alpha_p(t)$; this is consistent with the dispersion relation that we believe $\alpha_p(t)$ to satisfy on the basis of analyticity, namely:

$$\alpha_p(t) = \alpha_p(0) + t\alpha'_p(0) + \frac{t^2}{\pi} \int_{t_0}^{\infty} \frac{\text{Im } \alpha_p(t')}{t'^2(t' - t)} dt' . \quad (3.25)$$

Fixed poles, on the other hand, are usually assumed to be real [though there is no obvious reason why this has to be so]. Thus the statement that complex poles are allowed while real poles are not is frequently translated into the statement that moving poles are allowed, but fixed poles are not⁶⁾.

In a similar vein, branch points are frequently implicitly assumed to be real [i.e. $\alpha_c(t)$ is assumed to be real] because the simplest models giving rise to cuts have this property. The unitarity constraint on cuts is therefore often specialized to "soft cuts are O.K., but hard cuts are not". In particular, fixed cuts [for which $\alpha_c(t)$ is independent of t and is real] must be "soft", not "hard".

What do all these constraints tell us about diffraction? They really tell us rather little, except in one special case. We have found that they tell us that any fixed singularity in the j plane, if it is on the real axis, can only be a soft cut. Thus if the asymptotic amplitude factorizes, so that

$$T(s,t) \rightarrow F(s)f(t) ,$$

then we are forced to conclude that $F(s)$ must blow up more slowly than $s/\log s$ as $s \rightarrow \infty$; in other words, $\sigma_T(s)$ must vanish faster than $1/\log s$. Aside from the special case of factorization, however, we learn almost nothing from t -channel unitarity.

3.3 s-channel unitarity

The requirements on the high-energy elastic amplitude imposed by t -channel unitarity were rather simple to obtain, because we could fix t to be in the region between the elastic and inelastic thresholds, where elastic unitarity in the t -channel was exact. Therefore, no explicit knowledge of production amplitudes in the t -channel was required. In the s -channel, however, the situation is far more complex, because in the region of interest to us (namely high s) inelastic intermediate states necessarily play an important role in the unitarity relation. Nevertheless, we may at least formally write the s -channel unitarity relation down. It states that

$$A(s, t) = \frac{1}{2} \sum_{n=2}^{\infty} \left\{ \prod_{i=1}^n \int \frac{d^3 \vec{q}_i}{(2\pi)^3} \frac{1}{2\omega_i} (2\pi)^4 \delta^4 \left(P - \sum_{j=1}^n q_j \right) \right\} \\ T_{2 \rightarrow n} (p_1 + p_2 \rightarrow q_1 \dots q_n) T_{2 \rightarrow n}^* (p'_1 + p'_2 \rightarrow q_1 \dots q_n) . \quad (3.26)$$

Here the momenta are labelled as shown in Fig. 19. The intermediate states are written as states of n particles of momenta $q_1 \dots q_n$, where $q_{0i} = \omega_i = \sqrt{q_i^2 + 1}$; $T(p_1 + p_2 \rightarrow q_1 + \dots + q_n)$ stands for the production amplitude connecting the initial two particles of momenta p_1 and p_2 to the n particle state; similarly $T(p'_1 + p'_2 \rightarrow q_1 + \dots + q_n)$ is the amplitude to connect the final two particles of momenta p'_1 and p'_2 to this state. The bracket $\{ \}$ is the n -body phase space; we shall often abbreviate this by $\int d\Phi_n(P)$. Finally, $P^2 = (p_1 + p_2)^2 = (p'_1 + p'_2)^2 = s$, and $(p'_1 - p_1)^2 = (p'_2 - p_2)^2 = t$.

A special case of Eq. (3.26) has of course already been used; at $t = 0$ Eq. (3.26) just yields the optical theorem: $A(s, 0) = \text{Im } T(s, 0) = s \sigma_T(s)$.

It is often convenient to split off from the sum in Eq. (3.26), the elastic intermediate state, and to write its contribution explicitly. This gives, at high s ,

$$\begin{aligned}
 A(s,t) &= A_{inel}(s,t) \\
 &+ \frac{1}{16\pi^2 s} \iint \frac{dt_1 dt_2 \theta(-\lambda)}{\sqrt{-\lambda(t,t_1,t_2)}} T(s,t_1) T^*(s,t_2) \\
 &\equiv A_{inel}(s,t) + A_{el}(s,t) , \tag{3.27}
 \end{aligned}$$

where $\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$, and where $A_{inel} = (\text{Im } T)_{inel}$ stands for all the terms in the sum over n in Eq. (3.26) except the elastic one. If something is known about this function (which is, for some unfathomable reason, often called the "overlap integral") from other sources, then Eq. (3.27) becomes an integral equation for the elastic amplitude.

The first question of relevance is whether the elastic or inelastic contribution dominates in Eq. (3.27) at large s . We may again appeal to experiment for a hint. At $t = 0$, we of course have

$$A_{inel}(s,0) = [\text{Im } T(s,0)]_{inel} = s \sigma_{inel}(s) \tag{3.28}$$

and

$$A_{el}(s,0) = [\text{Im } T(s,0)]_{el} = s \sigma_{el}(s) . \tag{3.29}$$

Experimentally, values for $\sigma_{el}(s)/\sigma_T(s)$ for pp and π^-p scattering are plotted as functions of s in Fig. 20. As can be seen from the figure, at $t = 0$ at least, both the elastic and inelastic contributions seem to be present, though the inelastic one is three to four times larger.

Away from $t = 0$, the situation is less directly accessible to experiment; however, the following may be done. We may put an experimental fit to $T(s,t)$ into the integral giving A_{el} , assume $T(s,t) \approx iA(s,t)$ over the important range of t , and compare this with A_{inel} as given by $A(s,t) - A_{el}(s,t)$. This has been done for pp elastic scattering in

Fig. 21, which shows both $A(s,t)$ and $A_{inel}(s,t)$ computed by the above procedure⁷⁾. The most striking thing is the sign change taking place in $A_{inel}(s,t)$. Such a sign change is essentially inevitable given the rapid fall of the elastic peak in pp scattering, because the integral giving A_{el} necessarily spreads out a sharp peak.

Altogether, the indications from experiment are that both terms in Eq. (3.27) are likely to be of importance. In this event, we may ask, what kind of s -dependence is to be expected? In general, we cannot answer this question precisely, but we can get a hint by assuming a factored form for A : $A(s,t) = F(s)f(t)$. In so far as such a form is valid, and in so far as $T(s,t)$ is dominantly imaginary in the diffraction peak, we see that for A_{el} to be comparable to A we must have $F(s) = s$. Then if A_{inel} is also comparable to A it must also be proportional to s . Thus something like $T(s,t) = i s f(t)$ would be an acceptable form -- but of course other, more complicated, non-factorizable forms may be consistent as well.

One contribution to A_{inel} is from quasi-two-body intermediate states -- that is, from states produced by quasi-two-body diffraction dissociation processes. We can estimate the relative importance of these states by noting that at $t = 0$, their contribution to A_{inel} is

$$[A_{inel}(s,0)]_{\text{quasi-two-body contribution}} = s \sum_{A^*, B^*} \sigma_{AB \rightarrow A^* B^*}(s) , \quad (3.30)$$

where $\sigma_{AB \rightarrow A^* B^*}(s)$ is the cross-section to produce the resonances A^* and B^* from the initial two particles. By referring to the experimental values of such measured cross-sections, we can see what fraction of σ_{inel} is made up of quasi-two-body states. For pp scattering, for example, $\sigma_{inel} \sim 30$ mb, and the observed two-body dissociation cross-sections are of the order of 0.7, 0.2, 0.6, and 0.1 mb for production of the $N^*(1400)$, $N^*(1520)$, $N^*(1690)$ and $N^*(2190)$ resonances⁸⁾, respectively. We can thus conclude that unless there are many more hitherto unobserved resonances which can be diffractively reached from a proton, the two-body contribution to $(\sigma)_{inel}$ is nearly negligible. Therefore, some information on truly many-body intermediate states will probably be essential.

We can get a feeling for the importance of various multi-body intermediate state contributions by looking at experimental values for $\sigma_n(s)$, the total cross-section to produce n particles. Examples of such cross-sections are shown in Fig. 16. Clearly the data is insufficient for us to draw precise conclusions, but the suggestion is that the various $\sigma_n(s)$ die away, if at all, slowly with increasing s , and that each makes up a non-negligible fraction of σ_{inel} . This confirms the importance of multi-body processes in the s -channel unitarity relation.

Beyond these few remarks, little can be said on the basis of s -channel unitarity without some explicit dynamical assumptions, going far beyond the very basic principles of analyticity, unitarity and crossing, and which will allow us to calculate, or at least to estimate, the "overlap integral" A_{inel} . Any such assumption will of necessity involve guesses about the production amplitudes which make up this function. Various guesses will be described in some detail in the following section; however, let us comment here on two general properties of production which seem to be true and which will be relevant in constructing models. Firstly, it seems well verified experimentally that the transverse momenta $\vec{q}_{i\perp}$ in a production amplitude $T_{2\rightarrow n}(p_1 + p_2 \rightarrow q_1 + \dots + q_n)$ are bounded -- that is, T_n falls off very rapidly (exponentially?) when any $\vec{q}_{i\perp}$ becomes larger than perhaps 500 MeV or so⁹⁾. Secondly, over most of phase space, T_n may be independent of the longitudinal momenta $q_{i\parallel}$ ¹⁰⁾ (obviously, $\vec{q}_{i\perp}$ and $q_{i\parallel}$ are defined relative to the incident direction $\vec{p}_1$). This would mean that the dependence on the $q_{i\parallel}$ is basically just through the phase space factor $1/2\omega$. Hence the distribution in $q_{i\parallel}$ should be $dq_{i\parallel}/q_{i\parallel}$, when $q_{i\parallel}$ is large enough so that $\omega \approx q_{i\parallel}$.

From these two properties several conclusions follow which tend to be confirmed experimentally. One can show that average multiplicities are logarithmic in s [$\langle n(s) \rangle = a \log s + b$], and that doubly differential cross-sections scale¹¹⁾ [$d\sigma/d^3\vec{q} = 1/\omega F(\omega/\sqrt{s}, \vec{q}_\perp)$, where $d\sigma/d^3\vec{q}$ is the cross-section for the process $p_1 + p_2 \rightarrow q + \text{anything}$. This scaling behaviour is sometimes also called limiting fragmentation¹²⁾].

All this is by itself not enough to say much about A_{inel} ; it does, however, provide some constraints on the types of explicit models which we will find it worth while to consider later, and it is therefore well worth keeping in mind.

* * *

To complete the discussion of s-channel unitarity, let us translate what we have already said into the language of s-channel partial waves, and into the impact parameter representation. One or another of these representations, while they of course contain no new physics, may sometimes be more convenient for particular theoretical models.

First we define the s-channel partial wave amplitudes by

$$T(s,t) = \sum_{j=0}^{\infty} (2j+1) P_j(x_s) T(s,j) ; \quad (3.31)$$

or conversely, for integer j ,

$$T(s,j) = \frac{1}{2} \int_{-1}^1 dx_s T(s,t) P_j(x_s) , \quad (3.32)$$

where, at large s , the cosine of the s-channel centre-of-mass scattering angle is

$$x_s = 1 + \frac{2t}{s} . \quad (3.33)$$

$T(s,j)$ is a complex number and it is convenient to parametrize it by writing (for large s)

$$T(s,j) = 16\pi \left(\frac{\eta_j(s) e^{2i\delta_j(s)} - 1}{2i} \right) . \quad (3.34)$$

We call $\delta_j(s)$ the phase shift and $1 - \eta_j(s)$ the absorption in the j^{th} channel.

We have

$$\text{Im } T(s, j) = 16\pi \left(\frac{1 - \eta_j(s) \cos 2\delta_j(s)}{2} \right), \quad (3.35)$$

and we find, by substituting Eqs. (3.31) and (3.34) into our expression (3.27) for $A_{el}(s, t)$, that

$$[\text{Im } T(s, j)]_{el} = \frac{1}{16\pi} |T(s, j)|^2. \quad (3.36)$$

We obtain then, From Eq. (3.27) that

$$[\text{Im } T(s, j)]_{inel} = 16\pi \left(\frac{1 - \eta_j^2(s)}{4} \right), \quad (3.37)$$

and the s-channel unitarity relation in partial wave language has the form

$$\text{Im } T(s, j) = [\text{Im } T(s, j)]_{inel} + [\text{Im } T(s, j)]_{el}, \quad (3.38)$$

or

$$\text{Im } T(s, j) = 16\pi \left(\frac{1 - \eta_j^2(s)}{4} \right) + \frac{1}{16\pi} |T(s, j)|^2. \quad (3.39)$$

The quantity η is, of course, equal to unity below inelastic threshold, where the elastic unitarity condition is exact. At large s , however, we do not have any good idea of how it behaves; different theoretical models give different results. Examples of the limited experimental information are shown in Fig. 22.

There is one general relation between δ and η which follows from analyticity in s of $T(s, j)$ and which may be worth mentioning¹³⁾. It is that (again for large s)

$$\delta_j(s) = -\frac{\sqrt{s}}{2\pi} \oint_{s_{inel}}^{\infty} \frac{\log \eta_j(s')}{\sqrt{s'}(s' - s)} ds' + \frac{\sqrt{s}}{\pi} \int_{LHC} \frac{\text{Im } \delta_j(s')}{\sqrt{s'}(s' - s)} ds'. \quad (3.40)$$

This is derived directly by noting that $\log [T(s,j)/2iq_s]$ is an analytic function whose real part is $\delta_j(s)/\sqrt{s}$ and whose imaginary part on the right-hand cut is $-\frac{1}{2} \log \eta_j(s)/\sqrt{s}$. The value of such a relation, as is common with dispersion relations, depends on some knowledge of $\text{Im } \delta_j(s)$ on the left-hand cut, and this, of course, depends on dynamics.

* * *

Finally, let us remark on the impact parameter representation of the amplitude. Here we define the Bessel transform of $T(s,t)$ by

$$T(s,b) = \int_0^\infty \sqrt{-t} \, d\sqrt{-t} \, J_0(b\sqrt{-t}) \, T(s,t) , \quad (3.41)$$

which, when inverted, is the impact parameter representation:

$$T(s,t) = \int_0^\infty b \, db \, J_0(b\sqrt{-t}) \, T(s,b) . \quad (3.42)$$

This may be connected with the partial wave expansion by noting that when s and j are large the Legendre polynomial in Eq. (3.31) is well approximated by

$$P_j(\cos \theta) \approx J_0(b\sqrt{-t}) , \quad (3.43)$$

where $(2j + 1) = \sqrt{s} \, b$. The partial wave expansion may, therefore, be rewritten by replacing the sum over j by an integral over b :

$$T(s,t) = 8\pi \, s \int_0^\infty b \, db \, J_0(b\sqrt{-t}) \left(\frac{\eta(s,b) \, e^{2i\delta(s,b)} - 1}{2i} \right) , \quad (3.44)$$

where we have now expressed the phase shift and absorption as functions of s and b rather than of s and j . Comparing this with Eq. (3.42), we obtain the identification

$$T(s,b) = 8\pi s \left(\frac{\eta(s,b) e^{2i\delta(s,b)} - 1}{2i} \right), \quad (3.45)$$

which it is conventional to write in terms of the so-called "eikonal" $\chi(s,b)$ as

$$T(s,b) = 8\pi s \left(\frac{e^{2i\chi(s,b)} - 1}{2i} \right). \quad (3.46)$$

Thus

$$\chi(s,b) = \delta(s,b) - \frac{i}{2} \log \eta(s,b). \quad (3.47)$$

Note, finally, that the eikonal can be expressed directly through Eq. (3.41) in terms of the Bessel transform of the scattering amplitude itself.

While the connection with the partial wave expansion is useful, it should be emphasized that the impact parameter representation is, of course, a valid representation in its own right, and depends on partial waves only for its physical interpretation.

In potential theory, the eikonal can be expressed, at high energies, in terms of the potential itself, through the relation¹⁴⁾

$$\chi(s,b) = -\frac{1}{2} \int_{-\infty}^{\infty} dz V(\vec{b} + z \hat{e}_z).$$

Thus $\log \eta$ is related to the imaginary part of the potential, that which represents absorption. If we write $V = V_R + i V_I$, then

$$\log \eta(s,b) = \int_{-\infty}^{\infty} V_I(\vec{b} + z \hat{e}_z) dz.$$

For a real potential, then, $\log \eta = 0$ or $\eta = 1$ and only elastic scattering is possible.

As an illustration of the physics represented by these equations, suppose $V_I = -K$ (a constant) out to some radius R and vanishes beyond, and suppose $V_R = 0$. Then $\delta(s,b) = 0$ and

$$\log \eta(s,b) = \begin{cases} -2K\sqrt{R^2 - b^2} & b < R \\ 0 & b > R \end{cases}$$

If $K \rightarrow \infty$ (full absorption) then we find $\eta = 0$ for $b < R$ and $\eta = 1$ for $b > R$; this gives, from Eq. (3.42),

$$T(s,t) = 4\pi i s \int_0^R b db J_0(b\sqrt{-t})$$

$$= 4\pi i s R \left(\frac{J_1(R\sqrt{-t})}{R\sqrt{-t}} \right)$$

and hence $\sigma(s) = 2\pi R^2$. More generally, any absorbing potential (that is, with $V_R = 0$) of fixed radius will have zero phase shift and an absorption independent of s . For any such situation, we clearly have the form

$$T(s,t) = i s f(t) \quad (3.48)$$

with

$$f(t) = 4\pi \int_0^\infty b db J_0(b\sqrt{-t}) [1 - \eta(b)] \quad (3.49)$$

The resulting differential cross-section will generally have dips (diffraction dips), associated with the oscillations of J_0 , unless $\eta(b)$ is very cleverly contrived. These dips, as we will see later, occur in all of the models of diffraction based on the impact parameter representation.

In particle theory, of course, we have no idea *a priori* what we should use for the potential; hence these simple interpretations are not readily available to us. The "imaginary part of the potential" reflects the presence, in particle theory, of many inelastic channels, and the "absorption" represents the possibility of particle creation. Hence to compute V_I , or η , it would be necessary to have some model of how particle production takes place.

To complete our discussion of the impact parameter representation, we may insert the expansion (3.42) into our expression for A_{e1} , Eq. (3.27), to obtain the result that

$$A_{e1}(s,t) = \frac{1}{8\pi s} \int b \, db \, J_0(b\sqrt{-t}) |T(s,b)|^2, \quad (3.50)$$

where we have made use of the identity

$$\iint \frac{dt_1 \, dt_2 \, \theta(-\lambda)}{\sqrt{-\lambda(t,t_1,t_2)}} J_0(b_1\sqrt{-t_1}) J_0(b_2\sqrt{-t_2}) = \frac{\pi}{8} \frac{1}{b_1} J_0(b_1\sqrt{-t}) \delta(b_1 - b_2). \quad (3.51)$$

Therefore in the impact parameter representation the s-channel unitarity relation takes the form

$$A(s,t) = A_{e1}(s,t) + A_{inel}(s,t),$$

where

$$[A(s,t)]_{e1} = \frac{1}{8\pi s} \int_0^\infty b \, db \, J_0(b\sqrt{-t}) |T(s,b)|^2 \quad (3.52)$$

and

$$[A(s,t)]_{inel} = 8\pi s \int_0^\infty b \, db \, J_0(b\sqrt{-t}) \left(\frac{1 - \eta^2(s,b)}{4} \right). \quad (3.53)$$

To summarize the situation, further progress is possible only with some dynamical assumptions regarding production, which allow one to compute the overlap integral $A_{inel}(s,t)$, or the absorption $1 - \eta_j(s)$ and phase shift $\delta_j(s)$, or the eikonal $\chi(s,b)$. To calculate any one of these is sufficient; the same physics is encompassed in all. It is only a matter of convenience which language is chosen. In the following section various theoretical models for one, some, or all of these quantities are described, together with their results. All are quite arbitrary and involve very drastic assumptions; none works very well in all experimental situations. Nevertheless, they indicate the level at which the theory of diffraction stands.

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- 4) Note that if $T^{(-)}$ were to dominate over $T^{(+)}$, there would be no particular reason to expect the Pomeranchuk Theorem to hold even approximately. Note also that if diffraction were attributed to the subtraction term $sb(t)$ in Eq. (3.1), a possibility we mentioned and discarded earlier, then it would indeed be $T^{(-)}$ and not $T^{(+)}$ which contained diffraction.
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R. Oehme, Phys. Rev. Letters 18, 1222 (1967).
As always, it is of course possible to weasel out of these restrictions. For example, if there is a square root-type branch point moving through $j = 1$ at t -channel threshold, then it is in principle possible to put a fixed pole, which is present on the physical sheet for t below threshold, onto an unphysical sheet above threshold, and thereby avoid violating unitarity. [See R. Oehme, op. cit., and also C.I. Tan and J. Finkelstein, Phys. Rev. Letters 19, 1061 (1967)]. This escape is, however, unattractive in that there is no theoretical reason for the existence of such a cut, nor is there any experimental sign that it exists.

Another conceivable escape is that, while of course $T(s,t)$ must be analytic in t for a finite value of s , it is not rigorously necessary that the asymptotic form as $s \rightarrow \infty$ be analytic in t . Thus one could, in principle, have for example $T(s,t) \rightarrow F(s) f(t)$ with $f(t) \propto \theta(t_{\text{threshold}} - t)$; hence no violation of unitarity is present. This sounds pretty forced, however, and no plausible such model has yet been invented.
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It has, naturally, been suggested that many-body intermediate state effects can be approximated, in some duality sense, by a sum over a large number of quasi-two-body states, not necessarily associated with well defined resonances. Such models have, however, not been notably successful.

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4. THEORETICAL MODELS OF ELASTIC SCATTERING

4.1 Conventional Regge-type models

The underlying assumption in all Regge type models of diffraction scattering is that diffractive processes are not basically different from other types of reactions. The general validity of Regge theory as a description of processes with quantum-number exchange is now widely accepted¹⁾. It is, therefore, natural to attempt to describe diffraction in a normal Regge manner as well, and this is in fact what was done immediately with the advent of Regge theory in 1961²⁾.

4.1.1 Good old Regge poles

At the simplest level, the Regge theory of diffraction scattering assumes the existence of a Regge trajectory $\alpha_P(t)$, called the Pomeranchuk trajectory, in the vacuum channel. It is assumed to have vacuum quantum numbers, even signature, and to have an intercept of one: $\alpha_P(0) = 1$. In accord with the usual Regge rules, then, one can say that any amplitude in which the vacuum can appear as a cross-channel has the asymptotic behaviour associated with exchange of the Pomeranchuk trajectory, since it is higher than all other trajectories which can be exchanged. Thus for an elastic amplitude we expect, as $s \rightarrow \infty$, and for t fixed,

$$T(s,t) \rightarrow R(s,t) \equiv \beta_A(t) \beta_B(t) \left(\frac{s}{s_0} \right)^{\alpha_P(t)} \left(- \frac{1 + e^{-i\pi\alpha_P(t)}}{\sin \pi \alpha_P(t)} \right). \quad (4.1)$$

Factorization³⁾ of the residues has been explicitly indicated. To simplify the notation, we shall usually choose the scale s_0 to be equal to one in what follows, and we shall also frequently drop the index P on the trajectory function.

We shall also on occasion use slightly different forms for the Regge pole term (4.1), so let us standardize our notation now.

We shall write $\beta(t)$ for $\beta_A(t) \beta_B(t)$ when we do not want to bother to explicitly indicate factorization. The residue of the corresponding pole in the even signatured partial wave amplitude will be called $\bar{\beta}(t)$; thus

$$T^{(+)}(t, j) = \frac{\bar{\beta}(t)}{j - \alpha(t)},$$

where

$$\bar{\beta}(t) = -\beta(t) \frac{(q_t^2)^{\alpha(t)}}{2\alpha(t) + 1} \cdot \frac{\Gamma(\alpha(t) + 1)}{\Gamma(\alpha(t) + \frac{1}{2})} \cdot \frac{\sqrt{\pi}}{2}.$$

Note the identity

$$\frac{1 + e^{-i\pi\alpha}}{\sin \pi \alpha} = \frac{e^{-i\pi\alpha/2}}{\cos \pi \alpha/2}.$$

If there were no particles on the trajectory, or if we are not near even integer values of α , it is often convenient to absorb the $\cos \pi \alpha/2$ factor into the residue function $\beta(t)$. Thus we shall define

$$-\gamma(t) = \beta(t)/\cos(\pi \alpha(t)/2)$$

and write

$$R(s, t) = \gamma(t) e^{-i\pi\alpha(t)/2} s^{\alpha(t)}.$$

Since $\alpha_p(0) = 1$, we find that at $t = 0$ the amplitude is pure imaginary and linear in s ; consequently the total cross-section approaches a constant and we have

$$\sigma_{AB}(s) \rightarrow \beta_A(0)\beta_B(0). \quad (4.2)$$

Total cross-sections therefore factor³⁾; this is the first prediction of conventional Regge theory:

$$\sigma_{AB} \cdot \sigma_{CD} = \sigma_{AC} \cdot \sigma_{BC}. \quad (4.3)$$

Experimentally, this relation is difficult to check. Theoretically, it poses difficulties as well, in that it suggests that we should expect total cross-sections from nuclei, for example, to vary like A rather than the $A^{2/3}$ dependence which is both observed empirically and predicted by other theoretical models. The resolution of this difficulty probably lies in the existence of Regge cuts⁴⁾.

Other Regge trajectories are, empirically, straight lines; it is therefore natural to assume, in accord with the basic idea that diffraction is much the same as any other kind of scattering, that

$$\alpha_P(t) = 1 + \alpha'_P t, \quad (4.4)$$

at least near $t = 0$. For other trajectories, we know the slope is about $1/(\text{GeV})^2$; the first guess would therefore be that α'_P has this value too. The question then arises: what particles lie on the Pomeranchuk trajectory? The first physical particle should have spin-parity 2^+ , and of course $I = Y = B = 0$ since it must occur in the vacuum channel. If $\alpha'_P \sim 1/(\text{GeV})^2$, the mass of the particle should be about 1400 MeV. Two candidates exist experimentally, the f_0 meson at 1260 MeV and the f'_0 at 1515 MeV. There are, however, some slight difficulties. The particle on the Pomeranchuk trajectory should be an SU(3) singlet (since it is in the vacuum channel); yet the evidence seems to be that the f_0 is mostly an SU(3) octet. But if, therefore, the f'_0 goes with the Pomeron, what trajectory goes with the f_0 ? There is good evidence for another vacuum trajectory, called $\alpha_{P'}$, with $\alpha_{P'}(0) \sim 0.5$ ⁵⁾, and this might produce the f_0 . But then the P and P' trajectory must cross, which, while not impossible, is a little bit unsavoury theoretically. So perhaps the f_0 is on the P after all, and it is just that the SU(3) mixing angle changes rapidly from $t = 0$ (where P is pure singlet) to $t = m_{f_0}^2$ (where it is mostly octet)⁶⁾. Another possibility is that the P and P' do not exactly cross, but nevertheless exchange character, so that the picture is as shown in Fig. 23. There are even some theoretical arguments for this kind of behaviour, based on multiperipheral type models⁷⁾.

In any event, the question of which, if any, particles lie on the Pomeranchuk is, at present, in a somewhat confused state, and we shall drop it for now.

For negative t , the simple Regge pole model also makes some predictions. From Eq. (4.1) we see that near $t = 0$ the differential cross-section has the form

$$\frac{d\sigma}{dt} = \left(\frac{d\sigma}{dt} \right)_{t=0} \left| \frac{\beta(t)}{\beta(0)} \right|^2 e^{2\alpha'_P t \log s}. \quad (4.5)$$

We thus predict a shrinkage of the elastic peak with $\log s$, so that the slope parameter will have the form

$$b(s) = b_0 + \alpha' \log s . \quad (4.6)$$

The constant term b_0 evidently is to be attributed to whatever t -dependence exists in the residue functions.

It is amusing, as an aside, to notice the behaviour of the absorption η and phase shift δ in a pure Regge pole model. Let us write

$$R(s,t) = 8\pi s \int_0^\infty b \, db \, J_0(b\sqrt{-t}) \left(\frac{\eta(s,b) e^{2i\delta(s,b)} - 1}{2i} \right) , \quad (4.7)$$

or

$$\frac{\eta(s,b) e^{2i\delta(s,b)} - 1}{2i} = \frac{1}{8\pi s} \int_0^\infty \sqrt{-t} \, d\sqrt{-t} \, J_0(b\sqrt{-t}) R(s,t) . \quad (4.8)$$

If we write $R(s,t)$ in the form $R(s,t) = \gamma s^\alpha e^{-i\pi\alpha/2}$ and choose $\gamma(t) = \gamma_0 \exp b_0 t$ and $\alpha = 1 + \alpha' t$, then we can also write $R(s,t) = i \gamma_0 e^{b_1 t} s$, where $b_1 = b_0 + \alpha'(\log s - i\pi/2)$. The integral can easily be evaluated, and yields

$$\frac{\eta e^{2i\delta} - 1}{2i} = \frac{i}{8\pi} \left(\frac{\gamma_0}{2b_1} \right) e^{-b^2/4b_1} . \quad (4.9)$$

Equating the real and imaginary parts separately here, we find that at a fixed impact parameter

$$\delta \rightarrow 0 \quad \text{like} \quad (1/\log s)$$

and

$$\eta \rightarrow 1 \quad \text{like} \quad (1/\log s)^2 .$$

(4.10)

This corresponds, in the simple language of Section 3, to scattering from an absorbing disk whose radius grows like $\log s$, but whose absorption $1 - \eta$ vanishes like $(1/\log s)^2$.

Comparing Eq. (4.6) with the experimental data on diffraction peaks discussed in Section 1, we see that if this very simple picture of pure Pomeron exchange suffices at present energies, then $\alpha'_P \sim 0.5/(\text{GeV})^2$ ⁸⁾. However, this value may easily be considerably altered because of interference effects with lower-lying poles, or other garbage, which has not yet gone away at the energies now accessible to measurement.

Effects of lower-lying trajectories, or whatever, are in fact plainly visible. We have constructed α_P so as to give, asymptotically, constant cross-sections and to satisfy the Pomeranchuk theorems. At lesser than asymptotic energies, however, lower trajectories will give non-zero contributions too. These appear in cross-sections in the form

$$\sigma(s) \rightarrow \sigma(\infty) + \sum_R s^{\alpha_R(0)-1} \beta_R(0) , \quad (4.11)$$

where the sum is over the other than Pomeranchuk trajectories which may be exchanged in the reaction. Comparing this form with the data mentioned earlier does not give an entirely adequate fit; it seems, in fact, that the approach to asymptotia is slower than powers of s . This suggests that our simple description in terms of Regge poles alone cannot be correct, and other types of j -plane effects must be present too. We do not have to look far. Regge cuts have been known to exist for a long time, and indeed their effects are known to be important in many non-elastic scattering processes⁹⁾. It is, therefore, very natural to expect cuts to be present in diffraction as well. Various models of how cuts may appear exist, and we shall next turn to a discussion of some of these.

4.1.2 Cuts

4.1.2.1 AFS cuts

The simple Pomeranchuk pole model assumed that

$$T(s,t) \rightarrow R(s,t) = \beta(t) s^{\alpha(t)} \left(- \frac{1 + e^{-i\pi\alpha(t)}}{\sin \pi \alpha(t)} \right) . \quad (4.12)$$

Historically, the first alternative possibility to Eq. (4.12) to be considered was to assume instead that¹⁰⁾

$$A_{inel}(s,t) \rightarrow \text{Im } R(s,t) . \quad (4.13)$$

There is no very convincing justification for this assumption; the best that can be said is that if one assumes a model of production amplitudes in which production is described by simply a tree-type Feynman diagram (see Fig. 25a) then the entire imaginary part A would be just the sum of all ladder graphs, and this is known in simple field theories to generate a Regge pole. The assumption must, therefore, really be considered quite arbitrary.

Equation (4.13), when coupled with the s -channel unitarity relation, yields an integral equation for $T(s,t)$:

$$A(s,t) = \text{Im } R(s,t) + \frac{1}{16\pi^2 s} \iint \frac{dt_1 dt_2 \theta(-\lambda)}{\sqrt{-\lambda(t,t_1,t_2)}} T(s,t_1) T^*(s,t_2) . \quad (4.14)$$

Although this equation can be solved exactly analytically (we shall return to this shortly), we can get an idea of the content of the solution by solving it by iteration. Through one iteration, we find

$$A(s,t) = \text{Im } R(s,t) + \frac{\beta_0^2}{32\pi^2 \alpha} e^{b_0 t/2} \frac{s^{\alpha_c(t)}}{\log s} , \quad (4.15)$$

where we assume that $\text{Im } R$ dominates $\text{Re } R$ under the unitarity integral, and where we assume $\beta(t) = \beta_0 e^{b_0 t}$ and $\alpha(t) = 1 + \alpha' t$. The function $\alpha_c(t) = 1 + \alpha' t/2 = 2\alpha(t/4) - 1$.

The correction term in Eq. (4.15) mirrors the existence in the j -plane of a logarithmic branch cut in the cross-channel partial wave amplitudes $T(t,j)$ of both signatures at $j = \alpha_c(t)$. (This cut is conventionally known as the "AFS cut".)

While at $t = 0$ the first term $\text{Im } R(s,t)$ dominates (albeit only by a $\log s$ factor) as $s \rightarrow \infty$, for $t < 0$ the second term wins because

$\alpha_c(t) > \alpha(t)$ [that is, the branch point in $T(t,j)$ is further to the right in the j plane than the pole in $T^{(+)}(t,j)$]. Thus the asymptotic behaviour (4.12) is correct only at $t = 0$. The shrinkage of forward diffraction peaks will now be less than obtained with the pole alone, and the total cross-section behaves like $\text{const.} + 1/\log s$, rather than as given by Eq. (4.11), as $s \rightarrow \infty$.

The appearance of this j -plane cut can also be seen through the projection into t -channel partial waves of the s -channel unitarity equation. In general, we have from Eq. (3.27), that

$$A(s,t) = A_{\text{inel}}(s,t) + \frac{1}{16\pi^2 s} \iint \frac{dt_1 dt_2}{\sqrt{-\lambda}} \theta(-\lambda) \times \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{j_1} \tilde{T}(t_1 j_1) dj_1 \right\}^* \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{j_2} \tilde{T}(t_2 j_2) dj_2 \right\} . \quad (4.16)$$

[We have ignored signature here to simplify the appearance of this equation -- strictly speaking the functions $\tilde{T}(t_1 j_1)$ and $\tilde{T}(t_2 j_2)$ on the right-hand side should be replaced by

$$\tilde{T}^{(+)}(t,j) \left(\frac{1 + e^{-i\pi j}}{2} \right) + \tilde{T}^{(-)}(t,j) \left(\frac{-1 + e^{-i\pi j}}{2} \right)] .$$

Now by the same token we can write

$$A(s,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj s^j \tilde{A}(t,j) .$$

This expression can be inserted into Eq. (4.16), and the Mellin transform inverted to obtain

$$\begin{aligned}
 A(t, j) = & A_{inel}(t, j) + \\
 & + \frac{1}{16\pi^2} \iint \frac{dt_1 dt_2}{\sqrt{-\lambda}} \theta(-\lambda) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj_1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj_2 \quad (4.17) \\
 & \times \frac{\tilde{T}(t_1 j_1) \tilde{T}^*(t_2 j_2)}{j + 1 - j_1 - j_2} .
 \end{aligned}$$

Equation (4.17) is quite general; it only restates, in terms of t -channel partial waves, the s -channel unitarity relation at large s . To specialize it to the AFS model, we must now assume, as before, that the inhomogeneous term $A_{inel}(t, j)$ is given by a Regge pole. Therefore, corresponding to Eq. (4.13), we write $A_{inel}(t, j) = \text{Im } R(t, j)$. But

$$R(s, t) = \beta s^\alpha \left(- \frac{1 + e^{-i\pi\alpha}}{\sin \pi \alpha} \right) \quad (4.18)$$

so that

$$\text{Im } R(s, t) = \beta s^\alpha \quad (4.19)$$

and hence

$$\text{Im } R(t, j) = \int_0^\infty ds s^{-j-1} \text{Im } R(s, t) = \frac{\beta}{j - \alpha} . \quad (4.20)$$

Again then, we have an integral equation (modulo the assumption that $\text{Im } \tilde{T}$ dominates under the integral), this time for $A(t, j)$. The first iteration gives

$$\begin{aligned}
 A(t, j) = & \text{Im } R(t, j) \\
 & + \frac{1}{16\pi^2} \iint \frac{dt_1 dt_2}{\sqrt{-\lambda}} \theta(-\lambda) \frac{\bar{\beta}(t_1) \bar{\beta}(t_2)}{j + 1 - \alpha(t_1) - \alpha(t_2)} . \quad (4.21)
 \end{aligned}$$

Evidently we have a cut in $A(t, j)$. If $\alpha(t)$ is monotonic, the branch point occurs when $j + 1 = [\alpha(t_1) + \alpha(t_2)]_{\max} = 2\alpha(t/4)$; then we have $\alpha_c(t) = 2\alpha(t/4) - 1$ as before. We also note that including signature means that $\text{Im } \tilde{T}$ should be replaced by

$$\text{Im } \tilde{T}^{(+)} \left(\frac{1 + e^{-i\pi j}}{2} \right) + \text{Im } \tilde{T}^{(-)} \left(-\frac{1 + e^{-i\pi j}}{2} \right);$$

hence the cut term contributes to both signatures, also as we saw before.

The j -plane form of the AFS cut is quite convenient for future application, so we shall dignify it with a name. We rewrite Eq. (4.21) as

$$A(t, j) = \text{Im } R(t, j) + A_{\text{AFS}}(t, j), \quad (4.22)$$

thereby defining the function $A_{\text{AFS}}(t, j)$.

Further properties of the cuts in Eq. (4.15) which are worth remarking on are, firstly, that the pole and the cut interfere constructively; they have the same phase at $t = 0$. Secondly, the discontinuity across the cut in the j plane does not vanish at even integer values of j . Hence physically observable partial wave amplitudes [i.e. $\tilde{T}^{(+)}(t, j)$ with $j = 0, 2, 4 \dots$] have branch points in t at $t = t_c$, where $\alpha_c(t_c) = j$. These branch points do not, in general, coincide with the branch points required by unitarity, and therefore the amplitude should really not be expected to have them.

A final objection to the AFS cuts arises from the fact that field theory models based on Feynman diagrams do not have them¹¹⁾.

These remarks cast doubt on the correctness of the assumption, Eq. (4.13) (but since this assumption was completely arbitrary anyway that should not bother us much). Nevertheless, for the sake of completeness we may as well continue on to the exact solution to Eq. (4.14)¹²⁾.

The exact integral equation we have is of the form

$$\frac{A(s,t)}{s} = \beta_0 e^{b_1 t} + \frac{1}{16\pi^2} \iint \frac{dt_1 dt_2}{\sqrt{-\lambda}} \left(\frac{A(s,t_1)}{s} \right) \left(\frac{A(s,t_2)}{s} \right) \quad (4.23)$$

if, as before, we assume that $\text{Im } T$ dominates $\text{Re } T$ under the integral, and where we write

$$\text{Im } R(s,t) = s \beta_0 e^{b_1 t} \quad (4.24)$$

so that

$$b_1 = b_0 + \alpha' \log s .$$

The exact solution to this equation is¹²⁾

$$A(s,t) = 8\pi \beta_0 s \int_0^1 \frac{dy}{y} \left(1 - \sqrt{1 - \frac{\beta_0}{4\pi b_1} y} \right) J_0(2\sqrt{b_1 t \log y}) . \quad (4.25)$$

From Eq. (4.25), we see that at large t ,

$$A(s,t) \rightarrow 16\pi \frac{\sqrt{\log(4\pi b_1/\beta_0)}}{(-t)} e^{-2\sqrt{-b_1 t \log 4\pi b_1/\beta_0}} ; \quad (4.26)$$

thus in this region $\text{Im } T$ behaves like $(\log \log s)^{\frac{1}{2}} e^{-2\sqrt{-\alpha' t \log s(\log \log s)}}$ so far as its s -dependence is concerned. Equations (4.25) and (4.26) predict a differential cross-section disagreeing violently with experiment, in that the fall off with increasing $-t$ is far too slow to be compatible with the data. The total cross-section comes out to be

$$\sigma_T = 16\pi b_1 \left(1 - \sqrt{1 - \frac{\beta_0}{4\pi b_1}} + \frac{1}{2} \log \frac{\beta_0}{16\pi b_1} \frac{1 + \sqrt{1 - \beta_0/4\pi b_1}}{1 - \sqrt{1 - \beta_0/4\pi b_1}} \right) . \quad (4.27)$$

As $s \rightarrow \infty$, this yields the behaviour

$$\sigma_{e1}(s)/\sigma_T(s) \rightarrow 1 ,$$

which is, perhaps, somewhat surprising.

It is also of some interest to look at partial waves here. We find

$$\eta_j(s) = \left(1 - \frac{\beta_0}{4\pi} \sqrt{\frac{\pi s}{b_1}} e^{-b_1 s/2} I_{j+\frac{1}{2}}(b_1 s/2) \right)^{\frac{1}{2}} \quad (4.28)$$

so that if $s \rightarrow \infty$, we get

$$\eta_j(s) \rightarrow \sqrt{1 - \beta_0/4\pi b_1} \rightarrow 1 . \quad (4.29)$$

Thus the inelasticity has the qualitative behaviour of first dropping as s increases, and then rising again to unity asymptotically as s increases further.

4.1.2.2 Absorptive cuts

So much for the AFS model. Undeterred by its drawbacks, we shall proceed to other cut models.

Our next venture is based on the eikonal representation of the amplitude, Eq. (3.46). This representation is of value only if we have some knowledge of the eikonal $\chi(s,b)$. If T , and hence χ , were small, we could write

$$T(s,t) \approx 8\pi s \int_0^\infty b db J_0(b\sqrt{-t}) \chi(s,b) \quad (4.30)$$

and, by inverting the Bessel transform,

$$\chi(s,b) \approx \frac{1}{8\pi s} \int_0^\infty \sqrt{-t} d\sqrt{-t} J_0(b\sqrt{-t}) T(s,t) \quad (4.31)$$

calculate χ . If T is not small, these expressions are, of course, false. Nevertheless, suppose that we have some sort of Born approximation expression for $T(s,t)$; call it $B(s,t)$. Then if $T \approx B$, we could write

$$\chi(s,b) \approx \frac{1}{8\pi s} \int_0^\infty \sqrt{-t} d\sqrt{-t} J_0(b\sqrt{-t}) B(s,t) . \quad (4.32)$$

But now, even if B is not small, so that $T \neq B$, we can still try to use Eq. (4.32) to calculate χ , and then calculate T from the general expression in terms of χ , Eq. (3.42).

There is no justification for this procedure at all. It is completely analogous to the radiation damping approximation, or to the N/D method, both of which consist of using the Born approximation amplitude in some other role (in one case as $\tan \delta$ rather than $\sin \delta e^{i\delta}$, in the other as N rather than T itself) in order to attempt to improve the accuracy of the approximation. The attempt to do the same thing with the eikonal has its origin in the Glauber formula¹⁴⁾ in potential theory, where it is in fact correct to replace χ by the potential.

In our case, of course, we have no idea what the potential, or B , is. So what is more natural than to take it to be the simple Pomeranchuk Regge pole itself¹³⁾: $B(s,t) = R(s,t)$? Then we have

$$\chi(s,b) = \frac{1}{8\pi s} \int_0^\infty \sqrt{-t} d\sqrt{-t} J_0(b\sqrt{-t}) R(s,t) \quad (4.33)$$

and we can now calculate the amplitude from

$$T(s,t) = 8\pi s \int_0^\infty b db J_0(b\sqrt{-t}) \left(\frac{e^{2i\chi(s,b)} - 1}{2i} \right). \quad (4.34)$$

It is to be emphasized that this, like the AFS guess of equating $R(s,t)$ with $A_{inel}(s,t)$, is totally arbitrary, and there is no reason why it should be correct. It is nevertheless of some interest to compare the two models. Let us expand Eq. (4.34) in powers of χ , that is of R . We find

$$\begin{aligned} T(s,t) &= R(s,t) + 8\pi i s \int_0^\infty b db J_0(b\sqrt{-t}) [\chi(s,b)]^2 + \dots \\ &= R(s,t) + \frac{i}{16\pi^2 s} \iint \theta(-\lambda) \frac{dt_1 dt_2}{\sqrt{-\lambda}} R(s,t_1) R(s,t_2) + \dots \end{aligned} \quad (4.35)$$

In the AFS model, in contrast, to order R^2 we had

$$A(s,t) = \text{Im } R(s,t) + \frac{1}{16\pi^2 s} \iint \theta(-\lambda) \frac{dt_1 dt_2}{\sqrt{-\lambda}} R(s,t_1) R^*(s,t_2) + \dots \quad (4.36)$$

The difference is that in one case, there appears RR^* under the integral, and in the other case just RR . Since, for small t , R is mostly imaginary, this means that the two models have nearly opposite signs for the first correction term; where in the AFS model, the pole and cut interfere constructively, in the other the interference is destructive.

We may explicitly evaluate the integral in Eq. (4.35)¹⁵⁾

$$T(s,t) = R(s,t) + \frac{\gamma_0^2}{32\pi \alpha'} e^{b_0 t/2} \frac{s^{\alpha_c(t)}}{\log s} e^{-(i\pi/2)\alpha_c(t)} \quad (4.37)$$

There is, as in the AFS case, a logarithmic cut at $j = \alpha_c(t)$ with the same $\alpha_c(t)$. However, this cut occurs only in one signatured partial wave amplitude, namely $T^{(+)}(t,j)$, and furthermore its discontinuity vanishes at $j = 0, 2, 4 \dots$. Therefore it introduces no false branch points in t into physically observable partial wave amplitudes. Finally, cuts of this sort do occur in Field Theoretic Models¹¹⁾.

This first iteration can, in the same way as for the AFS case, also be expressed in the t -channel j plane, and it is perhaps instructive to do this. Everything is very similar to the calculation indicated for AFS, and we find that Eq. (4.21) becomes instead

$$T(t,j) = R(t,j) + \frac{1}{16\pi^2} \iint \theta(-\lambda) \frac{dt_1 dt_2}{\sqrt{-\lambda}} \frac{\gamma(t_1) \gamma(t_2)}{j + 1 - \alpha(t_1) - \alpha(t_2)} \times \quad (4.38)$$

$$\times \exp \left\{ -\frac{i\pi}{2} [\alpha(t_1) + \alpha(t_2) - 1] \right\}.$$

The cut is at the usual place, but, as we have already seen, it now appears only in the even signaturred amplitude.

The more obvious objections to the AFS model thus do not apply to this model (called, incidentally, the absorptive cut model). Let us see how well it works empirically.

Equations (4.34) and (4.37) allow one to calculate, numerically, the amplitude $T(s,t)$. The calculation is made easier if the exponential in Eq. (4.34) is expanded, because the Bessel transform of powers of χ can be evaluated analytically.

If we take $R(s,t)$ to have the form

$$R(s,t) = i s \gamma_0 e^{b_1 t}, \quad (4.39)$$

where $b_1 = b_0 + \alpha'(\log s - i\pi/2)$, then $\chi(s,b)$ is found from Eq. (4.33) to be

$$\chi(s,b) = \frac{i}{16\pi} \left(\frac{\gamma_0}{b_1} \right) e^{-b^2/4b_1}. \quad (4.40)$$

Now we expand the exponential in Eq. (4.13) to obtain

$$T(s,t) = -4\pi i s \int_0^\infty b db J_0(b\sqrt{-t}) \sum_{n=1}^\infty \frac{[2i \chi(s,b)]^n}{n!}. \quad (4.41)$$

The integrals can all be evaluated, and we find

$$T(s,t) = -4\pi i s \sum_{n=1}^\infty \frac{1}{n!} \left(\frac{-\gamma_0}{8\pi b_1} \right)^n \frac{2b_1}{n} e^{tb_1/n}. \quad (4.42)$$

The resulting differential cross-section is shown in Fig. 24¹⁶⁾. The agreement with experiment is poor.

Again, however, we should not be surprised. We have made, in Eq. (4.33), a totally unwarranted and arbitrary assumption; it should come as no shock if the resulting model does not work very well.

Before we leave this subject let us elaborate just a bit more on the relationship between the AFS and absorption models.

In the AFS model, we identified A_{inel} with a Regge pole. Thus we wrote

$$\text{Im } R(s,t) = 8\pi s \int_0^\infty b \, db \, J_0(b\sqrt{-t}) \left(\frac{1 - \eta^2(s,b)}{4} \right), \quad (4.43)$$

or inverting

$$\eta(s,b) = \left(1 - \frac{1}{2\pi s} \int_0^\infty \sqrt{-t} \, d\sqrt{-t} \, J_0(b\sqrt{-t}) \, \text{Im } R(s,t) \right)^{\frac{1}{2}}. \quad (4.44)$$

In the absorption model we identified the eikonal χ with a Regge pole. Thus we wrote

$$\begin{aligned} R(s,t) &= 8\pi s \int_0^\infty b \, db \, J_0(b\sqrt{-t}) \, \chi(s,b) \\ &= 8\pi s \int_0^\infty b \, db \, J_0(b\sqrt{-t}) \left[\delta(s,b) - \frac{i}{2} \log \eta(s,b) \right]. \end{aligned} \quad (4.45)$$

Now, inverting, we find

$$\log \eta(s,b) = - \frac{1}{4\pi s} \int_0^\infty \sqrt{-t} \, d\sqrt{-t} \, J_0(b\sqrt{-t}) \, \text{Im } R(s,t). \quad (4.46)$$

[Note that for small $\text{Im } R$, the two expansions are as follows:

$$\text{AFS:} \quad \eta = 1 - \frac{1}{2} X - \frac{1}{8} X^2 + \dots$$

$$\text{Abs} \quad \eta = 1 - \frac{1}{2} X + \frac{1}{8} X^2 + \dots$$

$$\text{where} \quad X \equiv \frac{1}{2\pi s} \int_0^\infty \sqrt{-t} \, d\sqrt{-t} \, J_0(b\sqrt{-t}) \, \text{Im } R(s,t).$$

The opposite sign of the first cut term is thus exhibited very clearly.]

Thus in one case $\text{Im } R \leftrightarrow 1 - \eta^2/4$; in the other case $\text{Im } R \leftrightarrow -\frac{1}{2} \log \eta$. When η is near one, these expressions agree:

$$\frac{1 - \eta^2}{4} \approx \frac{1 - \eta}{2} + \dots,$$

and

$$-\frac{1}{2} \log \eta \approx -\frac{1}{2}(\eta - 1) + \dots$$

When η is not near one, however, they differ.

Both expressions are arbitrary; both are *ad hoc*; neither agrees well with experiment. That fact is, perhaps, just as well.

4.1.3 MUREX constraints

In both the AFS and absorption models, we attempted to replace the effects of inelastic intermediate states with a simple Regge pole -- in one case by saying $(\text{Im } T)_{\text{inel}}$ was a Regge pole, in the other case by saying χ was a Regge pole. In neither case did we have any specific model of production in mind; the assumption we made was arbitrary. Let us begin to discuss possible ways of remedying this, in the hope that we can construct something a bit more fundamental.

Let us define, in a production process, the following $2n - 2$ variables: pair energies, or subenergies,

$$s_{i,i+1} = (q_i + q_{i+1})^2, \quad i = 1 \dots n - 1 \quad (4.47)$$

and momentum transfers

$$t_i = (p_1 - q_1 \dots - q_i)^2, \quad i = 1 \dots n - 1. \quad (4.48)$$

A complete set consists of $3n - 4$ variables; for our purposes however we do not need to specify the remainder explicitly¹⁷⁾.

Let us define the multi-Regge region of phase space to be the region where all pair energies are large and all momentum transfers are small.

Then the weakest form of the multi-Regge (MUREX) assumption is that in this region only the production amplitude looks like repeated exchange of the leading Regge singularity in the appropriate crossed channel. Specifically, if all leading exchanges are Regge poles, we write¹⁸⁾

$$\begin{aligned}
 T_{2 \rightarrow n}(p_1 + p_2 \rightarrow q_1 + \dots + q_n) &= \beta(t_1)(s_{12})^{\alpha(t_1)} \times \\
 &\times R(t_1) \beta(t_1, t_2)(s_{23})^{\alpha(t_1)} R(t_2) \dots \\
 &= R(s_{12}, t_1) R(s_{23}, t_2) \dots R(s_{n-1, n}, t_{n-1}) .
 \end{aligned}
 \tag{4.49}$$

[Here $R(t)$ stands for the signature factor

$$= \frac{1 \pm e^{-i\pi\alpha(t)}}{\sin \pi \alpha(t)} ,$$

or perhaps more conveniently, for $e^{-i\pi\alpha(t)/2}$.] If the leading singularity in one or more of the cross-channels is a cut, Eq. (4.49) is modified accordingly.

This most conservative form of MUREX cannot allow us to calculate A_{inel} , because we have assumed that we know $T_{2 \rightarrow n}$ only in a very limited region of phase space. It does, however, permit us to place an upper bound on the kind of Regge singularities we can have, in that this region of phase space by itself must not contribute so much to the total cross-section that the Froissart bound is violated.

We shall discuss the arithmetic in detail in a moment, in connection with the much stronger form of the MUREX assumption, the so-called multi-Regge bootstrap model. Suffice it to list, for the present, the conclusions which are obtained regarding these bounds^{19,20}:

- i) If a given Regge pole, fixed or moving, can be exchanged more than once, then we must have $\alpha(0) < 1$ for it; otherwise $\sigma \rightarrow \infty$ like a power of s ¹⁹.
- ii) If a "hard" cut is exchanged more than once, we must have $\alpha_c(0) < 1$ ¹⁹.

It is of some interest to combine these limitations with those (far more fundamental ones) obtained by imposing t -channel unitarity (see Section 3). Table 1 ensues:

Table 1

	t-channel	s-channel
Fixed pole	No	Yes if $\alpha < 1$
Moving pole	Yes	Yes if $\alpha(0) < 1$
Fixed cut	Yes if soft	Yes if soft
Moving cut	Yes if soft or complex	Yes if soft or if $\alpha_c(0) < 1$

We have, as yet, no answer from the weak MUREX model to the more complicated situation when combined cuts and poles comprise the leading singularity. It is also important to keep in mind that the t-channel limits are very basic, while the s-channel ones depend on the possibly false assumption that the MUREX model is true (albeit only in the MUREX region).

4.1.4 Factorized models of production

At this point, for the first time, we shall face up squarely to the fact that if we really want to understand diffraction, we must commit ourselves to some sort of model of particle production. Diffraction is, we believe, the result of the absorption which particle production causes; to calculate this absorption we need to know something about production. What, then, can we say about the amplitudes $T_{2 \rightarrow n}$?

The class of models which we shall describe in this section is that in which production amplitudes are assumed to factor. There are many versions of this assumption, as we will see below, differing in what the production amplitude is supposed to factor into. However, the arithmetic is very largely common to all of these versions, and so are the main features of the conclusions.

Let us first describe the class of factorized models in which the production amplitude is simply a product of some basic function: that is, in which the production amplitude can be written

$$T_{2 \rightarrow n}(p_1 + p_2 \rightarrow q_1 + \dots + q_n) = T_0\left(\frac{\bar{s}_0}{\bar{s}_1}, t_1\right) T_0\left(\frac{\bar{s}_1}{\bar{s}_2}, t_2\right) \dots$$

$$\dots T_0\left(\frac{\bar{s}_{n-2}}{\bar{s}_{n-1}}, t_{n-1}\right), \quad (4.50)$$

where $s_i = (q_1 + \dots + q_i)^2$ and $\bar{s}_i = (q_{i+1} + \dots + q_n)^2$. We can also phrase the assumption in another way:

$$T_{2 \rightarrow n}(p_1 + p_2 \rightarrow q_1 + \dots + q_n) = T_0(s_{12}, t_1) T_0(s_{23}, t_2) \dots$$

$$\dots T_0(s_{n-1, n}, t_{n-1}), \quad (4.51)$$

and indeed this choice is the more conventional one. In the MUREX region, where all pair energies $s_{i, i+1}$ are large and all momentum transfers are small, we know that $\bar{s}_{i-1}/\bar{s}_i$ and $s_{i, i+1}$ coincide²¹⁾. Thus both Eq. (4.50) and Eq. (4.51) reduce to MUREX in the MUREX region, provided $T_0(s, t) \rightarrow R(s, t)$ when s becomes large.

At lower pair energies, however, the two forms differ, and we have a choice to make. Since the same conclusions are obtained whichever choice is made, our choice will be made on the basis of arithmetical convenience: we choose to use Eq. (4.50)²²⁾.

Various specific versions of this model are the following:

i)

$$T_0(s, t) = \frac{g}{t - 1}. \quad (4.52)$$

This model yields simply a field theoretic ladder as shown in Fig. 25a. It is well known²³⁾ that inserting this version of $T_{2 \rightarrow n}$ into the s-channel unitarity relation (3.26) results in $A(s, t)$ behaving like a Regge pole, with $\alpha(t) = -1 + O(g^2)$.

ii)

$$T_0(s, t) = R(s, t), \quad (4.53)$$

where R is the leading Regge singularity which can occur in the cross-channel. This model, which is called the MUREX bootstrap (BS) model, amounts to believing that the MUREX form discussed pre-

viously is valid over all of phase space, instead of only in the limited MUREX region²⁴⁾. This extremely strong assumption is manifestly false if taken literally; it is therefore conventional to justify it by an appeal to "duality"²⁵⁾, and to say that while in detail the assumption is false, it is true "on the average", in that the approximate production amplitude, when integrated over phase space, gives nearly the same result as the integral of the true production amplitude. Even in this more limited form, however, this MUREX bootstrap model, as it is called, is probably wrong²⁶⁾.

iii) A model which has not yet been studied in detail is to assume that

$$T_0(s,t) = T(s,t) , \quad (4.54)$$

where $T(s,t)$ is the two-body amplitude itself. If duality were correct, we could replace T by R , its (presumed) asymptotic form, in which case this coincides with the MUREX BS model. If duality is not accurate, however, then it might be safer to leave it as T .

This list illustrates some of the kinds of factorized theories which can exist; no doubt other versions can be constructed; no doubt, also, their predictions will be similar.

* * *

In any event let us next proceed to outline the techniques by which these theories are handled. Let us return to Eq. (4.50), and insert this form for $T_{2 \rightarrow n}$ into the s-channel unitarity relation, Eq. (3.26).

We obtain

$$\begin{aligned} A(s,t) = & \frac{1}{2} \sum_{n=2}^{\infty} \int d\Phi_n(P) \times \\ & \times T_0\left(\frac{\bar{s}_0}{\bar{s}_1}, t_1\right) \dots T_0\left(\frac{\bar{s}_{n-2}}{\bar{s}_{n-1}}, t_{n-1}\right) \times \\ & \times T_0\left(\frac{\bar{s}_0}{\bar{s}_1}, t_1'\right)^* \dots T_0\left(\frac{\bar{s}_{n-2}}{\bar{s}_{n-1}}, t_{n-1}'\right)^* . \end{aligned} \quad (4.55)$$

To derive this, it is of course necessary to assume that none of the produced particles in the intermediate state cross each other, so that the i^{th} particle from the end in one production amplitude is also the i^{th} from the end in the other. It can be argued (not entirely convincingly) that contributions where lines cross are less important at high s . First let us separate out the $n = 2$ term, by defining (note for $n = 2$, $s_1 = \bar{s}_1 = 1$)

$$A_0(s, t) = \frac{1}{2} \int d\Phi_2(P) T_0(s, t_1) T_0^*(s, t'_1) . \quad (4.56)$$

(Evidently A_0 is the same as A_{e1} calculated in this model). In the various versions listed above, this function is recognizable as the following:

- i) If $T_0 = 1/(t - 1)$, then A_0 is just the (imaginary part of the) box diagram.
- ii) If $T_0 = R$, a Regge pole, then A_0 is precisely the AFS amplitude we defined earlier.
- iii) If $T_0 = T$, then A_0 is precisely $A_{e1}(s, t) = \text{Im } T(s, t)_{e1}$.

* * *

The remaining terms in the sum in Eq. (4.55) (which constitute A_{inel}) can be written in the form

$$A_{\text{inel}} = \int \frac{d^3 \vec{q}_1}{(2\pi)^3} \frac{1}{2\omega_1} \left\{ T_0 \left(\frac{s}{\bar{s}_1}, t_1 \right) T_0^* \left(\frac{s}{\bar{s}_1}, t'_1 \right) \right\} \times A(\bar{s}_1, t) .$$

Noting that $\bar{s}_1 = s + 1 - 2\sqrt{s}\omega_1$, we may rewrite this as

$$A_{\text{inel}} = \frac{1}{\pi s} \int d\bar{s}_1 \sqrt{s^2 - 2s(\bar{s}_1 + 1) + (\bar{s}_1 - 1)^2} \times \quad (4.57)$$

$$\times A_0(s/\bar{s}_1, t) A(\bar{s}_1, t) ;$$

the range of integration is over those $\bar{s}_1$ for which the argument of the square root remains positive.

With Eq. (4.57), the s-channel unitarity relation now becomes an integral equation for A; approximately, we can write

$$A(s,t) = A_0(s,t) + \frac{1}{\pi} \int_1^s ds_1 A_0(s/s_1,t) A(s_1,t) . \quad (4.58)$$

This equation can be solved by projecting it into t-channel partial waves, through the Mellin transform

$$A(s,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj s^j A(t,j) . \quad (4.59)$$

We obtain the solution

$$A(t,j) = \frac{A_0(t,j)}{1 - \frac{1}{\pi} A_0(t,j)} . \quad (4.60)$$

From this form all the various conclusions we shall describe below may be readily obtained. The important feature is the denominator in Eq. (4.60); it is here the j-plane behaviour of the "output" amplitude $A(t,j)$ will be visible.

Let us begin with the MUREX BS version of the factored model. Here, as we have noted earlier, we replace T_0 in Eq. (4.50) by some Regge object R.

First suppose the "input" R is simply a fixed pole:

$$R(s,t) = \beta(t) s^\alpha . \quad (4.61)$$

Then

$$A_0(s,t) = g(t) s^{2\alpha-1} , \quad (4.62)$$

where

$$g(t) = \frac{1}{16\pi^2} \iint \frac{dt_1 dt_2}{\sqrt{-\lambda}} \beta(t_1) \beta(t_2) . \quad (4.63)$$

Hence

$$A_0(t, j) = \frac{g(t)}{j - (2\alpha - 1)} , \quad (4.64)$$

so that finally

$$A(t, j) = \frac{g(t)}{j - \alpha(t)} , \quad (4.65)$$

where

$$\alpha(t) = 2\alpha - 1 + \frac{1}{\pi} g(t) . \quad (4.66)$$

Thus the output is a moving pole. Note, incidentally, that if $\alpha = 1$, then

$$\alpha(t) = 1 + \frac{1}{\pi} g(t) > 1$$

in violation of the Froissart bound. Thus, as mentioned earlier, repeated flat Pomeron exchange is impossible unless the Pomeron lies below one. Note also that version (i) of the factored model, where $T = g/(t - 1)$, is a special case of this with $\alpha = 0$ and $\beta(t) = g/(t - 1)$.

The partial cross-sections σ_n , for producing n particles, can also be calculated, and we find

$$\sigma_n(s) = g(0) s^{2\alpha-2} \frac{(g(0)/\pi \log s)^{n-2}}{(n-2)!} \theta(s - n^2) .$$

Thus we obtain a logarithmic multiplicity: $\langle n(s) \rangle = \frac{1}{\pi} g(0) \log s + 2$. This conclusion is in fact characteristic of all versions of factored production amplitudes, and, since it has experimental support, tends to argue in favour of the factorized assumption. It is also the case that the scaling of inclusive cross-sections, mentioned earlier as seeming to hold experimentally, is (approximately) a consequence of all factored models.

Next, suppose the input R is a moving pole:

$$R(s, t) = \beta(t) s^{\alpha_{in}(t)} . \quad (4.67)$$

Then A_0 becomes precisely the AFS function containing the logarithmic cut generated by this pole at $\alpha_c(t) = 2\alpha_{in}(t/4) - 1$. Thus the output

$A(t,j)$ contains both a moving pole (differing from the input pole) as well as the AFS cut.

By similar arguments to the above the following table can be constructed summarizing the conclusions of the MUREX BS model for various types of input²⁷⁾.

Table 2

Input	Output
Flat pole	Moving pole with $\alpha(0) < 1$.
Moving pole	Moving pole with $\alpha(0) < 1$ plus a moving cut.
Flat cut	Flat cut; can be consistent if cut is of form $(j-1)^{-2/3}$.
Moving cut	Moving cut; above input cut except at $t = 0$. Consistent at $t = 0$ if $\alpha_c(0) = 1$ and if cut is of form $(j-1)^{-1/3}$ at $t = 0$.

Only the flat cut case is consistent; however, it is a strong cut and so violates t -channel unitarity (see Section 3). Furthermore, to obtain consistency even in this case it is necessary to presume that there is some "background" present in the function A_0 in the denominator of Eq. (4.60) in addition to the AFS cut, and that this "background" can be arranged to exactly cancel out the one in the denominator at all t ²⁷⁾.

One may be more complicated if one wishes. One may, for example, use as input more than a single Regge pole -- so that

$$R(s,t) = \sum_i \beta_i(t) s^{\alpha_c(t)}. \quad (4.68)$$

Then the denominator in Eq. (4.60) will contain sums of AFS functions.

Another modification is to permit the "internal" and "external" coupling of a given Regge pole to be different; in that case the A_0 in the denominator of Eq. (4.60) contains "internal" couplings while that in the numerator has "external" ones.

All of this can be, and has been done. For the present, however, let us not elaborate further on the conclusions from this class of factored models, but let us rather go on to describe other classes of models. The conclusions are much the same in all of these, so we will reserve our discussion of results until we have finished with all of them.

A second class of factored models is that in which the production amplitude repeats itself alternately. Thus

$$\begin{aligned} T_{2 \rightarrow n}(p_1 + p_2 \rightarrow q_1 + \dots + q_n) = \\ = T_0(s_{12}, t_1) X\left(\frac{s}{s_2 \bar{s}_2}, t_2\right) T_0(s_{34}, t_3) \dots \\ \dots T_0(s_{n-1, n}, t_{n-1}) . \end{aligned} \quad (4.69)$$

Evidently, in such models n is always even.

The prime example of this class is the so-called multiperipheral model¹⁰⁾, in which $T_0 = T$ and $X = 1/(t - 1)$.

The arithmetic by which Eq. (4.69) is handled is not appreciably different from what has gone before; we again separate the $n = 2$ term, and then obtain from the s -channel unitarity relation

$$\begin{aligned} A(s, t) = A_0(s, t) + \frac{1}{s} \int ds_1 \int ds_2 \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \times \\ \times A_0(s_1, t) K\left(\frac{s}{s_1 s_2}, t\right) A(s_2, t) , \end{aligned} \quad (4.70)$$

where the Kernel K is

$$K(s, t) = \left(\frac{1}{2\pi}\right)^4 \iint \frac{\theta(-\lambda) dt_1 dt_2}{\sqrt{-\lambda(t, t_1, t_2)}} X(s, t_1) X^*(s, t_2) . \quad (4.71)$$

Again, with the Mellin transform, Eq. (4.70) may be solved by projecting in t-channel partial waves, and we find

$$A(t,j) = \frac{A_0(t,j)}{1 - A_0(t,j) K(t,j)} . \quad (4.72)$$

The result is rather similar to that in the earlier class of models. It is therefore no surprise that the conclusions are the same also.

It is worth noting, incidentally, that the diagrams summed in Eq. (4.72) are very like the Bethe-Salpeter ladder shown in Fig. 26, though, because of our assumption of factorization, it is not precisely the same. The Bethe-Salpeter equation shown in Fig. 26 has in fact also been solved, numerically, on a computer²⁸⁾, with results, as is only to be expected, again much the same.

All versions of the factored models, then, end up with basically the same form:

$$A(t,j) \sim [1 - A_0(t,j) K(t,j)]^{-1} . \quad (4.73)$$

Included in the function A_0 is, presumably, the AFS cut generated by two Pomerons; the remainder of A_0 , as well as K , has no significant behaviour in the vicinity of $j = 1$ and $t = 0$. It is, therefore, reasonable to expect that this remainder can be expanded around $j = 1$ and $t = 0$; thus we can write

$$A(t,j) \sim [j - \alpha_0(t) - A_{\text{AFS}}(t,j) K(t,j)]^{-1} , \quad (4.74)$$

where $\alpha_0(t)$ is linear in t -- i.e., $\alpha_0 = a + bt$. It is obviously convenient to interpret α_0 as a sort of unperturbed Regge pole, generated by whatever else (other lower-lying non-Pomeron Regge exchanges for example) occurs in $A_0(t,j)$. In so far as the AFS Pomeron-Pomeron cut is negligible, then, diffraction could be described simply by this single Regge pole.

It is worth remarking that the general form

$$A(t,j) \sim [j - \alpha_0(t) - f(t,j)]^{-1} , \quad (4.75)$$

where $f(t,j)$ is some function containing a branch point in j , can be obtained in a wide variety of ways²⁹⁾, and it is therefore likely that this form transcends the specific assumptions inherent in the factorized model of production amplitudes.

Whatever its origin, we may expect that in Eq. (4.75), the "cut term" f will contain a branch point at $j = \alpha_c(t)$, the AFS branch point. In all multiperipheral models this cut is logarithmic, though this need, perhaps, not be the case in general. Whatever the nature of the cut, we can find output Regge poles from the equation

$$\alpha(t) = \alpha_0(t) + f(t, \alpha(t)) . \quad (4.76)$$

Since f has a cut, the solution $\alpha(t)$ to this equation will, in general, be complex, and there will be poles, in general, on all sheets of $A(t,j)$ ³⁰⁾.

In particular, if the cut in f is logarithmic (as suggested by all of our multiperipheral models) so that we may write

$$f(t,j) = \beta_0(t) \log [j - \alpha_c(t)] , \quad (4.77)$$

then we obtain one pole on the physical sheet which cannot cross the cut (unless β_0 vanishes at the crossing point), but approaches it asymptotically as $t \rightarrow -\infty$. To avoid trouble with Froissart this pole must have $\alpha(0) < 1$; however, the amount by which it is less than one cannot be calculated, but is related to the (unknown) residue function of the input pole. The fact that, experimentally, $\alpha(0)$ must be near one cannot be predicted by these models. We also obtain poles on all other sheets, with the poles on the n^{th} sheet and $-n^{\text{th}}$ sheet in complex conjugate positions.

A further feature obtained from the multiperipheral models is that the residue of the physical sheet pole quickly becomes very small as t decreases from zero, while this is not the case for the unphysical sheet poles. The situation in the j -plane is somewhat as sketched in Fig. 27³¹⁾.

The natural identification, then, is that the Pomeron is the physical sheet output pole; its intercept is less than one; it curves sharply to

move parallel to the cut; and its residue at negative t is small. The nearest unphysical sheet poles (a complex conjugate pair) may, perhaps, be identified with a subsidiary vacuum channel Regge pole, such as the P' . The residue of the P' is then large for negative t . This set of conclusions is dignified by the name "Schizophrenic Pomeron Model" ³²⁾.

To summarize this section on factored models, let us emphasize the following points:

- i) The MUREX bound results do not involve extremely drastic assumptions, and therefore have more of a chance of being right than the more detailed predictions of factored models.
- ii) The general form, Eq. (4.75), is probably right, although the detailed behaviour of f predicted by the factored models may well be wrong. Therefore it is probably true that many (perhaps all) Regge poles (and perhaps even the Pomeron) are complex pairs, on some sheet or other of the partial wave amplitude.
- iii) The "Schizophrenic Pomeron Model" is highly unsavoury, in that the experimental constancy (or near constancy) of total cross-sections is attributed to a dynamical accident (namely that the intercept of the Pomeron is not far below one) rather than to any principle.
- iv) While factored models are arranged to satisfy s -channel unitarity for the elastic amplitude, they do not for amplitudes for $n \rightarrow m$ particles when n or $m \neq 2$.

4.1.5 Two-fireball model

We saw that we were in trouble with the MUREX bound if the Pomeron were a pole at, or passing through $\alpha = 1$, which could be exchanged more than once. Suppose, therefore, that we simply do not permit it to be exchanged many times, but only once ³³⁾. Then we might draw a diagram like Fig. 28, and say that a single Pomeron separates two "fireballs" into which the final particles in a production process can be divided.

Hence let us write, simply,

$$\begin{aligned}
 T_{2 \rightarrow n}(p_1 + p_2 \rightarrow q_1 + \dots + q_n) \\
 &= \sum_{i=1}^{n-1} V_i(p_1, q_1 \dots q_i) X(s, t_i; s_i, \bar{s}_i) V_{n-i}(p_2, q_{n-i} \dots q_n),
 \end{aligned}
 \tag{4.78}$$

where s , t_i , s_i and $\bar{s}_i$ are as defined previously, where X stands for the "Pomeron exchange", and where V_i and V_{n-i} are vertices describing the production of the two fireballs, containing, respectively, i and $n-i$ particles with the indicated momenta. We must sum over all places where the "Pomeron" can be exchanged; hence the sum over i .

It is of interest, first, to compare this with the factored form for the production amplitudes described earlier.

For example, look at Eq. (4.69). This may be rephrased as

$$T_{2 \rightarrow n} = V_i \times V_{n-i}, \quad \text{for any } i, \quad i = 1 \dots n-1, \quad (4.79)$$

[where

$$V_i = T_0(s_{12}, t_1) \times \left(\frac{s}{s_2 \bar{s}_2}, t_2 \right) \dots T_0(s_{i-1,i}, t_{i-1}) \quad]$$

or, equally well, as

$$T_{2 \rightarrow n} = \frac{1}{n-1} \sum_{i=1}^{n-1} V_i \times V_{n-i}. \quad (4.80)$$

Comparing this with Eq. (4.78), we see that the only difference between the factored model and the two-fireball model is the $1/n-1$ outside of the sum; all the rest is the same. The $1/n-1$ is, however, important, and the predictions of the two models are not identical.

Since the form for the production amplitude here is so similar to that in factored models, the same arithmetic may essentially be repeated verbatim. We easily obtain, from Eq. (4.78), the result that

$$\begin{aligned} A(s, t) = & 2 \int \frac{d^4 Q_1}{(2\pi)^4} \int \frac{d^4 Q_2}{(2\pi)^4} (2\pi)^4 \delta^4(P - Q_1 - Q_2) \times \\ & \times V(s_1, t, t_1, t_2) X(s, t_1, s_1, s_2) \times \\ & \times X^*(s, t_2, s_1, s_2) V(s_2, t, t_1, t_2), \end{aligned} \quad (4.81)$$

where

$$s_{1,2} = (Q_{1,2})^2, \quad t_1 = (p_1 - Q_1)^2, \quad t_2 = (p'_1 - Q_1)^2,$$

and where

$$V(s_1, t, t_1, t_2) = \frac{1}{2} \sum_{i=1}^{\infty} \int d\Phi_i(Q_1) \times \quad (4.82)$$

$$\times V_i(p_1, q_1 \dots q_i) V_i^*(p'_1, q_1 \dots q_i) .$$

As in the factored models, it is necessary to assume that particles produced in, say, the left-hand fireball in the first production process do not cross over to become part of the right-hand fireball in the second if one thinks of the two fireballs as being produced forward and backward in the centre-of-mass system, this assumption is, perhaps, not too unreasonable.

Equation (4.81) may be rewritten in the form

$$A(s, t) = \frac{1}{s} \int ds_1 \int ds_2 \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \times$$

$$\times \left(\frac{1}{2\pi}\right)^4 \iint \frac{\theta(-\lambda)}{\sqrt{-\lambda}} dt_1 dt_2 \times \quad (4.83)$$

$$\times V(s_1, t, t_1, t_2) X(s, t_1, s_1, s_2) \times$$

$$\times X^*(s, t_2, s_1, s_2) V(s_2, t, t_1, t_2) ,$$

where³⁴⁾

$$-\lambda = (2|\vec{p}| |\vec{Q}_1|) (1 - x^2 - x_1^2 - x_2^2 + 2xx_1x_2)$$

and where, as usual

$$|\vec{p}| = \sqrt{s/4 - 1}$$

$$|\vec{Q}_1| = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2s(s_1 + s_2) + (s_1 - s_2)^2}$$

$$t = -2|\vec{p}|^2 (1 - x)$$

$$t_{1,2} = 1 - \frac{s - s_1 - s_2}{2} + 2|\vec{p}| |\vec{Q}| x_{1,2} .$$

Equation (4.83) is of course useless unless we assume something about the vertices V . Let us outline a possible sequence of simplifying assumptions.

First, let us assume $V(s_2, t, t_1, t_2)$ is independent of t_1 and t_2 . If $X(s, t_1, s_1, s_2)$ cuts off sharply for large t_1 , so that the range of t_1 in Eq. (4.83) is severely restricted, this may be a reasonable guess. Then we find

$$A(s, t) = \frac{1}{s} \int ds_1 \int ds_2 \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) V(s_1, t) K(s, t, s_1, s_2) V(s_2, t) , \quad (4.84)$$

where

$$K = \left(\frac{1}{2\pi} \right)^4 \iint \frac{\theta(-\lambda)}{\sqrt{-\lambda}} dt_1 dt_2 X(s, t_1, s_1, s_2) X^*(s, t_2, s_1, s_2) . \quad (4.85)$$

Next, let us assume X is a function only of the combination $s/s_1 s_2$. This dependence is analogous to what we had in the various factored models, and would be suggested if X were Regge-like. Then K is also a function of $s/s_1 s_2$.

If, in addition, we assume that X cuts off exponentially, say, in t , then it is easy to see from Eq. (4.85) that K becomes exponentially small for large s unless s_1 and s_2 are restricted by $s_1 s_2 < s$. Thus we can effectively replace the θ function in Eq. (4.84) by $\theta(s - s_1 s_2)$.

Let us note that for the particular case $n = 2$, Eq. (4.78) reduces to

$$\begin{aligned} T(s, t) &= T_{2 \rightarrow 2}(p_1 + p_2 \rightarrow q_1 + q_2) \\ &= V_1(p_1, q_1) X(s/1 \cdot 1, t) V_1(p_2, q_2) ; \end{aligned}$$

furthermore, we may absorb the t -dependence in V_1 into X and chose $V_1 = 1$. Thus $X(s, t) = T(s, t)$. From this, and Eq. (4.85), we then find

$$K = \frac{1}{\pi^2} \left(\frac{s}{s_1 s_2} \right) A_{el} \left(\frac{s}{s_1 s_2} , t \right) .$$

Finally, let us note that the diagram (Fig. 28) suggests that V_1 is just the amplitude for the exchanged "Pomeron" (whatever that is) and the particle p_1 to produce the particles q_1 to q_i . Thus we might guess

$$V_i(p_1, q_1 \dots q_i) = T_{2 \rightarrow i}^{ND}(p_1, Q_1 - p_1 \rightarrow q_1 + \dots + q_i) , \quad (4.86)$$

where, since there is supposed to be only a single Pomeron, T^{ND} stands for the non-diffractive part of the Pomeron + p_1 production amplitude.

Then we find, from Eq. (4.82) that

$$V(s_1, t) = \pi \delta(s_1 - 1) + A^{\text{ND}}(s_1, t) ,$$

where A^{ND} is the non-diffractive absorptive part of the Pomeron + p_1 elastic scattering amplitude.

All of these assumptions change Eq. (4.83) into the equation

$$\begin{aligned} A(s, t) = & \frac{1}{s} \int ds_1 \int ds_2 \theta(s - s_1 s_2) \times \\ & \times [\pi \delta(s_1 - 1) + A^{\text{ND}}(s_1, t)] \frac{1}{\pi^2} \left(\frac{s}{s_1 s_2} \right) A_{\text{el}} \left(\frac{s}{s_1 s_2}, t \right) \times \\ & \times [\pi \delta(s_2 - 1) + A^{\text{ND}}(s_2, t)] . \end{aligned} \quad (4.87)$$

An alternative to this is to permit multiple Pomeron exchange; in this case the production amplitude in Eq. (4.86) should include the diffractive part, and hence we should replace A^{ND} in Eq. (4.87) by A itself.

If Eq. (4.87) holds as it stands, with A^{ND} in it, then [presuming $A^{\text{ND}}(s_1, t)$ to vanish with a power of s_1] consistency occurs if both A and A_{el} are linear in s . Thus we obtain the form $A(s, t) = s f(t)$, where

$$\begin{aligned} f(t) = & \frac{g(t)}{\pi^2} \int_0^\infty \frac{ds_1}{s_1^2} \int_0^\infty \frac{ds_2}{s_2^2} \times \\ & \times [\pi \delta(s_1 - 1) + A^{\text{ND}}(s_1, t)] [\pi \delta(s_2 - 1) + A^{\text{ND}}(s_2, t)] \end{aligned}$$

and where

$$g(t) = \frac{1}{16\pi^2} \iint \frac{\theta(-\lambda) dt_1 dt_2}{\sqrt{-\lambda}} f(t_1) f(t_2) .$$

If, on the other hand, Eq. (4.87) holds with A^{ND} replaced by A itself, then it becomes a (rather complicated) integral equation for A . [It must be noted that it is not entirely obvious that this equation makes

sense, in that with repeated Pomeron exchange it is hard to understand the sum over i in Eq. (4.78); it would then seem more reasonable to have the $1/n-1$ factor as in Eq. (4.80).]

Nevertheless, we may ask what happens. Let us specialize Eq. (4.87) to $t = 0$, and assume for the moment a factored form for the amplitude A : $A(s,t) = s \sigma(s) f(t)$. Then Eq. (4.87) becomes a relation among total cross-sections, of the form

$$\sigma(s) \sim \int \frac{ds_1}{s_1} \int \frac{ds_2}{s_2} \theta(s - s_1 s_2) \sigma(s_1) \sigma(s_2) \left[\sigma\left(\frac{s}{s_1 s_2}\right) \right]^2. \quad (4.88)$$

The behaviour of $\sigma(s)$ consistent with this equation is readily shown to be $\sigma(s) \sim (\log s)^{-2/3}$, corresponding to a $(j - 1)^{-1/3}$ fixed cut.

If, instead, we don't assume A factors, but that it has a logarithmically shrinking peak, then $A_0(s,0) \sim s[\sigma(s)]^2/\log s$. Then our constraint is

$$\sigma(s) \sim \int \frac{ds_1}{s_1} \int \frac{ds_2}{s_2} \theta(s - s_1 s_2) \sigma(s_1) \sigma(s_2) \left[\sigma\left(\frac{s}{s_1 s_2}\right) \right]^2 \left(\log \frac{s}{s_1 s_2} \right)^{-1},$$

from which we find $\sigma(s) \sim (\log s)^{-1/3}$. That is, we have a $(j - 1)^{-2/3}$ type cut.

It is remarkable that these results are the same as those obtained (see Table 2) from the MUREX BS model, in view of the extra $1/n$ factor in Eq. (4.80).

In any event, let us not pursue these possibilities further. Suffice it to say that the two-fireball model, which evidently is not very restrictive, provides a vehicle for playing many games involving diffraction, and may one day, once we know more about the various functions in Eq. (4.81), even allow us to deduce something useful.

4.2 Non-shrinking, or not really Regge-like, models

The next class of models to be discussed is that in which the asymptotic elastic amplitude factors:

$$T(s,t) \rightarrow F(s) f(t), \quad (4.89)$$

as well as other models of a non-Regge or at least non-conventional-Regge nature.

Most of these models have non-shrinking diffraction peaks, and the crossed-channel partial wave amplitude $T(t,j)$ has fixed singularities in j rather than moving ones. (The constraints associated with this property have already been remarked upon in Section 3.)

4.2.1 Models with constant cross-sections

The most popular special case of this class of theories assumes that

$$F(s) = i s ; \quad (4.90)$$

this is a plausible form because it is what the non-relativistic quantum mechanical scattering from an absorbing blob of fixed radius looks like. In this type of model, cross-sections are asymptotically constant [$\sigma_T = f(0)$], elastic amplitudes are purely imaginary, the Pomeranchuk theorem holds, and of course diffraction peaks do not shrink. All of these properties are more or less compatible with (if not required by) existing experiments.

In the j -plane, $T^{(+)}(t,j)$ has a fixed pole at $j = 1$: we have

$$T^{(+)}(t,j) \propto \frac{f(t)}{j-1} , \quad (4.91)$$

near $j = 1$, where $f(t)$ is the same function defined in Eq. (4.89).

[This is, of course, incompatible with t -channel unitarity: see Section 3. People who assume the form (4.90) unanimously ignore this fact.]

If Eq. (4.89) is indeed the correct form for the asymptotic amplitude, all that remains for the theoretician to do is to provide a means of calculating $f(t)$, the shape of the diffraction peak.

Before going on to do this, however, it is worth noting that the $i s f(t)$ form is the limit of a simple Regge pole type theory in which the Pomeron becomes flat and equal to one. We simply have to let $\alpha_P(t) \rightarrow 1$ in Eq. (4.1), and identify $\beta(t)$ with $f(t)$ to obtain $T(s,t) \rightarrow i s f(t)$. [Note that this suggests that the function $f(t)$ should factor; see the remarks on factorization at the outset of Section 4.1.]

4.2.1.1 The droplet model

One natural way to obtain $f(t)$ is to intuit it from analogies with the non-relativistic theory which suggested the form $i s f(t)$ in the first place.

Let us use the impact parameter picture. To obtain the $i s f(t)$ form, we assume $\delta(s,b)$ vanishes and that $\eta(s,b)$ becomes independent of s . Thus we have

$$f(t) = 4\pi \int_0^\infty b db J_0(b\sqrt{-t}) [1 - \eta(b)] . \quad (4.92)$$

The ensuing arithmetic will be somewhat simplified if we notice that this equation can be rewritten as a two-dimensional Fourier transform:

$$f(t) = 2 \int d^2\vec{b} [1 - \eta(b)] e^{iq \cdot \vec{b}} , \quad (4.93)$$

where $\vec{q}^2 = -t$, and $\vec{b}$ is a two-dimensional vector.

Now, how do we calculate the absorption? Let us use a very simple physical picture, namely that the two scattering particles simply consist of blobs of absorbing material, of some given spatial density distribution $\rho(\vec{r})$ ³⁵.

The absorption should then have the form:

$$\eta(b) = \exp \left[-K \int d^2\vec{b}' D(\vec{b}') D(\vec{b} - \vec{b}') \right] , \quad (4.94)$$

where K is some (assumed constant) absorption coefficient and where

$$D(\vec{b}) = \int_{-\infty}^{\infty} dz \rho(x,y,z) \quad (4.95)$$

is the two-dimensional density distribution of the particles at an impact parameter $b = \sqrt{x^2 + y^2}$.

For small absorption (η near 1) we can expand the exponential to get

$$\eta(b) = 1 - K \int d^2\vec{b}' D(\vec{b}') D(\vec{b} - \vec{b}') \quad (4.96)$$

which means that

$$\begin{aligned} f(t) &= 2K \int d^2\vec{b} e^{i\vec{q}\cdot\vec{b}} \int d^2\vec{b}' D(\vec{b}') D(\vec{b} - \vec{b}') \\ &= 2K |D(t)|^2 \end{aligned} \quad (4.97)$$

where

$$D(t) = \int d^2\vec{b} e^{i\vec{q}\cdot\vec{b}} D(\vec{b}) . \quad (4.98)$$

For the density distribution [or rather its Fourier transform $D(\vec{q})$] let us choose the experimentally measured electromagnetic form factor. Then we get, finally,

$$f(t) \propto |F(t)|^2 \quad (4.99)$$

or

$$\frac{d\sigma/dt}{(d\sigma/dt)_{t=0}} = |F(t)|^4 . \quad (4.100)$$

This function is plotted in Fig. 29 for proton-proton scattering [so that $F(t)$ is the measured proton form factor] together with the experimental values. The agreement is not bad.

If we do not wish to assume η to be near 1, then we can write an expression for $f(t)$ without expanding. We have

$$f(t) = 2 \int d^2\vec{b} e^{i\vec{q}\cdot\vec{b}} \left(1 - e^{-K \int d^2\vec{b}' D(\vec{b}') D(\vec{b} - \vec{b}')} \right) \quad (4.101)$$

with D defined by

$$D(\vec{b}) = \int \frac{d^2\vec{q}}{(2\pi)^2} e^{-i\vec{q}\cdot\vec{b}} F(t) . \quad (4.102)$$

This cannot be evaluated analytically, but it can be calculated on a computer³⁶⁾. The results are shown in Fig. 30. Now (as is more or less inevitable in any eikonal type model, of which this is one) the differential cross-section has oscillations in it.

Indeed, if K is real, it has zeros. Therefore, the fit shown in Fig. 30 includes the case of a complex K -- it is therefore a two-parameter fit. Even so, it is not as impressive as the approximate version given in Eq. (4.100)³⁷⁾.

While the intuitive approach to the calculation of $f(t)$ may be attractive, in that it makes use of a plausible physical picture of how diffraction occurs, we would have an even more attractive theory if we could deduce $f(t)$ systematically from some more basic knowledge of high-energy production processes.

If Eq. (4.90) is correct, the s-channel unitarity relation takes the form

$$f(t) = \left(\frac{A_{\text{inel}}(s, t)}{s} \right) + \frac{1}{16\pi^2} \iint \frac{dt_1 dt_2}{\sqrt{-\lambda}} f(t_1) f(t_2) . \quad (4.103)$$

It is therefore implicit that the inelastic contribution must behave like $s f_0(t)$, with some $f_0(t)$ to be determined from a knowledge of how production amplitudes behave.

4.2.1.2 Van Hove model

One thing we can do is just to guess $f_0(t)$ -- this is, of course, not really any closer to a fundamental theory of diffraction than the intuitive models based on classical pictures of what happens (which we described before). Furthermore, we know experimentally (see Section 3) that experiment requires $f_0(t)$ to change sign, and so far no "natural" guess of $f_0(t)$ is anywhere near as sophisticated as this. Nevertheless, we may illustrate the mathematics of how to calculate $f(t)$ from $f_0(t)$ by describing a theory based on a very simple guess for $f_0(t)$, the so-called Van Hove model³⁸).

To be specific, let us guess that

$$f(t) = a e^{bt} . \quad (4.104)$$

The unitarity relation (4.103) now becomes an integral equation for $f(t)$, which can, in fact, be solved exactly. We have met precisely this integral equation before, in our discussion of the AFS model. The only difference was that there the slope b was s dependent; here it is not. This difference, however, does not affect the mathematics. We may, therefore, write down the solution directly: we have

$$f(t) = 8\pi b \int_0^1 \frac{dy}{y} \left(1 - \sqrt{1 - \frac{a}{4\pi b} y} \right) J_0 \left(\sqrt{bt \log y} \right) . \quad (4.105)$$

For large t , this behaves like $1/t e^{-\sqrt{-t}}$ which is much too slow a fall off to agree with experiment. We can also read off the cross-section $f(0)$. We find

$$\sigma_T = 16\pi b \left(1 - \sqrt{1 - \frac{a}{4\pi b}} + \frac{1}{2} \log \frac{a}{16\pi b} \frac{1 + \sqrt{1 - a/4\pi b}}{1 - \sqrt{1 - a/4\pi b}} \right). \quad (4.106)$$

and, since $a = \sigma_{inel}$, we get

$$\frac{\sigma_{el}}{\sigma_T} = 1 - \frac{a/16\pi b}{1 - \sqrt{1 - \frac{a}{4\pi b}} + \frac{1}{2} \log \frac{a}{16\pi b} \frac{1 + \sqrt{1 - a/4\pi b}}{1 - \sqrt{1 - a/4\pi b}}}. \quad (4.107)$$

There is, then, a limit on the permitted ratio of a to b ; we must have $a < 4\pi b$. The maximum value of the right-hand side of Eq. (4.107) occurs at $a = 4\pi b$; at this value $\sigma_{el}/\sigma_T = 0.2$. Thus in the model we must have $\sigma_{el} < 0.2 \sigma_T$. (Experimentally, this limit is not obeyed, at least at present energies, by the pp scattering data, though it is by the πp data. See Fig. 20).

4.2.1.3 Modified multiperipheral-type models

If we do not merely want to guess $f_0(t)$ (and after all if we are going to guess f_0 we might as well save ourselves trouble and guess f itself) then we have to know something about production amplitudes. We have seen before that multiperipheral versions of production amplitudes cannot yield the $isf(t)$ form (see Section 4.1); however, modified versions of multiperipheral models can.

Let us describe a straightforward example of such a modified factored theory.

If we recall the multi-Regge theory described in Section 4.1, the type of modification necessary to make a consistent non-shrinking model is clear. There we found, with a flat input trajectory equal to α , the result

$$A(s, t) = s^{2\alpha-1} \sum_{n=2}^{\infty} [g(t)]^{n-1} \frac{(1/\pi \log s)^{n-2}}{(n-2)!} \theta(s - n^2). \quad (4.108)$$

Therefore, if the assumed form of the production amplitude is modified to³⁹⁾

$$T_{2 \rightarrow n} = \frac{is}{(\log s)^{n-2/2}} (\pi C)^{n-2} f(t_1) \dots f(t_{n-1}) , \quad (4.109)$$

then inserting this into the unitarity relation will yield

$$\begin{aligned} A(s,t) &= is \sum_{n=2}^{\infty} [g(t)]^{n-1} \frac{(C)^{n-2}}{(n-2)!} \theta(s-n)^2 \\ &= is g(t) e^{Cg(t)} \end{aligned} \quad (4.110)$$

where, as in the MUREX case,

$$g(t) = \frac{1}{16\pi^2} \iint \frac{dt_1 dt_2}{\sqrt{-\lambda}} f(t_1) f(t_2) . \quad (4.111)$$

Our assumed form of $T_{2 \rightarrow n}$ reduces, for the case $n = 2$, to

$$T(s,t) = is f(t) ; \quad (4.112)$$

thus we have consistency with s-channel unitarity provided only that $f(t)$ satisfies the integral equation

$$f(t) = g(t) e^{Cg(t)} . \quad (4.113)$$

What we have done is to modify the multiperipheral type model by inserting a $(\log s)^{-1/2}$ into each internal Pomeron vertex. The vertex then vanishes as $s \rightarrow \infty$, so that we are now permitted to let $\alpha_p \rightarrow 1$, without contradicting the Froissart bound.

We have, however, paid several prices. First, the model is no longer multiperipheral, although that this is a crime is not entirely obvious. Second, the average multiplicity is now constant asymptotically; specifically

$$\langle n(s) \rangle \rightarrow \log \sigma_T / \sigma_{e1} + 2 . \quad (4.114)$$

In addition, we find $\sigma_n \rightarrow \text{const.}$ as $s \rightarrow \infty$, σ_n is Poisson distributed, and the internal coupling strength C is given by

$$C = \frac{1}{\sigma_{el}} \log \frac{\sigma_T}{\sigma_{el}} . \quad (4.115)$$

It is not likely that a model so contrived makes sense; nevertheless it is of interest to show a solution (obtained numerically) to the integral equation (4.113) for the shape of the diffraction peak. This is exhibited in Fig. 31⁴⁰⁾.

The agreement is rather good, but in view of the *ad hoc* nature of the model, and of the difficulty with the multiplicity, this is likely to be a coincidence.

It is no doubt possible to construct other doctored versions of multiperipheral models which can be consistent with the $\text{isf}(t)$ form; however we suspect that in all of these the multiplicity will come out to be constant, and this seems to be a clear violation of experiment⁴¹⁾.

4.2.1.4 Complex Pomerons

However appealing they may be intuitively, all of these $\text{isf}(t)$ theories are incompatible with t -channel unitarity, and therefore are necessarily wrong; they can at best be only approximate.

We saw in Section III what is required of a non-shrinking theory -- i.e. a theory with fixed singularities in the j plane -- to make it compatible with t -channel unitarity: a fixed pole must be complex, and a fixed cut must be either complex or soft.

What happens if we make our Pomeron pole complex? Let us write $\alpha_P(t) = \alpha_R(t) + i \alpha_I(t)$; at this stage we will be general and not impose $\alpha_R = 1$ or anything of the sort.

Near $j = \alpha$, we have $T^{(+)}(t, j) \sim \beta / (j - \alpha)$; hence the unitarity relation requires, if α_I is small, that for t above threshold

$$\frac{\beta}{j - \alpha} - \frac{\beta^*}{j - \alpha^*} = 2i \rho \frac{\beta}{j - \alpha} \cdot \frac{\beta^*}{j - \alpha^*} \quad (4.116)$$

or

$$\alpha_I = \rho \beta . \quad (4.117)$$

Thus at least $\alpha_I(t)$ must vary with t ⁴²).

But then $T^{(+)}(t, j)$, in order to maintain its desired reality properties, must also have a pole at $j = \alpha_P^*(t)$; thus we are necessarily dealing with a complex conjugate pair of Pomeranchuk poles, and the imaginary part of the Pomeron must vary with t .

The asymptotic amplitude, due to this complex pair, will now have the form

$$T(s, t) \rightarrow |\beta(t)| s^{\alpha_R(t)} e^{-i\pi/2 \alpha_R(t)} \times F(s, t) \quad (4.118)$$

where the factor F (which is unity for a single pole) is

$$\begin{aligned} F(s, t) = & \cos [\phi(t) + \alpha_I(t) \log s] \cosh \frac{\pi}{2} \alpha_I(t) \\ & + i \sin [\phi(t) + \alpha_I(t) \log s] \sinh \frac{\pi}{2} \alpha_I(t) , \end{aligned} \quad (4.119)$$

where ϕ is the phase of the residue function β . If we take $\alpha_R(t) = 1$, for $t < 0$, to conform as closely as possible to our fixed pole idea, and to give no shrinkage, then

$$T(s, t) \rightarrow isf(t) F(s, t) \quad (4.120)$$

[with, as before, $f(t) = \beta(t)$]. Thus

$$\sigma_T(s) = f(0) \operatorname{Re} F(s, 0) \quad (4.121)$$

and

$$d\sigma/dt = \frac{1}{16\pi} |f(t)|^2 |F(s, t)|^2 . \quad (4.122)$$

Now these correction factors are, first

$$\operatorname{Re} F(s, 0) = \cos [\phi(0) + \alpha_I(0) \log s] \cosh \frac{\pi}{2} \alpha_I(0) . \quad (4.123)$$

Therefore, in order to avoid a negative total cross-section as s increases, we must have $\alpha_I(0) = 0$. Then $\operatorname{Re} F(s, 0) = 1$, so the total cross-section is just $f(0)$, as in the single fixed pole case. Second

$$|F(s, t)|^2 = 1 + \sin^2 \frac{\pi}{2} \alpha_I(t) - \sin^2 [\phi(t) + \alpha_I(t) \log s] . \quad (4.124)$$

The differential cross-section now develops, superimposed on the shrink-less form, some slow oscillations varying with $\log s$. How important these are depends, naturally, on the size of the ϕ and α_T . At present, experiment neither excludes nor supports such behaviour. The complex pole possibility is then something we should keep in the back of our minds, until such time as either experiment or theory makes it more compelling.

In the meantime, let us proceed to other "acceptable" non-shrinking models.

4.2.2 Flat cuts: falling cross-sections

So much for complex poles. We saw in Section III that another kind of fixed singularity in the j -plane which could satisfy unitarity was a "soft" cut. A second class of acceptable non-shrinking models is therefore of this nature. We may write

$$T^{(+)}(t, j) = g(j) f(t) + T_{\text{smooth}}^{(+)}(s, t) \quad (4.125)$$

where T_{smooth} is not singular at $j = 1$, and where $g(j)$ is a function with a branch point as $j = 1$, such that $g(1) = 0$.

Corresponding to Eq. (4.125) we have, in the s -channel, a total cross-section vanishing faster than $1/\log s$, which does not seem to be compatible with existing experiments. One is, in principle, permitted to take the view, however, that present energies are not yet sufficiently high to see the fall-off in $\sigma_T(s)$; if we believe this, we may proceed.

The assumed form (4.125) must satisfy elastic unitarity, in particular at and near $j = 1$, when t is between elastic and inelastic threshold. Therefore, in this region, we obtain

$$f - f^* = 2i\rho (f T_{\text{smooth}}^{(+)*} + f^* T_{\text{smooth}}^{(+)}) \quad (4.126)$$

or

$$f/f^* = \frac{1 + 2i\rho T_{\text{smooth}}^{(+)}}{1 - 2i\rho T_{\text{smooth}}^{(+)*}} \bigg|_{j=1} \quad (4.127)$$

Thus f has the same phase as $1 + 2ip T^{(+)}$, i.e.

$$f(t) = |f(t)| e^{2i\delta_1^{(+)}(t)} \quad (4.128)$$

where $\delta_1^{(+)}(t)$ is the phase of the even signatured vacuum channel partial wave amplitude at $j = 1$ ⁴³).

For comparison, we may note that the isovector electromagnetic form factor $F_V(t)$, in the elastic region, satisfies the relation

$$\text{Im } F_V(t) = \rho(t) F_V^*(t) T_{I=1}^{(-)}(t, j) \Big|_{j=1} \quad (4.129)$$

where $T_{I=1}^{(-)}(t, j)$ is the odd signatured partial wave amplitude with isospin 1 and odd C . Assuming exchange degeneracy, though, we may identify this with the even signature, isospin zero, even C amplitude $T^{(+)}(t, j)$ appearing in Eq. (4.126). Thus the phase of $f(t)$ in the elastic region is twice the phase of $F_V(t)$; hence insofar as the elastic region dominates the dispersion relations in t for f and F_V we conclude $f(t) \propto F_V(t)^2$, as in the droplet model.

The precise kind of soft cut cannot be determined by t -channel unitarity, and then we are, as usual, immediately faced with the question of what production amplitudes look like.

One possible model is our old friend the multi-Regge (MUREX) bootstrap model, in which now the dominant exchange is assumed to be a flat Regge cut. Requiring consistency between the input and output cut (in contrast to the more conventional moving singularity MUREX bootstrap, where consistency can only be obtained at $t = 0$, we can now achieve consistency at all t) results in the cut coming out to be of the form $(j - 1)^{-1/3}$, which translates into a cross-section behaving as $(\log s)^{-2/3}$ ²⁷). Unfortunately, this is a hard cut, and hence cannot conform to the demands of t -channel unitarity.

Another possible model is the "double fireball" model, also described previously, but with the diffractive exchange taken as one flat cut rather than a moving pole. Again we must decide what to do with the vertices producing the two blobs; under various assumptions ⁴⁴), of uncertain quality, we can achieve consistency with a cut like $(j - 1)^{1/3}$. This cut

is soft, as required by t-channel unitarity. No fundamental theoretical objections therefore exist; but we do have to live with a total cross-section falling like $(\log s)^{-4/3}$, which is quite uncomfortable experimentally. Furthermore, our choice for the blob vertices in the production amplitude allows us to predict the average multiplicity in production processes, and the result is $\langle n \rangle \sim s^{1/4}$ ⁴⁴⁾. This conclusion as well is not supported by experiment ⁴¹⁾.

4.2.3 Field theoretic models: rising cross-sections

The only theory in which we have any real confidence these days is quantum electrodynamics. This theory is unitary, crossing symmetric, and analytic, and a natural question is therefore to ask how this theory resolves the question of diffraction.

The high-energy behaviour of quantum electrodynamics has been studied extensively in recent years, with precisely the aim of answering this question ⁴⁵⁾. It is, of course, not obvious that quantum electrodynamics behaves in the same way as does hadron physics, so we do not know how many (if any) of the quantum electrodynamic conclusions can be carried over to the problem of primary interest to us. Nevertheless the quantum electrodynamics results are quite interesting, and we shall describe them below.

The first step in the analysis of QED consists of summing the most divergent terms as $s \rightarrow \infty$ in each order of perturbation theory. As in other types of field theories, in massive electrodynamics this amounts to summing all ladder graphs; however here it is necessary to include all twisted ladders as well, as indicated in Fig. 32, in order to preserve gauge invariance. The sum of all ladders plus twisted ladders is called a tower; the amplitude for a tower turns out to be of the form

$$T_T(s,t) = is \cdot \frac{s^a}{(\log s)^2} F(t) . \quad (4.130)$$

This form is rather similar to what is obtained from the same approximation in other simple field theories, or in other channels in electrodynamics. (For example, in that the power behaviour is s^{1+a} with a positive value of a). There are, however, some differences which are worth noting.

The first is that in QED, a is a constant, independent of t . In fact, in spinor electrodynamics, $a = (11/32) \pi \alpha^2$ (α is of course the fine structure constant) and in scalar electrodynamics, $a = (5/32) \pi \alpha^2$. Thus the value of a is model dependent; but in general it is positive and independent of t .

Second, the tower amplitude is purely imaginary.

Third, the s -dependence is not just a power (which would correspond to a Regge pole) but contains the factor $(1/\log s)^2$. Thus it represents a fixed soft Regge cut with a $(j - 1) \log(j - 1)$ type of branch point.

The most important feature of the tower is that it violates the Froissart bound. It is therefore impossible for it, by itself, to give the correct asymptotic behaviour of the entire amplitude. The tower was what resulted from summing the most divergent graphs in each order; evidently the true asymptotic form must also include graphs which, in a given order, are not the most divergent.

The conjecture^{4,5)} is that the additional graphs which are necessary consist of all repeated exchanges of towers, as shown in Fig. 33. (Note that the end photons in each tower may be mixed up in any manner.)

The sum of all these graphs, it turns out, can be written in the form

$$\begin{aligned} T(s,t) &= 8\pi s \int_0^\infty b db J_0(b\sqrt{-t}) \left(\frac{\eta(s,b) - 1}{2i} \right) \\ &= 8\pi s \int_0^\infty b db J_0(b\sqrt{-t}) \left(\frac{e^{\log \eta(s,b)} - 1}{2i} \right) \end{aligned} \quad (4.132)$$

where

$$T_{\text{Tower}}(s,t) = -4\pi i s \int_0^\infty b db J_0(b\sqrt{-t}) \log \eta(s,b) . \quad (4.133)$$

That is, the scattering is purely absorptive, and the tower gives the absorption through (4.133). The graph with n exchanged towers is the n th term in the expansion of $e^{\log \eta(s,b)}$ in (4.132).

As $s \rightarrow \infty$, $\log \eta \rightarrow -\infty$ for a fixed value of b . On the other hand, as $b \rightarrow \infty$, $\log \eta \rightarrow 0$ for a fixed s . In fact, from (4.133) and the known form

for $F(t)$ in Eq. (3.130), one finds that $\log \eta(s,b)$ falls off exponentially in b ; we have

$$\log \eta(s,b) \propto - \frac{s^a}{(\log s)^2} e^{-b/b_0} . \quad (4.134)$$

Furthermore, b_0 is a fixed (s independent) number, since the exponent a in the tower is independent of t .

Therefore in Eq. (4.132) we find that out to a radius R , $\eta(s,b) - 1 \approx -1$ (since $\log \eta \rightarrow -\infty$ so $\eta \rightarrow 0$) and we have complete absorption. Beyond the radius R , however, $\eta(s,b) - 1 \approx 0$, because here $\log \eta = 0$, or $\eta = 1$. Thus outside of R there is no scattering at all.

The value of R can conveniently be defined as the value of b where $\log \eta \sim 1$; from (4.134) we see that

$$R(s) \sim b_0 \log \frac{s^a}{(\log s)^2} .$$

For sufficiently large s , this becomes more simply $R \sim \log s$.

The behaviour of the cross-section is now immediately evident. We have

$$\begin{aligned} \sigma_T(s) &= 4\pi \int_0^R b \, db = 2\pi R^2 \\ &\sim \left[\log \frac{s^a}{(\log s)^2} \right]^2 \end{aligned} \quad (4.135)$$

so that eventually

$$\sigma_T(s) \sim (\log s)^2 . \quad (4.136)$$

This result reflects the physics: we have a fully absorbing scatterer with a radius R growing logarithmically with the energy.

Away from $t = 0$, the behaviour of the amplitude can also be read off from Eq. (4.132). We have

$$T(s,t) = 4\pi i s R(s) \frac{J_1(\sqrt{-t} R(s))}{\sqrt{-t}} \quad (4.137)$$

so that for very large s ,

$$T(s,t) \sim \left(s^{(1+i\sqrt{-t} b_0)} + s^{(1-i\sqrt{-t} b_0)} \right) \sqrt{\log s}. \quad (4.138)$$

In the j -plane, therefore, the structure is as follows: there are two cuts, with branch points at $j = 1 \pm i\sqrt{-t} b_0$. For negative t these branch points are complex, at complex conjugate positions, and with constant real part equal to one. For positive t both branch points are real; one is above $j = 1$ at $j = 1 + b_0\sqrt{t}$ and the other is below at $j = 1 - b_0\sqrt{t}$. Both cuts, for both signs of t , are hard: they are of the form $(j - \alpha_c)^{-3/2}$.

A few other predictions are, first, that there are zeros in $d\sigma/dt$ at $t_i = -(\beta_i/R)^2$, where β_i are the zeros of J_1 . [This is obvious from Eq. (4.137)]. Since $R \sim \log s$, the position of the zeros will move toward $t = 0$ as s increases. Numerically, we have $-t_1 \sigma_T(s) = 2\pi\beta_1^2 = 35.9$ [mb(GeV)²]. For pp scattering, then, where at present energies $\sigma_T \sim 40$ mb, the first zero is near $t \sim -1$ (GeV)². It is tempting to identify this with the apparent structure in the present pp data near $t = -1$ (GeV)². (see Fig. 29).

The second prediction is also trivial: since the scattering is purely absorptive, we expect $\sigma_{el}/\sigma_T = \frac{1}{2}$.

Finally, we note that the asymptotic amplitude is purely imaginary; thus we predict $\text{Re}T/\text{Im}T \rightarrow 0$ both at $t = 0$ and at $t \neq 0$.

These various predictions are certainly not clearly apparent in the existing hadron data, especially the predictions that $\sigma_T \sim (\log s)^2$ and that $\sigma_{el}/\sigma_T \rightarrow \frac{1}{2}$. To understand this apparent discrepancy it is necessary to recall Eq. (4.135); it is only as $s \rightarrow \infty$ that $\sigma_T \sim (\log s)^2$. At lower energies (4.135) holds, and because of the $(1/\log s)^2$ factor, at these lower energies the $(\log s)^2$ growth should not be so apparent. Beyond what energy it should be noticeable depends on the value of a , and as we have seen this is model dependent. Nevertheless, reasonable guesses for a certainly suggest that the $(\log s)^2$ behaviour should begin to become visible before $s \sim 1000$ (GeV)².

Similar results can be obtained in more trivial field theories, such as the ϕ^3 theory⁴⁶⁾. In this theory it is well known that the tower is

just a ladder and generates simply a Regge pole with trajectory $\alpha(t) = -1 + g^2 f(t)$, where $f(t)$ is a simple, and known, function of t . The -1 reflects the fact that the basic particle in the theory is spinless, in contrast to the case of QED. (In general, one expects the -1 to be replaced by $s_1 + s_2 - 1$, where s_1 and s_2 are the spins of the two exchanged particles in the ladder). Thus in this instance we are permitted three cases, according as g^2 is such that $\alpha(0) < 1$, $\alpha(0) = 1$, and $\alpha(0) > 1$. One finds then, as in QED, that repeated ladder exchange leads to an eikonal formula, with the following results: If $\alpha(0) < 1$, then $\sigma(s) \rightarrow 0$. If $\alpha(0) = 1$, then $\sigma(s) \rightarrow \text{const.}$ If $\alpha(0) > 1$, then the results are as in QED: $\sigma(s) \sim (\log s)^2$.

It is, of course, not clear that the $\alpha(0) = 1$ and $\alpha(0) > 1$ cases make sense here, since they require such a large value of g that the approximate form of α , which involves calculating α only to order g^2 , is unlikely to be reasonable.

How much of all this (which we emphasize is learned from QED) can we reasonably expect to be true in hadron physics? It is difficult to say. The basic physical ingredient in the picture was that there existed full absorption in a region with a radius growing like $\log s$. The fact that the radius varied like $\log s$ was a result of the exponential drop in $\log \eta(s, b)$ [see Eq. (4.134)], and this in turn depended on the kind of t dependence in the $F(t)$ in Eq. (4.130). The fact that there was full absorption followed from the violation of the Froissart bound by the tower; that is, from the fact that $\log \eta$ blew up with s .

When we go to hadron physics, it is perhaps not unlikely that $\log \eta$ grows very rapidly, (in fact like a power of s). For example, the various factorizable models described earlier gave amplitudes violating the Froissart bound by a power of s if repeated Pomeron exchange, with a Pomeron intercept of one, was allowed. If these amplitudes were to determine $\log \eta$ *a la* Eq. (4.133), then we would again have a $\log \eta$ which grows with a power of s .

The question of how $\log \eta$ varies with b is perhaps more delicate, since this depends on how the Bessel transform of $\log \eta$ cuts off with t .

For example, suppose the hadronic analogue of T_{Tower} were exponential in t with no shrinkage. Then we would find $\log \eta$ cutting off like $e^{-(b/b_0)^2}$,

and hence the criterion determining R tells us that $R^2 \sim \log s$. Thus we end up with $\sigma_T \sim \log s$. Or, suppose that the analogue of T_{Tower} cuts off exponentially in t but the peak shrinks logarithmically with s . Then the $\log \eta$ cut off is

$$\exp \left[- \left(\frac{b}{b_0} \right)^2 \frac{1}{\log s} \right],$$

and we find

$$\frac{R^2}{\log s} \sim \log s .$$

This yields $\sigma_T \sim (\log s)^2$, which is the same result as in the QED case, even though it arises in a different way.

Various other possibilities are easily constructed; all in all the best we can say is that the QED model is very suggestive, and extremely interesting; but it does not yet allow us to make an unambiguous set of predictions for hadron physics.

As a final bit of wet blanketism, we should also keep in mind the fact that even the behaviour of QED itself described above has not been proved to be correct; we do not know with certainty that other graphs cannot alter the result (4.133) on which everything we concluded was based. Furthermore, QED is based on the existence of elementary particles, while hadrons, we like to believe, are all composite. Therefore it is by no means obvious that, even if we do know the behaviour of QED, we can expect to carry over QED results to hadron physics.

REFERENCES AND FOOTNOTES

- 1) Perhaps it should not be. Regge phenomenology is, of course, notoriously vague in its predictions, and is slippery enough so that it is difficult, by any experiment, to prove it wrong. Nevertheless, aside from πN charge-exchange scattering, its ability to fit data is not all that impressive. [See, for example, G. Fox, High-Energy Collisions (Gordon and Breach, NY, 1969), p. 367.] In spite of this, the Regge idea does seem to have achieved wide acceptance .
- 2) R. Blankenbecler and M. Goldberger, Phys. Rev. 126, 766 (1962).
G. Chew, S. Frautschi and S. Mandelstam, Phys. Rev. 126, 1202 (1962).
S. Frautschi, M. Gell-Mann and F. Zachariasen, Phys. Rev. 126, 2204 (1962).
- 3) M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962).
V. Gribov and I. Pomeranchuk, Phys. Rev. Letters 8, 343 (1962).
- 4) M. Gell-Mann and B. Udgaonkar Phys. Rev. Letters 8, 346 (1962).
- 5) K. Igi, Phys. Rev. Letters 9, 76 (1962).
- 6) N. Cabibbo, X. Horwitz and Y. Ne'eman, Phys. Letters 22, 236 (1966).
- 7) G. Chew and D. Snider, UCRL preprint (1970) (to appear in Physical Review). We will return to this model later.
- 8) V. Barger and D. Cline, Proc. Conf. on Expectations from High-Energy Accelerators, Madison, Wisconsin (1970).
- 9) See, for example, V. Barger and D. Cline, Phenomenological Theories of High-Energy Scattering; an Experimental Evaluation (W.A. Benjamin, Inc., NY, 1969).
- 10) D. Amati, S. Fubini and A. Stanghellini, Nuovo Cimento 26, 896 (1962). Strictly speaking, one must assume only that $(\text{Im } T)_{inel} \rightarrow (\text{Im } R)$; however, one must also assume that $\text{Im } T$ dominates $\text{Re } T$ under the integral defining $(\text{Im } T)_{el}$ in Eq. (3.27). Altogether this amounts pretty much to assuming Eq. (4.13).
- 11) S. Mandelstam, Nuovo Cimento 30, 1127 and 1148 (1963).
- 12) D. Amati, M. Cini and A. Stanghellini, Nuovo Cimento 30, 193 (1963).
- 13) The only supporting argument which can be given for this approximation is to say that an exchanged ladder graph gives rise to a Regge pole; hence exchange of lots of ladders, including those which cross, yields Eq. (4.33). [See, for example, Abarbanel and Itzykson, Phys. Rev. Letters 23, 53 (1969).] This is not much of an argument. It may be noted, for example, that it can be shown that corrections to the eikonal formula are more important, at

13 cont'd.

large s , than the higher order terms in χ obtained from Eq. (4.34). [See, for example, Blankenbecler and Sugar, Phys. Rev. 183, 1387 (1969).] It is also entirely unclear what sort of assumption about the production amplitudes this corresponds to.

14) R. Glauber, Lectures in Theoretical Physics, Boulder (1958), Vol. I, p. 315.

15) Note that here we use, for convenience, one of our alternative forms for R . We write

$$R = \gamma s^{\alpha} e^{-i\pi\alpha/2}$$

and assume that

$$\gamma(t) = \gamma_0 e^{b_0 t}.$$

16) S. Frautschi, B. Margolis, Nuovo Cimento 56, 1155 (1968). The only hint that this model might have some connection to reality is a slight wiggle in the experimental p-p differential cross-section near $t \sim -1$ (GeV) ; it is conceivable that this wiggle is associated with the first diffraction dip shown in Fig. 24.

17) A complete set may be made by adding the so-called Toller angles to the $s_{i,i+1}$ and the t_1 . See, for example, N. Bali, G. Chew and A. Pignotti, Phys. Rev. Letters 19, 614 (1967).

18) The internal vertex could, in principle, be a function of the "Toller angles". However, in the region of presumed validity of the multi-Regge formula, the internal vertices cannot actually depend on these [see, for example, J.M. Wang and C.Y. Tan, Phys. Rev. 185, 1899 (1969), and G. Luxton, Ph.D. Thesis, Calif. Inst. Technology (1969)]. Furthermore, experimentally then there is little or no dependence on the "Toller angle". See R. Lipes, G. Zweig and W. Robertson, Phys. Rev. Letters 22, 433 (1969).

19) J. Finkelstein and K. Kajantie, Phys. Letters 26 B, 305 (1968).

20) It should be remembered that these results fail if the internal vertices $\beta(t_1, t_2)$ vanish when both t_1 and t_2 vanish. See I. Verdiev et al., Soviet Phys. JETP 19, 1148 (1964).

21) F. Zachariasen and G. Zweig, Phys. Rev. 160, 1326 (1967).

22) Our choice of Eq. (4.50) is not really necessary in deriving the important results of the multiperipheral model. Many other techniques, based on either Eq. (4.50) or Eq. (4.51) or mixtures thereof, also exist; however, they all end up at Eq. (4.65). See, for example, G. Chew, M. Goldberger and F. Low, Phys. Rev. Letters 22, 208 (1969), and W. Frazer and C. Mehta, Phys. Rev. Letters 23, 258 (1969).

- 23) There are vast numbers of references to ladders. See, for example, M. Gell-Mann et al., Phys. Rev. 133 B, 145 and 161 (1964), R.J. Eden et al., The Analytic S-Matrix (Cambridge Univ. Press, 1966), and other references therein.
- 24) G. Chew and A. Pignotti, Phys. Rev. Letters 20, 1078 (1968).
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- 26) G. Chew, T. Rogers and D. Snider, Phys. Rev. D2, 765 (1970).
- 27) J. Finkelstein, private communication.
- 28) G. Chew and D. Snider, Physics Letters 31 B, 71 (1970) and ref. (7). J. Ball and G. Marchesini, Phys. Rev. 188, 2209 and 2508 (1969).
- 29) See for example, V. Gribov and G. Migdal, Soviet J. Nuclear Phys. 8, 784 (1969).
P. Kaus and F. Zachariasen, Phys. Rev. D1, 2962 (1970).
- 30) Note that this property is true not only for diffraction scattering, but for any Regge pole.
- 31) This type of behaviour is first outlined in Frazer and Mehta (Ref. 22) and in Ball and Marchesini (Ref. 28). The same conclusions were subsequently arrived at by Chew and Snider (Ref. 7).
- 32) This situation is described by Chew (Ref. 7) as a "schizophrenic Pomeron", because the Pomeron is steep and strong for $t > 0$, but weak and flatter for $t < 0$.
- 33) There is even some experimental evidence which lends support to this restriction. See R. Lipes, G. Zweig and W. Robertson, Phys. Rev. Letters 22, 433 (1969).
- 34) S. Mandelstam, Phys. Rev. 115, 1741 (1959).
- 35) T.T. Chou and C.N. Yang, Phys. Rev. Letters 20, 1213 (1968), and Phys. Rev. 170, 1591 (1968).
- 36) L. Durand and R. Lipes, Phys. Rev. Letters 20, 637 (1968).
- 37) There have, in fact, been attempts to put the model with $f(t) \propto |F(t)|^2$ on a deeper theoretical basis [see Abarband, Drell and Gilman, Phys. Rev. Letters 20, 280 (1968) and Phys. Rev. 177, 2458 (1969)]. One postulates a fundamental vector interaction, to which diffraction is attributed; this leads naturally to $d\sigma/dt \propto |F(t)|^4$. However, this also leads naturally to $T(s,t) \propto s |F(t)|^2$ real, not imaginary. Diffraction arises from the subtraction term $sb(t)$ in an s-channel dispersion relation, or a δ_{ji} term in $T^{(-)}(t,j)$. Such models contradict the idea that the physical origin of diffraction lies in absorption through particle production; we therefore find them repellent.

- 38) L. Van Hove, Rev. Mod. Phys. 36, 655 (1964). At first it was thought possible to deduce this form for $f_0(t)$ from the "uncorrelated jet" model of production amplitudes; however this was later realized to be incorrect. See Van Hove and Zalewski, Nuovo Cimento 46, 806 (1966). (The "uncorrelated for" model is one in which the production amplitude is simply taken to be a product $f(q_{1L}) \cdots f(q_{nL})$; some features of this model are similar to the factorized models mentioned earlier such as the multiplicity and scaling; but overall it is certainly much too simple. For example, it cannot yield Multi-Regge behaviour even in the MUREX region.)
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- 40) J. Ball and F. Zachariasen, Univ. of Utah preprint, November, 1970.
- 41) L. Jones, Proc. Conf. on Expectations from High-Energy Accelerators, Madison, Wisconsin (1970).
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- 43) R. Hwa, Caltech preprint (1970).
- 44) R. Hwa, Phys. Rev. D1, 1790 (1970). Note that even Hwa does not want to take his estimate of a $(j - 1)\frac{1}{2}$ cut seriously; we may therefore be excused for doing likewise.
- 45) H. Cheng and T.T. Wu; innumerable papers. Particularly Phys. Rev. Letters 24, 1456 (1970).
- 46) S.J. Chang and T.M. Yan, Phys. Rev. Letters 25, 1586 (1970).
B. Hasslacher et al., Phys. Rev. Letters 25, 1591 (1970).

5. CONCLUSION

Let us summarize where we are.

First, our basic premise throughout was the physical idea that diffraction is the result of absorption produced by the possibility of particle creation; we rejected out of hand the alternative idea that diffraction somehow represents some new type of fundamental interaction. We were therefore led to try to understand something about particle production.

Models of production were grouped into two classes. The first of these, which we called "Conventional Regge models" consisted of a sequence of more and more sophisticated models, culminating in the group of "factorizable" models. The earlier stages in this sequence, namely "Good old Regge poles", and either AFS or absorptive Regge cuts, were unlikely to be exactly right by themselves, since to be so they would require rather peculiar and artificial looking forms for production amplitudes in order to have the elastic amplitude have the very simple form of, say, just a Regge pole and yet satisfy s-channel unitarity. Thus if the Regge-like class of models is correct, it presumably must be its highest achievement, the multiperipheral models, which is the winner. However, as we have seen, the multiperipheral models suffer from two very serious drawbacks. One is that they produce a total cross-section falling with a power of the energy. Furthermore, the fact that this power is very small does not follow from any principle; it has to be put in. Thus what seems to be one of the most fundamental pieces of experimental information, the near constancy of total cross-sections, appears in these models as a dynamical accident. Surely this is a most unsavory state of affairs. The other drawback is that while these models do satisfy s-channel unitarity for the elastic amplitude, they are not unitary for more complicated amplitudes, for $n \rightarrow m$ particles, where n or m is not two. Nevertheless, other features of these models, such as the logarithmic multiplicity, the scaling of inclusive cross-sections, and their simplicity, remain attractive. They cannot, therefore, be discarded without qualms.

The second class of models were labelled "non-shrinking models", and this was really a catch-all for any non-Regge type of model. (One must keep in mind, of course, that shrinking versus non-shrinking may not be the clearest distinction to make; Regge models with fixed, or flat, singularities produce non-shrinking peaks). The most primitive models in this class suffered from their inability to satisfy t-channel unitarity; these, of course, cannot conceivably be exactly correct. Some more sophisticated versions of these, such as Hwa's model, were arranged to be compatible with t-channel unitarity, but gave decidedly unattractive results for multiplicities, for example. Clearly, the most interesting and most hopeful model in this class is the one based on Quantum Electrodynamics that was developed by Cheng and Wu. This yields a cross-section as big as one is permitted by Froissart to have, and thereby provides a natural understanding of the observed (near) constancy of total cross-sections. It has no shrinkage; yet it is compatible with t-channel unitarity. The only obvious drawback is that it is based on a field theory containing elementary particles, and it is always a question how far such a theory simulates what we believe to be the true situation, one in which all hadrons are composite.

Perhaps we are thus led to a rather hopeful position. Perhaps the next step should be to attempt to combine multiperipheral type models with the hints from QED. Perhaps one should think of the multiperipheral ladder not as the answer in itself, but as the QED ladder which goes into an eikonal. This would permit one to have Pomerons with intercept exactly unity, thereby obviating the necessity for cross-sections falling like powers and permitting an understanding of their observed near constancy. It would also, in all likelihood, permit a theory which is unitary not only for elastic scattering, but for $n \rightarrow m$ particles as well. In any event, some progress along these lines may be possible, and we do not, therefore, necessarily need to feel ourselves hopelessly at a loss. Diffraction may end up being understandable, after all.

Acknowledgements

I would like to express my gratitude to J. Finkelstein, for guidance on the methodology and conclusions of multiperipheral and multi-Regge models; to H. Cheng and T.T. Wu, for a very helpful summary of the results of their studies of Quantum Electrodynamics; to J. Ball for many expositions of what has later come to be called the "Schizophrenic Pomeron" model; and to G. Zweig and K. Gottfried for assorted philosophical remarks and opinions regarding the origin of diffractive phenomena.

Figure captions

Fig. 1a : Total cross-sections $\sigma(s)$ versus s for:

- i) pp scattering
- ii) $\bar{p}p$ scattering
- iii) π^+p scattering
- iv) π^-p scattering
- v) K^+p scattering
- vi) K^-p scattering.

[From G. Fox and C. Quigg, "Compilation of Elastic Scattering Data", UCRL-20001, January 1970]

Fig. 1b : Total cross-sections $\sigma(s)$ versus s for π^-d and K^-d scattering. [From J.V. Allaby et al., Phys. Letters 30 B, 500 (1969).]

Fig. 2 : Differential cross-sections for pp elastic scattering. [From J.V. Allaby, XV Int. Conf. on High-Energy physics, Kiev, 1970.]

Fig. 3 : Differential cross-sections for $\bar{p}p$ elastic scattering. [From J.V. Allaby, see caption to Fig. 2.]

Fig. 4 : Differential cross-sections for π^+p elastic scattering. [From J.V. Allaby, see caption to Fig. 2.]

Fig. 5 : Differential cross-sections for π^-p elastic scattering. [From J.V. Allaby, see caption to Fig. 2.]

Fig. 6 : Differential cross-sections for K^+p elastic scattering. [From J.V. Allaby, see caption to Fig. 2.]

Fig. 7 : Differential cross-sections for K^-p elastic scattering. [From J.V. Allaby, see caption to Fig. 2.]

Fig. 8 : Slope parameters for pp elastic scattering. [From E. Lillethun, Proc. Int. Conf. on Elementary Particles, Lund (1969), p. 167.]

- Fig. 9 : Slope parameters for π^+p and π^-p elastic scattering.
[From J.V. Allaby, see caption to Fig. 2.]
- Fig. 10 : Slope parameters for K^+p and K^-p elastic scattering.
[From D. Morrison, Proc. Int. Conf. on Elementary Particles, Lund (1969), p. 237.]
- Fig. 11 : Elastic total cross-section $\sigma_{e1}(s)$ versus s for pp and $\bar{p}p$ scattering. [From G. Giacomelli, CERN-HERA 69-3, December (1969).]
- Fig. 12 : Elastic total cross-section $\sigma_{e1}(s)$ versus s for π^+p and π^-p scattering. [Taken from G. Giacomelli, CERN Internal Report NP 70-11, April (1970).]
- Fig. 13 : Elastic total cross-section $\sigma_{e1}(s)$ versus s for K^+p and K^-p scattering. [From G. Giacomelli, CERN-HERA 69-3, December (1969).]
- Fig. 14 : Cross-sections $\sigma(s)$ versus s for various quasi-two-body diffraction dissociation reactions. [From Anderson et al., Phys. Rev. Letters 16, 855 (1966).] Note for contrast the non-diffractive cross-section for $N^*(1238)$ production.
- Fig. 15 : $d\sigma/dt$ versus t for several diffraction dissociation processes: $p + p \rightarrow p + N^*$, $\pi^\pm + p \rightarrow \pi^\pm + N^*$, where N^* is $N^*(1400)$ and $N^*(1690)$. [From Foley et al., Phys. Rev. Letters 19, 398 (1967).]
- Fig. 16 : Cross-section $\sigma(s)$ versus s for various diffractive production processes: shown are the cross-sections for 2, 4, 6, 8, and 10 prongs in π^-p collisions. [From W.D. Walker, High-Energy Collisions, (Gordon and Breach, NY, 1969), p. 295.]

- Fig. 17 : Cross-sections $\sigma(s)$ versus s for the non-diffractive production process $\pi^- p \rightarrow \rho^0 n$. [From W.D. Walker, Fig. 16.]
- Fig. 18 : $\alpha(s) \equiv \text{Re } T(s,0)/\text{Im } T(s,0)$ versus s for
a) $\pi^- p$ elastic scattering
b) pp elastic scattering.
[From J.V. Allaby, see caption to Fig. 2.]
- Fig. 19 : Diagram of the s-channel unitarity relation Eq. (3.26), showing the various momentum labels.
- Fig. 20 : Experimental values for the ratio $\sigma_{\text{el}}(s)/\sigma_{\text{T}}(s)$ versus s for $\pi^+ p$ and $\pi^- p$ scattering. [From G. Giacomelli, CERN Internal Report NP 70-11, April, (1970).]
- Fig. 21 : $A(s,t)$ and $A_{\text{inel}}(s,t)$ versus t for pp elastic scattering, at $p_{\text{lab}} = 29.3 \text{ (GeV)}^2$. [From F. Zachariasen, Boulder Summer School, 1969 (to be published by Gordon and Breach, NY).]
- Fig. 22 : Experimental values of $\eta_j(s)$ for some representative processes. Shown are $\eta_j(s)$ for pion-nucleon scattering in:
a) the $I = 3/2, J = 3/2$; and b) the $I = 1/2, J = 3/2$ states. [From C. Lovelace, Proc. Int. Conf. on Elementary Particles, Heidelberg (1968), p. 79.]
- Fig. 23 : Schematic diagram of "repelling" Pomeron and P' . The P' is presumably on the unphysical sheet of the indicated cut.
- Fig. 24 : $d\sigma/dt$ versus t for pp elastic scattering, at various energies, as calculated from the absorption model. [From Ref. 16.]
- Fig. 25 : Diagrams describing some versions of factored production amplitudes:
a) a field theoretic ladder;
b) a multiperipheral diagram.

- Fig. 26 : A diagrammatic representation of the integral equation, Eq. (4.70).
- Fig. 27 : Schematic diagram of a physical sheet Regge pole and cut. The positions of the physical sheet pole (solid dot), branch cut (dotted line) and unphysical sheet poles (hollow dot) are shown at a given value of t .
- Fig. 28 : Diagram representing the production amplitude in the two-fireball model.
- Fig. 29 : $d\sigma/dt$ versus t for pp elastic scattering at $p_{lab} = 19.6$ (GeV/c), and $d\sigma/dt$ as calculated from $|F(t)|^4$, where $F(t)$ is the proton electromagnetic form factor. [What is actually used is the magnetic form factor $G_M(t)$. The experimental points (dots) are from J.V. Allaby et al., Phys. Letters 38 B, 67 (1968).]
- Fig. 30 : $d\sigma/dt$ versus t for pp elastic scattering as calculated from Eqs. (4.101) and (4.102. [From Ref. 36.]
- Fig. 31 : $d\sigma/dt$ versus t for pp elastic scattering at $p_{lab} = 19.6$ (GeV/c), and $d\sigma/dt$ as calculated from Eq. (4.113). [From Ref. 40.] The dots represent the data, and the line the theoretical prediction.
- Fig. 32 : Diagrams comprising a "tower" in massive quantum electrodynamics. [From Ref. 45.]
- Fig. 33 : Diagrams showing examples of repeated tower exchange. [From Ref. 45.]

FIG. 1(a) (i) + (ii)

FIG.1(a) (iii) +(iv)

FIG. 1(a) (v) + (vi)

|t| GeV²

FIG.8

FIG. 9(b)

FIG. 9(a)

FIG.10

FIG.12(a)

FIG.12(b)

FIG.13

FIG. 14

FIG.15

FIG. 16

FIG.17

FIG. 20

FIG. 21

FIG. 22(a)

FIG. 22(b)

FIG. 23

FIG. 24

FIG. 25

FIG. 26

FIG. 27

FIG.28

FIG.29

FIG.30

FIG.31

FIG.33

FIG.32