Interaction Energy of ‘t Hooft-Polyakov Monopoles

B. Kleihaus *
Department of Mathematical Physics, National University of Ireland Maynooth

J. Kunz
Fachbereich Physik, Universität Oldenburg, D-26111 Oldenburg, Germany

and

D. H. Tchrakian
Department of Mathematical Physics, National University of Ireland Maynooth
and
School of Theoretical Physics – DIAS, 10 Burlington Road, Dublin 4, Ireland
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The dependence of the energies of axially symmetric monopoles of magnetic charges 2 and 3, on the Higgs self-interaction coupling constant, is studied numerically. Comparing the energy per unit topological charge of the charge-2 monopole with the energy of the spherically symmetric charge-1 monopole, we confirm that there is only a repulsive phase in the interaction energy between like monopoles.
It is well known that like monopoles [1,2] of the Georgi-Glashow (GG) model exhibit only a repulsive phase. In the Bogomol’nyi-Prasad-Sommerfield (BPS) limit [3,4], when the dimensionless coupling constant $\lambda$ parametrising the strength of the Higgs self-interaction potential vanishes, this interaction disappears. This is a consequence of the vanishing of all components of the stress tensor [5] in this limit. In the BPS limit the Higgs field becomes massless and mediates a long range attractive force which cancels the long range repulsive magnetic force of the $U(1)$ field, exactly. The force between like monopoles in the BPS limit was studied in detail by Manton [6] and Nahm [7], who showed that it decreases faster than any inverse power. When $\lambda > 0$ however, the Higgs field becomes massive and as a result decays exponentially. Consequently the long range magnetic field dominates at large distances, leading to the repulsion of like monopoles of the GG model. This was concluded by Goldberg et al. [8], using the time-rate of change of the stress tensor for the field configuration of two exponentially localized monopoles, situated apart at a distance much larger than the sizes of the monopole cores.

Inspite of the well known scenario described above, to our knowledge, there has not been any detailed numerical verification of it to date. It is the purpose of the present work to supply this.

A numerical study of the energy of the unit charge spherically symmetric ‘t Hooft-Polyakov monopole was carried out by Bogomol’nyi and Marinov long ago [9], where the dependence of the energy on $\lambda$ was studied in detail. No such study was carried out for axially symmetric monopoles of higher charges. The study of axially symmetric monopoles was confined to the BPS limit only [10,11]. In the present work we will analyse the $\lambda > 0$ case, and in particular for the $n = 2$ and $n = 3$ monopoles of the GG model.

Our procedure is analogous to the work of Jacobs and Rebbi [12], where they study the energy per unit charge of the topologically stable charge-$n$ vortices of the Abelian Higgs model (AHM) [5], and demonstrate that in these theories there are bound states of vortices or more generally, that there are both attractive and repulsive phases.

In both the AHM and the GG model at hand, the masses of the solitons, the vortex (resp.) the monopole, depend on the strength of the dimensionless Higgs self-interaction coupling constant $\lambda$. In both cases, topological stability is guaranteed if $\lambda > 0$. The difference between these two systems is that the values of $\lambda$ for which the topological inequalities are saturated, namely the critical values $\lambda_c$, are different. For the AHM $\lambda_c = 1$ while for the GG model $\lambda_c = 0$. It is clear that in the BPS limit ($\lambda = \lambda_c$) the masses per unit charge of charge-$n$ solitons in either of these two theories are equal to 1, for all $n$.

It follows that in the case of the AHM the energy per unit charge vs. $\lambda$ curves for the $n = 1$ and $n = 2$ vortices cross at least at one point, namely at $\lambda_c = 1$, and hence that there will be both attractive/repulsive phases according as to whether the mass per unit charge of the $n = 2$ vortex is lower/higher than that of the $n = 1$ vortex. That there is only one such intersection point can be speculated on the grounds that we expect such curves for solitons to be monotonic. This scenario was confirmed by the numerical analysis of Ref. [12].

In the case of the GG model, the $n = 1$ and $n = 2$ curves meet at $\lambda_c = 0$. Speculating again that that there is only one such intersection, we would conclude that there is only an attractive/repulsive phase according as to whether the mass per unit charge of the $n = 2$ monopole is lower/higher than that of the $n = 1$ monopole. Speculating further, on the basis of the results found in Refs. [6,7], we would expect that there is only a repulsive phase in the GG models. This expected result is precisely what we will verify by numerical analysis in the present work.

In addition to confirming this expected conclusion, we will find the detailed dependences of the monopole masses per unit charge of the axially symmetric charge-$n$ monopole on the Higgs coupling constant $\lambda$. The numerical analysis will be carried out for $n = 2$ and $n = 3$. In this sense, the present work is a charge-$n$ version of the work of Ref. [9].

The static Hamiltonian of the GG model is

$$
\mathcal{H} = \frac{1}{2} Tr(|F_{ij}|^2) + Tr(|D_i \phi|^2) + \frac{1}{16} \lambda (2 Tr(\phi^2) + \eta^2)^2,
$$

(1)

with the field strength tensor
\[ F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \]  

(2)
of the gauge field \( A_i \), and the covariant derivative

\[ D_i \phi = \partial_i \phi + [A_i, \phi] \]  

(3)
of the Higgs field \( \phi \), and \( \eta \) denotes the vacuum expectation value of the Higgs field. The mass of the vector fields is \( M_W = \eta \) (we have set the gauge coupling constant to one), and the Higgs coupling constant \( \lambda \) is related to the Higgs mass \( M_H \) by

\[ \lambda = \frac{2}{M_H^2} \]  

Employing spherical coordinates, we parametrize the non-vanishing components of the gauge field by the Ansatz

\[ A_r = \frac{H_1(r, \theta)}{r} \frac{i\tau_r^{(n)}}{2}, \quad A_\theta = (1 - H_2(r, \theta)) \frac{i\tau_\theta^{(n)}}{2}, \]  

\[ A_\varphi = -n \sin \theta [H_3(r, \theta) \frac{i\tau_\varphi^{(n)}}{2} + (1 - H_4(r, \theta)) \frac{i\tau_\varphi^{(n)}}{2}] \]  

(4)
and the Higgs field by

\[ \phi = \phi_1(r, \theta) \frac{i\tau_\varphi^{(n)}}{2} + \phi_2(r, \theta) \frac{i\tau_\theta^{(n)}}{2} \]  

(5)
The matrices \( \tau_i^{(n)} \) are defined by

\[ \tau_r^{(n)} = \sin \theta n \varphi \sigma_1 + \sin \theta n \varphi \sigma_2 + \cos \theta \sigma_3 , \]  

\[ \tau_\theta^{(n)} = \cos \theta n \varphi \sigma_1 + \cos \theta n \varphi \sigma_2 - \sin \theta \sigma_3 , \]  

\[ \tau_\varphi^{(n)} = -\sin n \varphi \sigma_1 + \cos n \varphi \sigma_2 , \]  

where \( \sigma_i \) denote the Pauli matrices and the integer \( n \) refers to the winding number which equals the topological charge of the solutions. This Ansatz (based on the Ansatz used in Ref. [10]) allows for axially symmetric solutions with \( n > 1 \) and reduces to the Ansatz for spherically symmetric solutions if \( n = 1 \). Due to a local \( U(1) \) invariance of the Ansatz it is necessary to impose a gauge fixing condition which guarantees the uniqueness of solutions. Here we take the gauge condition \( r \partial_r H_1 - \partial_\theta H_2 = 0 \) [13].

The energy of a solution with winding number \( n \)

\[ E_n(\lambda) = \int \mathcal{H} d^3 r \geq 4\pi Q \eta \]  

(6)
is bounded from below by the magnetic flux

\[ \varepsilon_{ijk} \int Tr(F_{ij} D_k \phi) d^3 r = \varepsilon_{ijk} \int Tr(\phi F_{ij}) dS_k = 4\pi Q \eta , \]  

(7)
where \( Q \) represents the topological charge, and \( Q = n \) for the Ansatz given above, provided that the solutions satisfy the appropriate boundary conditions (stated in the following).

The boundary conditions follow from the requirements of finite energy and analyticity as well as symmetry. They are given by

\[ H_1(0, \theta) = H_3(0, \theta) = \phi_1(0, \theta) = \phi_2(0, \theta) = 0 , \quad H_2(0, \theta) = H_4(0, \theta) = 1 \]
at the origin,

\[ H_i(\infty, \theta) = \phi_2(\infty, \theta) = 0, \quad \phi_1(\infty, \theta)/\eta = 1 \]

at infinity,

\[ H_1(r, 0) = H_3(r, 0) = \phi_2(r, 0) = 0, \quad \partial_\theta H_2(r, 0) = \partial_\theta H_4(r, 0) = \partial_\theta \phi_1(r, 0) = 0 \]
on the \( z \)-axis and

\[ H_1(r, \pi/2) = H_3(r, \pi/2) = \phi_2(r, \pi/2) = 0, \quad \partial_\theta H_2(r, \pi/2) = \partial_\theta H_4(r, \pi/2) = \partial_\theta \phi_1(r, \pi/2) = 0 \]
on the \( \rho \)-axis.

![Figure 1](image-url)

**Fig. 1** The normalized energy per unit charge as a function of \( \lambda \) is shown for \( n = 1 \) (dashed line) and \( n = 2 \) (solid line). The inlet shows the excess of the energy per unit charge for small values of \( \lambda \) for \( n = 1 \) (*) , \( n = 2 \) (+) and \( n = 3 \) (×). The straight lines with slope 1/2 are fitted to the data values corresponding to \( \lambda = 1 \times 10^{-5} \) for \( n = 1 \) and \( n = 2 \) and to \( \lambda = 2 \times 10^{-5} \) for \( n = 3 \).

The partial differential equations for the functions \( H_i(r, \theta) \) and \( \phi_i(r, \theta) \) are obtained by inserting the Ansatz into the Euler-Lagrange equations of the static Hamiltonian with gauge fixing term added. We have solved the partial differential equations numerically, subject to the above boundary conditions for the winding numbers \( n = 1 - 3 \) and for a large range of values of the Higgs coupling constant \( \lambda \).

In Fig. 1 we show the normalized energy per unit charge \( \tilde{E}_n(\lambda)/n = E_n(\lambda)/(4\pi n\eta) \) for \( n = 1 \) and \( n = 2 \) as a function of \( \lambda \in [10^{-5}, 10^3] \). In the BPS limit \( \lambda = 0 \), both curves coincide since \( \tilde{E}_n(0)/n = 1 \) for all \( n \) (self-dual solutions). For \( \lambda > 0 \) both curves increase monotonically, with \( \tilde{E}_2(\lambda)/2 > \tilde{E}_1(\lambda) \) in agreement with the results of Ref. [8]. In the limit of infinitely large Higgs coupling constant the curves for \( n = 1 \) and \( n = 2 \) converge to the constants \( \tilde{E}_1(\infty) = 1.787 \) [9,14] and \( \tilde{E}_2(\infty)/2 = 2.293 \), respectively. Thus the two curves coincide only at \( \lambda = 0 \), implying the existence of only a repulsive phase.

In the limit \( \lambda \to \infty \) the Higgs potential acts like a constraint, forcing the norm of the Higgs field

\[ |\phi(\vec{r})| = \sqrt{-2Tr(\phi^2(\vec{r}))} = \sqrt{\phi_1^2(r, \theta) + \phi_2^2(r, \theta)} \]
From this unacceptable behaviour at the origin, we conclude that these solutions are unphysical. For \( n = 1 \) the function \( \phi_2 \) vanishes and the Higgs field is parametrised by a single function which corresponds to its norm. In the limit \( \lambda \to \infty \), in this case the Higgs field itself is completely fixed to its vacuum expectation value for all \( r \) except for \( r = 0 \). For \( n > 1 \) on the other hand, \( \phi_2(r, \theta) \) is non-trivial and only the norm of the Higgs field is fixed to a constant for all \( r \) except for \( r = 0 \) in the limit \( \lambda \to \infty \). Thus the Higgs field itself remains non-trivial in this limit. It can be parametrised by the phase function

\[
\Phi(r, \theta) = \arctan(\phi_2(r, \theta)/\phi_1(r, \theta)) \, .
\]

As in the \( n = 1 \) case, the contributions of the Higgs potential to the energy are negligible in this limit, and consequently the energy becomes independent of the Higgs coupling constant.

The gauge field functions do not develop a discontinuity in the limit of infinitely large Higgs coupling constant. Indeed, in this limit they can be identified with solutions of the same model with modified boundary conditions for the Higgs field at the origin, which allow for Higgs fields with constant norm. We have constructed these solutions with \( \phi \equiv \eta \) numerically for \( n = 1 \) and \( n = 2 \). Denoting these solutions by \( \tilde{H}_i, \tilde{\phi}_i \), etc., their gauge field functions \( \tilde{H}_i \) correspond to the limiting functions of the gauge fields of the multimonopoles, and, for \( n > 1 \), their phase function \( \Phi \) corresponds to the limiting function of the phase function \( \Phi \) for all \( r \) except for \( r = 0 \). Their energy \( \tilde{E}_n \) is finite and coincides with the energy \( E_{\infty}(\lambda \to \infty) \) of the multimonopoles. At the origin, however, the energy density diverges and the gauge field functions are not analytic. Also for \( n > 1 \), the phase function is multivalued at the origin, i.e., \( \Phi(r = 0, \theta) \) is a non-trivial function of \( \theta \).

Expanding the differential equation for the phase function at \( r = 0 \) we find

\[
\hat{\Phi}(r = 0, \theta) = 2 \arctan(\tan^{n}(\theta/2)) - \theta \, .
\]

From this unacceptable behaviour at the origin, we conclude that these solutions are unphysical.

Let us next consider the monopoles for small values of the Higgs coupling constant. Near \( \lambda = 0 \) the normalized energy per unit charge can be written in the form

\[
\tilde{E}_n(\lambda)/n \sim 1 + c_n \lambda^n \, .
\]

We show \( \tilde{E}_n(\lambda)/n - 1 \) for \( n \leq 3 \) and small values of \( \lambda \) in Fig. 1 (inlet). The figure suggests that the exponent \( \alpha \) does not depend on the winding number \( n \), and that it has the value 1/2 and not 1 as might have been expected naively from the static Hamiltonian. This result is very natural, however, since \((\lambda/2)^{1/2}\) is just the mass of the Higgs field (in dimensionless units). The physical interpretation is therefore, that the increase of the energy with increasing Higgs coupling constant \( \lambda \) (for small \( \lambda \)) is proportional to the mass of the Higgs field. We remark, that the increase of the energy for small values of \( \lambda \) cannot be obtained by substituting the Prasad-Sommerfield solution into the energy functional. In that case, due to the power law decay \((\sim 1/r)\) of the Higgs field the integration of the Higgs potential would diverge. By considering a Taylor expansion of \( \tilde{E}_n(\lambda)/n - 1 \) in terms of \( \lambda \) at \( \lambda = 0 \) and taking into account that the gauge fields and Higgs field depend on \( \lambda \) implicitly, we find that the first term is again the divergent integral of the Higgs potential evaluated with the Prasad-Sommerfield solution. This implies, that even for arbitrarily small values of \( \lambda \) the model with finite \( \lambda \) cannot be treated as a perturbation of the model with \( \lambda = 0 \).

Let us now compare with the Weinberg-Salam model of the electroweak interactions, where the Higgs field is in the fundamental representation of \( \text{SU}(2) \). The non-perturbative solutions of this model are saddlepoints of the energy functional (sphalerons and multisphalerons). Here sphalerons

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\(^{1}\)This is also true for \( n = 1 \)
represent the top of the energy barrier between adjacent vacua \[15,13\], and, like multimonopoles, multisphalerons are also characterised by their winding number \(n\). Numerical analysis shows, that for small values of the Higgs coupling constant, the energy per winding number \(n > 1\) is smaller than the energy of sphalerons \((n = 1)\), whereas for large values of the Higgs coupling constant it is larger, with equality occurring roughly at \(M_H \approx M_W\) [16]. Because of their instability, however, one cannot speak of attractive and repulsive phases of sphalerons.

Finally we remark that, for \(n > 2\), multimonopoles need not be axially symmetric. Indeed, in the BPS limit also multimonopole solutions with only discrete symmetries have been found for the higher topological charges \[17\]. It remains an open challenge to construct such solutions also for finite Higgs coupling constant. In particular, a numerical analysis would reveal, which of the possible shapes of a multimonopole with a given higher topological charge would be energetically favoured.

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