Maximal Acceleration Corrections to the Lamb Shift of Hydrogen, Deuterium and He$^+$

G. Lambiase$^{a,}$, G. Papini$^{b,}$ and G. Scarpetta$^{a,}$

$^a$Dipartimento di Scienze Fisiche “E.R. Caianiello”
Università di Salerno, 84081 Baronissi (SA), Italy

$^b$Istituto Nazionale di Fisica Nucleare, Sezione di Napoli

$^c$Department of Physics, University of Regina,
Regina, Sask. S4S 0A2, Canada

$^d$International Institute for Advanced Scientific Studies
Vietri sul Mare (SA), Italy

Abstract

The maximal acceleration corrections to the Lamb shift of one-electron atoms are calculated in a non-relativistic approximation. They are compatible with experimental results, are in particularly good agreement with the $2S - 2P$ Lamb shift in hydrogen and reduce by $\sim$ 50% the experiment-theory discrepancy for the $2S - 2P$ shift in He$^+$.

PACS: 04.90.+e, 12.20.Ds.
Keywords: Maximal acceleration, Lamb shift

$^a$E-mail: lambiase@vxsa.csied.unisa.it
$^b$E-mail: papini@cas.uregina.ca
This paper presents the calculation of maximal acceleration (MA) corrections to the Lamb shift of one-electron atoms and ions, according to the model of Caianiello and collaborators [1], [2]. The view frequently held [3], [4] that the proper acceleration of a particle is limited upwardly finds in this model a geometrical interpretation epitomized by the line element

\[ ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = ds^2 \left( 1 - \frac{\dot{x}^2}{A^2_m} \right) = \sigma^2(x) ds^2 \]  

(1)

experienced by the accelerating particle along its worldline. In (1) \( A_m = 2mc^3/\hbar \) is the proper MA of the particle of mass \( m \), \( \ddot{x}^\mu \) its acceleration and \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) is the metric due to a background gravitational field. In the absence of gravity, \( g_{\mu\nu} \) is replaced by the Minkowski metric tensor \( \eta_{\mu\nu} \). Similar results have also been obtained in the context of Weyl space [5] and of a geometrical analogue of Vigier’s stochastic theory [6].

Eq. (1) has several implications for relativistic kinematics [7], the energy spectrum of a uniformly accelerated particle [8], the periodic structure as a function of momentum in neutrino oscillations [8], the Schwarzschild horizon [9], the expansion of the very early universe [10] and the mass of the Higgs boson [11]. It also makes the metric observer-dependent, as conjectured by Gibbons and Hawking [12], and leads in a natural way to hadron confinement [13].

The extreme large value that \( A_m \) takes for all known particles (\( A_m \approx 0.9 \times 10^{12} m \) m s\(^{-2} \) MeV\(^{-1} \)) makes a direct test of Eq. (1) very difficult. Nonetheless a realistic test that makes use of photons in cavities has been recently suggested [14] and attempts in this direction will hopefully lead to conclusive results.

Recent advances in high resolution spectroscopy are now allowing Lamb shift measurements of unprecedented precision, leading in the case of simple atoms and ions to the most stringent tests of quantum electrodynamics (QED). MA corrections due to the metric (1) appear directly in the Dirac equation for the electron that must now be written in covariant form [15] and referred to a local Minkowski frame by means of the vierbein field \( e^a_\mu(x) \). From (1) one finds \( e^a_\mu(x) = \sigma(x) \delta^a_\mu \), where Latin indices refer to the locally inertial frame and Greek indices to a generic non-inertial frame. The covariant matrices \( \gamma^\mu(x) \) satisfy the anticommutation relations \( \{ \gamma^\mu(x), \gamma^\nu(x) \} = 2g^{\mu\nu}(x) \), while the covariant derivative \( D_\mu \equiv \partial_\mu + \omega_\mu \) contains the total
connection \( \omega_\mu = \frac{1}{2} \sigma^{ab} \omega_{\mu ab} \), where \( \sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b] \), \( \omega_{\mu b} = (\Gamma^\lambda_{\mu \nu} \gamma^a - \partial_\mu \gamma^a) \epsilon^\nu_b \) and \( \Gamma^\lambda_{\mu \nu} \) represent the usual Christoffel symbols. For conformally flat metrics \( \omega_\mu \) takes the form \( \omega_\mu = \frac{3}{2} \sigma^{ab} \eta_{\mu \nu} \sigma_{b} \). By using the transformations \( \gamma^\mu(x) = e^\mu_a(x) \gamma^a \) so that \( \gamma^\mu(x) = \sigma^{-1}(x) \gamma^\mu \), where \( \gamma^\mu \) are the usual constant Dirac matrices, the Dirac equation can be written in the form

\[
\left[ i \hbar \gamma^\mu \left( \partial_\mu + i \frac{e}{\hbar c} A_\mu \right) + i \frac{3 \hbar}{2} \gamma^\mu (\ln \sigma)_{,\mu} - m \sigma(x) \right] \psi(x) = 0. \tag{2}
\]

From (2) one obtains the Hamiltonian

\[
H = -i \hbar c \vec{\alpha} \cdot \vec{\nabla} + e \gamma^0 \gamma^\mu A_\mu(x) - i \frac{3 \hbar}{2} \gamma^0 \gamma^\mu (\ln \sigma)_{,\mu} + m c^2 \sigma(x) \gamma^0, \tag{3}
\]

which is in general non-Hermitian [15]. However when one splits the Dirac spinor into large and small components, the only non-Hermitian term is \( (\ln \sigma)_{,0} \). If \( \sigma \) varies slowly in time, or is time-independent, as in the present case, this term can be neglected and Hermiticity is recovered.

A recent attempt to estimate the Lamb shift corrections due to (1) was carried out in the local frame of the electron and did not therefore take into account properly the electromagnetic field experienced by the electron [16]. Hamiltonian (3) corrects this inadequacy. The calculations are also extended to include the Lamb shift in deuterium and \( \text{He}^+ \). Here, as in the previous MA calculations [16], the nucleus is considered to be pointlike and its recoil is neglected.

In QED the Lamb shift corrections are usually calculated by means of a non-relativistic approximation [17]. This is also done here. For the electric field \( E(r) = k Z e / r^2 (k = 1 / 4 \pi c \epsilon_0) \), the conformal factor becomes \( \sigma(r) = (1 - \left( \frac{r_0}{r} \right)^{1/2}) \), where \( r_0 \equiv (k Z e^2 / m A_m)^{1/2} \sim \sqrt{Z} \cdot 10^{-14} \text{m} \) and \( r > r_0 \). The calculation of \( \tilde{\gamma}^\mu \) is performed classically in a non-relativistic approximation. This is justified because for the electron \( v / c \) is at most \( \sim 10^{-3} \). Neglecting contributions of the order \( O(A_m^{-1}) \), \( \sigma(r) \sim 1 - (1/2)(r_0 / r)^4 \). This expansion requires that in the following only those values of \( r \) be chosen that are above a cut-off \( \Lambda \), such that for \( r > \Lambda > r_0 \) the validity of the expansion is preserved. The actual value of \( \Lambda \) will be selected later. The length \( r_0 \) has no fundamental significance in QED and depends in general on the details of the acceleration mechanism. It is only the distance at which the electron would attain, classically, the acceleration \( A_m \) irrespective of the probability of getting there.
By using the expansion for $\sigma(r)$ in (3) one finds that all MA effects are contained in the perturbative terms

$$H_m = -\frac{mc^2}{2} \left( \frac{r_0}{r} \right)^4 \beta + i \frac{3\hbar c}{4} \frac{r_0^4 \hat{\sigma}}{r^4} \cdot \nabla \frac{1}{r} \equiv \mathcal{H} + \mathcal{H'}.$$  

(4)

By splitting $\psi(x)$ into large and small components $\varphi$ and $\chi$ and using $\chi = -i(h/2me)\hat{\sigma} \cdot \nabla \varphi \ll \varphi$ one obtains for the perturbation due to $\mathcal{H}$

$$\delta \mathcal{E}_{nlm} \simeq -\frac{mc^2}{2} r_0^4 \int d^3 r \frac{1}{r^4} \varphi_{nlm} \varphi_{nlm}. $$

(5)

The perturbation due to $\mathcal{H'}$ vanishes. In (5) $\varphi_{nlm}$ are the well known eigenfunctions for one-electron atoms. The integrations over the angular variables in (5) can be performed immediately and yield

$$
\begin{align*}
\delta \mathcal{E}_{20} &= -\frac{mc^2}{16} \left( \frac{r_0}{a_0} \right)^4 \left\{ \left[ 4 \left( \frac{a_0}{\Lambda} \right) + 1 \right] e^{-\Lambda/a_0} - 8 E_1 \left( \frac{\Lambda}{a_0} \right) \right\}, \\
\delta \mathcal{E}_{21} &= -\frac{mc^2}{48} \left( \frac{r_0}{a_0} \right)^4 e^{-\Lambda/a_0}, \\
\delta \mathcal{E}_{10} &= -2mc^2 \left( \frac{r_0}{a_0} \right)^4 \left\{ \left( \frac{a_0}{\Lambda} \right) e^{-2\Lambda/a_0} - 2 E_1 \left( \frac{2\Lambda}{a_0} \right) \right\},
\end{align*}
$$

(6)–(8)

where $E_1(x) = \int_1^\infty dy \ e^{-xy}/y$ and $a_0$ is the Bohr radius divided by $Z$. In order to calculate the $2S - 2P$ Lamb shift corrections it is now necessary to choose the value of the cut–off $\Lambda$. While in QED Lamb shift and fine structure effects are cut–off independent, the values of the corresponding MA corrections increase when $\Lambda$ decreases. This can be understood intuitively because the electron finds itself in regions of higher electric field at smaller values of $r$. $\Lambda$ is a characteristic length of the system. It must also represent a distance from the nucleus that can be reached by the electron whose acceleration and relative perturbations depend on the position attained. One may tentatively choose $\Lambda \sim a_0$. According to the wave functions involved, the probability that the electron be at this distance ranges between 0.1 and 0.5. Smaller values of $\Lambda$ lead to larger acceleration corrections, but are reached with much lower probabilities. This is the case of the Compton wavelength of the electron whose use as a cut–off is therefore ruled out in the present context. For $\Lambda \sim a_0$, Eqs. (6)–(8) give the corrections to the levels $2S, 2P$ and $1S$ ($Z = 1$)
\( \delta \mathcal{E}_{20} \sim -22.96 \text{kHz}, \delta \mathcal{E}_{21} \sim -33.42 \text{kHz}, \delta \mathcal{E}_{10} \sim -325.45 \text{kHz} \), yielding the Lamb shift correction \( \delta \mathcal{E}_L = \delta \mathcal{E}_{20} - \delta \mathcal{E}_{21} \sim +10.46 \text{kHz} \). The results are summarized below [18].

a) 2S – 2P Lamb shift for Hydrogen. The most recent experimental and theoretical values of the classic Lamb shift are reported in Table I and compared with the theory with MA corrections. These amount to \( \delta \mathcal{E}_L \) above. \( r_p \) is the rms charge radius of the proton [19], [20]. The MA corrections are in very good agreement with all experimental results reported and the value \( r_p = 0.862 \text{fm} \). They also appear to be consistently in the right direction. The coefficients of (6)–(8) are proportional to powers \( (Z \alpha)^6 \) from which it follows that the MA corrections are comparable in magnitude with those obtained from perturbative QED up to order \( \alpha^7 \). Further improvements in experimental sensitivity might indeed be able to distinguish between the MA and QED contributions. Unfortunately, higher experimental precision seems difficult to achieve because of the 100MHz natural linewidth of the 2P state.

b) 1S ground state Lamb shift \( L_{1S} \) in Hydrogen. Measurements of \( L_{1S} \) have recently become very precise by comparison of the 1S – 2S resonance with four times the frequencies of the 2S – 4S and 2S – 4D two-photon transitions. The MA corrections are given by \( \delta \mathcal{E}_{10} \) above. The results are compared in Table II. The first line repeats the results before 1992. Experiment and theory were known to agree (within 0.1MHz) for \( r_p = 0.805 \text{fm} \). The MA corrections also agree within 0.2MHz. More recently a discrepancy has appeared between experiment and theory with the adoption of the more reliable value \( r_p = 0.862 \text{fm} \) increasing the theoretical estimate to \( 8173.12(6) \text{MHz} \). The agreement is improved in this instance by the MA corrections for the choice of the new radius. More recent experimental and theoretical data are compared on the last three lines of Table II. The MA corrections would restore by themselves the agreement between experiment and QED without two loop corrections. However the agreement between experiment and QED improves significantly when the two-loop corrections calculated by Pachucki [25] are included in the theoretical estimate and the MA corrections are excluded. These latter effects shift the theoretical estimate by \( \sim 0.3 \text{MHz} \) below the experimental results. It is interesting to observe that the dominant MA correction, of order \( \alpha^6 \), is approximately of the same magnitude of the two-loop correction of order \( \alpha^7 \) which must therefore be considered as truly large. While the Paducki calculation restores the agreement between theory and experiment for hydrogen, it upsets that of the 2S – 2P splitting of \( He^+ \) [28].
The MA contributions (6)-(8) are particularly sensitive to the choice of \( \Lambda \). For instance, a 10% increase in the value of \( \Lambda \) shifts upward the MA correction from \(-325\text{kHz}\) to \(-230\text{kHz}\), and improves the agreement between experiment and MA theory considerably. This is largely due to the presence of a ground state wave function peak at \( r = a_0 \).

c) Lamb shift \( \frac{1}{4}L_{1S} - \frac{5}{4}L_{2S} + L_{4S} \) in Hydrogen and Deuterium. These are measurements of the \( L_{1S} \) Lamb shift by direct comparison of the \( 1S \rightarrow 2S \) with the \( 2S \rightarrow 4S \) two-photon transitions. The MA corrections are determined from (5) and the corresponding hydrogenic eigenfunctions and are \( L_{4S} = -2.54\text{kHz} \) and \( \frac{1}{4}L_{1S} - \frac{5}{4}L_{2S} + L_{4S} = -55\text{kHz} \). The results are compared in Table III, where \( r_{ch} \) is the rms charge radius of the nucleus. The agreement between experiment and theory is good for hydrogen and remains good with the introduction of MA. For deuterium the agreement is still reasonable because of uncertainties in the measurement of charge and matter radii. The introduction of MA lowers theoretical estimates by 55kHz, which is in the right direction. The MA estimate based on the earlier calculation [30] still falls within the experimental error of the most recent measurement.

d) \( L_{1S} \) for Deuterium. The MA correction is \( \delta E_{10} \) and the results are summarized in Table IV. The agreement of the MA theory with experiment is again better in the absence of two-loop corrections. When these are included, the theory falls short by approximately 270kHz.

e) Lamb shift \( 2S \rightarrow 2P \) for \( He^+ \). The MA corrections are here \(+0.527\text{MHz}\). The results are given in Table V. While the agreement between experiment [32] and theory was good (~10kHz) before the introduction of two-loop corrections, the latter have introduced a discrepancy of ~ 1.27MHz [28] to ~ 1.190MHz [33]. The method of measurement used to obtain the \( He^+ \) result [32] has been recently verified by a parallel high-precision measurement of the Lamb shift in \( H \) [33]. The discrepancy must be treated seriously and is unresolved. The MA contributions reduce significantly the disagreement with theory to ~ 0.74MHz and ~ 0.66MHz, respectively. If the \( He^+ \) experiments will confirm the predictions of QED, then the Lamb shift measurements in hydrogen will determine the proton radius to within a few percent [31].

In conclusion, the agreement between MA corrections and experiment is at present very good for the \( 2S \rightarrow 2P \) Lamb shift in hydrogen (~7kHz) and comparable with the agreement of experiments with standard QED with
and without two-loop corrections. The agreement is also good for the $\frac{1}{4}L_{1S} - \frac{5}{4}L_{2S} + L_{4S}$ in Lamb shift in hydrogen and comparable, in some instances, with that between experiment and QED ($\sim 30\text{kHz}$). The corresponding MA corrections for deuterium fare worse than the conventional theory, but no worse than the disagreement ($\sim 38\text{kHz}$) between the two QED estimates considered. For the $L_{1S}$ case in deuterium, the MA theory is worse ($\sim -270\text{kHz}$) than the standard one in reproducing the experimental data when two-loop corrections are included, but better than QED alone when these are excluded. The latter statement also applies to the $L_{1S}$ shift in hydrogen. Finally, the MA corrections improve the agreement between experiment and theory by $\sim 50\%$ for the $2S-2P$ shift in $He^+$. While the two-loop corrections have been independently confirmed by two groups [31], there seems room for improvement on the experimental side regarding the sizes of proton and deuterium and the nuclei. At the same time new experiments, now in planning stages [27], should resolve some of the discrepancies now existing between experiment and QED and ultimately provide stringent tests of the MA theory.

Research supported by MURST fund 40\% and 60\%, DPR 382/80, the Natural Sciences and Engineering Research Council of Canada and NATO Collaborative Research Grant No. 970150. G.P. gladly acknowledges the continued research support of Dr. K. Denford, Dean of Science and Dr. L. Symes V. President Research, University of Regina. G.L. wishes to thank Dr. K. Denford for his kind hospitality during a stay at the University of Regina and V.V. Nesterenko for useful discussions.


Table I. $2S - 2P$ Lamb shift for Hydrogen

<table>
<thead>
<tr>
<th>Experiment (kHz)</th>
<th>Theory (kHz)</th>
<th>$r_p$ (fm)</th>
<th>MA (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1057845(9)[21]</td>
<td>1057810(4)(4)a[22]</td>
<td>0.805(11)</td>
<td>1057820.46</td>
</tr>
<tr>
<td>1057851.4(19)[23]</td>
<td>1057829(4)(4)a[22]</td>
<td>0.862(12)</td>
<td>1057839.46</td>
</tr>
<tr>
<td>1057839(12)[24]</td>
<td>1057838(6)b[25]</td>
<td>0.862(12)</td>
<td>1057848.46</td>
</tr>
<tr>
<td>1057842(12)c[26]</td>
<td>1057839(4)[27]</td>
<td>0.862(12)</td>
<td>1057849.46</td>
</tr>
</tbody>
</table>

a: correction to order $\alpha^2(\alpha Z)^5m$
b: two loop corrections, $\alpha^2(\alpha Z)^5m$
c: result of Ref. [24] amended to take into account a new value of $\alpha$
Table II. $L_{1S}$ in Hydrogen

<table>
<thead>
<tr>
<th>Experiment (MHz)</th>
<th>Theory (MHz)</th>
<th>$r_p$ (fm)</th>
<th>MA (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8172.82(11)[29]</td>
<td>8172.94(9)[29]</td>
<td>0.805</td>
<td>8172.615</td>
</tr>
<tr>
<td>8172.86(6)[30]</td>
<td>8172.97[30]</td>
<td>0.805</td>
<td>8172.645</td>
</tr>
<tr>
<td></td>
<td>8172.654(40)[31]</td>
<td>0.805</td>
<td>8172.329</td>
</tr>
<tr>
<td></td>
<td>8173.12(6)[30]</td>
<td>0.862</td>
<td>8172.795</td>
</tr>
<tr>
<td>8172.874(60)[28]</td>
<td>8173.097(40)$^a$[28]</td>
<td>0.862</td>
<td>8172.772</td>
</tr>
<tr>
<td></td>
<td>8172.802(40)$^b$[25]</td>
<td>0.862</td>
<td>8172.477</td>
</tr>
<tr>
<td>8172.827(51)[31]</td>
<td>8172.802(30)$^b$[31]</td>
<td>0.862</td>
<td>8172.477</td>
</tr>
</tbody>
</table>

$^a$: without two-loop corrections
$^b$: with two-loop corrections
Table III. $\frac{1}{4}L_{1S} - \frac{5}{4}L_{2S} + L_{4S}$

<table>
<thead>
<tr>
<th>Experiment (MHz)</th>
<th>Theory (MHz)</th>
<th>$r_{ch}$ (fm)</th>
<th>MA (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>868.61(3)[29]</td>
<td>868.64(2)</td>
<td>0.805</td>
<td>868.585</td>
</tr>
<tr>
<td>868.66(2)</td>
<td></td>
<td>0.862</td>
<td>868.605</td>
</tr>
<tr>
<td>868.630(12)[25],[30]</td>
<td>868.623(5)</td>
<td>0.862</td>
<td>868.568</td>
</tr>
<tr>
<td>868.656</td>
<td></td>
<td>0.862</td>
<td>868.601</td>
</tr>
<tr>
<td>Deuterium</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table IV. $L_{12}$ for Deuterium.

<table>
<thead>
<tr>
<th>Experiment (MHz)</th>
<th>Theory (MHz)</th>
<th>$r_{ch}$ (fm)</th>
<th>MA (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8184.00(8)[30]</td>
<td>8184.13(6)[30]</td>
<td>2.115</td>
<td>8183.805</td>
</tr>
<tr>
<td>8183.807(78)[28]</td>
<td>8184.080(47)$^a$[28]</td>
<td>2.115</td>
<td>8183.755</td>
</tr>
<tr>
<td></td>
<td>8183.785(47)$^b$[28]</td>
<td></td>
<td>8183.460</td>
</tr>
</tbody>
</table>

a: without two-loop corrections  
b: with two-loop corrections
Table V. $2S - 2P$ Lamb shift for He$^+$

<table>
<thead>
<tr>
<th>Experiment (MHz)</th>
<th>Theory (MHz)</th>
<th>MA (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1404.252(16) [32]</td>
<td>1404.251(20)a [32]</td>
<td>1404.037</td>
</tr>
<tr>
<td>1404.125[28]</td>
<td>1404.1777</td>
<td>1404.1777</td>
</tr>
<tr>
<td>1404.333[33]</td>
<td>1404.857</td>
<td>1404.857</td>
</tr>
</tbody>
</table>

a: without two-loop corrections  
b: with two-loop corrections
References


[18] It has also been suggested that the MA be referred to Planck's mass [4]. Then \(A_\mathcal{P} = m_P c^3/\hbar \sim 5 \cdot 10^{31} m/s e^2\) and \(\Lambda \sim a_0\) gives \(\delta \xi_L \sim 7.8 \cdot 10^{-4} \text{Hz}\). A cut-off that yields seemingly reasonable values of \(\delta \xi_L\) may be derived from the requirement that in the expansion of \(\sigma(r)\) terms \((r_0/r)^4\) be at least \(\sim 10^{-2}\). This leads to \(A \sim 100^{1/4}r_0 \sim 4.8 \cdot 10^{-2}\text{m}\) and \(\delta \xi_L \sim 1.6\text{Hz}\), which must be construed as an upper limit. However, the probability that the electron be so close to the origin (indeed well inside the nucleus) is vanishingly small.

[19] L.N. Hand, D.J. Miller, R. Wilson, Rev. Mod. Phys. 35 (1963) 335


