INITIAL DATA AND THE END STATE OF SPHERICALLY SYMMETRIC
GRavitATIONAL COLLAPSE

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Abstract

Generalizing earlier results on the initial data and the final fate of dust collapse, we study here the relevance of the initial state of a spherically symmetric matter cloud towards determining its end state in the course of a continuing gravitational collapse. It is shown that given an arbitrary regular distribution of matter at the initial epoch, there always exists an evolution from this initial data which would result either in a black hole or a naked singularity depending on the allowed choice of free functions available in the solution. It follows that given any initial density and pressure profiles for the cloud, there is a non-zero measure set of configurations leading either to black holes or naked singularities, subject to the usual energy conditions ensuring the positivity of energy density. We also characterize here wide new families of black hole solutions resulting from spherically symmetric collapse without requiring the cosmic censorship assumption.
1. INTRODUCTION

The role of initial data towards determining the end state of gravitational collapse of a spherically symmetric dust cloud has been analyzed in some detail recently [1]. It is understood now, for example, that given any initial density profile for such a cloud, one can always choose the velocity function describing the infall of the matter shells in such a manner that either a black hole or a naked singularity will result as the final state for such a collapse. Such results, examining the relevance of the regular initial data, from which the collapse of the cloud commences, towards its evolution in terms of a black hole or a naked singularity should lead us to a better understanding of the genericity and stability aspects of naked singularities. This is most essential and will have important implications on any possible formulations of the cosmic censorship hypothesis (see e.g. [2] for a review of the recent developments). Even if evolutions permitting naked singularity were allowed by general relativity, if these were non-generic in some well-defined sense, that could lead to a suitable formulation for the cosmic censorship principle.

Our purpose here is to generalize the results such as those in [1] to collapsing clouds with a more general form of initial data and matter. While dust could possibly be regarded as a good approximation to the state of matter in the final stages of collapse [3], it is an idealized form which does not take into account the pressures and stresses, which could play an important role towards determining the end state of gravitational collapse. We consider here general type I matter fields [4], which include most of the physically important forms such as dust, perfect fluids, massless scalar fields and so on. In fact, almost all observed forms of matter and equations of state would fall within this general class. Having defined the regular initial data for the matter cloud in terms of its initial density and pressures on an initial spacelike surface, we examine to what possible final states the collapse of such a cloud could develop, while being subject to the usual energy conditions requiring the positivity of energy density. It is shown that given any regular distribution of matter at the initial epoch, there always exists an evolution from this initial data which could result
either into a black hole or a naked singularity, depending on the allowed choice of free
functions available in the solution. We also consider here a model describing collapsing
shells, which reveals some interesting features of gravitational collapse, and the structure
of the singularity when it is visible.

Further, our analysis here identifies wide new families of black hole solutions in terms of
the parameters given in the initial data space or the allowed evolutions considered. Since
the cosmic censorship hypothesis is widely useful for black hole physics, it is of crucial
importance to isolate and identify the causes in the gravitational collapse phenomena in
general relativity which may be responsible for the formation of naked singularities or black
holes. The role of the initial data in this context can not be overemphasized. In fact, it
is already known in several recent naked singularity examples that the initial data at the
onset of the collapse does categorize the naked singular or black hole spacetimes, resulting
from collapse. A particularly transparent example, apart from the dust collapse, is that
of the Vaidya-Papapetrou radiation collapse models, where the rate of implosion of the
mass \( \lambda \equiv dm(u)/du \) before the formation of singularity determines whether the collapse
would end in a naked singularity or a black hole (see e.g. [5] for a discussion). For such
collapse models, there exist non-zero measure sets in the space of initial data, which evolve
in either a naked singularity or a black hole.

In other words, given the initial matter profiles in terms of the densities and pressures
distributions, our results here indicate a specific procedure to fine tune the evolutions so
as to necessarily produce a black hole as the end product of collapse. Such a conclusion
no longer requires the assumption of cosmic censorship, which is turning out to be a
difficult proposition to establish rigorously. From such a perspective, ours is a more specific
characterization, in terms of the space of initial data and the possible evolutions, in order
to generate black holes.

2. EINSTEIN EQUATIONS AND REGULARITY CONDITIONS

We are interested here in the problem of how a given regular initial data set evolves
dynamically in a spherically symmetric spacetime in the context of the occurrence and nature of singularities. Our main purpose is to analyze the Einstein field equations, with a given initial data set such as the state of matter and the velocities of the spherical shells at the onset of collapse for a compact object, in order to determine the possibilities of this configuration evolving into either a black hole or a naked singularity. We therefore consider the gravitational collapse of a matter cloud that evolves from a regular initial data defined on an initial spacelike surface. The energy-momentum tensor has a compact support on this initial surface where all the physical quantities such as densities and pressures are regular and finite. For such a cloud of sufficiently high total mass, there will not be any stable equilibrium configurations available, and a continual gravitational collapse must ensue resulting finally into a space-time singularity of infinite densities and curvatures.

For the matter distribution of the spacetime within the cloud undergoing gravitational collapse, we shall not restrict ourselves to any specific form of matter but consider general type I matter fields. In fact, all the known and observed physical matter fields are of this type, except the directed radiations which are of type II. It thus follows that as far as the form of the matter is concerned, we have included quite general collapse scenarios which include all the physically reasonable matter fields, and possible equations of state. The type I and type II matter fields are characterized by the existence of two real orthogonal eigenvectors, or one double-null real eigenvector respectively, in a timelike invariant 2-plane in the spacetime. The remaining fields of type III and type IV are considered physically unreasonable, as they necessarily violate the energy conditions ensuring the positivity of the mass-energy density. Furthermore, such fields have not been observed in nature so far, all the observed matter fields being of type I. Thus, we would not attribute any physical significance or interpretation to the same presently. A type II matter distribution corresponds to zero-rest mass fields representing a directed radiation. The radiation collapse, corresponding to such a field can be described by a Vaidya spacetime and has been studied in detail to show that both the black holes and naked singularities could result in such a
Thus we concentrate here on type I matter fields, for spherically symmetric spacetimes. Such a matter field, in a general coordinate system, can be expressed as

\[ T^{ab} = \lambda_1 E_1^a E_1^b + \lambda_2 E_2^a E_2^b + \lambda_3 E_3^a E_3^b + \lambda_4 E_4^a E_4^b \]  

(1)

where \((E_1, E_2, E_3, E_4)\) is an orthonormal basis with \(E_4\) and \((E_1, E_2, E_3)\) being timelike and spacelike eigenvectors respectively, with \(\lambda_i\)s \((i = 1, 2, 3, 4)\) being the eigenvalues. For such a spherically symmetric matter distribution we can choose coordinates \((x^i = t, r, \theta, \phi)\) adopted to the above orthonormal frame, and the metric is written as,

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\psi} dr^2 + R^2 d\Omega^2 \]  

(2)

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\) is the line element on a two-sphere. Here \(\nu, \psi\) and \(R\) are functions of \(t\) and \(r\), and the stress-energy tensor \(T^a_b\) given by equation (1) has only diagonal components in this coordinate system (i.e. we are using a comoving coordinate system), given by

\[ T^t_t = -\rho, \quad T^r_r = p_r, \quad T^\theta_\theta = p_\theta = T^\phi_\phi, \quad T^t_r = T^r_t = 0 \]  

(3)

The quantities \(\rho, p_r, \) and \(p_\theta\) are the eigenvalues of \(T^a_b\) and are interpreted as the density, radial pressure, and tangential stresses respectively. We take the matter fields to satisfy the weak energy condition, i.e. the energy density as measured by any local observer must be non-negative, and so for any timelike vector \(V^a\) we must have

\[ T_{ab} V^a V^b \geq 0 \]  

(4)

which amounts to

\[ \rho \geq 0, \quad \rho + p_r \geq 0, \quad \rho + p_\theta \geq 0 \]  

(5)

From the point of view of the dynamical evolution of the initial data at an epoch of time from which the collapse commences, one has a total of six arbitrary functions of \(r\), namely

\[ \nu(t_i, r) = \nu_o(r), \quad \psi(t_i, r) = \psi_o(r), \quad R(t_i, r) = R_o(r), \]
\[ \rho(t_i, r) = \rho_o(r), \quad p_r(t_i, r) = p_{r_o}(r), \quad p_\theta(t_i, r) = p_{\theta_o}(r) \]  

These functions constituting the initial data are to be specified at some initial spacelike surface at an initial time \( t = t_i \). The dynamical evolution of such a set of initial data is determined by the Einstein equations, and for the metric (2) these are given by,

\[ T^t_t = -\rho = -\frac{F'}{k_o R^2 R'}, \quad T^r_r = p_r = -\frac{\dot{F}}{k_o R^2 \dot{R}} \]  

\[ \nu'(\rho + p_r) = 2(p_\theta - p_r) \frac{R'}{R} - p'_r \]  

\[ -2\dot{R}' + R' \frac{\dot{G}}{G} + \dot{H}' \frac{H'}{H} = 0 \]  

\[ G - H = 1 - \frac{F}{\dot{R}} \]  

where () and (′) represent partial derivatives with respect to \( t \) and \( r \) respectively, \( F = F(t, r) \) is an arbitrary function of \( t \) and \( r \), and we have put

\[ G = G(t, r) = e^{-2\psi}(R')^2, \quad H = H(t, r) = e^{-2\nu} \dot{R}^2 \]  

The initial data represented by the functions \( \nu_o, \psi_o, \rho_o, p_{r_o}, p_{\theta_o} \) and \( R_o \) are not all independent. From equation (8) it follows that there is a relation between them as given by

\[ \nu'_o(\rho_o + p_{r_o}) = 2(p_{\theta_o} - p_{r_o}) \frac{R'_o}{R} - p'_{r_o} \]  

\[ \Rightarrow \nu_o(r) = \int \left( \frac{(2p_{\theta_o} - 2p_{r_o}) R'_o}{R_o(\rho_o + p_{r_o})} - \frac{p'_{r_o}}{\rho_o + p_{r_o}} \right) dr \]  

Furthermore, there is a coordinate freedom left in the choice of the scaling of the coordinate \( r \), which can be used to reduce the number of independent initial data to four. It follows that there are only four independent arbitrary functions of \( r \) constituting the initial data. Evolution of this data is governed by the field equations, and we have in all five equations with seven unknowns, namely \( \rho, p_r, p_\theta, \nu, \psi, R \) and \( F \), giving us freedom of choice of two functions. Selection of these two free functions, subject to the given initial data and the
weak energy condition above, determines the matter distribution and the metric of the spacetime and thus leads to a particular evolution for the initial data.

In spherically symmetric spacetimes, the function $F(t, r)$ is treated as the mass function for the cloud with $F \geq 0$. In order to preserve the regularity of the initial data at $t = t_i$, we must have $F(t_i, 0) = 0$, that is, the mass function vanishes at the center of the cloud.

In this paper, we are interested only in the spacetimes which at some initial epoch $t = t_i$ are free from any singularities in the initial data, i.e. the evolution of the collapse must develop from a regular initial data. Our interest is in the evolutions of such data sets, which develop from this initial singularity free state, into a possible singularity of the spacetime. We then study the nature and structure of this singularity from the perspective of the cosmic censorship. The initial data is considered to be singularity free if the curvatures and densities are all finite, that is the Kretschmann scalar is bounded on the initial surface. The Kretschmann scalar $K = R^{abcd} R_{abcd}$ for a spherically symmetric spacetime, as given by the metric in (2), can be put in the following form,

$$K = C^2 - \frac{1}{3} T^2 + 2T^2$$

where

$$C^2 = C^{abcd} C_{abcd} = \frac{4}{3} \left( \frac{F}{R^3} + p_r - p_\theta - \rho \right)^2$$

$$T^2 = (T^a_a)^2 = \left( p_r + 2p_\theta - \rho \right)^2$$

$$T^2 = T^{ab} T_{ab} = \rho^2 + p_r^2 + 2p_\theta^2$$

Thus, a singularity will appear on the initial surface if either the density $\rho$, or one of the pressures become unbounded at any point on the initial surface, or $(F/R^3) \to \infty$ at any point. We require that at the initial surface $t = t_i$, the density and pressures are finite and bounded. Further more, we have

$$F(t_i, r) = \int \rho_o(R_o) R_o^2 dR_o$$

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and hence the spacetime is singularity free initially in the sense that the Kretchmann scalar, density and pressures are finite. But as the collapse evolves, a singularity could develop at a later time whenever either of the density or one of the pressures become unbounded. We shall consider below such specific evolutions of the initial data which model a gravitationally collapsing matter cloud.

3. COLLAPSING MATTER CLOUDS

The initial data for a collapsing matter cloud basically means the initial densities and pressures describing the initial state of matter at the onset of collapse, namely, $\rho_o, p_{\theta_o}, p_{\phi_o}$, and the function $\psi_o$, which is related to the initial velocity of the collapsing shells. From the point of view of the cosmic censorship hypothesis, all regular, physically reasonable initial data sets should evolve into a black hole. It is known, of course, that there are initial data sets with reasonable forms of matter such as the Lemaitre-Tolman-Bondi dust collapse, radiation collapse, and self-similar adiabatic perfect fluid scenarios, which result into either a black hole or a naked singularity, depending on the nature of initial data and initial distributions of densities and pressures. Thus, the above broad concept of cosmic censorship has to be fine tuned and made more precise in order to arrive at a genuine formulation and proof for censorship (see e.g. [2] or [6]).

It follows that either for such a purpose of cosmic censorship, or to understand the naked singularities of gravitational collapse better, we need to examine the evolutions of an arbitrary, but regular and physically reasonable set of initial data undergoing the gravitational collapse. This requires defining more precisely as to what constitutes a regular, physically reasonable, initial data. As far as the form of the matter is concerned, since all the observed forms of matter have been of type I as discussed earlier, we have considered here general type I matter fields. Next, it is widely accepted that all physically reasonable matter forms must satisfy the energy conditions ensuring the positivity of mass-energy densities. Therefore, one of the conditions we must impose naturally is that at the onset of the collapse, all the initial data sets specifying the density and pressures profiles of the
cloud must satisfy an energy condition, and the same must hold during the later evolution of collapse. The regularity of the initial data also means that at the onset of collapse at some \( t = t_i \), the spacetime must be singularity free, as discussed earlier. Further more, from the point of view of general relativity the data should be at least \( C^2 \) differentiable. Therefore we require that the functions \( \rho_o, p_{r_o}, p_{\theta_o} \) be atleast \( C^2 \) function of \( r \). The same applies to the metric functions \( \nu_o \) and \( \psi_o \) which must be \( C^2 \) functions of \( r \) throughout the spacelike section \( t = t_i \) for all \( r \geq 0 \).

As stated above, for the initial data to be physically reasonable the weak energy condition is to be satisfied. This imposes the following restrictions on the initial data,
\[
\begin{align*}
\rho_o(r) \geq 0, & \quad \rho_o + p_{r_o} \geq 0, \quad \rho_o(r) + p_{\theta_o}(r) \geq 0 \\
\end{align*}
\] (19)

Further, for the evolution of the matter in the spacetime to be physically realistic, energy conditions (5) have to be satisfied. Therefore we assume that throughout the cloud, at the initial and at all later epochs the weak energy condition is satisfied during the entire evolution of the collapsing cloud till the formation of the spacetime singularity at \( R = 0 \).

Since the function \( \nu_o \) is related to the initial matter data by equation (13), and is a \( C^2 \) function of \( r \), this puts certain restrictions on the arbitrariness of the choice of the functions \( \rho_o, p_{r_o}, p_{\theta_o} \). For example, we have
\[
[p_{\theta_o} - p_{r_o}]_{r=0} = 0
\]
(20)

For the sake of physical reasonableness, we require the center \( r = 0 \) to be the regular center for the cloud, which means \( R(t,0) = 0 \). Also, one would like to have the initial density \( \rho_o(0) > 0 \) at the center \( r = 0 \). This implies that we have
\[
\rho_o(0) + p_{r_o}(0) > 0, \quad \nu_o(r) = r^2 h(r)
\]
(21)
where \( h(r) \) is at least a \( C^1 \) function of \( r \) for \( r = 0 \), and at least a \( C^2 \) function for \( r > 0 \). This means that the pressure gradients vanish at the center \( r = 0 \), basically meaning that
the forces vanish at the center. In this section we consider the above scenario for the sake of physical reasonableness, however, it is possible to give a more general formalism independent of requirements such as above, as we shall discuss in section 6.

In fact, it would be reasonable to require that pressures be positive at the onset of the collapse, since for astrophysical bodies physically we would prefer the pressures rather than tensions. Further more, to make the scenario physically more appealing, we may require the density to be decreasing as we move away from the center $r = 0$. In that case, for various reasonable equations of state such as $p = k\rho, 0 < k < 1$, (a perfect fluid), or $p = k\rho^\gamma$, the pressure also will decrease away from the center together with the decreasing density. This will be typically the case in the massive bodies such as stars and such other astrophysical systems. Then as such the energy conditions could impose restrictions on the maximum size of the matter cloud with such an initial density and pressure distribution.

We use the coordinate freedom available in rescaling the radial coordinate $r$ such that

$$R(t, r) = R_0(r) = r$$

The physical area radius $R$ then monotonically increases with the coordinate $r$, and there are no shell-crossings on the initial surface, with $R' = 1$. Since we are considering gravitational collapse, we also have $\dot{R} < 0$.

Consider now the gravitational collapse of a matter cloud with a general initial data as prescribed above. We supply the two free functions $F$ and $\nu$ as below,

$$\nu = c(t) + \nu_0(R), \quad F = f(r) + F_0(R)$$

From equations (7) to (13), the evolution of the collapse is described by the equations,

$$\nu_0(R) = R^2g(R) = \int_0^R \left( \frac{2p_{\theta_0} - 2p_{r_0}}{r(\rho_0 + p_{r_0})} - \frac{p_{r_0}'}{\rho_0 + p_{r_0}} \right) dr$$

$$G = b(r)e^{2\nu_0}$$

$$\sqrt{R}\dot{R} = -a(t)e^{\nu_0} \sqrt{b(r)Re^{\nu_0} - R + f + F_0}$$
\[ \rho = \frac{f'}{R^2 R'} + \frac{F_{o,R}}{R^2} \]  
\[ p_r = -\frac{F_{o,R}}{R^2} \]  
\[ 2p_\theta = R\nu,_{R}(\rho + p_r) + 2p_r + Rp_r,_{R} \]  
\[ F_o(R) = -\int_0^R r^2 p_{r_o} dr \equiv -R^3 p(R), \quad f(r) = \int_0^r r^2 (\rho_o + p_{r_o}) dr \equiv r^3 \epsilon(r) + r^3 p(r) \]  

Here \( F(t, r) = 2r^3 \epsilon(r) \). The quantities \( \epsilon(r) \) and \( p(r) \) are to be treated as the average mass and pressure densities of the cloud, and are decreasing functions of \( r \). Since \( p_{r_o} \) is a positive function on the initial surface, it follows that \( F_o < 0 \) throughout the spacetime, and as such the radial pressure is non-negative throughout the spacetime. The arbitrary function \( b(r) \) characterizes the velocity of the spherical shells at the initial time \( t = t_i \). We are dealing with the collapse situation with \( \dot{R} < 0 \), therefore the arbitrary function \( a(t) > 0 \).

It should be noted that for the case \( p_r = p_\theta = 0 \), the above set of equations, and the collapse models reduces to the Tolman-Bondi-Lemaitre case of general inhomogeneous dust collapse. In that case, the results in [1] apply as far as the role of the initial data towards determining the final fate of collapse is concerned in terms of either a black hole or a naked singularity. Thus, the models here may also be viewed as directly generalizing the Tolman-Bondi-Lemaitre results to include both the radial and tangential pressures in order to investigate again the role of initial data towards the final fate of collapse. Certain examples have been examined which include non-zero pressures, and it is seen that again both black holes and naked singularities arise as final fate of collapse [7].

Since \( r = 0 \) is the regular center of the cloud meaning \( R(t, 0) = 0 \), it follows from equation (26) that

\[ \sqrt{\nu} \dot{\nu} = -a(t)e^{\nu} \sqrt{v^3 (h(R)b(r) - p(R)) + b_o(r)v + \epsilon(r) + p(r)} \]  

where the arbitrary function \( b(r) = 1 + r^2 b_o(r) \), such that \( b_o(r) \) is at least a \( C^1 \) function of \( r \) for \( r = 0 \), and a \( C^2 \) function for \( r > 0 \), and we have put

\[ R = rv(t, r), \quad v(t_i, r) = 1 \]  

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\[ h(R) = h(rv) = \frac{e^{r^2 v^2 g(rv)} - 1}{r^2 v^2} \]  

(33)

The functions \( b_o(r), h(rv), v(t, r), u(rv) \), and \( f_o(r) \) are all at least \( C^1 \) functions of their arguments. Note that at \( t = t_i \) we have \( v = 1 \) and since \( \dot{v} < 0 \), we have \( v < 1 \) throughout the spacetime.

We note that the quantity \( R(t, r) \geq 0 \) here is the area radius in the sense that \( 4\pi R^2(t, r) \) gives the proper area of the mass shells at any given value of the comoving coordinate \( r \) for a given epoch of time. The area of such a shell at \( r = \text{const.} \) goes to zero when \( R(t, r) = 0 \). In this sense, the curve \( t = t_s(r) \) such that \( R(t_s, r) = 0 \) describes the singularity in the spacetime where the mass shells are collapsing to a vanishing volume, where the density and pressures diverge. This shell-focusing singularity occurs along the curve \( t = t_s(r) \) such that \( v(t_s, r) = 0 \), the Kretchmann scalar diverges at such points.

Using the remaining degree of freedom left in the scaling of the time coordinate \( t \) we could put \( a(t) = 1 \). Equation (31) can then be integrated with the initial condition \( v(t_i, r) = 1 \) to obtain the function \( v(t, r) \). Note that the coordinate \( r \) is treated as a constant in the equation. We in fact get

\[
\int_0^1 \frac{\sqrt{v} dv}{\sqrt{h_o(r) v e^{3\nu_o} + e^{2\nu_o}(v^3(h(rv) - p(rv)) + \epsilon(r) + p(r))}} = t
\]

(34)

where we have chosen for the sake of simplicity \( t_i = 0 \). The time \( t = t_s(r) \) corresponds to the occurrence of singularity is then given by,

\[
t = t_s(r) = \int_0^1 \frac{\sqrt{v} dv}{\sqrt{h_o(r) v e^{3\nu_o} + e^{2\nu_o}(v^3(h(rv) - p(rv)) + \epsilon(r) + p(r))}}
\]

(35)

For such a collapsing cloud, a singularity can also occur at points where \( R' = 0 \), which are termed the shell-crossing singularities. Though these are singularities of a weaker nature in general through which the spacetime can possibly be extended using a suitable extension procedure [8], the comoving coordinate system we have used here may break down and the metric might possibly become degenerate at the points where \( R' = 0 \). However,
our purpose here is to study the shell-focusing singularity at \( R = 0 \), which is essentially different and could be much stronger gravitationally as compared to the shell-crossings which are possibly delta-function like singularities, caused by different shells crossing each other where the density momentarily blows up. Hence, we choose the evolution of the initial data in such a manner that any shell-crossings are avoided in the collapse, except possibly at the singularity. Here it may be useful to mention that a similar situation regarding the occurrence of shell-crossings arises in Tolman-Bondi-Lemaître dust collapse models also. However, as has been pointed out in earlier works, for a given initial density profile one can always choose appropriate initial velocity of the dust shells such that during the evolution no shell-crossings are encountered or visa-versa.

In a similar fashion, this can be achieved in the present general case as well by a suitable choice of the functions involved, as specified below. At a given epoch of time, the functions \( \epsilon(r) \), \( p(r) \), \( \nu_o(R) \) and \( h(R) \) are at least \( C^1 \) functions and further more \( \epsilon(r) \) and \( p(r) \) are decreasing functions of \( r \). The function \( b_o(r) \) is an arbitrary function representing the initial velocities of the collapsing spherical shells. From equation (35) it is clear that the singularity time \( t_s(r) \) is an explicit function of the velocity function \( b_o(r) \) which is a free function, and one can choose it in such a way that \( t_s(r) \) is an increasing function of \( r \), i.e. \( dt_s/dr > 0 \). The exact nature of such velocity functions \( b_o(r) \) for which \( t_s(r) \) in equation (35) is an increasing function of \( r \), depends upon the exact behavior of initial density and pressures within the cloud. For example, for a matter cloud initially satisfying an equation of state of the type \( p = a\rho^\gamma \) one of the many possibilities is the function \( b_o(r) > 0 \) such that \( b_o'(r) < 0 \) and is less than a certain minimum for \( r_b \geq r \geq 0 \). For all such functions \( b_o(r) \), therefore, the singularity curve \( t = t_s(r) \) is an increasing curve for all allowed values of coordinate \( r \), and the successive spherical shells within the cloud collapse to singularity successively, and shell crossings do not occur. Thus \( R' = v + rv' > 0 \) (note that \( R' = 1 \) initially) throughout the spacetime. Furthermore, if \( [r^2b_o]' \geq 0 \) than \( \sqrt{\nu}R' \leq 1 \) within the cloud for \( 1 \geq v \geq 0 \).
The shell-focusing singularity $R = 0$ occurs first at $r = 0$ and the time of occurrence of such a singularity from equation (35) is given by,

$$t_s = t_s(0) = \int_0^1 \frac{\sqrt{v}dv}{\sqrt{v^4(h_o - p_o) + b_{oo}v + \epsilon_o + p_o}}$$

where $h_o = h_o(0), p_o = p(0), \epsilon_o = \epsilon(0), b_{oo} = b_o(0)$ are constants related to the central density and pressures. In fact, near the center $r = 0$ we have

$$t_s(r) = t_s + rX(0) + O(r^2)$$

where the function $X = X(v)$ is given by

$$X(v) = \sqrt{\frac{\epsilon_o + p_o + b_{oo}v + v^3(h_o - p_o)}{\epsilon_o + p_o}} \int_v^1 \frac{\sqrt{v}(\epsilon_1 + p_1 + b_1v - v^4h_1)dv}{(v^3(h_o - p_o) + b_{oo}v + \epsilon_o + p_o)^{3/2}}$$

where $\epsilon_1 = -\epsilon'(0), p_1 = -p'(0), b_1 = -b_o'(0), h_1 = h, R(0)$. For the case that we are considering where the pressures have been taken to be positive, the central singularity at $r = 0$ could be naked, all subsequent singularities with $r > 0$ are covered as the quantity $F/R \to \infty$ and the trapped surfaces and the apparent horizon develop prior to the formation of the singularity. (When the pressures are allowed to be negative, still subject to the validity of the weak energy condition, the other parts of the singularity can be visible in principle, see e.g. Cooperstock et al in [7]). It thus remains only to examine the nature of the central singularity.

We now show that the weak energy condition is satisfied by the class of collapse models under consideration. For the models to be physically reasonable, it is necessary that the positivity of energy density is preserved at the initial epoch, and also throughout the entire evolution of the collapse, till the formation of the spacetime singularity at $R = 0$. At the initial epoch, for the regularity and physical reasonableness of the initial data, we require the initial matter density $\rho_o(r)$ to be positive, and also the initial radial and tangential pressures $p_{r_o}$ and $p_{\theta_o}$ to be positive, or at least non-negative. Thus we have,

$$\rho_o(r) \geq 0, \quad p_{r_o} \geq 0, \quad p_{\theta_o} \geq 0$$
The initial density $\rho_o(0) = \rho_c$ is maximum at the regular center $r = 0$, and decreasing away from the center, and the central pressures $p_r, p_\theta$ are also positive. It is therefore intuitively clarifying to understand the behavior of these physically relevant quantities in the neighborhood of the regular center $r = 0$, and near the singularity, during the evolution of the collapsing cloud. Firstly, it is clear from equation (28) that the radial pressure $p_r$ will be positive throughout out the subsequent evolution of the cloud. It then follows from equation (27) and (28) that $\rho + p_r \geq 0$ throughout the spacetime, since it is that way to begin with. Next, note that in the near regions of the center at $r = 0$, and that of the singularity at $v = 0$, we have $\rho \geq 0, \rho + p_r \geq 0, \rho + p_\theta \geq 0$, if the same is satisfied at the initial surface, regardless of any restrictions on the form of initial densities and pressures. Actually, the initial density $\rho_o(0) = \rho_c$ is maximum at the regular center $r = 0$, and decreasing away from the center, and the central pressures $p_r, p_\theta$ are also positive there. In the near regions of the center $r = 0$, $\sqrt{v} R' = v^{3/2} + r X(v) + O(r^2)$, $X(v) < 1$, and we have from equation (27) to (29)

$$v^2 R' \rho = \rho_c - \rho_1 r + p_{rc}(1 - v^2 - v^{3/2} X(v) r) + O(r^2)$$  \hspace{2cm} (39)$$

$$v^2 R'(\rho + p_\theta) = \rho_c + p_{\theta c} - \rho_1 r \frac{\rho_c + p_{\theta c}}{\rho_c + p_{rc}} + O(r^2)$$ \hspace{2cm} (40)$$

where $\rho_c, p_{rc}$ and $p_{\theta c}$ represent the central density and pressures at the initial epoch $t = t_i$, and $\rho_1$ is the first derivative of $\rho_o(r)$ at the center. As $v \to 0$ we have $\rho \to +\infty$, $\rho + p_r \to +\infty$, and $\rho + p_\theta \to +\infty$. Thus $\rho \geq 0, \rho + p_r \geq 0$ and $\rho + p_\theta \geq 0$, and the energy conditions are clearly satisfied in a certain neighborhood of the regular center $r = 0$ throughout the evolution as seen from the above. It is seen that at the shell focusing singularity $v = 0$ the density becomes infinite where the curvature singularity occurs. In the case when the densities and pressures are initially decreasing functions of $r$, then throughout the cloud they have maximum values at the center, and are decreasing within the cloud.

In fact, a sufficient condition that the energy conditions be satisfied for a matter cloud of a finite size $r_c \geq r \geq 0$ at all times is that $(f(r) - f(R))' \geq 0$ within the coordinate range
$r_c \geq r \geq 0$. Note that the inequality is satisfied both initially for $v = 1$ (i.e. at $t = 0$), in near central region throughout the evolution, and in the neighborhood of the entire singularity curve $v = 0$. In fact, as $v \to 0$ we have $\rho \to +\infty, \rho + p_r \to +\infty, \rho + p_\theta \to +\infty$.

Further, it follows from equation (30) that for any given value of the radial coordinate $r$,

$$f(r) = \int_0^r r^2(\rho_o + p_r)dr \geq f(R(r,t)) = \int_0^R r^2(\rho_o + p_r)dr$$

Hence the mass function $F \geq 0$ for all $r$ at all times throughout the cloud. Since $f' = r^2(\rho_o + p_r) > 0$ and is an increasing function of $r$ for $r_c \geq r \geq 0$ where $f''(r_c) = 0$, it follows that $f'(r) \geq f(R(R)R'$, and energy conditions are satisfied within $r_c \geq r \geq 0$. Thus, for a given initial matter distribution as above, there is a finite cloud where the energy conditions are satisfied throughout the evolution of the initial data. Next, for a cloud of arbitrary size and for situations with completely arbitrary initial density and pressures which are not necessarily decreasing (or even including negative pressures, or pressures and density increasing with increasing $r$ etc.), note that the negative contribution in the expressions for $\rho$ and $\rho + p_\theta$ is finite throughout the evolution while the positive terms are maximum at the center and become unbounded at the singularity. The behavior of the term $R'f_{,R} = r^2v^2R'(\rho(R) + p_r(R))$ (specificly $R' \propto b_o^{-1}(r)$ for $v > 0$) depends explicitly on the velocity function $b_o(r)$ which is a free function, while $f'(r) = r^2(\rho_o(r) + p_r(r))$ is independent of the same. Therefore a suitable choice of $b_o(r)$ such that

$$\frac{\rho_r(r) + p_r(r)}{\rho_o(R) + p_r(R)} \geq v^2(v + rv')$$

within the range $r_b \geq r \geq 0$ would result in the energy conditions being satisfied for such an arbitrary size cloud for a given initial data.

Clearly, depending upon the conditions and values on the boundary of the cloud for the density and pressures, the exterior matching for the general class of models given here could be Schwarzschild, or one of the other suitable spacetimes, such as for example a Vaidya model spacetime. We shall not enter into these details of the matching presently,
as our main concern here is the local nakedness or otherwise of the central singularity at $R = 0$.

4. APPARENT HORIZON AND THE NATURE OF SINGULARITY

The apparent horizon within the collapsing cloud is given by $R/F = 1$, which gives the boundary of the trapped surface region in the spacetime. It is the behavior of the apparent horizon curve (which meets the central singularity at $R = r = 0$) near the center which essentially determines the visibility, or otherwise, of the central singularity. For example, it is known within the context of the Tolman-Bondi-Lemaitre models that the apparent horizon can be either past pointing timelike or null, or it can be spacelike, as can be seen by examining the nature of the induced metric on this surface. This is unlike the event horizon curve which is always future pointing null. If the neighborhood of the center gets trapped earlier than the singularity, then it is covered, and if that is not the case the singularity can be naked [9], with families of nonspacelike trajectories escaping from it.

In order to consider the possibility of existence of such families, and to examine the nature of the central singularity occurring at $R = 0, r = 0$ in the general class of models considered in the previous section, let us consider the equation of the outgoing radial null geodesics which is given by,

$$\frac{dt}{dr} = e^{\psi - \nu}$$  \hspace{1cm} (42)

The singularity appears at the point $v(t_s, r) = 0$ which corresponds to $R(t_s, r) = 0$, therefore if there are future directed outgoing radial null geodesics, terminating in the past at the singularity, then along these trajectories we have $R \to 0$ as $t \to t_s$.

Writing the equation for these radial null geodesics in terms of the variables $(u = r^{5/3}, R)$ we get,

$$\frac{dR}{du} = \frac{3}{5} \left( \frac{R}{u} + \sqrt{v v'} \frac{1}{u} \right) \frac{1 - F}{R} \sqrt{G(\sqrt{G} + \sqrt{H})}$$ \hspace{1cm} (43)

If the null geodesics terminate in the past at the singularity with a definite tangent, then at the singularity the tangent to the geodesics $dR/du > 0$ in the $(u, R)$ plane, and must have
a finite value. In the case of collapsing ball of matter we are considering, all singularities at
\( r > 0 \) are covered since \( F/R \rightarrow \infty \), and therefore \( dR/du \rightarrow -\infty \), and only the singularity
at center \( r = 0 \) could be naked. As mentioned earlier, for the case when \( R' > 0 \) near the
central singularity, we have

\[
x_o = \lim_{t \to t_0, r \to 0} \frac{R}{u} = \lim_{t \to t_0, r \to 0} \frac{dR}{du} \Rightarrow x_o^{3/2} = \frac{3}{2} X(0)
\]  

(44)

where \( X(0) \) is given by equation (37). Since \( X(0) > 0 \) the singularity is at least locally
naked. The behavior of outgoing radial null geodesics in the neighborhood of the singularity
are described by \( R = x_o u \) in \((R, u)\) plane and in \((t, r)\) plane it is given by

\[
t - t_s(0) = x_o r^{5/3}
\]  

(45)

The global visibility of such a singularity will depend on the overall behavior of the
various functions concerned within the matter cloud and we shall not go into those details
presently. It has been seen, however, from the study of various examples so far, that once
the singularity is locally naked, one can always make it globally visible by a suitable allowed
choice of functions. Note that in cases where the choice of \( b_o(r) \) is such that \( X(0) < 0 \) the
singularity would be covered. When \( X(0) = 0 \), one has to consider the next higher order
expansion term which is nonvanishing in equation (37), and that will then determine the
nature of the singularity by means of essentially a similar analysis.

One can also write the equation for these radial null geodesics in terms of the variables
\((t, R)\) to see how the area radius \( R \) grows along these outgoing null geodesics with increasing
values of time. As mentioned earlier, for the case where \( R' > 0 \) near the central singularity,
we get

\[
X_o = \lim_{t \to t_0, r \to 0} \frac{R}{t - t_s(0)} = \lim_{t \to t_0, r \to 0} \frac{dR}{dt} \Rightarrow \lim_{t \to t_0, r \to 0} \left[ e^{\nu} \frac{1 - \frac{F}{\sqrt{G + \sqrt{H}}}}{\sqrt{G + \sqrt{H}}} \right] = 1
\]  

(46)

Again this shows that the singularity is naked at least locally. In fact, as pointed out
above, it follows that for \( b_o'(0) \neq 0 \) the area coordinate behaves as \( R = \text{const.} r^{5/3} \) near the
singularity.
5. COLLAPSING SHELLS

We next consider here collapsing thick shells of matter, subject to the weak energy condition, but otherwise with a general form of matter, as this reveals some interesting features regarding the structure of the singularity. We rescale the coordinate $r$, contrary to the earlier case of collapsing matter clouds, such that at some initial $t = t_i$ the density distribution of shells is expressed as

$$\rho_o(r) = \rho_c (1 - \rho_1 r^\alpha)$$

(47)

where $\rho_c > 0$, $\rho_1 > 0$ and $\alpha \geq 1$ are constants. It is seen that in the scaling given as above, the area radius $R$ again increases with the increasing coordinate $r$.

We choose functions $F$ and $G$ such that

$$F = f(t)R, \quad G = a(r)e^{\Phi(R)}$$

(48)

where $f(t)$ and $a(r)$ are arbitrary $C^2$ functions of $t$ and $r$ only. From field equations (7) to (11) we get the matter distribution in spacetime being characterized by $\rho$, $p_r$ and $p_\theta$

$$\rho = \frac{f(t)}{R^2}$$

(49)

$$p_r = -\frac{\dot{f}}{RR} - \frac{f(t)}{R^2}$$

(50)

$$p_\theta = -\frac{\dot{f}}{2RR} \left(1 - \frac{R}{2} \Theta_{,R}\right)$$

(51)

$$e^\Theta = ae^\Phi + 1 - f$$

(52)

$$\dot{R} = -b(t)e^{\frac{\Phi}{2}} \sqrt{ae^\Phi + 1 - f}$$

(53)

where $H = e^\Theta$ and $\Theta(R, t)$ is treated as function of $R$ and $t$ and $(,R)$ represents partial derivative with respect to $R$ with $t$ kept constant. Also, $b(t) > 0$ is an arbitrary function of $t$. If the spacetime characterized by the solutions of the field equations given by these
equations does indeed evolve from the initial matter data \( \rho_o(r), p_{r_o}(r) \) and \( p_{\theta_o}(r) \), one must have at the initial epoch \( t = t_i \)

\[
\rho(t_i, r) = \frac{f_o}{R_o^2}, \quad f_o = f(t_i), \tag{54}
\]

\[
p_r(t_i, r) = p_{r_o} = \frac{f_1}{R_o v_o} - \frac{f_o}{R_o^2}, \quad f_1 = [\dot{f}]_{t=t_i}, v_o(r) = -v(t_i, r), \tag{55}
\]

\[
p_\theta(t_i, r) = \frac{f_1}{2R_o v_o} (1 - R_o \mu, R_o), \quad 2\mu(R_o) = \Theta(R_o, t_i) - \log c_o \tag{56}
\]

where \( c_o \) is a constant. On further simplification we get

\[
R_o^2(r) = \frac{f_o}{\rho_c - \rho_1 r^\alpha} \tag{57}
\]

\[
e^{\Phi_o} = \frac{f_1^2}{b_o^2 R_o^2 \epsilon^2(R_o)} c_o e^{-\mu(R_o)}, \quad \Phi(R_o) = \Phi_o \tag{58}
\]

\[
a(r) = (c_o e^{2\mu(R_o)} - 1 + f_o) e^{-\Phi_o} \tag{59}
\]

where \( \epsilon(R_o) = \rho_o + p_{r_o} \) and \( b(t_i) = b_o, c_o, f_o, f_1 \) are constants. Free constant \( c_o > 0 \) is such that \( a(r) > 0 \) for all allowed \( r \). The functional form of \( \Phi(R) \) is given by

\[
e^{\Phi(R)} = \frac{f_1^2}{b_o^2 R^2 \epsilon^2(R)} c_o e^{-2\mu(R)}, \tag{60}
\]

Note that \( \Phi(R) \) is a \( C^2 \) function of \( R \) for all \( R > 0 \) and \( a(r) \) is also a \( C^2 \) function of \( r \) within the cloud. These conditions further imply that \( e^\Theta \) is also a \( C^2 \) function of \( R \) and \( t \) for \( R > 0 \). Differentiating (53) with respect to \( r \) we get

\[
R' = \frac{a'(r) e^\Phi}{(e^\Theta)'(R) - (e^\Phi)'(R)} \tag{61}
\]

At \( t = t_i, R_o' > 0 \), and so from the above equation it follows that \( R' \) could vanish only when either \( e^\Phi = 0 \) or at points where \( e^\Theta \to \infty \). Since both \( e^\Phi > 0 \) and \( e^\Theta > 0 \) are \( C^2 \) functions for \( R > 0 \), it follows that \( R' \) could vanish only at \( R = 0 \). Thus for \( t < t_s \), we have \( R' > 0 \). The inner shell of the matter is labeled by \( r = 0 \) and the outermost shell by \( r = r_b \). At \( t = t_i \) the density and pressures are finite throughout the cloud and the Kretchmann scalar
is bounded at $t = t_i$. The singularity appears first when the innermost shell collapses to zero volume in the sense that $R(t_s, 0) = 0$. The time of collapse of further successive shells to singularity is given by $R(t_s(r), r) = 0$. The weak energy condition is satisfied if $f(t) > 0$ and $\dot{f}(t) > 0$. The interesting feature of such an example is that at the singularity $R = 0$ the mass function $F = 0$, and as such for $f(t) < 1$ we have $R > F$ and the entire singularity curve is not covered by the apparent horizon. Thus we have a naked singularity where not just the central point but the entire singularity curve is visible. On the other hand, for $f(t) > 1$ the singularity is covered and we have a black hole. Of course, in either case, the mass function $F$ vanishes on the entire singularity curve and so this is a “massless” singularity in that sense, but what is important is this example illustrates the possibilities inherent within the Einstein field equations towards the determination of the final fate of gravitational collapse of a massive matter cloud.

6. CAUCHY PROBLEM AND THE INITIAL DATA

In previous sections different classes of evolutions for the initial data leading to either a black hole or a naked singularity were discussed. We shall make now some general comments on all possible dynamical evolutions of a given regular initial data in the context of the occurrence of the singularities, naked or otherwise, for general matter fields. This is related in a way to the Cauchy problem in the context of general relativity. The difference, however, is that while most discussions on Cauchy problem in relativity avoid singularities, our purpose is precisely to examine the possible evolutions of an initial data into a spacetime singularity, in terms of its visibility or otherwise.

We pointed out in [10] that a naked singularity would occur generically in spherically symmetric collapse situations. The point of emphasis here would be to examine this issue in the context of a given arbitrary initial data. We examine if the in built freedom of choice in selection of an equation of state in general relativity is sufficient enough to allow generic evolutions of an arbitrary initial data into a naked singularity or a black hole subject to the energy conditions being satisfied throughout the evolution. The basic issue is, firstly,
given an arbitrary physically reasonable initial data describing the state of the matter at the onset of gravitational collapse, does general relativity generically allow the evolution of this data into a singularity, subject to the energy conditions being satisfied throughout, and if so, do all these evolutions have to necessarily end either as black hole or a naked singularity, or is there still some degree of freedom available in the field equations which would allow the formation of either as desired.

To address this issue along the lines of [10], the field equations given in equations (7) to (11) can be put in the following form if one uses $R$ and $t$ as variables instead of $t$ and $r$,

$$\rho = \frac{F_{,R}}{k_0 R^2}, \quad p_r = -\frac{F_{,R}}{k_0 R^2} - \frac{F_{,t}}{R^2 T}$$  \hspace{1cm} (62)

$$e^{2\nu} = \left( \frac{Q_{,R}}{F_{,RR} + 2p_\theta} \right)^2$$  \hspace{1cm} (63)

$$\dot{R} = -\frac{Q_{,t}}{F_{,t} Q (F_{,RR} + 2p_\theta R)}$$  \hspace{1cm} (64)

$$\left( \frac{Q^2 (1 - \frac{F}{R}) + F_{,t}}{F_{,t} (F_{,RR} + 2p_\theta R)} \right) Q_{,RR} - Q_{,t} = f_1(R, t, Q, Q, R)$$  \hspace{1cm} (65)

$$H = G - 1 + \frac{F}{R}$$  \hspace{1cm} (66)

where $,R$ and $,t$ represent partial derivatives with respect to $R$ and $t$ keeping $t$ and $R$ as constants respectively, and we have put

$$Q = \frac{F_{,t}}{\sqrt{G - 1 + \frac{F}{R}}}, \quad p = F_{,RR} + 2p_\theta R$$  \hspace{1cm} (67)

$$f_1(R, t, Q, Q, R) = \frac{Q_{,R}}{Qp} \left( Q_{,R} + Q^2 \left( \frac{G[p^2 F]_{,R}}{2p^2 F^2} + \left[ \frac{F}{R} \right]_{,R} + \frac{\dot{F}_{,R}}{2F} \right) + \frac{Q^3}{2R} - \frac{\dot{F} Q}{2F} \right)$$  \hspace{1cm} (68)

The mass function $F$ and the tangential pressure $p_\theta$ are free functions of $R$ and $t$, and give the distribution of matter throughout the spacetime. As usual $R$ at the initial $t = 0$ is scaled as $R(0, r) = r$ using the scaling freedom available. Knowing $F(R, t)$ and $p_\theta(R, t)$ one integrates the second order parabolic partial differential equation (65) with appropriate initial and boundary data to obtain $Q(R, t)$. Then $e^{2\nu}$ is immediate
from equation (63). Equation (64) is then integrated for $R(t,r)$ with the initial condition $R(0,r) = r$. Equations (66) is then solved for the remaining unknown $\psi$. The initial data, as pointed out in section 2, consists of four independent functions of coordinate $r$ at an initial spacelike slice $t = 0$ when the collapse commences. These are basically the density distribution $\rho_o(r)$, radial pressure $p_{ro}(r)$, and tangential pressure $p_{\theta o}(r)$, which describe the state of matter at the onset of collapse, and the velocity distribution function $V_o = V_o(r) = -\dot{R}(0,r)$, which describes the radial velocities of the spherical shells towards the center. Thus, for an arbitrary initial data at $t = 0$ we have

$$F(R,0) = F(r,0) = F_o(r) = \int_0^r r^2 \rho_o dr, \quad F_{,t}(R,0) = F_1(R) = F_1(r) = r^2 p_{ro} V_o$$ (69)

$$p_{\theta}(R,0) = p_{\theta}(r,0) = p_{\theta o}(r)$$ (70)

The functions $F$ and $p_{\theta}$ are free functions to be chosen, subject to their initial values at $t = 0$. The functional behavior of $F$ and $p_{\theta}$ for $t > 0$ is completely free except that they should at least be $C^2$, and such that energy conditions are satisfied throughout the collapse, i.e. $\rho \geq 0, \rho + p_{,t} \geq 0$. This will be the case when

$$F_{,R} \geq 0, \quad F_{,t} \geq 0, \quad p_{\theta} \geq -\frac{F_{,R}}{k_o R^2}$$ (71)

Thus energy conditions are satisfied for all the evolutions where the mass function is a monotone increasing function of both $R$ and $t$, with non-negative tangential pressures.

We therefore consider only those sets of arbitrary functions $F(R,t)$ and $p_{\theta}(R,t)$ for which $F \geq 0, F_{,R} > 0, F_{,t} > 0, p_{\theta} > -\rho$, and $p = F_{,RR} + 2p_{\theta} R > 0$ everywhere except perhaps at the singularity $R = 0$ where it could vanish. Note that in case $p = 0$ the parabolic partial differential equation (65) is replaced by an second order ordinary differential equation for $T = \dot{R}$ with $Q = c(t) F_{,t}$, and all our conclusions here would again apply in this case. We would further consider only those sets of functions for which $f_1$ in equation (68) is at least $C^0$ function of its arguments. From equation (14) which gives the expression for the Kretschmann scalar, it is clear that at the initial time $t = 0$ it is finite since the initial
data in the form of density, pressures, and velocity functions are all at least $C^2$ functions, and therefore at $t = 0$ we have $F(R, 0) = R^3 h(R)$, where $h(R)$ is at least $C^1$ function of $R$. Therefore, for all sets $F(R, t)$ a shell-focusing singularity $R(t_s, r) = 0$ would develop at a later time $t = t_s(r)$ if $F \simeq R^n, n < 3$. Similar situations would occur when we have $R'(t, r) = 0$, and a shell-crossing singularity would occur. Thus for a given set of arbitrary initial data of matter and velocity functions at the initial spacelike hypersurface, there are infinite many evolutions satisfying the energy conditions, characterized by the free functions $F$ and $p_{\theta}$, resulting into a singularity. The question that needs to be answered is for a particular evolution determined by a particular choice of functions $F$ and $p_{\theta}$, would there be a naked singularity or a black hole.

Note that the second order partial differential equation above for $Q$ is a quasi-linear parabolic equation in variables $R$ and $t$. The coefficient of $Q_{,RR}$ is positive and at least $C^0$ (depending upon the choice of $F$ and $p_{\theta}$), while that of $Q_{,t}$ is $-1$. Furthermore, the driving term $f_1(Q, R, t, Q, R)$ is also at least $C^0$ function of its argument. The initial value of $Q(R, 0) = Q_o(R)$ at $t = 0$ is fixed by the initial data as mentioned earlier. Thus we have a well posed Cauchy problem with the boundary condition at some $t = t_s$ still free to choose. The quasi-linear parabolic partial differential equations, such as (74), have been studied quite extensively for existence of solutions for an initial condition at $t = 0, Q(R, 0) = Q_o(R)$, and the boundary condition at some $t = t_s$, and the solutions do exist with quite general form of boundary conditions $Q(R, t) = g(R, t)$. Therefore, for all sets of functions $(F, p_{\theta})$ for which the coefficient of $Q_{,RR}$ in equation (65) is positive, and together with $f_1$ is at least $C^0$ function of its argument, the solution with quite general boundary conditions would exist which would not only evolve from a given initial data but also satisfy the energy conditions throughout evolution. This is the case as shown in examples considered in earlier sections. The point is that with the choice of a particular initial data set, namely $(\rho_o, p_{ro}, p_{\theta o}, V_o)$, and a particular set of evolutions given by $(F(R, t), p_{\theta}(R, t)$ we still have the choice of the boundary condition at $t = t_s$ at our
disposal. The freedom left in the form of the choice of $Q(R, t_s)$, once chosen, completes the solution of the field equation and it is this choice which effects the nature of the singularity occurring at $t = t_s, R = 0$ in terms of being naked or otherwise. This is precisely what happens in the classes considered in earlier sections where an evolution from an arbitrary initial matter data can form either a black hole or a naked singularity.

To clarify this, let us briefly consider the radial null geodesics equation, which is,

$$\frac{dt}{dr} = e^{\psi - \nu} \Rightarrow \frac{dR}{dt} = \frac{QQ, R (1 - \frac{E}{R})}{p(\sqrt{F} + \sqrt{\hat{F} + Q^2 (1 - \frac{E}{R})})} \equiv U(X, t) \quad (72)$$

where $X = R/t - t_s$. Since at the initial $t = 0$, $R = r$ and $R' = 1$, and $\dot{R} < 0$ throughout the spacetime, the singularity would occur at a later time $t = t_s > 0$. From equation (74) it follows that for a choice of functions $(F, p_0)$ such that $\dot{F} > 0, p > 0$ throughout the spacetime, $Q$ has at least two continuous $R$-derivatives and that $p$ has at least one continuous $R$-derivative, then $R' > 0$ and is finite throughout the spacetime if it starts that way, except possibly at the singularity $R = 0$. Thus in such cases shell-crossings would not occur, and the first shell-focusing singularity would occur at the center at $r = 0, t = t_s$. Note that in case $F(0, t_s) > 0$ the singularity would be covered. The singularity would be naked and there would be outgoing radial null geodesics if the equation $V(X) = U(X, t_s) - X = 0$, where

$$X_o = \lim_{(R \to 0, t \to t_s)} \left( \frac{R}{t - t_s} \right) = \lim_{(R \to 0, t \to t_s)} \left( \frac{dR}{dt} \right) = U(X_o, t_s) \quad (73)$$

has a real positive root $X = X_o$. In the case when there are no real positive roots for $V(X) = 0$, then the singularity is covered. The value of the function $Q(R, t_s)$ which appears in the root equation above thus becomes important. For all functions $Q(R, t_s)$ chosen such that the equation has a root gives rise to the naked singularity, the choice otherwise leads to a black hole. $Q(R, t_s)$ is precisely the boundary condition in equation (65) which one has a freedom to choose for obtaining a particular solution. Therefore, as illustrated by the two examples in the preceding sections, there would be a non-zero
measure set of evolutions from an arbitrary initial data specified at the onset of the collapse, which could always develop into a singularity, naked or covered.

7. CONCLUDING REMARKS

We considered here various classes of collapsing evolutions from arbitrary initial data and it is shown that these can be chosen so as to form either a black hole or a naked singularity. Such an initial data, prescribed in terms of the initial density and pressure profiles of the cloud, and the initial velocities of the matter shells, can evolve into either of these outcomes depending on the evolution chosen. In other words, there are permissible classes of evolutions, subject to the energy conditions and other conditions ensuring the physical reasonability, which give rise to either a black hole as the final product of collapse, or a naked singularity.

It appears that the scenario presented above is repeated in nearly all known exact physically reasonable solutions of the field equations describing spherically symmetric gravitational collapse. In the known classes, such as radiation collapse [5], dust [11], perfect fluids and more general forms of matter [7], and massless scalar field collapse [2], singularities both naked or covered do occur. Therefore, from the point of view of cosmic censorship, the important question currently is not the occurrence of naked singularities but whether these occurrences are permitted by the general relativity on a generic enough basis, and whether they are stable according to a suitably defined stability criteria. The issue of genericity is significant from the point of view that these phenomena could be impossibly rare in the space of solutions of the general theory of relativity. The known literature does not indicate this clearly, because for all known important classes of solutions in the spherically symmetric gravitational collapse both the naked singularities and black holes occur for wide ranging situations.

An important indicator in this direction, as pointed out earlier, is the imploding Vaidya model, where the singularity is naked for $\lambda \leq 1/8$ and a black hole for $\lambda > 1/8$, where the parameter $\lambda$ represents the initial data in the form of the rate of mass loss.
Thus, the singularities, both naked and covered, are stable against the perturbation of the parameter \( \lambda \), and the point \( \lambda = 1/8 \) is the critical point indicating the transition from one phase to the other. A similar situation also occurs in the case of dust collapse where the perturbation in the initial density or velocity distribution within a certain domain does not alter the nature of the singularity. The analysis here in general has a significance in that the nature of the singularity is seen to be stable in a certain sense against the perturbation of the initial data.

In this sense, we have characterized here wide new families of black hole solutions forming in spherically symmetric gravitational collapse, without using the assumption of the cosmic censorship hypothesis, but in terms of the possible evolutions of the initial data for the collapsing object. What we still do not know is the actual measure of each of these classes in the space of all possible evolutions which are allowed from a given general and arbitrary but physically reasonable initial data set. This is a problem related closely to the issue of stability of naked singularities. As is well-known, the stability in general relativity is a complicated issue because there is no well-defined formulation or specific criteria to test for stability. Fast evolving numerical codes for core collapse models may possibly provide further insights into this aspect. All the same, these classes appear to be generically arising in the collapse models considered here, at least within spherical symmetry, in that they are not an isolated phenomena but belong to a general family. Because, given any density and pressure profiles for the cloud, there exists an evolution which will lead to either a black hole or a naked singularity as desired as the end product of collapse. In this sense, both black holes and naked singularities do seem to arise generically as the end product for spherically symmetric gravitational collapse. Given the complexity of the field equations, if a phenomena occurs so widely in spherical symmetry, it is not unlikely that the same would be repeated in more general situations.

It thus appears from the above considerations that the occurrence of singularities, naked or otherwise, is inherent in the theory of general relativity, and a distinction be-
tween these cases may not be possible through general relativity alone. What is essential is to examine how the perturbations and departures from sphericity would alter these conclusions. Some efforts have been made to examine this issue [12]. It is also possible that the quantum effects near the naked singularities may help to preserve some kind of a quantum cosmic censorship, or these quatum effects could give rise to interesting signatures for naked singularities [13].
References


